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ON A COVERING PROBLEM
FOR PARTIALLY SPECIFIED SWITCHING FUNCTIONS

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Abstract

We consider the problem of finding the minimum number $K(n,c)$ of total switching functions of $n$ variables necessary to cover the set of all switching functions which are specified in at most $c$ positions. We find an exact solution for $K(n,2)$ and an upper bound for $K(n,c)$ which is better than a previously known upper bound by an exponential factor.

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1. Introduction

The problem considered here can be stated as follows:

P1: Given the set $F$ of all $c$-specified boolean functions of $n$ variables, i.e., all functions which are specified in at most $c$ positions, to find the cardinality $K(n,c)$ of a set $G$ of total functions such that

- $P1-1$: For all $f$ in $F$, there is a $g$ in $G$ such that $g$ covers $f$, i.e., if $f(x)$ is specified then $g(x) = f(x)$.

- $P1-2$: $K(n,c) = |G|$ is minimal.

This problem relates the number of additional exterior connections (besides input and output) that are required in a circuit which is to be $c$-universal. (A circuit is $c$-universal if it is capable of simulating the behavior of any partial function which is specified in $c$ or less points of its domain.)

This problem was studied in [1] in connection with adaptive networks, where an upper bound for $K(n,c)$ was shown to be

$$K(n,c) \leq \sum_{k=m}^{m+c} \left( \begin{array}{c} m+c \cr k \end{array} \right)$$

where $m = 2^n$, $p = \lfloor c/2 \rfloor \mod 8$, $8 = m+1-c$.

This upper bound agrees with the exact solutions for $c=1$ (i.e., $K(n,1)=2$) and $c=2^{n-1}$ (i.e., $K(n,2^{n-1})=2^{2n-1}$). For $c=2$ we have $\delta=2^{n-1}$ and, for any $n > 1$, $p=1$ so

$$K(n,2) \leq \sum_{k=1}^{2n} \left( \begin{array}{c} 2n \cr k \end{array} \right) \approx \left( \begin{array}{c} 2n \cr n \end{array} \right) = 2^n + 1$$

and in general, for small $c$, this bound is of the order of $2^{nc/2}$.

In this note we show that for $c=2$, $K(n,2) = O(n)$ and present an upper bound which, for fixed $c$, is a power of $n$. 
2. An Exact Solution for $K(n,2)$

Consider the following problem:

P2: Given $n$ and $c$, find the dimension $s(n,c)$ of a vector space over $\mathbb{F}_2$ such that there is a set $P$ of at least $2^n$ vectors in it satisfying:

P2-1: $(V_{p_1}, p_2, \ldots, p_c) \in P$, $(\forall b_1, b_2, \ldots, b_c \in \{0,1\})$, $p_1 b_1 p_2 b_2 \ldots p_c b_c \neq 0$

P2-2: $s(n,c)$ is minimal

Notation: We will use the following convention

1) $(a,b,\ldots,z) \in M$ means for all elements $a,b,\ldots,z$ in $M$.
2) $p^b = 1$ if $b=1$ then $p$ else $\neg p$

The first result we present shows that essentially, P1 and P2 are equivalent problems.

Lemma 1: For all $c > 1$, $K(n,c) \leq s(n,c)$.

Proof: We show that any solution to P1 satisfying P1-1 is a solution to P2 satisfying P2-1 and conversely. This implies that the minimality conditions are also satisfied.

Let $G = \{g_1, g_2, \ldots, g_K(n,c)\}$ be a solution to P1 satisfying P1-1. Consider the set $P = \{p(x) = (g_1(x), g_2(x), \ldots, g_K(n,c)(x)) | x \in \{0,1\}^n\}$. Let $x,y \in \{0,1\}^n$ with $x \neq y$. Then $p(x) = p(y) \Rightarrow (\forall g) \in G$, $g(x) = g(y)$. But since $c > 1$, this implies that there is a c-specified function $f$ with $\emptyset = f(x) \neq f(y) = 1$ which is not covered by any $g \in G$ which is a contradiction. Thus $p(x) \neq p(y)$, which shows that $|P| = 2^n$.

