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Homotopical Patch Theory

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Abstract
Homotopy type theory is an extension of Martin-Löf type theory, based on a correspondence with homotopy theory and higher category theory. The propositional equality type becomes proof-relevant, and acts like paths in a space. Higher inductive types are a new class of datatypes which are specified by constructors not only for points but also for paths. In this paper, we show how patch theory in the style of the Darcs version control system can be developed in homotopy type theory. We reformulate patch theory using the tools of homotopy type theory, and clearly separate formal theories of patches from their interpretation in terms of basic revision control mechanisms. A patch theory is presented by a higher inductive type. Models of a patch theory are functions from that type, which, because function are functors, automatically preserve the structure of patches. Several standard tools of homotopy theory come into play, demonstrating the use of these methods in a practical programming context.

1. Introduction
Martin-Löf’s intensional type theory (MLTT) is the basis of proof assistants such as Agda [23] and Coq [8]. Homotopy type theory is an extension of MLTT based on a correspondence with homotopy theory and higher category theory [3, 10, 12, 13, 25, 32, 33]. In homotopy theory, one studies topological spaces by way of their points, paths (between points), homotopies (paths or continuous deformations between paths), and so on. In type theory, a space corresponds to a type A. Points of a space correspond to elements a, b : A. Paths in a space are modeled by elements of the identity type (propositional equality), which we notate p : a =b A. Homotopies between paths p and q correspond to elements of the iterated identity type p =_{a =b A} q. The rules for the identity type allow one to define the operations on paths that are considered in homotopy theory. These include identity paths refl : a = a (reflexivity of equality), inverse paths ! p : b = a when p : a = b (symmetry of equality), and composition of paths g ◦ p : a = c when p : a = b and q : b = c (transitivity of equality), as well as homotopies relating these operations (for example, refl ◦ p = p), and homotopies relating these homotopies, etc. This correspondence has suggested several extensions to type theory. One is Voevodsky’s univalence axiom [16, 34], which describes the path structure of the universe (the type of small types). Another is higher inductive types [24, 25, 30], which are a new class of datatypes, specified by constructors not only for points but also for paths. Higher inductive types were originally introduced to allow basic topological spaces such as circles and spheres to be defined in type theory, and have had significant applications in a line of work on using homotopy type theory to give computer-checked proofs in homotopy theory [18, 19, 22, 32].

The computational interpretation of homotopy type theory as a programming language is a subject of active research, though some special cases have been solved, and work in progress is promising [4, 5, 21, 31]. The main lesson of this work is that, in homotopy type theory, proofs of equality have computational content, and can influence how a program runs. This suggests investigating whether there are programming applications of computationally relevant equality proofs. Some preliminary applications have been investigated. For example, Licata and Harper [20] apply ideas related to homotopy type theory to modeling variable binding. Altenkirch [2] shows that containers [1] in homotopy type theory can be used to represent more data structures than in MLTT, such as sets and bags. However, at present, the programming side is less well-developed than the mathematical applications.

In this paper, we present an extended example of applying higher inductive types in programming. The example we consider is patch theory [6, 9, 14, 15, 27, 29], as developed for the version control system Darcs [29]. Intuitively, a patch is a syntactic representation of a function that changes a repository. A patch ("delete file f") applies in certain repository contexts (where the file f exists), and results in another repository context (where the file f no longer exists)—so the contexts act as types for patches. Patches are closed under identity (a no-op), composition (sequencing), and inverses (undo). These satisfy certain general laws—composition is associative; inverses compose to the identity. Moreover, there are domain-specific patch laws about the basic basic patches ("the order of edits to independent lines of a file can be swapped"). The semantics of a patch explains how to apply it to change a repository.
A pseudocommutation operation on patches allows for merging divergent edits to a repository \([11]\); this is an example of a syntactic transformation on patches. The semantics and syntactic transformations satisfy certain laws, such as the fact that applying a composition of patches is the same as the composition of applying the patches, and that the patches produced by merging two edits leave the two repositories in the same state.

Building on this work, we develop patch theory in the context of homotopy type theory, using paths to model aspects of patch theory. First, we use paths to model the laws that patches and transformations must satisfy. However, we go further than this, and model \textit{patches themselves as paths}, making use of the proof-relevant notion of equality in homotopy type theory. We make an explicit distinction between patch theories and \textit{higher inductive types}. In Section 3, we review Darcs and begin to consider applications of this new class of datatypes.

The goal of this paper is to introduce higher inductive types and composition (e.g. sending composition of patches to composition of paths) in a practical programming setting. For example, our first examination of patch theory is actually the circle. Defining the semantics of a patch theory is analogous to calculations of homotopy groups in homotopy type theory. We make an explicit distinction between patch theories and \textit{higher inductive types}, which are modeled by proofs of equality in identity types. For example, there are homotopies expressing that the path operations satisfy the group(oid) laws:

\[
\text{refl} \circ p = p = p \circ \text{refl},
\]

Any simply-typed function \(f : A \rightarrow B\) determines a function \(ap f : x = y \rightarrow f(x) = f(y)\)

that takes paths \(x \rightsquigarrow y\) to \(f(x) \rightsquigarrow f(y)\). Logically, this expresses that propositional equality is a congruence; homotopically, it expresses that any function has an action on paths. \(ap f\) preserves the path operations, in the sense that there are homotopies

\[
\begin{align*}
\text{ap f} (\text{refl}(a)) & = \text{refl}(f a) \\
\text{ap f} (! p) & = ! (\text{ap f} p) \\
\text{ap f} (p \circ q) & = (\text{ap f} p) \circ (\text{ap f} q)
\end{align*}
\]

For a family of types \(B : A \rightarrow \text{Type}\) and a dependent function \(f : (x : A) \rightarrow B(x)\), there is

\[\text{apd} : (p : x = y) \rightarrow \text{PathOver B p f x} (f y)\]

\(\text{PathOver}\) represents a path in the dependent type \(B\) between \(b1 : B(a1)\) and \(b2 : B(a2)\) that "lies over" \(p : a1 = a2\); logically, it is a kind of heterogeneous equality \([26]\) relative to a particular path relating the type of its endpoints. For \(\text{apd}\), heterogeneous equality is necessary because \(f x : B(x)\) whereas \(f y : B(y)\).

### 2.2 \(n\)-types

A type \(A\) is a \textit{set}, or 0-type, iff any two paths in \(A\) are equal—

for any two elements \(m,n : A\), and any two proofs \(p,q : m = n\), there is a homotopy \(p = q\). Similarly, a type is a 1-type if any two paths between paths are equal. A type is a proposition, or \((-1)-\)type, iff any two elements are equal. A type is contractible if it is a proposition and moreover it has an element.

### 2.3 Univalence

Writing \(\text{Type}\) for a type of (small) types, Voevodsky’s univalence axiom states that, for sets \(A\) and \(B\), the paths \(A \simeq B\) are given by bijections between \(A\) and \(B\). That is, define \text{Bijection} \(A, B\) to be the type of quadruples

\[\begin{align*}
(f : A \rightarrow B, g : B \rightarrow A, \\
p : (x : A) \rightarrow g (f x) = x, q : (y : B) \rightarrow f (g y) = y)
\end{align*}\]

consisting of two functions that are mutually inverse up to paths. Then one consequence of univalence is that there is a function \(ua : \text{Bijection} A, B \rightarrow A = B\)

which says that a bijection between \(A\) and \(B\) determines a path between \(A\) and \(B\). The force of this is to stipulate that \textit{all constructions respect bijection}; for example, if \(C[x]\) is a parametrized type (e.g. \(C\) could be \text{List}, \text{Tree}, \text{Monoid}, etc.), then given a bijection \(b : \text{Bijection} A, B\), we have

\[ua : \text{Bijection} A, B \rightarrow A = B\]

\(\forall x, y. b(x) = b(y)\).

