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INTEGRALS WITH A KERNEL IN THE SOLUTION OF NONLINEAR EQUATIONS IN N DIMENSIONS

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December 1975
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This work was supported in part by the Office of Naval Research under Contract N0014-67-0314-0010, NR 044-422 and by the National Science Foundation under Grant GJ 32111.
We consider iterations for solving the nonlinear equation $F(x) = 0$ in the $N$ dimensional Banach space, $1 \leq N \leq +\infty$, which use "integral information with a kernel". This information consists of the "standard information" $F^{(j)}(x_d)$, $j = 0, 1, \ldots, s$ and the integral $\int_0^1 g(t) F(x_d + ty_d) dt$ where $s \geq 1$, $x_d$ is an approximation to the solution and $y_d$ depends on the standard information. We show there exists an iteration with order $2s + 1 + s N$ and prove its optimality.
1. INTRODUCTION

We want to approximate the simple solution \( \alpha \) of the nonlinear equation

\[
F(x) = 0
\]

where

\[ F: D \to B_2, \quad D \text{ is an open convex subset of } B_1, \quad B_1 \text{ and } B_2 \text{ are } N\text{-dimensional Banach spaces, } 1 \leq N \leq +\infty \text{ and } [F'(\alpha)]^{-1} \text{ is a bounded operator.} \]

This problem is often solved by construction of the sequence of successive approximations to \( \alpha \) using the standard information on \( F \)

\[ M_s = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)\}, \]

where \( x_d \) is a close approximation to \( \alpha \).

In previous papers we investigated another kind of information, namely the integral information

\[ M_{-1,s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 F(x_d + ty_d)dt\}, \]

where \( s \geq 1 \) and \( y_d \) depends only on the standard information (see Kacewicz [75a] and [75b]),

We showed there exists an iteration of maximal order \( s + 3 - \delta \) (for optimally chosen \( y_d \)), where

\[
\delta = \begin{cases} 
0 \text{ if } N = 1 \text{ or } s \geq 2 \\
1 \text{ otherwise}
\end{cases}
\]
Since the maximal order of iterations using the standard information is equal to \( s + 1 \), the use of the integral increases the maximal order by \( 2 - \delta \).

In this paper we consider more general kind of integral information, namely integral information with a kernel.

\begin{equation}
\mathfrak{M}^g_{-1,s} = \mathfrak{M}^g_{-1,s}(x_d; F) = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 g(t)F(x_d + ty_d)dt\},
\end{equation}

where

\[ s \geq 1, \quad y_d = (x_d, F(x_d), F'(x_d), \ldots, F^{(s)}(x_d)), \quad g = g(t) \text{ is a complex function of a complex variable such that } \int_0^1 |g(t)|dt < +\infty. \]

Note that if \( g(t) \equiv 1 \) then \( \mathfrak{M}^g_{-1,s} = \mathfrak{M}^g_{-1,s} \). The question is how the maximal order of iteration depends on \( g \).

In Section 2 we define the iteration \( I^g_{-1,s} \) which uses \( \mathfrak{M}^g_{-1,s} \) for optimally chosen \( y_d \) (see Section 4) and is of order \( \min(s+1+m, 2s+1+\delta^-N, 1) \) (see Section 3 and Corollary 1 in Section 4), where \( m \) is an integer depending on \( g \) (defined in Section 2) and \( \delta^-N, 1 \) is the Kronecker delta. In Section 4 we prove the iteration \( I^g_{-1,s} \) is maximal. Furthermore we show there exists a polynomial \( g = g(t) \) independent on \( F \) such that \( m = s+\delta^N, 1 \). Since for such \( g \) the order is equal to \( 2s+1+\delta^N, 1 \), the value of the integral with a kernel, which is represented by the vector of size \( N \), increases the maximal order by \( s+\delta^N, 1 \).

In Section 5 we show that for \( N \) sufficiently large the iteration \( I^g_{-1,1} \) has smaller complexity index than any interpolatory iteration \( I_{0,k} \), which uses the information \( \mathfrak{M}^g_k, k \geq 1 \), under some assumptions on the cost of computing the value of function, its derivatives, and the integral. In
Section 6 we give examples of the function $g$ and show some connections between information $S_{-1,s}^g$ and certain two-point information without memory.

