A priori bounds for solutions of quasi-linear elliptic
differential equations

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A PRIORI BOUNDS FOR SOLUTIONS OF QUASI-LINEAR
ELLIPTIC DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Let $\Omega$ be a region in $\mathbb{R}^n$, $n = 1, 2, \ldots, 5$, $\partial \Omega$ denote its boundary. We consider quasi-linear elliptic differential equations of the form

$$ (1) \quad L[u] = - \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\beta}(x)D^\beta u) = f(x,u), \quad x \in \Omega, $$

subject to the boundary conditions

$$ (2) \quad u(x) = 0, \quad x \in \partial \Omega, $$

where we have freely used the standard multi-index notation, cf. [15]. For example, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and if $\alpha = (\alpha_1, \ldots, \alpha_n)$ is any index whose components are non-negative integers, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$ and

$$ D^\alpha = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n} $$

A basic question, both in proving the existence of a solution of (1), (2) by means of the Schauder Fixed Point Theorem, cf. [1], [8], and in approximating that solution numerically, cf. [5], [6], [7], [11], [12], and [13] is whether or not we can obtain an a priori bounds for classical solutions of (1), (2) in the uniform norm over $\Omega$.

The special case of $n = 2$, $L = -\Delta$ has been studied by many people. In [6] a uniform norm a priori bound was obtained for the case in which

$$ \frac{\partial f}{\partial u} \leq \gamma < \frac{1}{\rho}, \quad \text{where} \quad \rho = \max_{x \in \Omega} |\psi(x)| \quad \text{and} \quad \Delta \psi(x) = -1, \quad x \in \Omega, \quad \psi(x) = 0, \quad x \in \partial \Omega. $$

In [9] such an a priori bound was obtained for the case in which
\[ \liminf \frac{f(x,u)}{|u|} \geq 0, \quad \text{as} \ |u| \to \infty, \text{and in} \ [11] \text{such an a priori bound was obtained for} \]
\[ \text{the case in which there exists a positive constant} \ u_0 \ \text{such that} \]
\[ \frac{f(x,u)}{|u|} \leq - \left( \frac{1}{a} \right)^2 \ \text{for all} \ |u| \geq u_0 \ \text{where} \ \Omega \ \text{is contained in the strip} \]
\[ |x_1| \leq a. \]

In this paper, we give new conditions on the problem (1), (2) which guarantee the existence of a uniform norm a priori bound.

2. MAIN RESULTS

We assume throughout this paper that the coefficients, \( a(\cdot), \) \( |\alpha|, |\beta| \leq 1, \) are real-valued, bounded, measurable functions in \( \Omega \) and that there exists a positive constant \( C \) such that if
\[ a(u,v) = \int_{\Omega} \sum_{|\alpha|,|\beta| \leq 1} a_{\alpha\beta}(x) \partial^\alpha u(x) \partial^\beta v(x) \, dx, \]
then
\[ a(u,u) \leq C \| u \|^{2}_{H^2,1}(\Omega), \]
for all \( u \in W^{1,2}(\Omega), \) the completion of the real-valued \( C^0(\Omega) \) functions with respect to \( \| \cdot \|_{W^{1,2}(\Omega)} \). We remark that if \( L \) is strongly elliptic, i.e., there exists a positive constant \( \theta \) such that
\[ \sum_{|\alpha|,|\beta| \leq 1} a_{\alpha\beta}(x) \xi^\alpha \xi^\beta \geq \theta |\xi|^2 \]
for all \( x \in \Omega \) and all real \( n \)-vectors \( \xi \), then by Gårding's inequality, cf. [15, pg. 175] there exists a constant \( S \) such that the quadratic form associated with the differential operator \( L + SI \) satisfies (4) and we may consider the problem
\[ (L + SI)[u] = f(x,u) + Su, \ x \in \Omega, \]
\[ u(x) = 0, \ x \in \partial \Omega, \]
which is equivalent to (1), (2).