1 There is an unfortunate terminological coincidence here: “Patch theory” means “the study of patches,” just as “group theory” is the study of groups. “A patch theory” means “a specific language of patches,” just as “a theory in first-order logic” is a specific collection of terms and formulae.

2 Composition is in function-composition order, \((p : y = x) \circ (q : x = y)\).

3 For non-sets, univalence requires a notion of equivalence that generalizes bijection. However, here we will only use it for sets.
ap C (ua b) : C[A] = C[B]
which is a bijection between C[A] and C[B]. In plain MLTT, one would need to spell out how a bijection lifts to a bijection on lists or monoids; with univalence, this lifting is given by a new generic program in the form of ap. This generic program is one of the sources of computational applications of homotopy type theory.

We can define the identity, inverse, and composition of bijections directly (focusing on the underlying functions):

reflb : Bijection A A
reflb = (\ (x -> x) , \ (x -> x) , ... )

1b : Bijection A B -> Bijection B A
1b (f,g,p,q) = (g,f,q,p)

\ob_ : Bijection B C -> Bijection A B -> Bijection A C
(f1,g1,p1,q1) \ob (f2,g2,p2,q2) = (f1 . f2, g2 . g1, ...)

Applying path operations to univalence is homotopic to applying the corresponding operations to bijections:

ua reflb = refl
ua reflb = (ua 1b)
ua b1 o ua b2 = ua (b1 ob b2)

When p : A = B, we write coe p : A -> B for the function, defined by identity type elimination, that "coerces" along the path p. coe is functorial, in the sense that

coe refl x = x
coe (p o q) x = coe p (coe q x)
coe p is a bijection, with inverse 1p; we write coe-biject p : Bijection A B when p : A = B. The univalence axiom additionally asserts that there is a computation rule

coe (ua (f,g,p,q)) x = f(x)

That is, coercing along a path constructed by univalence applies the given bijection. Because ! (ua (f,g,p,q)) = ua (1b (f,g,p,q)), we also have that

coe (! (ua (f,g,p,q))) x = g x

Because of these rules, in the presence of univalence, paths can have non-trivial computational content. A bijection (f,g,p,q) determines a path ua(f,g,p,q), and coercing along this path applies f. Thus, two different bijections (f,g,p,q) and (f',g',p',q') determine two paths ua(f,...) and ua(f',...) that behave differently when coerced along.

2.4 Higher Inductive Types

Ordinary inductive types are specified by generators; for example, the natural numbers are have generators zero and successor: zero : Nat and succ : Nat -> Nat. Higher-dimensional inductive types (or just higher inductive types) generalize inductive types by allowing generators not only for points (terms), but also for paths. For example, one might draw the circle like this:

This drawing has a single point, and a single non-identity loop from this base point to itself. This translates to a higher inductive type with two generators:

base : Circle
loop : base = base

base is like an ordinary constructor for an inductive type, which takes no argument. loop generates a path on the circle, which is an element of the identity type base = Circle base—think of this as "going around the circle once clockwise". The paths of higher inductive types are constructed from generators, such as loop, using the path operations described above. The intuition is that refl stays still at the base point, whereas loop o loop goes around the circle twice clockwise, and ! loop goes around the circle once counter-clockwise.

2.4.1 Circle Recursion

The fact that the type of natural numbers is inductively generated by zero and successor is encoded in its elimination rule, primitive recursion. Primitive recursion says that to define a function f : Nat -> X, it suffices to map the generators into X, giving x0 : X and x1 : X -> X. Then the function f satisfies the equations

f zero = x0
f (succ n) = f n

Similarly, the circle is inductively generated by base and loop, so to define a function from the circle into some other type, it suffices to map these generators into that type, which means giving a point and a loop in that type. That is, to define a function f : Circle -> X, it suffices to give b' : X and l' : b' -> b'.

For an inductive type, the β-reduction rules state that applying the elimination rule to a generator computes to the corresponding branch. Thus, by analogy, the computation rules for the circle should say that, for a function f : Circle -> X that is defined by giving b' and l',

f base = b'
f loop = l'

However, the second equation does not quite make sense, because f is a function Circle -> X but loop is a path on the circle. It is therefore necessary to use ap (defined above) to denote f's action on paths:

ap f loop = l'

This computation rule preserves types because its left-hand side is a proof of f base = f base, which by the first computation rule equals b' = b', which is the type of loop'.

Example 2.1. As a first example, we write a function to "reverse" a path on the circle—to send the path that goes around the circle n times clockwise to the path that goes around the circle n times counter-clockwise, and vice versa. Because a path on the circle is represented by the identity type base = base, we seek a function

revPath : (base = base) -> (base = base)
such that, for example, revPath (loop o loop) = ! loop o ! loop and revPath (! loop o ! loop) = loop o loop. We could define this function by revpath p = ! p, but because the goal is to illustrate circle recursion, we instead give an equivalent definition that analyzes p.

To define this function using circle recursion, we need to rephrase the problem as constructing a function Circle -> X for some type X. The key idea is to define a function rev : Circle -> Circle and then to define revPath to be ap rev. That is, to define a function on the paths of the circle, we define a function on the circle itself, whose action on paths is the desired function. In this case, we define

rev : Circle -> Circle
rev base = base
ap rev loop = ! loop

revPath p = ap rev p

One technical issue about higher inductive types is whether the computation rule ap f loop = l' is a definitional equality or a
path/propositional equality. Current models and implementations justify only the latter, so we will take it to be a propositional equality. When we illustrate how programs run in this paper, we will do it by giving a sequence of propositional equalities relating a program to a value, so the rule still functions as a "computation" step—as do the rules mentioned above, which state that \( ap \) behaves homomorphically on paths built from the group operations. For example, one can calculate

\[
\begin{align*}
\text{revPath}(\text{loop} \circ \text{loop}) &= \text{ap rev}(\text{loop} \circ \text{loop}) \\
&= (\text{ap rev loop}) \circ (\text{ap rev loop}) \\
&= \text{! loop} \circ \text{! loop}
\end{align*}
\]

Just as the recursion principle for the natural numbers can be generalized to an induction principle, the full form of the circle elimination rule is a principle of "circle induction": to define a dependent function \( f : (x : \text{Circle}) \rightarrow C(\text{base}) \) and \( l' : \text{PathOver C loop b' b'} \), it suffices to give \( b' : C(\text{base}) \) and \( l' : \text{PathOver C loop b' b'} \). We refer the reader to [22, 32] for topological intuition.

### 3. Patch Theory

The developers of the Darcs distributed revision control system [29] have proposed a partially formalized theory of versioned repositories, called patch theory [6,8,11,14,15,27], which specifies properties of patches under operations such as composing, reverting and merging. Patch theory provides a general framework for describing the behavior of various version control systems.

Here, we formulate Darcs patch theory in the context of homotopy type theory, clearly separating its semantic aspects (what repositories and patches are) from its purely algebraic properties (how they behave). We describe how to present a theory of version control as a higher inductive type whose structure encodes both the generic aspects common to all such theories (as set out in Darcs patch theory) as well as the aspects particular to a given theory, specifying the types of patches available and the specialized laws that they obey. In the following sections, we illustrate this method with a number of examples.