2. DEFINITION OF THE ITERATION $I_{-1,s}^g$

We shall use the notation

$$I_j = \frac{\partial}{\partial t} g(t) t^{s+j} dt, \quad \forall j \geq 1.$$

Let us define

$$B_0 = \{ g = g(t): I_1 = 0 \}$$

$$B_1 = \{ g = g(t): I_1 \neq 0, I_2 = 0 \},$$

$$B_m = \left\{ g = g(t): I_1 \neq 0, I_2 \neq 0, \frac{I_k}{I_1} = \left( \frac{I_2}{I_1} \right)^{k-1} \right\} \quad k = 2, 3, \ldots, m,$$

$$\frac{I_{m+1}}{I_1} \neq \left( \frac{I_2}{I_1} \right)^m \right\} \quad \text{for } m \geq 2.$$

Note that $B_m \neq 0, \forall m$. Indeed, the function

$$g(t) = t - \frac{s+2+m}{s+3+m}$$

belongs to $B_m$ for $m = 0, 1$. For $m \geq 2$ we can find a function $g$ for which

$$I_j = 1, j = 1, 2, \ldots, m, I_{m+1} = 2.$$

Suppose $g$ is of the form
(2.6) \( g(t) = \sum_{i=0}^{m} g_i t^i \)

Then the equalities (2.5) give us the system of linear equations on \( g_i \), \( i = 0, 1, \ldots, m \)

\[
(2.7) \sum_{i=0}^{m} \frac{1}{s+j+i+1} g_i = 1 + \delta_{j,m+1} \quad j = 1, 2, \ldots, m+1.
\]

Since the matrix \( \begin{bmatrix} \frac{1}{s+j+i+1} \end{bmatrix} \) is symmetric and positive definite, the coefficients \( g_i \) exist and hence \( B_m \neq 0 \), \( \forall m \).

In the remaining part of this paper we often use the notation \( h = \varphi(x; F) \) which means that \( h \) is the approximation of \( \alpha \) obtained by one step of the iteration \( \varphi \) based on \( x \) and a certain information on \( F \). Recall that if \( z = I_{0,s}(x; F) \), where \( I_{0,s} \) means the maximal interpolatory iteration which uses the standard information \( \mathcal{I}_s \) for \( s \geq 1 \), then

\[
\lim_{x \to \alpha} \frac{z - \alpha}{(\alpha - x)^{s+1}} = \frac{F(s+1)(\alpha)}{(s+1)! F'(\alpha)} \quad \text{for } N = 1
\]

and

\[
\lim_{x \to \alpha} \frac{||z - \alpha||}{||\alpha - x||^{s+1}} \leq \frac{||F'(\alpha)||^{-1} F(s+1)(\alpha)||}{(s+1)!} \quad \text{for } N \geq 2.
\]

We define now the iteration \( I_{-1,s} \), which uses the information \( \mathcal{I}_{-1,s} \) for \( y_d \) given by

\[
y_d = \begin{cases} \text{arbitrary} & \text{if } m = 0 \\ z_d - x_d & \text{if } m = 1 \\ \frac{1}{I_1(z_d - x_d)} & \text{if } m \geq 2, \end{cases}
\]

where \( x_d \) is an approximation to the solution \( \alpha \), \( z_d = I_{0,s}(x_d; F) \) and \( g \in B_m \).
The next approximation \( h_d = l_{-1,s}^{I}(x_d; F) \) in \( l_{-1,s}^{I} \) is defined as a zero of the polynomial \( w = w(x) = w(x; x_d, F) \),

(2.9) \( w(h_d; x_d, F) = 0 \)

(with a criterion of its selection, e.g., the nearest zero to \( x_d \)), where \( w \) is given as follows.

**Case I.** \( N = 1 \).

(2.10) \( w(x; x_d, F) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s + A(x_d, F)(x-x_d)^{s+1}, \)

where

(2.11) \( A(x_d, F) = \begin{cases} 0 & \text{if } m = 0 \\ \frac{1}{y_d^{s+1}} \int_0^1 g(t)F(x_d+ty_d)dt - \sum_{i=0}^{s} \frac{1}{i!} F^{(i)}(x_d)y_d^{i} \int_0^1 g(t)t^i dt & \text{otherwise} \end{cases} \)

**Case II.** \( 2 \leq N \leq +\infty \).

(2.12) \( w(x; x_d, F) = F(x_d) + F'(x_d)(x-x_d) + \ldots + \frac{1}{s!} F^{(s)}(x_d)(x-x_d)^s + \ldots \)

\[
+ c \left[ \int_0^1 g(t) F(x_d+ty_d)dt - \sum_{i=0}^{s} \frac{1}{i!} F^{(i)}(x_d)y_d^{i} \int_0^1 g(t)t^i dt \right],
\]

where

\[
c = \begin{cases} 0 & \text{if } m = 0 \\ \frac{1}{y_d^{s+1}} & \text{if } m = 1 \\ \frac{1}{y_d^{s+1}} \frac{I_2}{I_1} & \text{if } m \geq 2 \end{cases}
\]
Note that to find a good approximation of \( h_d \) in numerical practice it is possible to perform a few Newton steps on the equation (2.9).