We define the real number...
(7) \[ \Lambda = \inf_{w \in W^1,2(\Omega)} \frac{\int_\Omega |\partial_x^{|\beta| \leq 1} \alpha_{\partial \beta} (x) D^\beta w(x) d^\beta w(x) dx}{\int_\Omega [w(x)]^2 dx} \] 

Inequality (4) yields the result that \( \Lambda > 0 \).

We assume that \( f(x,u) \) is a measurable function with respect to \( x \in \Omega \), continuous with respect to \( u \) for almost all \( x \in \Omega \) and there exists a constant \( \gamma < \Lambda \) such that

(8) \[ f(x,u) - f(x,0) \leq \gamma < \Lambda \] for almost all \( x \in \Omega \) and all \( u \neq 0 \). Clearly (8) is satisfied if \( f(x,u) \) is continuously differentiable with respect to \( u \) for almost all \( x \in \Omega \) and \( \frac{\partial f(x,u)}{\partial u} \leq \gamma < \Lambda \) for almost all \( x \in \Omega \). Our first result gives an a priori bound in the \( \| \cdot \|_{W^{1,2}(\Omega)} \) norm for generalized solutions of (1), (2). It improves and extends Lemma 4 of [7], which considers the case of \( n = 1 \).

**Theorem 1.** Let \( u(x) \) be a generalized solution of (1), (2), i.e.,

\[ \int_\Omega |\partial_x^{||\beta| \leq 1} \alpha_{\partial \beta} (x) D^\beta u d^\beta u dx = \int_\Omega f(x,u) u dx \]

for all \( \phi \in W^1,2_0(\Omega) \). Then

(10) \[ \| u \|_{W^{1,2}(\Omega)} \leq \frac{1}{C} \left( \int_\Omega |\partial_x^{||\beta| \leq 1} \alpha_{\partial \beta} (x) D^\beta u d^\beta u dx \right)^{1/2} \leq \frac{1}{C} \left[ a(u,u) \right]^{1/2} \]

\[ \leq \frac{1}{C(\Lambda - \gamma)^{1/2}} \left( \int_\Omega [f(x,0)]^2 dx \right)^{1/2} = M. \]

**Proof.** Setting \( \phi = u \) in (9) and using inequality (8) we obtain

\[ a(u,u) = \int_\Omega f(x,u) u dx \leq \int_\Omega f(x,0) u dx + \gamma \int_\Omega u^2(x) dx \leq \int_\Omega u^2(x) dx \leq (\int_\Omega u^2(x) dx)^{1/2} \left( \int_\Omega f^2(x,0) dx \right)^{1/2} + \frac{\gamma}{\Lambda} a(u,u). \]
Inequality (10) follows from the fact that \( y < A \). QED.

The next result follows directly from Theorem 1 and the Sobolev Imbedding Theorem cf. [15].

**Corollary.** If \( n = 1 \) and \( u(x) \) is a generalized solution of (1), (2) then \( u(x) \in C^0([0,1]) \) and \[ |u(x)|_{C^0([0,1])} = \max_{x \in [0,1]} |u(x)| \leq \frac{1}{2} M, \] where \( M \) is the positive constant defined in (8).

The situation for \( n > 1 \) is not quite as simple. We say that the differential operator \( L \) given in (1) is regular in \( \Omega \) if and only if the following condition is satisfied. If \( g \in L^p(\Omega) \) (1 < \( p < \infty \)) and \( u \in W^{1,2}_0(\Omega) \cap L^p(\Omega) \) (1 < \( p < \infty \)) is such that \[ a(u,v) = \int_\Omega g(x)v(x) \, dx \] for all \( v \in W^{1,2}_0(\Omega) \), then \( u \in W^{2,p}(\Omega) \) and
\[
|u|_{W^{2,p}(\Omega)} \leq k \left( \|g\|_{L^p(\Omega)} + |u|_{L^p(\Omega)} \right)
\]
where \( k \) is a positive constant independent of \( u \). We remark that any differential operator, \( L \), satisfying (4) is regular in \( \Omega \) if \( \Omega \) and the coefficients \( a_{ij}(x) \) are sufficiently smooth, cf. [2, Theorem 8.2].