In Darcs patch theory, each patch has well-defined domain and codomain contexts, which represent, respectively, the states of the repository on which a patch is applicable, and the states resulting from such an application. For example, a patch that deletes a file is applicable only to states in which the file exists, and results in a state in which it does not. In addition, patches respect certain laws that relate sequences of patches to equivalent sequences of patches—equivalent, in the sense that the two sequences have the same effect on the state of a repository.

One of the properties of patches in Darcs patch theory is that they are all invertible. Applying the inverse of a patch after applying the patch itself (\(<p \cdot \text{!p}> \) undoes the effect of the patch, leaving the repository in its original state. This seems very natural. But it is also possible to apply the inverse patch first (\(<\text{!p} \cdot p> \), to an appropriate repository state, and this composition should also be equivalent to doing nothing. This seems less natural, and forces us to use some care when defining contexts and patches. The reason that Darcs patch theory requires inverses— as opposed to just retractions—for patches is that doing so is the basis for the Darcs approach to reordering of patches, a critical ingredient in defining the merge of two disparate patches.

However, the requirement that patches be invertible fits very nicely into homotopy type theory, where the path structure of types is undirected. We exploit this coincidence to encode theories of version control as higher inductive types. Contexts are represented as pointed types of a type. Patches are represented as paths between points, with the path operations \( \text{refl} \) and \( p \cdot q \) and \( ! p \) representing a no-op patch, patch composition, and undo, respectively. Patch laws are represented as 2-dimensional paths between paths. Patch laws are necessary to reason about syntactic transformations on patches, such as an optimizer, which should compute a patch equal to the one it is given, or a merge, which given two divergent edits, should compute two additional patches that reconcile them.

Encoding a theory of patches as a higher inductive type immediately imposes some reasonable laws on patches, namely, the groupoid laws. But a theory of version control is more than an arbitrary groupoid. We would like a version control system to provide operations such as the "cherry picking" of only selected patches from a sequence and the merging of divergent patches. In Darcs patch theory these operations are derived from an operation known as \( \text{pseudocommutation} \) that reorders adjacent patches. Intuitively, a pair of composable patches \( f \cdot g \) pseudocommutates with a parallel pair of composable patches \( h \cdot k \), if \( h \) has the same effect as \( g \), but in the domain context of \( f \), whereas \( k \) has the same effect as \( f \) but in the codomain context of \( h \). In general there appears to be no canonical way to pseudocommute a composable pair of patches. Instead, we give some criteria (adapted from Darcs) that a choice of a pseudocommutation of a composable pair of patches should satisfy.

We define \( \text{pseudocommutation} (pc) \) to be a function on composable pairs of paths that yields a parallel pair of composable paths such that:

- the two compositions are equal:
  \[
  pc(f,g) = (h,k) \implies f \cdot g = h \cdot k,
  \]
- the function is an involution:
  \[
  pc(f,g) = (h,k) \implies pc(h,k) = (f,g),
  \]
- the function respects path inverses:
  \[
  pc(f,g) = (h,k) \implies pc(g,\text{!k}) = (\text{!f},h).
  \]

Because of the involution requirement, we adopt the symmetric notation \( "(f,g) \leftrightarrow (h,k)" \) to mean that \( pc(f,g) = (h,k) \).

Each of these properties can be expressed diagrammatically. Composition equality means that the two composites form a commutative square, involution means that the square can be reflected through its start and end points, and respect for path inverses means that the square can be rotated through an edge:

\[
\begin{array}{ccc}
A & \overset{g}{\rightarrow} & B \\
\downarrow \text{refl} & \downarrow \text{!} & \downarrow \text{rot} \\
D & \overset{f}{\rightarrow} & C
\end{array}
\]

Together, these properties ensure that any isometry of a pseudocommuting square (reversing edge orientations as needed) is also pseudocommuting. Based on their diagrammatic representations we will refer to the latter properties as reflection and rotation, respectively. The reflection and rotation properties are in fact two-way implications because the composition of two reflections, respectively, four rotations, in the identity function. Thus a reflection is also an unreflection and an unrotation is just the composition of three rotations. Note that it is always possible to define a pseudocommutation function, because the identity is one such.

Using pseudocommutation we can implement an abstract merge operation. Given a span \( (h,f) \), we seek a cospan \( (k,g) \) such that

\[\text{4} \text{ In this section we composition in diagrammatic order \("f \cdot g\" for g \cdot f\) to better match the diagrams to follow. \]

\[\text{5} \text{ or sometimes just \"commutation\", though this term is technically inaccurate. \]
\[(f, g) \leftrightarrow (h, k):\]
\[\begin{array}{c}
\text{A} \\
\text{f} \\
\text{B} \\
\text{C} \\
\text{\rotate (g)} \\
\text{D} \\
\text{k} \\
\end{array} \]

We can regard this span as a composition either as \(!h \cdot f\) or as \(!f \cdot h\).

In both cases, if we call the result of applying pseudocommutation \(\text{merge}(f, g)\), then:

\[(!h, f) \leftrightarrow (k, g) \leftrightarrow (f, g) \leftrightarrow (h, k)\]

In the second case, if we call the result of applying pseudocommutation \(\text{merge}(g, !k)\), then:

\[(f, h) \leftrightarrow (g, !k) \leftrightarrow (h, k) \leftrightarrow (f, g) \leftrightarrow (h, k)\]

So we may compute \(g\) and \(k\) by pseudocommuting either \((!h, f)\) or \((f, h)\), and the fact that reflection and rotation are invertible ensures that the result is well-defined.

Pseudocommutation gives us a merge operation that is well-defined, symmetric (\(\text{merge}(h, f) = \text{merge}(f, h)\)) and reunites the two branches of a span, but this is not enough to guarantee that we get the merges that we might expect. For example, if we take \(\text{pseudocommutation to be the identity function then the merge of the two branches of a span, but this is not enough to guarantee that we get the merges that we might expect. For example, if we take \(\text{pseudocommutation to be the identity function then the merge of}
\]
\(\text{span (}
\)
\(\text{f, h, f, k)}\)
\(\text{and}
\)
\(\text{h, k)}\)
\(\text{that is, the induced merge}
\]
\(\text{operation simply undoes both changes, reverting the repository}
\)
\(\text{to the state from which it diverged (which is, after all, a valid}
\)
\(\text{reconciliation of two competing changes, if not the most desirable}
\)
\(\text{one in general!}
\)
\(\text{Next, we present several examples of patch theories as higher}
\)
\(\text{inductive types. We show how to implement their semantics, and}
\)
\(\text{additionally some examples of patch optimization and merging, to}
\)
\(\text{illustrate syntactic transformations.}
\)

4. Patches as Paths

First, we define a very simple language of patches, to illustrate the basic technique: we take the repository to be a single integer, and the patches to be adding or subtracting some number \(n\) from it. Because all patches apply in any context, we need only a single patch context, which we call \(\text{num}\). Patches will then be represented as patches \(\text{num} = \text{num} = \text{num} = \text{num}\).

This is, of course, just a renaming of the circle!

**Remark 4.1.** By presenting it using a higher inductive type, the patch theory automatically includes identity, inverses, and composition. Without higher inductive types, one would need syntax constructors for identity, composition, and inverses; e.g. using a datatype as follows:

```haskell
data Patch where
  add1 : Patch
  id : Patch
```

Then, to achieve the correct equational theory of patches, one would need to impose the group laws on this type; this could be done using a quotient type \([7]\) to assert that

```haskell
assoc : compose p (compose q r) = compose (compose p q) r
invr : compose (compose p id) = p
invl : compose (compose id p) = p
unitr : compose (compose p) = id
unitr : compose (compose p) = id
```

By representing a patch theory as a higher inductive type, the group operations and laws are provided by the ambient type theory, so the definition need not include these boilerplate constructors.