We see that for \( m = 0 \) \( I_{-1,s}^g \) is equal to the well known interpolatory iteration \( I_{0,s} \) which uses the standard information \( R_s \) and is of order \( s+1 \). Hence we assume that \( m \geq 1 \).

One can verify that the polynomial \( w \) satisfies the following interpolatory conditions. For \( N = 1 \),

\[
\frac{1}{s+1} \int_0^1 g(t) w(x_d + ty_d) dt = \int_0^1 g(t) F(x_d + ty_d) dt.
\]

For \( 2 \leq N \leq +\infty \),

\[
w(x_d) = F(x_d) + O(||x-x_d||^{s+1})
\]

\[
\frac{1}{s+1} \int_0^1 g(t) w(x_d + ty_d) dt = \int_0^1 g(t) F(x_d + ty_d) dt + O(||x-x_d||^{s+1}).
\]

3. CONVERGENCE OF THE ITERATION \( I_{-1,s}^g \)

If the function \( F \) is sufficiently smooth in the neighborhood of the zero \( \alpha \), then from (2.10), (2.11), (2.12) and due to the special form of \( y_d \) given by (2.8) we have

\[
(3.1) \quad F(x) - w(x; x_d, F) = R(x),
\]

where
for \( N = 1 \)

\[
R(x) = \begin{cases} 
\frac{1}{(s+2)^3} F^{(s+2)}(x_d) (x-x_d)^{s+2} + O((x-x_d)^{s+3}) + \\
+ O((z_d-x_d)^2 (x-x_d)^{s+1}) & \text{if } m = 1 \\
& + \sum_{k=2}^{m} \frac{1}{(s+k)^3} F^{(s+k)}(x_d) (x-x_d)^{s+1} \frac{[(x-x_d)^{k-1} - (z_d-x_d)^{k-1}]}{k} + \\
& + \frac{1}{(s+1+m)^3} F^{(s+1+m)}(x_d) (x-x_d)^{s+1} \left[(x-x_d)^m - \frac{I_{m+1}}{l_1^m} \right] (z_d-x_d)^m \\
& + O((x-x_d)^{s+2+m}) + O((z_d-x_d)^{m+1} (x-x_d)^{s+1}) & \text{if } m \geq 2,
\end{cases}
\]

for \( 2 \leq N \leq + \infty \)

\[
R(x) = \begin{cases} 
\frac{1}{(s+1)^3} [F^{(s+1)}(x_d) (x-x_d)^{s+1} - F^{(s+1)}(x_d) (z_d-x_d)^{s+1}] + \\
+ \frac{1}{(s+2)^3} F^{(s+2)}(x_d) (x-x_d)^{s+2} + O(||x-x_d||^{s+3}) + \\
& + O(||z_d-x_d||^{s+3}) & \text{if } m = 1 \\
& + \sum_{k=0}^{m-1} \frac{1}{(s+1+k)^3} [F^{(s+1+k)}(x_d) (x-x_d)^{s+1+k} - F^{(s+1+k)}(x_d) (z_d-x_d)^{s+1+k}] + \\
& + \frac{1}{(s+1+m)^3} F^{(s+1+m)}(x_d) (x-x_d)^{s+1+m} - F^{(s+1+m)}(x_d) (z_d-x_d)^{s+1+m} + \\
& \left[\frac{I_{m+1}}{l_1^m} \right] (z_d-x_d)^m \\
& + O(||x-x_d||^{s+2+m}) + O(||z_d-x_d||^{s+2+m}) & \text{if } m \geq 2
\end{cases}
\]

From the Brouwer fix point theorem for \( N < + \infty \) or the Schauder fix point theorem for \( N = + \infty \) (see Ortega and Rheinboldt [70], p.164), from the definition (2.9) of \( I_{-1,s}^G \) and (3.1), (3.2) and (3.3) we get the following theorem about convergence of \( I_{-1,s}^G \). In Section 4 we shall use the result below to establish the order of \( I_{-1,s}^G \).
Theorem 1

Let the iteration $I^g_{-1,s}$ be defined by (2.9) and $g \in B_m$. If the function $F$ is sufficiently smooth in the neighborhood of its simple zero $\alpha$, then the approximation $h_d = I^g_{-1,s}(x_d; F)$ is well defined for $x_d$ sufficiently close to $\alpha$ and