Assume now that there are $c$ different elements $p_1, p_2, \ldots, p_c$ in $P$ such that, for some $b_1, b_2, \ldots, b_c \in \{0,1\}$, $p_1 b_1 p_2 b_2 \ldots p_c b_c = \emptyset$. Let $p_j = p(x_j) = (g_1(x_j), g_2(x_j), \ldots, g_K(n,c)(x_j))$ for some $n$-tuple $x_j \in \{0,1\}^n$. Let $f$ be a c-specified function such that $f(x_j) = b_j$ for $j = 1,2,\ldots,c$. Since $p_1 b_1 p_2 b_2 \ldots p_c b_c = \emptyset$ for each $k = 1,2,\ldots,K(n,c)$, there is a $j$, $1 \leq j \leq c$ such that $g_k(x_j) = 1 - b_j$. Thus, for this value of $j$ we have $g_k(x_j) \neq f(x_j)$ so $g_k$ does not cover $f$. Since
this holds for all $k$, we have that $G$ does not satisfy P1-1, a contradiction. Thus, P2-1 is satisfied.

Conversely, let $P$ be a set of $2^n$ $s$-dimensional vectors $P = \{p_0, p_1, p_2, \ldots, p_{2^n-1}\}$ satisfying P2-1. Consider the set $G = \{g_1, g_2, \ldots, g_s\}$ of boolean functions of $n$ variables defined as follows:

For each $1 \leq j \leq s$, $(V_i) \in \{0,1,\ldots, 2^n-1\}$, $g_j((i_2)_1, (i_2)_2, \ldots, (i_2)_n) = (p_j)_j$ where $i_2$ denotes the binary representation of $i$ with $n$ bits, $(i_2)_r$ denotes the $r$-th bit and for an $s$-dimensional vector $p$, $(p)_r$ denotes the $r$-th component.

Let $f$ be a $c$-specified function of $n$ variables. Without loss of generality, assume that $f$ is specified at $(i_2)_1, (i_2)_2, \ldots, (i_2)_n$ for $i = 0,1,\ldots,c-1$. We claim there is at least one $g$ which covers $f$. Define, for $i = 0,1,\ldots,c-1$, $b_i = f((i_2)_1, (i_2)_2, \ldots, (i_2)_n)$. Since $P$ satisfies P2-1, $p_0b_0p_1b_1\ldots p_{c-1}b_{c-1} \neq \emptyset$. Thus, there is a $j \in \{1,2,\ldots,s\}$ such that, for all $i \in \{0,1,\ldots,c-1\}$, $(p_i)_j = 1$. (Note that $(p_i)_j$ is either $p_i$ or its complement, and this means the $j$-th component of this vector is 1.) This means that $(p_j)_j = b_j$. By the definition of $b_j$ and the definition of $G$ we have

$$g_j((i_2)_1, (i_2)_2, \ldots, (i_2)_n) = f((i_2)_1, (i_2)_2, \ldots, (i_2)_n)$$

for all $i \in \{0,1,\ldots,c-1\}$. Thus, $g_j \in G$ covers $f$. This completes the proof of Lemma 1.

Now we focus our attention to Problem 2. In what follows, we assume $s$ is restricted to be even and we will show that $K(n,2)$ can be determined exactly (to within 1).

We first prove an auxiliary result. Since P2 can be interpreted as: Find the smallest $s$ such that there are at least $2^n$ points in the $s$-cube satisfying P2-1, we will now show that the search for points in the $s$-cube satisfying P2-1 can be reduced to the set of all points in the middle plane (i.e., having weight $s/2$).
Lemma 2: Let $c = 2$, $s$ be an even positive number, and $P$ be a set of $s$-dimensional vectors satisfying $P2-1$. Then, there is a set $Q$ of $s$-dimensional vectors, each of which has weight $s/2$ and such that $|Q| = |P|$, satisfying $P2-1$.