4.1 Interpreter

Next, we define an interpreter, which explains how to apply a patch to a repository. Because the intended semantics is that the repository is an integer, we would like to interpret the repository context \(\text{num as the type Int of integers. Because patches are invertible, we would like to interpret each patch as an element of the type Bijection Int Int.}

**Remark 4.2.** To build intuition, consider writing the interpreter “by hand”, for the quotient type Patch defined in Remark \([4, 7]\) which includes constructors for identity, inverse, and composition.

We would first define:

```haskell
interp : Patch \rightarrow Bijection Int Int
interp add1 = successor
interp id = id
interp (compose p1 p2) = interp p1 ob interp p2
interp (inv p) = !b (interp p)
```

where \(\text{successor : Bijection Int Int is the bijection given by}
\)
\(\text{(} \lambda x \rightarrow x + 1, \ \lambda x \rightarrow x - 1, \ldots\text{...)}\), then, to show that this definition is well-defined on the quotient of patches by the group laws, we would need to do a proof with 5 cases for the 5 group laws, where in each case we appeal to the inductive hypotheses and the corresponding group law for bijections.

Returning to our higher-inductive representation of patches, we define the interpreter using the recursion principle for \(\text{R, which is of course the same as circle recursion, as discussed in Section [2].}

We want to interpret each point of \(\text{R, which represent a repository context, the type of repositories in that context, and each path as a bijection between the corresponding types. In this case, that means we would like to interpret num as Int and add1 as the successor bijection. R-recursion says that to define a function \(f : \text{R} \rightarrow \text{X}, it suffices to find a point \(x_0 : \text{X} and a loop \(p : x_0 = x_0\). Thus, we can represent the interpretation by a function \(R \rightarrow \text{Type}, because a point of Type is a type, and a loop in Type is, by univalence, the same as a bijection! This motivates the following definition:}

```haskell
I : R \rightarrow Type
I num = Int
interp p = coe-bijection (ap I p)
```

Up to propositional equality, this definition satisfies the defining equations of \(\text{interp as defined in Remark [4, 2]. First, we can calculate that}

```haskell
interp add1 = successor
```

```haskell
interp add1 = coe-bijection (ap I add1) [definition]
interp add1 = coe-bijection (ua successor) [ap I on add1]
interp add1 = successor [coe on ua successor]
```

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using the computation rules for \( \text{ap} \) \( I \) on \( \text{add} \) (from higher inductive elimination) and \( \text{coe} \) on \( u a \) \( b \) (from univalence).

Moreover, even though we have not included them as equations, it takes path operations to the corresponding operations on bijections. For example,

\[
\begin{align*}
\text{interp} (p \circ q) &= \text{coe-biject} (\text{ap} I (p \circ q)) \\
                      &= \text{coe-biject} (\text{ap} I p \circ \text{ap} I q) [\text{ap on } q] \\
                      &= \text{ interp } p \circ \text{ interp } q
\end{align*}
\]

\( \text{interp refl} = \text{idb} \) and \( \text{interp } (! p) = ! b \text{ interp } b \) are similar. That is, the semantics is functorial.

For example, if we apply a patch \( \text{add1} \circ \text{add1} \) to a repository whose contents are \( 0 \), we have

\[
\begin{align*}
(\text{interp} (\text{add1} \circ (! \text{add1}))) &\circ 0 \\
&= ((\text{interp} \text{add1}) \circ \text{ interp } (\text{! add1}) \circ 0 \\
&= (\text{successor} \circ \text{! successor}) 0 \\
&= \text{successor \ ! successor} \\
&= \text{successor \ ! successor} \\
&= \text{successor \ ! successor} \\
&= \text{successor \ ! successor} \\
&= 0
\end{align*}
\]

Comparing this definition of \( \text{interp} \) with Remark 4.2 we see that the recursion principle for the higher-inductive representation of patches provides an elegant way to express the semantics of a patch theory, where much of the code in Remark 4.2 is provided “for free”. We needed to give only the key case for \( \text{add1} \), and not the inductive cases for the group operations—the semantics of the basic patches is automatically lifted functionally to the patch operations. Moreover, we did not need to prove that bijections satisfy the group laws—this fact is necessary for the univalence axiom to make sense, so it is effectively part of the metatheory of homotopy type theory, rather than our program. Moreover, the example illustrates that \textit{univalence can be used to extract computational content from a path}, by mapping the path into a path in the universe, which by univalence can be given by a bijection.

Because \( R \) is the circle, one may wonder about the topological meaning of this interpreter. In fact, the type family \( I \) defined here is called the \textit{universal cover of the circle}, and is discussed further in \cite{Coq, Coq-book}. \( \text{interp} \) computes what is called the \textit{winding number} of a path on the circle, which can be thought of as a normal form that counts how many times that path goes around the circle, after “detours” such as \( \text{loop} \circ ! \text{loop} \) have been reduced.

It is also worth noting that, although we were thinking of \( \text{num} \) as an integer and \( \text{add1} \) as successor, there is nothing forcing this interpretation of the syntax: we can give a sound interpretation \( I \) in any type with a bijection on it. For example,

\[
I^* : R \rightarrow \text{Bool}
I^* \text{ num} = \text{Bool}
I^* \text{ add1} = u a \text{ notb}
\]

where \( \text{notb} : \text{Bijection} \text{ Bool} \rightarrow (\text{not} , \text{ not} , \text{ ...}) \).

That is, we interpret the patches in \( \text{Bool} \) instead of \( \text{Int} \), and we interpret \( \text{add1} \) as adding \( 1 \) modulo \( 2 \). This semantics satisfies additional equations that are not reflected in the patch theory, such as

\[
\text{ap} I^* \text{ add1} \circ \text{ap} I^* \text{ add1} = u a (\text{notb} \circ \text{notb}) = \text{refl}
\]

In the next section we show how to augment a patch theory with equations such as these—but doing so would of course rule out the previous semantics in \( \text{Int} \), because adding \( 1 \) to an integer is not self-inverse. The equational theory of \( R \) is \textit{complete} for the interpretation as \( \text{Int} \), which in homotopy theory is known as the fact that the fundamental group of the circle is \( \mathbb{Z} \) (see \cite{Coq, Coq-book}).

4.2 Merge

As discussed in Section 3, merge follows from a pseudocommutation operation. Writing \( \text{Patch} \) for \( \text{doc} \rightarrow \text{doc} \), and specializing the interface to the setting where we have only one context, we need to implement the following:

\[
\begin{align*}
\text{ pcom } : \text{Patch} \times \text{Patch} &\rightarrow \text{Patch} \\
\text{square} : (f \ g h k : \text{Patch}) &\rightarrow \text{pcom} (f , g) = (h , k) \\
\text{ rot} : (f \ g h k : \text{Patch}) &\rightarrow \text{pcom} (f , g) = (h , k) \\
\text{reflect} : (f \ g h k : \text{Patch}) &\rightarrow \text{pcom} (f , g) = (f , g)
\end{align*}
\]

In this simple setting, any two patches commute, essentially because addition is commutative. Thus, we define

\[
\text{pcom}(f , g) = (g , f)
\]

For \( \text{rot} \), because we know \( h = g \) and \( k = f \), we need to show that \( \text{pcom}(g , f) = (f , g) \), which is true by definition. For \( \text{reflect} \), because \( h = g \) and \( k = f \), we need to show that \( \text{pcom}(g , f) = (f , g) \), which is also true by definition.