(i) For $N = 1$

\[
\lim_{x_d \to \alpha} \frac{h_d - \alpha}{(\alpha - x_d) \min(s+1+m, 2s+2)} = \begin{cases} 
D_{s+1+m} & \text{if } m = 0,1 \\
- \delta_{m,s+1} D_{s+1} \cdot D_{s+2} + & \\
+ \left(1 - \frac{I_1}{I_2}\right)^m \frac{I_{m+1}}{I_1} \cdot D_{s+1+m} & \text{if } 2 \leq m \leq s+1 \\
- D_{s+1} \cdot D_{s+2} & \text{if } m > s+1
\end{cases}
\]

(ii) For $2 \leq N < +\infty$

\[
\lim_{x_d \to \alpha} \frac{\|h_d - \alpha\|}{\|(\alpha - x_d) \min(s+1+m, 2s+2)\|} = \begin{cases} 
\|D_{s+1}\| & \text{if } m = 0 \\
\delta_{s,1} \cdot 2 \|D_2\|^2 + \|D_{s+2}\| & \text{if } m = 1 \\
\delta_{m,s} \cdot (s+1) \|D_{s+1}\|^2 + & \\
+ \left|1 - \frac{I_1}{I_2}\right|^m \frac{I_{m+1}}{I_1} \cdot \|D_{s+1+m}\| & \text{if } 2 \leq m \leq s \\
(s+1) \cdot \|D_{s+1}\|^2 & \text{if } m > s
\end{cases}
\]

where $D_k = \frac{1}{k!}[F'(\alpha)]^{-1}F(k)(\alpha)$.

Since $x_d$ is an arbitrary point, the theorem above describes the behavior of the function $h = I^g_{-1,s}(x; F)$ in the neighborhood of the zero $\alpha$ of $F$. 

4. ORDER OF INFORMATION $\theta_{-1,s}^g$ AND MAXIMALITY OF THE ITERATION $\tau_{-1,s}^g$

In this section we show that the iteration $\tau_{-1,s}^g$ has order equal to 
\[\min(s+1+m, 2s+1+\delta_{N,1})\] whenever $g \in B_m$. We prove that this order is maximal and $y_d$ given by (2.8) is optimal.

For this purpose we define the order of iteration and the order of information as in Wozniakowski [75b].

Let $\mathcal{F}$ be a class of functions $F$,

$$F: D_F \to B_2, \quad D_F \subset B_1, \quad \dim(B_1) = \dim(B_2) = N$$

which have a simple zero $\alpha = \alpha(F)$ and are analytic in its neighborhood. Let 
$\{x_d\}$ be a sequence converging to $\alpha$, $\lim x_d = \alpha$. We shall say that $\{F_d\} \subset \mathcal{F}$
is equal to $F \in \mathcal{F}$ with respect to $\theta_{-1,s}^g$ iff

\[(4.1) \quad \forall d \quad F_d(\alpha_d) = 0, \quad \lim d \alpha_d = \alpha,\]

\[(4.2) \quad \lim d F_d^{(k)}(\alpha_d) = G^{(k)}(\alpha), \quad k = 0,1,\ldots,\]

where $G \in \mathcal{F}$, $G(\alpha) = 0$,

\[(4.3) \quad \theta_{-1,s}^g(x_d;F) = \theta_{-1,s}^g(x_d,F_d) \quad \forall d, \quad \text{i.e.,} \]

$$F_d^{(k)}(x_d) = F_d^{(k)}(x_d), \quad k = 0,1,\ldots,s,$$

$$\int_0^1 g(t)F(x_d + ty_d)dt = \int_0^1 g(t)F_d(x_d + ty_d)dt.$$

The order of information $p = p(\theta_{-1,s}^g)$ is a real number such that

$$p(\theta_{-1,s}^g) = \begin{cases} 
\sup A & \text{if } A \neq \emptyset \\
0 & \text{otherwise}
\end{cases}$$
where
\[ A = \{ p \geq 1 : \forall \{ x_d \}, \lim_{d} x_d = \alpha, \forall F \in \mathcal{F}, F(\alpha) = 0, \forall \{ F_d \} \text{ equal to } F \} \text{ it is true that} \]
\[
\lim_{d} \frac{||x_d - \alpha||}{||x_d - \alpha||^{1 - \varepsilon}} = 0, \forall \varepsilon > 0 \}.
\]

Let \( \varphi_{-1,s}^g \) be an iteration which uses the information \( \gamma_{-1,s}^g \). The order of iteration \( \varphi_{-1,s}^g \), \( p = p(\varphi_{-1,s}^g) \) is a real number such that
\[
p(\varphi_{-1,s}^g) = \begin{cases} \sup B & \text{if } B \neq \emptyset \\ 0 & \text{otherwise} \end{cases},
\]
where
\[ B = \{ p \geq 1 : \forall \{ x_d \}, \lim_{d} x_d = \alpha, \forall F \in \mathcal{F}, F(\alpha) = 0, \forall \{ F_d \} \text{ equal to } F \}
\]
it is true that
\[
\lim_{d} \frac{||h_d - \alpha||}{||x_d - \alpha||^{1 - \varepsilon}} = 0, \forall \varepsilon > 0 \text{ where } h_d = \varphi_{-1,s}^g(x_d ; F) \}
\]
(see Woźniakowski [75b]).