Proof: We can assume, without loss of generality, that all vectors in $P$ have weight $\geq s/2$. (It is clear that changing a vector by its complement in any set satisfying $P2-1$ also produces a set satisfying $P2-1$.) If all vectors have weight $s/2$ we have proved the lemma. Assume then that $P$ contains $t$ vectors $p_1, p_2, ..., p_t$ with maximal weight $u > s/2$. We will construct a set $P'$ such that all vectors in it will have weights $w$ such that $s/2 \leq w < u$. Since $u - s/2$ is finite this will prove the lemma.

Choose any set of $t$ vectors $q_1, q_2, ..., q_t$ with the property that $q_i < p_i$ for $i = 1, 2, ..., t$ and such that the weight of each $q_i$ is $u-1$.

Claim: The set $P' = P \cup \{q_1, q_2, ..., q_t\} - \{p_1, p_2, ..., p_t\}$ is the required set.

To show the claim, we first note that there are always $t$ vectors $q_i$ as above. This follows directly from the relationship which exists between points in the $s$-cube.

Next we show that for any $p_j$, $j = 1, 2, ..., t$ and for any $p^b$, $p \in P - \{p_1, p_2, ..., p_t\}$, $w(p_j p^b) \geq 2$, where $w(p)$ denotes the weight of a boolean vector $p$. This follows because $w(p_j p^b) = w(p_j) + w(p^b) - w(p_j p^b) \geq u - (s-u+1) - (s-1) = 2$. We then have that $w(g_j p^b) = w(p_j a_j p^b) = w(p_j p^b) + w(-a_j) - w(p_j p^b + a_j) \geq 2 + (s-1) - s = 1$ and so $g_j p^b \neq 0$. (Here $a_j$ is an atom such that $a_j < p_j$ and $q_j = p_j(\neg a_j)$. Similarly, $w(-q_j p^b) = w((-p_j + a_j) p^b) = w(-p_j p^b + a_j) \geq w(-p_j p^b) \geq 1$.

This means that any vector $q$ and any vector in $P - \{p_1, ..., p_t\}$ satisfies $P2-1$. Clearly, any two vectors in $P - \{p_1, ..., p_t\}$ satisfy $P2-1$, so it remains to be shown that any two vectors in $\{q_1, q_2, ..., q_t\}$ satisfy $P2-1$.

We have $w(-q_i \wedge q_j) = w((-p_i \wedge a_j)(-p_j \wedge a_j)) \geq w(-p_i \wedge p_j) \geq 1$. 
Also \( w(q_i q_j) \geq 1 \) since \( q_i \neq q_j \) and \( w(q_i) = w(q_j) > s/2 \). Finally,
\[
w(q_i q_j) = w(p_i (q_i \sim q_i) p_j (q_j \sim q_j)) = w(p_i p_j) \neq w(q_i \sim q_i) - w(p_i p_j \sim q_i \sim q_j).
\]

Since
\[
w(p_i p_j) = w(p_i) + w(p_j) - w(p_i p_j) \geq (s/2 + 1) + (s/2 + 1) - (s - 1) = 3
\]

So \( w(q_i q_j) \geq 3 + (s - 2) - s = 1 \). This completes the proof of the lemma.

Finally, \( w(p_i) = w(p_j) = u > s/2 \),
\[
w(p_i) = w(p_j) = u \cdot 2^{s/2} - (s - 1) \approx 3 \cdot (s - 2) - s \approx 1.
\]

This completes the proof of the lemma. 

Lemma 2 makes the conditions in P2-1 to reduce \( w \),
\[
(V p_1, p_2) \in P, p_1 p_2 \neq q \text{ and } \sim p_1 p_2 \neq q
\]

(The other two conditions which imply \( p_1 < p_2 \) or \( p_2 < p_1 \) are satisfied trivially if \( w(p_1) = w(p_2) \)). But these conditions are equivalent to saying that \( p_1 \) or \( p_2 \) are each the complement of the other. Since the maximum number of points with weight \( s/2 \), satisfying this condition is
\[
1/2(\frac{s}{s/2})
\]
we have shown:

Theorem 1: The solution to problem P2, for \( c=2 \), is given by \( s \) satisfying
\[
s = \min[1/2(\frac{s}{s/2}) \geq 2^n].
\]

Since \( 1/2(\frac{s}{s/2}) \approx \frac{2^n}{(2^{1/2})^{0.5}}, \) \( s = O(n) \)

Thus we get \( K(n, 2) = O(n) \) as was to be shown.