For \( \text{square} \), we need to prove that \( g \circ f = f \circ g \) for any two loops \( \text{num} = \text{num} \) on the circle. It is not immediately obvious how to do this, because homotopy type theory does not provide a direct induction principle for the loops in a type. That is, there is no built-in elimination rule that allows one to, for example, analyze a loop \( f \) as either \( \text{add1} \), or the identity, or an inverse, or a composition—because such a case-analysis would additionally need to respect all equations on paths, which differ from type to type. Instead, such induction principles for paths are \textit{proved} for each type from the basic induction principles for the higher inductive types—roughly analogously to how, for the natural numbers, course-of-values induction is derived from mathematical induction.

Moreover, proving these induction principles is sometimes a significant mathematical theorem. In homotopy theory, it is called calculating the homotopy groups of a space, and even for spaces as simple as the spheres some homotopy groups are unknown. However, we have developed some techniques for calculating homotopy groups in type theory \cite{Coq, Coq-book}, which can be applied here.

In fact, for this particular example, the calculation has already been done: we know that the fundamental group of the circle is \( \mathbb{Z} \). Specifically, we know that the type \( \text{num} = \text{num} \) of loops at \( \text{num} \), which we use to represent patches, is in bijection with \( \text{Int} \). That is, the integers give normal forms (“\( \text{add} \) \( x \), for \( x \in \mathbb{Z} \)” for patches in the above patch theory. This is proved by giving functions back and forth that compose to the identity. The function \( \text{num} = \text{num} \rightarrow \text{Int} \) is exactly \( \lambda p \rightarrow \text{ interp } p \circ 0 \), for \( \text{ interp } p \) as defined above. The function \( \text{repeat} : \text{Int} \rightarrow \text{num} = \text{num} \) is defined by induction on \( x \), such that

\[
\begin{align*}
\text{repeat} 0 &= \text{refl} \text{repeat} (+ n) &= \text{add1} \circ \text{add1} \circ \ldots \circ \text{add1} (n \text{ times}) \\
\text{repeat} (- n) &= \text{!add1} \circ \text{add1} \circ \ldots \circ \text{!add1} (n \text{ times})
\end{align*}
\]

The proof that these two functions are mutually inverse is described in \cite{Coq, Coq-book}. Moreover, they define a group homomorphism, which means that \( \text{repeat} (x + y) = \text{repeat} x \circ \text{repeat} y \).

The bijection between \( \text{num} = \text{num} \) and \( \text{Int} \) induces a derived induction principle, which says that to prove \( P(p) \) for all paths \( p : \text{num} = \text{num} \), it suffices to prove \( \text{P}(\text{repeat } n) \) for all integers \( n \)—any patch can be viewed as \( \text{repeat } n \) for some \( n \). Applying this (twice) to the goal \( f \circ g = g \circ f \), it suffices to show \( \text{repeat } x \circ \text{repeat } y = \text{repeat } y \circ \text{repeat } x \).
This is proved as follows:

\[
\begin{align*}
\text{repeat } x & \circ \text{repeat } y \\
= \text{repeat } (x + y) & [\text{group homomorphism}] \\
= \text{repeat } (y + x) & [\text{commutativity of addition}] \\
= \text{repeat } y & \circ \text{repeat } x
\end{align*}
\]

Thus, for this language of patches, the correctness of pseudo-commutation follows from the fact that the fundamental group of the circle is $\mathbb{Z}$—our first example of a software correctness proof being a corollary of a theorem in homotopy theory!

5. Patches with Laws

In this section, we consider a slightly more complex patch theory, to illustrate how patch laws are handled. In the intended semantics of this theory, the repository consists of one document with a fixed number of lines, and there is one basic patch, which modifies the string at a particular line. To fit such a basic into a framework of bijections, we take the patch $s1 \leftrightarrow s2 \odot i$ to mean “permute $s1$ and $s2$ at position $i$”. That is, applying this patch replaces line $i$ with $s2$ if it is $s1$, or with $s1$ if it is $s2$, or leaves it unchanged otherwise. We impose some equational laws on this patch—e.g., edits at independent lines commute.

5.1 Definition of Patches

This patch theory is represented by the following higher inductive type:

\[
\begin{align*}
R : & \text{Type} \\
\text{doc} : & R \leftrightarrow_\mathbb{Q} R. (s1 \leftrightarrow s2 : \text{String}) (1 : \text{Fin } n) \rightarrow (\text{doc} = \text{doc}) \\
\text{indep} : & (i \neq j) \\
& (s \leftrightarrow t @ i) \circ (u \leftrightarrow v @ j) \\
& = (u \leftrightarrow v @ i) \circ (s \leftrightarrow t @ j) \\
\text{noop} : & s \leftrightarrow s @ i \equiv \text{refl}
\end{align*}
\]

The interpretation $\text{interp} : (doc = \text{doc})$ is defined as follows:

\[
\begin{align*}
\text{interp} : & (doc = \text{doc}) \\
& \rightarrow \text{Bijection} (\text{Vec String } n) (\text{Vec String } n)
\end{align*}
\]

The final two computation rules are “approximate” because they require some massaging by propositional equality to type check. For example, $\text{ap} (\text{ap } f) (\text{noop } s @ i)$ has type $\text{ap } f (s \leftrightarrow s @ i) = \text{ap } f \text{refl}$, but $\text{noop}'$ has type $\text{swap}' \ s \leftrightarrow s' = \text{refl}$. While $\text{ap } f \text{refl}$ is definitionally equal to $\text{refl}$, $\text{ap } f (s \leftrightarrow s @ i)$ is only propositionally equal to $\text{swap}' \ s \leftrightarrow s'$. Because the prior computation rule is only propositional, however, we will not actually need these two computation rules in what follows, so we elide the details. We will use clausal function notation for maps out of $R$, but keep in mind that the types of the right-hand sides of the equations are those of doc’ and swap’ and indep’ and noop’ above, which (in the latter two cases) are only propositionally equal to the types of the left-hand sides.

The induction principle for $R$ states that to define a function $f : (x : R) \rightarrow C(x)$, it suffices to give

\[
\begin{align*}
& c' : C(\text{doc}) \\
& s' : \text{Path}Over C (s1 \leftrightarrow s2 @ i) c' c'
\end{align*}
\]

A 2-dimensional path over a path as the image of indep.

A 2-dimensional path over a path as the image of noop.

