Online Appendix to "Coordination Equilibrium, Price Stickiness, and Markup Variations in the Market Game"

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1 Introduction

In this appendix, we provide technical details for basic results in the Shapley-Shubik market game that we use in "Coordination equilibrium, price stickiness, and markup variations in the market game" paper. Since online storage is cheap, we replicate the setup of the model used in this paper for reference here as well.

2 The Model

To study imperfectly competitive production economies, we use the market game model. In market games, agents trade goods at trading posts. There is a trading post for each good where agents can make bids to buy and make offers to sell the good. These bids are in terms of units of account and offers are in terms of physical commodities. Agents make bids and offers based on their expectations of prices. Prices are formed by simultaneous actions (bids and offers) of all agents who buy or sell at the corresponding trading post. Equilibrium occurs when agents' price expectations come true.

We consider a static and deterministic model populated with output goods, input goods, firms, and households. There are $J < \infty$ sectors that produce different types of output goods. There are $N < \infty$ input goods used to produce output goods, and input goods are indexed by $n \in \{1, \ldots, N\}$. There are $K_j < \infty$ firms that are endowed with the technology to produce output good $j \in \{1, \ldots, J\}$, and firm k in sector j is indexed by k_j where $k \in \{1, \ldots, K_j\}$. There are also H households who are endowed with input goods (and ownership shares of firms), and they are indexed by $h \in \{1, \ldots, H\}$.

We assume that firms sell all of output goods they produce and households sell all of their endowments of input goods. In the absence of this assumption, i.e., when agents can make bids and offers simultaneously at the same trading posts, a well-known coordination indeterminacy arises. To avoid this indeterminacy, we assume that households are endowed only with input goods (from which they receive no utility), and receive utility only from output goods (with which they are not endowed). This is a common assumption in the international trade literature (e.g., Ohlin 39) and see Peck et al. (1992) for further discussion of the importance of this assumption for determinacy of equilibrium in market games. There is also an economic rationale behind the Hecksher-Ohlin assumption. In competitive markets, it is well-known that Walrasian

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tatonnement process may not converge to the competitive equilibrium (Scarf 1960). However, the Hecksher-Ohlin assumption guarantees that the Walrasian tatonnement process will converge to the competitive equilibrium because it eliminates endowment-induced income effects in product markets. In market games, Kumar and Shubik (2004) provide an example in which tatonnement-like process may not converge to the equilibrium (analogous to Scarf (1960)'s counter example in competitive markets). However, we conjecture that the Hecksher-Ohlin assumption will induce the tatonnement-like process to converge to the equilibrium in market games as well.

2.1 Agents

This section proceeds as follows. First, we discuss a firm's endowments and actions. Second, we discuss a household's endowments and actions. Third, we elaborate on production technology.

First, firm k_j (firm k in sector j) produces output good j by using input goods, and his production technology is specified by $q_{k_j} = f^j(\phi_{k_j})$, where q_{k_j} is the output, $f^j : \mathbb{R}^N_+ \rightarrow \mathbb{R}_+$ is the production function, $\phi_{k_j} \in \mathbb{R}^N_+$ is the vector of input goods of firm k_j, and $f(0) = 0$. Because firms need input goods for production but they are not endowed with these goods, firms purchase input goods from households. To purchase input good n, firm k_j makes a bid $w^n_{k_j}$ at input trading post n, where $w_{k_j} = (w^1_{k_j}, \ldots, w^N_{k_j})$ is the vector of firm k_j's bids. To finance these bids, firms sell (all of) output goods they produce. In particular, firm k_j sells his output q_{k_j} at output trading post j, and as a result of this sale, makes profit π_{k_j}.

Second, household h is endowed with input goods, and his vector of endowments is $g_h = (g^1_h, \ldots, g^N_h) \in \mathbb{R}^N_+$. Households do not have access to the technology to produce output goods, but they get utility from consuming output goods. Household h's utility function u_h is at least twice continuously differentiable, strictly increasing, strictly concave, and his vector of (output goods') consumption is $x_h \in \mathbb{R}^J_+$. Because households get utility from consuming output goods but they are not endowed with these goods, households purchase output goods from firms. To purchase output good j, household h makes a bid b^j_h at output trading post j, where $b_h = (b^1_h, \ldots, b^J_h)$ is the vector of household h's bids. Because households need to finance these bids, they sell their endowments of input goods. Moreover, households are owners of firms - in particular, household h is endowed with ownership shares $\theta^{k_j}_h$ of firm k in sector j.