3. A Polynomial Bound on \( K(n, c) \)

In this section we will show that for each \( c \), \( K(n, c) \) grows not more than with a polynomial of \( n \), namely \( K(n, c) \leq 2^{c^2 n^{c-1}} \). This is a substantial improvement over the previously mentioned bound. To obtain this bound we will construct a set \( G \) of functions satisfying P1-1. The construction is a modification of one suggested to the author by R. Rivest who pointed out the existence of polynomial bounds for this problem.
Let $U$ and $V$ be sets of functions of $n-1$ variables. Let $U \times V$ be the set of functions of $n$ variables defined as $U \times V = \{ f | \exists u \in U, \exists v \in V, \forall (b_2, \ldots, b_n) \in \{0,1\}, f(\emptyset, b_2, \ldots, b_n) = u(b_2, \ldots, b_n), f(1, b_2, \ldots, b_n) = v(b_2, \ldots, b_n) \}$. Note that $|U \times V| = |U||V|$. Let $U = \{ u_1, u_2, \ldots, u_p \}$ and $V = \{ v_1, v_2, \ldots, v_p \}$ be sets of functions of $n-1$ variables with $p = |U| = |V|$. Let $U + V$ be the set of $p$ functions of $n$ variables defined as $U + V = \{ f_i(\emptyset, b_2, b_3, \ldots, b_n) | \forall (b_1, b_2, b_3, \ldots, b_n) \in \{0,1\}, f_i(\emptyset, b_2, b_3, \ldots, b_n) = u_i(b_2, b_3, \ldots, b_n), f_i(1, b_2, b_3, \ldots, b_n) = v_i(b_2, b_3, \ldots, b_n) \}$.

Let $G(n,c)$ be a set of functions satisfying PI-1 for some $n$ and $c$. $G(n,c)$ can be constructed as follows:

1) Find all $G(n-1,d)$, for $d = 1, \ldots, c$.

2) $G(n,c) = \{ G(n-1,c) \times G(n-1,c) \} \cup \bigcup_{k=1}^{c-1} G(n-1,k) \times G(n-1,c-k)$.

The following is an immediate consequence of this definition.

**Lemma 3:** The set $G(n,c)$ constructed as above satisfies PI-1.

From the above construction we get the following recurrence for $K(n,c)$:

$$K(n,c) \leq K(n-1,c) + \sum_{1 \leq k \leq c-1} K(n-1,k) \cdot K(n-1,c-k)$$

Using this recurrence we now show

**Theorem 2:** $K(n,c) \leq 2^c n^{c-1}$.

**Proof:** For $c=1$ we know $K(n,1) = 2$ so the theorem holds. Assume the result holds for all values of the second parameter less than $c$. Then, using the above recurrence,

$$K(n,c) \leq K(n-1,c) + \sum_{1 \leq k \leq c-1} 2^k (n-1)^{k-1} \cdot 2^c \cdot K(n-1,c-k)$$

Since the term inside the summation does not depend on $k$ we get a new recurrence:

$$K(n,c) \leq K(n-1,c) + 2^c (c-1)(n-1)^{c-2}$$

so

$$K(n,c) \leq 2^c (c-1) \sum_{j=1}^{n-1} j^{c-2} \leq 2^c (c-1) (n-1)^{c-1} / (c-1) \leq 2^c n^{c-1}$$
which proves the theorem.

Since the number of control lines to select any of the $K(n,c)$ functions is $\log K(n,c)$ we get as a corollary:

**Corollary 1:** The number of exterior connections (besides those used for input) to a $c$-universal circuit is no more than $(c-1) \log n + c$.

**Conclusions**

In this note we have reexamined the problem of the number of exterior connections needed to control a circuit which is to be $c$-universal. For $c = 2$ we have found an exact solution and shown an upper bound for this number in the general case. The small bound found (of the order of $c \log n$ for the number of exterior connections) makes the implementation of these circuits very practicable.

**References**