5.2 Interpreter

Because patches are represented by the type $\text{doc} = \text{doc}$, the interpreter for patches is a function

\[
\begin{align*}
\text{interp} : & (doc = \text{doc}) \\
& \rightarrow \text{Bijection} (\text{Vec String } n) (\text{Vec String } n)
\end{align*}
\]

As above, we generalize this to an interpretation of the whole path language $R$, and define a function $I : R \rightarrow \text{Type}$ such that

\[
\text{interp } p = \text{coe-biject} (\text{ap } I \ p)
\]

The interpreter $I$ is defined as follows:

\[
\begin{align*}
I : & R \rightarrow \text{Type} \\
I \ \text{doc} = & \text{Vec String } n \\
ap \ I (s1 \leftrightarrow s2 @ i) & = \text{ua} (\text{swapat } (s1,s2) i) \\
ap (ap \ I) (\text{indep } i \neq j) & = \\
?0 : & \text{ua} (\text{swapat } (s,t) i) \circ \text{ua} (\text{swapat } (u,v) j) \\
\equiv & \text{ua} (\text{swapat } (u,v) j) \circ \text{ua} (\text{swapat } (s,t) i) \\
ap (ap \ I) \ \text{noop} = ?1 : & \text{ua} (\text{swapat } (s,s) i) \equiv \text{refl}
\end{align*}
\]

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by equational properties of bijections, combined with the rules about the interaction of univalence with identity and composition described in Section 5.1. For example, $s1 = s2 \circ i$ is the the identity bijection, and then using the fact that $ua \circ db = refl$. $70$ is solved by turning both sides into a composition of bijections using the fact that $ua \circ ub \circ db = ua \circ (b1 \circ ob \circ b2)$, and then proving the corresponding fact about swapat:

\[
\text{swapat-independent : (i \neq j) \rightarrow (swapat (s,t) i) \circ (swapat (u,v) j) = (swapat (u,v) i) \circ (swapat (s,t) j)}
\]

As above, we do not need to give cases for the group operations or prove the group laws—these come for free, from functoriality.

5.3 Optimizer

To illustrate using the patch laws, we write a simple optimizer

\[
\text{optimize : (p : doc = doc) \rightarrow \Sigma (q : doc = doc). p = q}
\]

The type of optimize says that it takes a patch $p$ and produces a patch $q$ that behaves the same, according to the patch laws, as $p$. The goal is to optimize $s \leftrightarrow s \circ i$ to $refl$, saving ourselves two unnecessary string comparisons when the patch is applied. The optimizer requires analyzing the syntax of patches.

We show two definitions of optimize, to illustrate some different aspects of programming in homotopy type theory.

Program then prove. In this definition, we first write a function $\text{optimize1 : doc = doc = doc}$, and then prove that this function returns a path that is equal, according to the patch laws, to its input. The idea is to apply the following function $\text{opt0}$ to each patch $s1 \leftrightarrow s2 \circ i$:

\[
\text{opt0 : String \rightarrow String \rightarrow Fin n \rightarrow doc=doc}
\]

\[
\begin{align*}
\text{opt0 s1 s2 i} & = \text{if String.equals s1 s2} \\
& \quad \text{then refl} \\
& \quad \text{else (s1 \leftrightarrow s2 \circ i)}
\end{align*}
\]

To define $\text{optimize1}$, we generalize the problem to defining a function $\text{opt1}$ that acts on all of $R$, and then derive $\text{optimize1}$ as its action on paths (the same technique as reversing the circle in Section 5.1). This is defined as follows:

\[
\begin{align*}
\text{opt1 : R \rightarrow R} \\
\text{opt1 doc = doc} \\
\text{ap opt1 (s1 \leftrightarrow s2 \circ i) = opt0 s1 s2 i} \\
\text{ap (ap opt1) noop = ?0 : opt0 s s i = refl} \\
\text{ap (ap opt1) (indep i \neq j) = ?1 : opt0 s1 s2 i \circ opt0 s3 s4 j} \\
& \quad = opt0 s3 s4 j \circ opt0 s1 s2 i
\end{align*}
\]

We map doc to doc, and apply $\text{opt0}$ to $s1 \leftrightarrow s2 \circ i$. However, to complete the definition, we must show that the optimization respects the patch laws, via the goals $?0$ and $?1$ whose types are given above. The goal $?0$ is true because $\text{String.equals s s}$ will be true, so, after case-analysis, $\text{refl}$ proves that $\text{opt1 s s i = refl}$. The goal $?1$ requires case-analyzing both $\text{String.equals s1 s2}$ and $\text{String.equals s3 s4}$. If both are true, the goal reduces to $\text{refl \circ refl = refl \circ refl}$, which is true by $\text{refl}$. If the former but not the latter is true, the goal reduces to $\text{refl \circ s3 \leftrightarrow s4 \circ j} = \text{s3 \leftrightarrow s4 \circ j \circ refl}$, which is true by unit laws. The third case is symmetric. Finally, if neither are true, then the goal holds by $\text{indep}$.

Next, we prove this optimization correct. This is an example of $R$-induction:

\[
\begin{align*}
\text{opt1-correct : (x : R) \rightarrow x = opt1 x} \\
\text{opt1-correct doc = refl}
\end{align*}
\]

In the case for doc, we need to give a path $doc = opt1 doc$, but $\text{opt1 doc}$ is doc, so we give $\text{refl}$. In the case for $s1 \leftrightarrow s2 \circ i$, the induction principle requires an element of the type listed above. It turns out that, by rules for $\text{PathOver}$, this type is equivalent to $s1 \leftrightarrow s2 \circ i = opt0 s1 s2 i$

So this is where we prove that $\text{opt0}$ preserves the meaning of a patch. This requires two cases, one where $s1$ is equal to $s2$, in which case we use $\text{noop}$, and one where it is not, in which case we use $\text{refl}$.

The remaining two cases require proving that $\text{this proof of correctness of opt1}$ respects the patch laws. In each case, the goal asks us to prove the equality of two proofs of equality of patches. That is, the goal has the form $f1 \circ p = p \circ q \circ f2$, where $p$ and $q$ are two patches, and $f1$ and $f2$ are two proofs that the two patches are equal—which homotopically can be thought of as paths-between-paths, or, in more geometrically evocative terminology, as faces between edges.

One might think that such a goal would be trivial, because $f1$ and $f2$ are representing proofs that two patches are equal according to the patch laws, and we think of patch equality as a proof-irrelevant relation. But for the definition we have given above, there is nothing that actually forces any two such faces to be identified. For example, we can compose $\text{indep i \neq j} \circ (u \leftrightarrow v \circ j)$, a proof that $(s \leftrightarrow t \circ i) \circ (u \leftrightarrow v \circ j)$ is equal to itself, but there is no reason that this proof, which swaps twice, is necessarily the identity. Thus, although we have not considered any applications of this so far, we could potentially consider proof-relevant identifications between patches—proof-relevant patch laws. If we wished to do so, then these goals would need to be proved.

An alternative, which requires neither truncation nor proving any equations between faces, is to simultaneously implement the optimizer, and prove that it returns a patch equal to its input. To define

\[
\begin{align*}
\text{optimize : (p : doc = doc) \rightarrow \Sigma (q : doc = doc). p = q}
\end{align*}
\]

we need to define a function on all of $R$, and derive $\text{optimize}$ via its action on paths. However, $\text{optimize}$ is dependently typed, and $\text{ap f}$ for a simply-typed function $f$ always has such a dependent type. Thus, we define a dependently typed function and use the dependent form of $\text{ap}$, $\text{apd}$. Specifically, we define

\[
\begin{align*}
\text{opt : (x : R) \rightarrow \Sigma (y : R). y = x}
\end{align*}
\]

This type has the same shape as the type of $\text{optimize}$ above, except it is at the level of the points of $R$ rather than the paths. Its action on paths has the following type:
When the family B is known, the type PathOver B p b1 b2 can be “reduced” (via propositional equalities) to another type. In the case where B is x, y:R, y = x, as above, the rules for path-over-a-path in Σ-types, constant families, and patch types, yield an identification e as follows:

\[
e : \text{PathOver}(x, y:R, y = x) \vdash (\text{doc}, \text{refl}) \rightarrow (\text{doc}, \text{refl})
\]