Woźniakowski [75a] proved that the order of information is equal to the maximal order of convergence. We shall use this property to show \( I_{-1,s}^g \) is maximal.

We now prove the theorem about order of information \( \gamma_{-1,s}^g \).

Theorem 2

Let \( \gamma_{-1,s}^g \) be the integral information with a kernel
\[
\gamma_{-1,s}^g(x_d ; F) = \{ F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), \int_0^1 g(t)F(x_d + ty_d)dt \}
\]
where
\[ s \geq 1, y_d = y_d(x_d, F(x_d), \ldots, F^{(s)}(x_d)), g = g(t) \]
is a complex function of a complex variable such that $\int_0^1 |g(t)| \, dt < +\infty$ and $g \in \mathcal{B}_m$. Then

$$p(m^g_{-1,s}(t)) \leq \min(s+1+m, 2s+1+\delta_{N,1})$$

Furthermore, if

$$y_d = \begin{cases} 
\text{arbitrary} & \text{if } m = 0, \\
z_d - x_d & \text{if } m = 1, \\
\frac{I_1(z_d - x_d)}{I_2} & \text{if } m \geq 2,
\end{cases}$$

where $z_d = I_{0,s}(x_d:F)$

then

$$p(m^g_{-1,s}) = \min(s+1+m, 2s+1+\delta_{N,1})$$

**Proof**

We shall prove the first part of Theorem 2, i.e., we shall show that there exist $F \in \mathcal{F}$, $F(\alpha) = 0$, $\{x_d\}$, $\lim_{d} x_d = \alpha$ and $\{F_{d}\}$ equal to $F$, $F_d(\alpha_d) = 0$ such that

$$\lim_{d} \frac{||x_d - \alpha_d||}{\min(s+1+m, 2s+1+\delta_{N,1})} > 0.$$  \hfill (4.4)

We consider two cases.

**Case I.** $N = 1$

Let $F \in \mathcal{F}$, $F(\alpha) = 0$ and $e_d = \alpha - x_d$, where $\lim_{d} e_d = 0$. We set

$$F_d(x) = F(x) + (x-x_d)^{s+1}[(x-x_d)^{m-\gamma} - b_d], \forall d,$$

where

$$\gamma = \begin{cases} 
0 & \text{if } \lim_{d} \frac{y_d}{e_d} \cdot \frac{I_2}{I_1} = 0, \\
\gamma - 1 & \text{otherwise,}
\end{cases}$$

if $m = 0$ and

if $m \neq 0$. 


\[ b_d = y_d^{m-\nu} \frac{I_{m+1-\nu}}{I_1} \quad \text{for } m \geq 1. \]

One can verify that \([F_d] \) is equal to \( F \). Moreover,
\[
|\alpha - \alpha_d| = c_d |F_d(\alpha)| = \begin{cases} 
 0 & \text{if } m = 0 \\
 |e_d|^{s+1} + |e_d|^{m-\nu} |y_d| |I_{m+1-\nu}| & \text{otherwise,}
\end{cases}
\]
where
\[ F_d(\alpha_d) = 0 \text{ and } \lim_{d \to \infty} c_d = c > 0. \]

From above we have
\[ (4.6) \quad \lim_{d \to \infty} \frac{|\alpha - \alpha_d|}{|e_d|^{s+1+m}} > 0, \text{ for all } \nu_m. \]

This proves (4.4) for \( m = 0 \) or 1. Hence assume \( m \geq 2 \). Let us now consider the functions \([F_d] \) given by (4.5) with \( \nu = m-1 \). This means that
\[ F_d(x) = F(x) + (x-x_d)^{s+1} (x-x_d)^{m-\nu} |y_d| |I_{m+1-\nu}|. \]

Let \( z_d \) be defined by
\[ z_d = x_d + \frac{I_2}{I_1} y_d, \quad \forall_d, \]
and let the function \( F \) and the sequence \([x_d] \) be such that
\[ \lim_{d \to \infty} \frac{|z_d - \alpha|}{|e_d|^{s+1}} > 0. \]