Third, because households own the firms, they pay for the fixed costs of the firms they own. The fixed cost that household h pays is $0 \leq \delta \leq 1$ of his endowment g^n_h for all $n \in \{1, \ldots, N\}$. Because households do not get utility from input goods (i.e., primary factors of production), they sell all of their endowments of input goods after paying for the fixed costs of the firms they own. In particular, household h sells $e^n_h = g^n_h(1-\delta)$ at input trading post n, and $e_h = (e^1_h, \ldots, e^N_h) \in \mathbb{R}^N_+$. When fixed costs are too high ($\delta = 1$), no endowments are left to use as input for production. Then, firms make zero production, and hence households make zero bids on outputs. Therefore, we have a trivial equilibrium in which all agents make zero bids, and they do not engage in trade. When fixed costs are zero (i.e., $\delta = 0$), increasing returns appear in the form of decreasing marginal costs. When fixed costs are positive (i.e., $0 < \delta < 1$) and when technology exhibits decreasing (or constant) returns, decreasing (or constant) returns approximate increasing returns, that is, increasing returns appear in the form of fixed costs. When fixed costs are positive (i.e., $0 < \delta < 1$) and when technology exhibits increasing returns, increasing returns appear in the form of both fixed costs and decreasing marginal costs.

2.2 Prices and Allocations

This section explains how prices are formed and how goods are allocated. First, by using total bids and total offers, we calculate prices. Second, by using prices, we define allocation rules and budget constraints. Third, by using allocation rules, we verify that markets clear.

First, in the market game, the price of each good is equal to the ratio of the total bid (which represents the market demand) to the total offer (which represents the market supply).[2] The total bid is equal to the sum of all individual bids, and the total offer is equal to the sum of all individual offers at the corresponding trading post. At input trading post n, the total bid W^n and the total offer E^n are

$$W^n = \sum_{j=1}^{J} \sum_{k=1}^{K_j} w_{k_j}^n \text{ and } E^n = \sum_{h=1}^{H} e_h^n.$$

At output trading post j, the total bid B^j and the total offer Q^j are

$$B^j = \sum_{h=1}^{H} b_h^j \text{ and } Q^j = \sum_{k=1}^{K_j} q_{k_j}.$$

The price r^n of input good n and the price p^j of output good j are

$$r^n = \frac{W^n}{E^n} \text{ and } p^j = \frac{B^j}{Q^j},$$

where $r = (r^1, \ldots, r^N)$ is the vector of input prices, and $p = (p^1, \ldots, p^J)$ is the vector of output prices.

Second, in the market game, each agent's allocation is equal to the ratio of his own bid to the price of the good he purchases.[3] Firm k_j's allocation of input good n is

$$\phi_{k_j}^n = \frac{w_{k_j}^n}{r^n} = w_{k_j}^n \frac{E^n}{W^n},$$

and $\phi_{k_j} = (\phi_{k_j}^1, \ldots, \phi_{k_j}^N)$ is the vector of (input good) allocations of firm k_j. Household h's allocation of output good j is

$$x_h^j = \frac{b_h^j}{p^j} = b_h^j \frac{Q^j}{B^j},$$

where $x_h = (x_h^1, \ldots, x_h^J)$ is the vector of (output good) allocations of household h.

[2] Adopting the convention of Shapley and Shubik (1977), if all offers at a trading post are zero, all bids are lost, and the price of the good is defined to be zero.

[3] Following the lead of Shapley and Shubik (1977), if all bids at a trading post are zero, all offers are lost, and the allocation of the good is defined to be zero.

Each agent faces a budget constraint that restricts his total bid to his total income. This income is the product of his offer and the price of the good he sells. In particular, firm k_j's budget constraint is

$$\sum_{n=1}^{N} w_{k_j}^n \leq q_{k_j} p^j = q_{k_j} \frac{B^j}{Q^j}.$$

As mentioned in §2.1, households are endowed with input goods and ownership shares of firms, so households receive income from sales of input goods and shares of firms' profits. Moreover, because households own the firms, households pay for the fixed costs of firms they own. Hence, household h's budget constraint is

$$\sum_{j=1}^{J} b_h^j \leq \sum_{n=1}^{N} \frac{W^n}{E^n} e_h^n + \sum_{j=1}^{J} \sum_{k=1}^{K_j} \theta_h^{k_j} \pi_{k_j},$$

where π_{k_j} is firm k_j's profit.