Thus, if we define opt such that

\[
\text{opt doc} = (\text{doc}, \text{refl})
\]

then

\[
ap \text{opt} (p : \text{doc} = \text{doc}) : \text{PathOver}(x, y:R, y = x) \vdash (\text{doc}, \text{refl}) \rightarrow (\text{doc}, \text{refl})
\]

and we can define optimize by composing with e:

\[
\text{optimize : } (p : \text{doc} = \text{doc}) \rightarrow \Sigma(q : \text{doc} = \text{doc}). p = q
\]

This reduces the problem to defining opt, which we do as follows:

\[
\text{opt doc} = (\text{doc}, \text{refl})
\]

\[
ap \text{opt} (p : \text{doc} = \text{doc}) : \text{PathOver}(x, y:R, y = x) \vdash (\text{doc}, \text{refl}) \rightarrow (\text{doc}, \text{refl})
\]

For each of the noop and indep cases, we need to give a face between two specific paths between two specific points in the type Σ y:R. y = x (for some x). However, the type Σ y:R. x = y is in fact contractible—it is equivalent to unit. Intuitively, any pair \((y, p)\) can be continuously deformed to \((x, \text{refl})\) by sliding \(y\) along \(p\); see [22] Lemma 3.11.8. The path space of any contractible type is a proposition, so any two paths in it are connected by a face. Thus, because we formulated the problem as mapping into a contractible type, we can easily discharge the remaining goals.

This definition of opt, consisting of only the three cases given above, is much shorter than our previous attempt. Moreover, for comparison, suppose we instead wrote this optimizer for a datatype of patches that included identity, inverses, and composition as constructors (analogous to the one in Remark [4.1]). Then, in addition to giving the “interesting” case for optimizing \(s1 \leftrightarrow s2 \in 1\), we would need to give inductive cases describing how the optimizer acts on identity, inverses, and composition. Here, because the optimizer can be defined as a group homomorphism, we need to give only the “interesting” case; the inductive cases are handled by the framework.

\[
\text{Singleton Types and Computation}
\]

Because the type \(\Sigma(y : A). x = y\) is contractible, we can think of it as a singleton type, written \(S(x)\). It consists of “everything in \(A\) that is equal to \(x\),” or, more precisely, a point in \(\Sigma\) with a path to \(x\). One may well wonder what is the point of writing a function into a contractible type? Using the singleton notation we have

\[
\text{optimize : } (p : \text{doc} = \text{doc}) \rightarrow S(p).
\]

Because \(S(p)\) contractible, and hence equivalent to \(\text{unit}\), isn’t this just a triviality? The answer is “no” because even if two elements of a type are connected by a path (and hence cannot be distinguished by any other operation of type theory), the type nevertheless has meaningful computational content in that we may observe its output when it is run and thereby make distinctions that are obscured within the theory. Thus, even though the optimize function that we wrote above was equal (i.e., homotopic) to the function that simply returns \(p\) itself—or, indeed, any other function with that type—we expect, based on work on the computational interpretation of homotopy type theory, that it will in fact compute appropriately—e.g., \(\text{optimize} (s \leftrightarrow s \in 1)\) will in fact return \(\text{refl}\) because of the way it is programmed.

6. Patches with Types

In the previous sections we have only considered patches of the form \(s1 \leftrightarrow s2 \in 1\), which naturally form total bijections on the type of \(n\)-line documents. In Section [5] we exploited this fact to model these patches as paths in a higher inductive type, using univalence to map them to bijections on \(\text{Vec String} n\).

Now we will consider more realistic patches—inserting a string \(s\) as the 1th line in a file (ADD \(s\in 1\)), and removing the 1th line of a file (RM \(1\)). For example, the only patch applicable to an empty file is ADD \(s\in 0\); to the resulting file we may apply one of ADD \(s\in 0\), ADD \(s\in 0\), or RM \(0\), which respectively add \(s\) before or after \(\text{refl}\), or delete \(s\).

These new patches significantly complicate our definition of the patch theory \(R\) in a number of ways, most obviously because each patch only applies to files of at least a certain length: unlike in our previous patch languages, not all patches are composable. A first cut might be to classify repositories by the number of lines in their file—that is, index the points of \(R\) by \(\text{Nat}\) and say that addition of a line is a path \(\text{doc n} = \text{doc n+1}\), and deletion is a path \(\text{doc n+1} = \text{doc n}\).

This approach fails because addition and deletion aren’t bijections between \(n\) and \(n+1\)-line files. For example, the function which deletes the first line of a file is not in Bijection (\(\text{Vec String} n+1\)) (\(\text{Vec String} n\)), because it sends two files differing only in their first lines to the same file.

Univalence dictates that all the paths in \(R\) must be interpreted as bijections, so all the points of \(R\) must be interpreted as isomorphic sets. Because one of the points in \(R\) ought to represent the unique empty file, all its points must uniquely identify files—only one-element sets are isomorphic to one-element sets.

A solution, therefore, is to index contexts by file contents, i.e., \(\text{doc n file : } R\) where file : \(\text{Vec String} n\) and ADD \(s\in 1\) : \(\text{doc n file} = \text{doc n+1 file}'\), where file' is the result of adding \(s\) at line 1 in file. We reject this approach because it causes 10 We start our numbering at 0, so the positions in an \(n\)-line file coincide with \(\text{Fin n+1}\).

11 Although this patch language may still seem unusually simplistic, the approach taken in this section can scale to many features of real-world version control, in particular multi-file repositories.
the codomains of patches to depend on the concrete implementation of those patches, linking patches’ specifications to their intended implementations.

We will instead index contexts by patch histories, i.e., sequences of composable patches starting at the empty file. With respect to any particular implementation of patches, histories uniquely identify files, so we still sidestep the issue with bijections described above. As an added benefit, histories also reify sequences of patches in a way which facilitates certain operations on repositories, such as skipping forward or backward in time.

6.1 Definition of Patches

Let History n be the type of patch histories (sequences of patches) resulting in n-line files. We will define History n as a quotient higher inductive type to equate sequences of patches which result in the same changes to a file. For example, two additions in sequence can be commuted if the line numbers are shifted.

```
History : Nat → Type
[0] : History zero
ADD-_< : {n : Nat} (s : String) (l1 : Fin n+1) → History n → History n
RM-_< : {n : Nat} (l : Fin n+1) → History n → History n

ADD-ADD-< : {n : Nat} (l1 : Fin n+1) (l2 : Fin n+2) → History n → History n
ADD-ADD-≥ : {n : Nat} (l1 : Fin n+1) (l2 : Fin n+2) → History n → History n
```

(For the sake of clarity we have omitted some coercions between different Fin types.) To simplify the code in the remainder of this section, we have omitted the paths commuting ADD-RM, RM-ADD, and RM-RM, which can be defined in exactly the same way.

Here, histories serve as the “types” of patches. If we think of a patch as the formal representation of a change to a repository, then the domain of the patch is a history corresponding to a repository to which that change is applicable, and the codomain is the domain history extended by the patch which was just applied.

```
R : Type
doc : {n : Nat} → History n → R
adP : {n : Nat} (s : String) (l : Fin n+1) → History n → History n
rmP : {n : Nat} (l : Fin n+1) → History n → History n
```

Next, we would like to insert faces equating commuting sequences of patches, but our definition of histories means that no differing sequences of paths will ever be parallel! For example, when

```
11 < 12, the two paths
```

ought to be “equal” as patches, but it does not even make type sense to state this equation. We rely on the fact that histories are quotiented by the same commutation laws—that is, we already equated those exact elements of History n with the path ADD-ADD-<. Therefore, we can stipulate that the above two paths are equal

```
over
```

the ADD-ADD-< equation from History n, with respect to the type family x.h = x. Thus the faces of R are defined as follows:

- ADD-ADD-< : {n : Nat} (l1 : Fin n+1) (l2 : Fin n+2) → History n → History n
- ADD-ADD-≥ : {n : Nat} (l1 : Fin n+1) (l2 : Fin n+2) → History n → History n

6.2 Interpreter

Assume we have functions add and rm which implement our patches on concrete vectors of String's.

```
add : {n : Nat} (s : String) (l : Fin n+1) → Vec String n → Vec String n
rm : {n : Nat} (l : Fin n+1) → Vec String n → Vec String n
```

As before, we want to define a function I : R → Type which interprets points of R (histories) as types, and paths of R (patches) as bijections between those types. Then we can define interp p = coe-biject (ap I p) and obtain

```
interp : {n1 n2 : Nat} {h1 : History n1} {h2 : History n2} → doc h1 → History n1 → History n2
```

with the idea that interp (addP s l h) should in some sense be add s l, and interp (rmP l h) should be rm l.