Then \([F_d] \) is equal to \( F \) and
\[ (4.7) \quad \lim_{d \to \infty} \frac{|\alpha - \alpha_d|}{|e_d|^{2s+2}} > 0. \]

Hence, (4.6) and (4.7) prove (4.4) for \( N = 1 \).
Case II. $2 \leq N \leq +\infty$

Since the inequality $p(m, s) \leq \min(s+1+m, 2s+2)$ holds for $N = 1$ it also holds for any $2 \leq N \leq +\infty$. Hence, we want to now show that for $2 \leq N \leq +\infty$ $p(m, s) \leq 2s+1$, i.e., that (4.4) holds for $m > s$. It suffices to consider the case $N < +\infty$. Let $z_d = x_d + \frac{I_2}{I_1} y_d$, $\forall d$, $z_d = z_d(x_d, F(x_d), \ldots, F(s)(x_d))$.

If there exist $F \in \mathcal{F}$, $F(0) = 0$ and $\{x_d\}$, $\lim_{d} x_d = \alpha$ such that

$$\lim_{d} \frac{\|x_d - \alpha\|}{\|x_d - \alpha\|} > 0,$$

then the family of functions

$$F_d(x) = F(x) + [(x_1 - x_1d)^{s+1}(x_2 - z_2d), 0, \ldots, 0]^T,$$

is equal to $F$ with respect to $m_{-1,s}$ and (4.4) holds for zeros $\alpha_d$ of $F_d$.

In the formula above, $x_1d$, $z_2d$ denote the components of vectors $x_d$, $z_d$ respectively such that

$$\lim_{d} \frac{\|x_d - \alpha\|}{\|x_d - \alpha\|} > 0 \text{ and } \lim_{d} \frac{\|\alpha - z_d\|}{\|\alpha - z_d\|} > 0,$$

and

$$x = [x_1, \ldots, x_N]^T.$$

Hence assume

$$\lim_{d} \frac{\|x_d - \alpha\|}{\|x_d - \alpha\|} = 0 \text{ for any } F \text{ and } \{x_d\}.$$

Let the sequence $\{x_d\}$ satisfy the conditions

(i) $\lim_{d} x_d = \alpha$, $x_1d \neq \alpha_1$, $x_2d \neq \alpha_2$, $\lim_{d} \frac{\alpha_1 - x_1d}{\alpha_2 - x_2d} = 1$, $x_id = \alpha_i$ for $i = 3, 4, \ldots, N$,

where $F(\alpha) = 0$. 

From the assumptions above, it follows that $y_{2d}$ can be equal to zero only for a finite number of $d$, hence without loss of generality we can assume that $y_{2d} \neq 0 \forall d$.

Let us define

$$F_d(x) = F(x) + \left[ (x_1 - x_{1d})^{s+1} - \frac{y_{1d}^{s+1}}{y_{2d}} (x_2 - x_{2d})^{s+1}, 0, \ldots, 0 \right]^T, \quad (4.9)$$

One can verify that $\{F_d\}$ is equal to $F$. From (4.9) it follows that

$$||\alpha - \alpha_d|| = h_d \|F_d(\alpha)\| = \tilde{h}_d |a_d(z_{2d} - \alpha_2) - (z_{1d} - \alpha_1)| \cdot$$

$$\cdot \left| a_d^s(z_{2d} - x_{2d})^s + a_d^{s-1}(z_{2d} - x_{2d})^{s-1}(z_{1d} - x_{1d}) + \ldots \right.$$  

$$\ldots + a_d(z_{2d} - x_{2d}) (z_{1d} - x_{1d})^{s-1} + (z_{1d} - x_{1d})^s, \right|,$$

where

$$\lim_{d \to \infty} \tilde{h}_d = h > 0, F_d(\alpha_d) = 0, a_d = \frac{\alpha_1 - x_{1d}}{\alpha_2 - x_{2d}} (1 \lim a_d = 1).$$

It can be verified that there exists a function $F$ and $\{x_d\}$ satisfying the (i) condition such that

$$\lim_{d \to \infty} \frac{|a_d(z_{2d} - \alpha_2) - (z_{1d} - \alpha_1)|}{\|\alpha - x_d\|^{s+1}} > 0. \quad (4.11)$$

Indeed, otherwise (due to the similar argument which was used by Kacewicz [75b]) the iteration $\varphi$ for the solution of the nonlinear scalar equation $f(y) = 0$ defined as follows

$$B_{d+1} = \varphi(B_d; f) = z_{2d}(x_d, F(x_d), \ldots, F^{(s)}(x_d)) - z_{1d}(x_d, F(x_d), \ldots, F^{(s)}(x_d))$$
where $\beta_d$ is close to the solution (but not equal),

$$F(x) = [x_1, f(x_2), x_3, \ldots, x_N]^T$$

and

$$x_d = [\beta_d - I_0, s(\beta_d; f), \beta_d, 0, \ldots, 0]^T$$

has the order of convergence greater than $s+1$, i.e., greater than the order of used information, which is a contradiction.