Third, in the market game, allocation rules are designed in a way that markets always clear, and generate feasible allocations. Thus, we verify that the total use (for consumption or as production input) of each good equals its total endowment or total production. The market for input good $n \in \{1, \ldots, N\}$ clears as follows

$$\sum_{j=1}^{J} \sum_{k=1}^{K_j} \phi_{k_j}^n = \sum_{j=1}^{J} \sum_{k=1}^{K_j} w_{k_j}^n \frac{E^n}{W^n} = E^n = \sum_{h=1}^{H} e_h^n.$$

The market for output good $j \in \{1, \ldots, J\}$ clears as follows

$$\sum_{h=1}^{H} x_h^j = \sum_{h=1}^{H} b_h^j \frac{Q^j}{B^j} = Q^j = \sum_{k=1}^{K_j} q_{k_j}.$$

2.3 Nash Equilibrium

We adopt the standard definition of Nash equilibrium, that is, each agent makes a best response to other agents' actions. We denote other agents' actions as follows. At input trading post n, the sum of bids of all firms other than firm k_j is $W_{-k_j}^n$, the sum of offers of households other than household h is E_{-h}^n. At output trading post j, the sum of bids of households other than household h is B_{-h}^j, the sum of offers of firms other than firm k_j in sector j (which is also the total offer of other firms for output good j) is $Q_{-k_j}^j$. Before formally defining Nash equilibrium, we derive agents' best response functions stemming from their optimization problems. In particular, firm k_j solves the following problem

$$\max_{w^1_{k_j},\dots,w^N_{k_j},q_{k_j}} \frac{B^j}{Q^j_{-k_j}+q_{k_j}} q_{k_j} - \sum_{n=1}^{N} w^n_{k_j} \tag{1}$$

$$\text{s.t. } \sum_{n=1}^{N} w^n_{k_j} \le \frac{B^j}{Q^j_{-k_j}+q_{k_j}} q_{k_j}, \tag{2}$$

$$q_{k_j} = f^j(\phi_{k_j}), \tag{3}$$

where $\phi_{k_j} = (w^1_{k_j}\frac{E^1}{W^1},\dots,w^N_{k_j}\frac{E^N}{W^N})$. The objective function of firm k_j given in (1) is to choose $w^1_{k_j},\dots,w^N_{k_j}$ that maximize firm k_j's profit π_{k_j}. Budget constraint (2) guarantees that firm k_j's total cost does not exceed its revenue, i.e., (2) ensures that firm k_j's profit π_{k_j} is nonnegative. The constraint (3) specifies firm k_j's production technology. If we substitute (3) back into (1), firm k_j's optimization problem collapses to

$$\max_{w^1_{k_j},\dots,w^N_{k_j}} \frac{B^j}{Q^j_{-k_j}+f^j(\phi_{k_j})} f^j(\phi_{k_j}) - \sum_{n=1}^{N} w^n_{k_j}. \tag{4}$$

On the other hand, household h solves the following problem

$$\max_{b^1_h,\dots,b^J_h} u_h(x^1_h,\dots,x^J_h) \tag{5}$$

$$\text{s.t.} \sum_{j=1}^{J} b^j_h \le \sum_{n=1}^{N} \frac{W^n}{E^n} e^n_h + \sum_{j=1}^{J}\sum_{k=1}^{K_j} \theta^{k_j}_h \left(\frac{b^j_h + B^j_{-h}}{Q^j_{-k_j}+f^j(\phi_{k_j})} f^j(\phi_{k_j}) - \sum_{n=1}^{N} w^n_{k_j} \right). \tag{6}$$

The objective function of household h given in (5) is to choose $b^1_h,\dots,b^J_h$ that maximize household h's utility. Because households sell all of their endowments, offers of input goods are not decision variables for households. Budget constraint (6) guarantees that household h's total bid does not exceed his total income. We next define Nash equilibrium as follows.