The type of interp has gotten more complex than before, because patches now have many different domains and codomains, instead of a single doc object. As a result, we must decide on the interpretation of each doc h into the type universe—between which types does each patch induce a bijection?

As we discussed at the beginning of this section, we cannot simply interpret doc h as Vec String n, because adding and removing lines are not bijections on these types. Instead, we will essentially interpret doc h as the exact file which arises from applying the patches in h. That is, we will record in its type exactly how the file came into existence, rather than simply regarding it as a plain text file.

We can specialize any function f : A → B to a function between singleton types, as follows:

```
to-singleton : {M : A} (f : A → B) → S(M) → S(f M)
```

Because singleton types are contractible (contain exactly one point, and have trivial higher structure), every function between singleton types is automatically a bijection. Call this fact single-biject. Then we can define the interpretation I : R → Type

```
I (doc h) = S(replay h)
ap I (addP s l h) = ua (single-biject (to-singleton (addP s l h)))
ap I (rmP l h) = ua (single-biject (to-singleton (rmP l h)))
apd' (ap I) (addP s l h) = ?0
apd' (ap I) (rmP l h) = ?1
```

where apd' is a function which gives the action of a function on a PathOver, replay is a function which steps through a history to compute the file specified by that history, and ?0 and ?1 are proofs that the implementations of patches satisfy the commutation laws.

```
replay : {n : Nat} → History n → Vec String n
```
We know from Section 3 that merge follows from pseudocommutation. This patch language, it is worth illustrating how the patch laws described here would be involved in proving its correctness. In this setting \( pcom \) should have the following type:

\[
pcom : \{n1 \ n2 \ n3 : \text{Nat}\} \\
\{h1 : \text{History n1}\} \ \{h2 : \text{History n2}\} \ \{h3 : \text{History n3}\} \\
\quad \rightarrow \ \text{doc h1 = doc h2} \times \text{doc h2 = doc h3} \\
\quad \rightarrow \ \Sigma (n2' : \text{Nat}), \Sigma (h2' : \text{History n2'}) \\
\quad \rightarrow \ \text{doc h1 = doc h2'} \times \text{doc h2' = doc h3}
\]

For example, the function \( pcom \) should act as follows:\[12\]
\[
pcom (\text{addP} "b" 0 \ 0, \text{addP} "b" 1) = \\
(\text{addP} "b" 0 0, (\text{addP} \text{ADD-ADD-<} 0, \text{addP} "a" 0))
\]

In this call to \( pcom \), \( h1 = [] \) and \( h3 = \{\text{ADD "b"@0 :: ADD "a"@00}\} \). This says that pseudocommuting “add a at line 0” with “add b at line 1” gives “add b at line 0” and “add a at line 0” (composed with \text{ADD-ADD-<} to make its output history match that of the input).

The square law, which states that the top and bottom composites of the pseudocommutation square are equal, says that

\[
\text{pcom} \ (p,q) = (r,s) \rightarrow q \circ p = r \circ s
\]

Thus, for this example of how \( pcom \) should execute, the square law requires that

\[
\text{addP} "b" 0 \circ \text{addP} "a" 0 = \\
(\text{addP} \text{ADD-ADD-<} 0, \text{addP} "b" 0 0, \text{addP} "a" 0)
\]

Modulo expanding the definition of \text{PathOver} in the type family \( \text{doc h = doc x} \), this is exactly what the \text{addP-ADD-<} law states. This illustrates the role that the patch laws defined here would play for verifying the square law for \( pcom \).

## 7. Related and Future Work

We have shown how patch theory \([11]\) in the style of Darcs \([29]\) can be developed in homotopy type theory. The principal contribution of the present work is to reformulate patch theory using the tools of homotopy type theory, namely identity types, higher inductive types, and univalence, to clearly separate the formal theory of patches from its interpretation in terms of basic revision control mechanisms. Many standard tools of homotopy theory come into play, demonstrating the use of these methods in a practical programming context.

Several prior category-theoretic analyses of version control have been considered. Jacobson \([13]\) interprets patches as inverse semigroups, which are essentially partial bijections. Mimram and Di Giusto \([27]\) analyzes merging as a pushout, an alternative to the pseudocommutation-based merging we consider here. Houston \([14]\) also discusses merge as pushout, and a duality with exceptions. Our contribution, relative to these analyses, is to present patch theory in a categorical setting that is also a programming formalism, so it directly leads to an implementation. These analyses consider settings where not all maps are invertible. In homotopy type theory the path space of every type is symmetric, and to fit patch theories into this symmetric setting, we either considered a language where all patches were naturally total bijections on any repository (Section\[4\] and \[5\]), or used types to restrict patches to repositories where they are bijections (Section\[9\]). If we had a directed homotopy type theory (see \[20\] for some initial work towards this)—with, for example, a universe of partial bijections—we could perhaps apply these analyses more directly.

Dagit \([9]\) present an approach to proving some invariants of a version control implementation using advanced features of Haskell’s type system. Camp (Commute And Merge Patches) \([6]\) is

---

\[12\] We leave some additional arguments implicit, relative to the types given above.
an experimental version control system based on Darcs; the Camp project aims to prove the correctness of its patch theory in Coq. It would be interesting to formalize the programs we have written here in Coq or Agda, to investigate the effect of using homotopy type theory relative to this prior work.

To define merging on the patch theory with ADD and RM, we must complete the definition of pcom discussed in Section 5. As described in Section 4, to map out of a path type we must provide an induction principle for patches \( \text{doc} \circ h = \text{doc} \circ h' \). Intuitively, this should be possible because a history \( h \) is essentially a reified path \( \text{doc} \circ [] = \text{doc} \circ h \). In particular, we would certainly have to define clauses of pcom for pairs of composable primitive patches \((p, q)\). (The square and rotation laws guide the behavior of pcom on compositions or inverses of primitive patches, as described in 13.) We believe we can define pcom \((p, q)\) by using the function \( \text{interp}'(q \circ p) : \text{S}(h) \rightarrow \text{S}(\cdots : \cdots : \cdots : h) \) to compute \((\cdots, \cdots, \cdots)\), a length-2 suffix for the history \( h \), corresponding precisely to \((p, q)\). This fits with a long-range goal for homotopy type theory, which is to develop general characterizations of identity types—which, if it is possible, may have significant applications in algebraic topology.

An additional line of future work would be to consider different patch theories. For example, the patch languages considered here are either unityped, in that they have only one context (Section 4 and Section 5) or singleton-typed, in that each patch context has only one repository in it (Section 6). It would be interesting to consider whether there are any interesting alternatives in between.

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**References**