Finally, from (4.11) and (4.10) follows the inequality (4.4) for $m > s$, which means that $p(\Omega_{-1, s}) \leq 2s+1$. This proves Case II and also the first part of Theorem 2.

We shall prove the second part of Theorem 2. We want to show that for arbitrary $F \in \mathcal{F}$, $F(a) = 0$, $\{x_d\}$, $\lim_{d} x_d = a$, $\{F_d\}$ equal to $F$, $F_d(a_d) = 0$ we have

$$\lim_{d} \frac{||x_d - a||}{\min(s+1+m, 2s+1+\delta_N, 1)} = +\infty.$$ 

Since $||x_d - a||$ is at least of order $s+1$, (4.12) holds for $m = 0$. Assume $m > 1$.

Since $\{F_d\}$ is equal to $F$ we have

$$||F_d(a)|| \leq ||w(a; x_d, F)|| + ||F_d(a) - w(a; x_d, F_d)||$$

where the polynomial $w = w(\alpha; x_d, F)$ is given by (2.10) for $N = 1$ and (2.12) for $2 \leq N \leq +\infty$.

From (3.2) for $N = 1$ and (3.3) for $2 \leq N \leq +\infty$ we get

$$||x_d - a|| = O(||F_d(a)||) = O(||x_d - a|| \min(s+1+m, 2s+1+\delta_N, 1)).$$

Hence (4.12) holds which completes the proof of the Theorem 2.
Since

\[ ||I_{-1,s}^g(x_d:F) - \alpha_d|| \leq ||I_{-1,s}^g(x_d:F) - \alpha|| + ||\alpha - \alpha_d|| \]

we get from Theorem 1 and (4.14)

\[ \lim_{d} \frac{||I_{-1,s}^g(x_d:F) - \alpha_d||}{||x_d - \alpha|| \min(s+1+m,2s+1+\delta_N,1)} < +\infty \]

for any \( F \in \beta, F(\alpha) = 0, \{x_d\}, \lim_{d} x_d = \alpha, \) and \( \{F_d\} \) equal to \( F, F_d(\alpha_d) = 0. \)

Hence, from the definition of the order of iteration and Theorem 2 we have

**Corollary 1**

Let \( g \in B_m. \) Then

\[ p(I_{-1,s}^g) = \min(s+1+m,2s+1+\delta_N,1). \]

From Corollary 1 and Theorem 2 there follows immediately

**Corollary 2**

Let \( \Phi_{-1,s}^g \) be the class of iterations which use information \( I_{-1,s}^g. \) Then

\[ p(I_{-1,s}^g) = \sup_{\varphi_{-1,s}^g \in \Phi_{-1,s}^g} p(\varphi_{-1,s}^g), \]

i.e., the iteration \( I_{-1,s}^g \) is maximal.

Note that the order of information and at the same time order of iteration \( I_{-1,s}^g \) is maximized and equal to \( 2s+1+\delta_N,1 \) iff \( m \geq s+\delta_N,1. \) Thus, for the function \( g \) chosen such that \( m = s+\delta_N,1 \) (see (2.6) and (2.7)) one additional value of the integral which is represented by \( N \) new data increases the order by \( s+\delta_N,1. \)
5. COMPLEXITY INDEX

We want to compare the complexity indices of the iterations \( I^{g}_{-1,s} \) and \( I^{g}_{0,k} \). The complexity index \( z \) is defined by

\[
z = z(\varphi;F) = \frac{c(\mathcal{M};F) + c(\varphi)}{\log p}
\]

where \( \varphi \) is an iteration of order \( p \) which use the information \( \mathcal{M} \), \( c(\mathcal{M};F) \) is the information cost and \( c(\varphi) \) is the combinatory cost (see Traub and Woźniakowski [75]). For the integral information with a kernel the cost \( c(\mathcal{M}^{g}_{-1,s};F) \) consists of the costs of the standard information \( c(\mathcal{M}_{s};F) \) and the computed integral \( c(I) \). Let us assume that \( m = s+\delta_{N,1} \). Then \( p(I^{g}_{-1,s}) = 2s+1+\delta_{N,1} \) and one can verify that \( z(I^{g}_{-1,s};F) < z(I^{g}_{0,k};F) \) iff

\[
(5.1) \quad c(I) < \frac{\log(2s+1+\delta_{N,1})}{\log(2s+1+\delta_{N,1})} c(\mathcal{M}_{k};F) - c(\mathcal{M}_{s};F) + \frac{\log(2s+1+\delta_{N,1})}{\log(2s+1+\delta_{N,1})} c(I_{0,k}) - c(I^{g}_{-1,s}).
\]