**Theorem 2.** Let \( n = 2, \ldots, 5 \) and \( L \) be regular in \( \Omega \). If there exists positive constants \( A, k, t, \) and \( \epsilon \) such that

\[
|f(x,u)| \leq A + k |u|^t, \text{ if } n = 2, \text{ for all } x \in \Omega, -\infty < u < \infty,
\]

\[
|f(x,u)| \leq A + k |u|^{\frac{4n}{n-2(n+2\epsilon)}}, \text{ if } n = 3, \ldots, 5, \text{ for all } x \in \Omega, -\infty < u < \infty, \text{ and}
\]

\[
\frac{n}{2} + \epsilon \leq \frac{2n}{n-2}
\]

and \( u(x) \) is a generalized solution of (1), (2), then \( u(x) \in C^0(\Omega) \) and
satisfies an a priori bound in the uniform norm.

Proof. We first consider the case $n = 2$. By Sobolev's Imbedding Theorem, it suffices to show that any generalized solution, $u(x)$, satisfies an a priori bound in $W_0^{2,2}(\Omega)$. Hence, by inequality (11) it suffices to show that $u$ satisfies an a priori bound in $L^2(\Omega)$ and $g(x,u)$ satisfies an a priori bound in $L^2(\Omega)$. By Sobolev's Imbedding Theorem $u$ satisfies such an a priori bound and by a result of Vainberg, cf. [14] $g(x,u)$ satisfies such an a priori bound if and only if inequality (12) holds.

For the cases $n = 3, 4$ and $5$, it suffices by Sobolev's Imbedding Theorem, to show that $u$ satisfies an a priori bound in $W_0^{2,n} (\Omega)$ for some $\varepsilon > 0$. By inequality (11), it suffices to show that $u$ satisfies an a priori bound in $L^2(\Omega)$ and $g(x,u)$ satisfies an a priori bound in $L^2(\Omega)$. $u$ satisfies such an a priori bound since by Sobolev's Imbedding Theorem it satisfies such an a priori bound in $L^{2n/(n-2)}(\Omega)$ and $\frac{n}{2} + \varepsilon < \frac{2n}{n-2}$.

$g(x,u)$ satisfies such an a priori bound by a result of Vainberg, cf. [14].

QED.

We remark that for the case $n = 2$, $L = -\Delta$, Theorem 2 extends the result of [6] since $\frac{1}{2} < 1$, cf. [3].

3. AN APPLICATION

Theorem 3. If $L$ and $f(x,u)$ satisfy the hypotheses of Theorem 2, then (1), (2) has a generalized solution.

Proof. By Theorem 2, if $u$ is a generalized solution of (1), (2) then $u \in C^0(\Omega)$ and $\|u\|_{C^0(\Omega)} \leq B$ for some positive constant $B$. Consider the modified boundary value problem
(15) \( L[u] = f(x, u) = \begin{cases} f(x, B+1) & \text{if } u \geq B+1 \\ f(x, u) & \text{if } |u| \leq B+1 \\ f(x, -B-1) & \text{if } u \leq -B-1 \end{cases} \), \( x \in \Omega \).

(16) \( u(x) = 0 \), \( x \in \partial \Omega \).

It is easy to see the generalized solutions of (15), (16) satisfy the same a priori bound as those of (1), (2), and hence it suffices to show that (15), (16) has a generalized solution. However, the existence of such a solution follows by applying the Schauder Fixed Point Theorem to the mapping \( L^{-1} \) of \( W_0^{1,2}(\Omega) \) into itself, cf. [1], [8]. QED.
References