Definition 1 *The Nash equilibrium* $\{\widehat{w}^n_{k_j}, \widehat{b}^j_h, j \in \{1,\dots,J\}, k \in \{1,\dots,K_j\}, n \in \{1,\dots,N\}, h \in \{1,\dots,H\}\}$ *is such that for all* $j \in \{1,\dots,J\}$ *and* $k \in \{1,\dots,K_j\}$, $\widehat{w}^n_{k_j}$, $n \in \{1,\dots,N\}$ *solves (4) given* $\widehat{w}^n_{-k_j}$, $n \in \{1,\dots,N\}$ *and for all* $h \in \{1,\dots,H\}$, $\widehat{b}^j_h$, $j \in \{1,\dots,J\}$ *solves (5) - (6) given* $\widehat{b}^j_{-h}$, $j \in \{1,\dots,J\}$.

3 Analysis

This section is organized as follows. In §3.1, we show the existence of interior solution; in §3.2, we present the main existence theorem; in §3.3, we characterize the equilibrium with an extended example.

Before presenting our results, we derive first-order conditions we will use. The first-order condition of (4) with respect to $w^n_{k_j}$ (noting that $\phi_{k_j} = (\phi^1_{k_j}, \ldots, \phi^N_{k_j})$ and $\phi^n_{k_j} = w^n_{k_j}\frac{E^n}{W^n}$) is

$$\frac{B^j Q^j_{-k_j}}{(Q^j)^2}\frac{\partial f^j(\phi_{k_j})}{\partial \phi^n_{k_j}}\left[\frac{W^n_{-k_j}E^n}{(W^n)^2}\right] - 1 = 0, \forall n,$$

which boils down to (noting that $r^n = W^n/E^n$ and $p^j = B^j/Q^j$)

$$\frac{p^j}{r^n}\frac{\partial f^j(\phi_{k_j})}{\partial \phi^n_{k_j}}\frac{Q^j_{-k_j}}{Q^j}\frac{W^n_{-k_j}}{W^n} - 1 = 0 \tag{7}$$

$$\sum_{n=1}^{N}\frac{W^n}{E^n}e^n_h + \sum_{j=1}^{J}\sum_{k=1}^{K_j}\theta^{k_j}_h\left(\frac{b^j_h + B^j_{-h}}{Q^j_{-k_j} + f^j(\phi_{k_j})}f^j(\phi_{k_j}) - \sum_{n=1}^{N}w^n_{k_j}\right) - \sum_{j=1}^{J}b^j_h = 0.$$

Note that if the market consists of a very large number of firms, (7) boils down to the statement that the value of the marginal product of the n^{th} input is equal to the price of the n^{th} input because $Q^j_{-k_j}/Q^j$ and $W^n_{-k_j}/W^n$ will be almost one. Letting λ be the Lagrange multiplier of (6), the first-order condition of (5) - (6) with respect to b^j_h (noting that $x_h = (x^1_h, \ldots, x^J_h)$, $x^j_h = b^j_h\frac{Q^j}{B^j}$) is

$$\frac{\partial u_h(x_h)}{\partial x^j_h}\left[\frac{Q^j}{B^j}\frac{B^j_{-h}}{B^j}\right] + \lambda\left[\frac{\sum_{k=1}^{K_j}\theta^{k_j}_h f^j(\phi_{k_j})}{Q^j} - 1\right] = 0. \tag{8}$$

Note that if the market consists of a very large number of firms and households, (8) collapses to the statement that the marginal utility divided by the price is equal to the Lagrange multiplier because B^j_{-h}/B^j will be almost one and $f^j(\phi_{k_j})/Q^j$ will be almost zero.