Let \( 2 \leq N < +\infty \) and \( c(F^{(i)}(x)) \) denote the cost of computing \( F^{(i)}(x) \). \( c(F^{(i)}(x)) \) depends on the total number of arithmetical operations as well as on the cost of data access (which is usually greater than the cost of single arithmetical operation). Let \( c(F) = N \). Then we assume that \( c(I) = 0(N) \) and since \( F^{(i)}(x) \) can be represented in general by \( O(N^{i+1}) \) scalar function evaluations, assume that \( c(F^{(i)}) = 0(N^{i+1}) \). Since the information costs \( c(\mathcal{M}_{k};F) \) and \( c(\mathcal{M}^{g}_{-1,s};F) \) are of order \( N^{k+1} \) and \( N^{s+1} \) respectively and the combinatory costs \( c(I_{0,k}) \) and \( c(I^{g}_{-1,s}) \) are increasing functions of \( k \) and \( s \) respectively, we have for large \( N \)

\[
(5.2) \quad \min_{k \geq 1} z(I_{0,k};F) = z(I_{0,1};F) \text{ and}
\]

...
(5.3) \[
\min_{s \geq 1} z(I_{-1,s}^g;F) = z(I_{-1,1}^g;F).
\]

However, it should be stressed that if \( c(F^{(i)}) \) is essentially less than \( N^{i+1} \) then (5.2) and (5.3) are not necessarily true. Under our assumptions

\[
(\log 3-1)c(I_{-1,1}^g;F) + \log 3 \ c(I_{0,1}^g) - c(I_{-1,1}^g) = O(N^2)
\]

which means that (5.1) holds for large \( N \). From here, (5.2) and (5.3), it follows that \( I_{-1,1}^g \) has smaller complexity index than any iteration \( I_{0,k}, k \geq 1 \) and any \( I_{-1,s}, s \geq 2 \).

6. EXAMPLES

1. Let \( g(t) \equiv 1 \). Then \( m = 2 \) and order \( p(I_{-1,s}^g) = \min(s+3, 2s+1+\delta_{N,1}) = s+3-\delta \) where

\[
\delta = \begin{cases} 
0 & \text{if } N = 1 \text{ or } s \geq 2 \\
1 & \text{otherwise,}
\end{cases}
\]

which agrees with Kacewicz's [75b] result.

2. Let \( N = 1 \) and \( g(t) = \delta(t-1) \), where \( \delta \) is a generalized function such that

\[
\int_{-\infty}^{+\infty} \delta(t-1)F(t)dt = F(1)
\]

for any function \( F \) with bounded support (see Gel'fand and Shilov [64]). Then the information is of the form

\[
\mathfrak{m}_{s}^{g}_{-1} = \{F(x_d), \ldots, k^{(s)}(x_d), F(x_d+y_d)\}.
\]

Note that \( I_j = 1 \), \( \forall j \) and hence \( \frac{I_k}{I_1} = \left(\frac{I_2}{I_1}\right)^{k-1} \), \( \forall k \). Then formally we can set
m = +∞ and the order of information \( p(m) \) is equal to \( \min(s+1, 2s+2) = 2s+2 \), which agrees with the optimal order of this special Hermitian information (see Woźniakowski [75b]).

3. Let \( N = 1 \) and \( g(t) = \delta_k(t-1) \), where \( \int_{-\infty}^{+\infty} \delta_k(t-1)f(t)dt = F^{(k)}(1) \) for any sufficiently smooth \( F \) with bounded support.

Then the information is of the form

\[
\mathfrak{m}^G_{-1, s} = \{F(x_d), F'(x_d), \ldots, F^{(s)}(x_d), F^{(k)}(x_d+y_d)\}
\]

and it was considered by Brent [74]. It is easy to see that if \( k > s+1 \) then \( I_j = 0 \), hence \( m = 0 \) and the order is equal to \( s+1 \). If \( k \leq s+1 \) then \( I_j = \frac{(s+j)!}{(s+j-k)!} \), hence \( m = 2 \) and order is equal to \( s+3 \) which agrees with Brent's result.

ACKNOWLEDGMENTS

I would like to thank J. F. Traub and H. Woźniakowski for their helpful comments and assistance during the preparation of this paper.
7. REFERENCES