3.1 Existence of Interior Solution

In this section, we derive the condition under which firms seek interior profit maximum. We first show that as long as the firm's production function is strictly quasi-concave, so is its revenue function. We then show that the firm's cost function is strictly convex. For this analysis, we suppress consideration of the specific production sector, and hence we will drop index j from the notation and keep the notation as is otherwise. However, the analysis can easily be extended to the case with multiple production sectors. On the revenue side, let $h_k \equiv D_{\phi_k}(pq_k) = D_{\phi_k}f(\phi_k)\left[\frac{d(pq_k)}{dq_k}\right]$, where

$$\frac{d(pq_k)}{dq_k} = \frac{BQ_{-k}}{Q^2} \geq 0.$$

Letting production function f be homogeneous of degree $\gamma \geq 0$, Euler's theorem implies that $\phi_k \cdot D_{\phi_k}f(\phi_k) = \gamma f(\phi_k)$. Now consider

$$\phi_k \cdot h_k = \phi_k \cdot D_{\phi_k}f(\phi_k)\left[\frac{d(pq_k)}{dq_k}\right] = \gamma q_k\frac{BQ_{-k}}{Q^2} \geq 0,$$

and as $q_k \to \infty$, $\phi_k \cdot h_k \to 0$. Next, differentiating $\frac{d(pq_k)}{dq_k}$ with respect to q_k, we get

$$\frac{d^2(pq_k)}{dq_k^2} = \frac{d}{dq_k}\left[\frac{BQ_{-k}}{Q^2}\right] = -2\frac{BQ_{-k}}{Q^3} < 0.$$

Then, it follows that

$$\begin{aligned} D_{\phi_k}h_k &= \frac{BQ_{-k}}{Q^2}D^2_{\phi_k}f(\phi_k) - 2D_{\phi_k}f(\phi_k)\frac{BQ_{-k}}{Q^3}D_{\phi_k}f(\phi_k)^T \\ &= \frac{BQ_{-k}}{Q^2}\left[D^2_{\phi_k}f(\phi_k) - \frac{2}{Q}D_{\phi_k}f(\phi_k)D_{\phi_k}f(\phi_k)^T\right]. \end{aligned}$$

Using Euler's theorem yields

$$\phi_k \cdot D_{\phi_k}f(\phi_k) = \gamma f(\phi_k).$$

Differentiating both sides with respect to ϕ_k gives

$$\begin{aligned} &D^2_{\phi_k}f(\phi_k)\phi_k = (\gamma - 1)D_{\phi_k}f(\phi_k) \text{ or} \\ &D^2_{\phi_k}f(\phi_k)\phi_k D_{\phi_k}f(\phi_k)^T = (\gamma - 1)\, D_{\phi_k}f(\phi_k)D_{\phi_k}f(\phi_k)^T. \end{aligned}$$

Hence, denoting the identity matrix by I, we have

$$\begin{aligned} D^2_{\phi_k}f(\phi_k) - \frac{2}{Q}D_{\phi_k}f(\phi_k)D_{\phi_k}f(\phi_k)^T &= D^2_{\phi_k}f(\phi_k) + \frac{2}{Q\,(1-\gamma)}D^2_{\phi_k}f(\phi_k)\phi_k D_{\phi_k}f(\phi_k)^T \\ &= D^2_{\phi_k}f(\phi_k)\left[I + \frac{2}{Q\,(1-\gamma)}\phi_k D_{\phi_k}f(\phi_k)^T\right]. \end{aligned}$$

Now let production function f be strictly quasi-concave.[4] Then, along any direction ρ orthogonal to $D_{\phi_k}f$, we obtain

$$\rho^T D_{\phi_k}h_k\rho = \frac{BQ_{-k}}{Q^2}\rho^T D^2_{\phi_k}f(\phi_k)\rho < 0$$

because $BQ_{-k}/Q^2 > 0$ and f is strictly quasi-concave. Thus, the firm's revenue function is quasi-concave.

On the cost side, firm k's allocation of input good n, $\phi_k^n = w_k^n \frac{E^n}{W^n}$ implies $\frac{W^n_{-k}+w^n_k}{w^n_k} = \frac{E^n}{\phi^n_k}$, which leads to $\frac{W^n_{-k}}{w^n_k} + 1 = \frac{E^n}{\phi^n_k}$. Then, we get

$$w_k^n = \frac{\phi_k^n W^n_{-k}}{E^n - \phi_k^n}.$$

As a result, total cost of firm k is

$$C(\phi_k) = \sum_{n=1}^{N} w_k^n = \sum_{n=1}^{N}\frac{\phi_k^n W^n_{-k}}{E^n - \phi_k^n}.$$

[4]Note that a strictly quasi-concave production function can exhibit any returns to scale, encompassing decreasing, constant, and increasing returns.

Then, $D^2_\phi C(\phi_k)$ is a diagonal matrix consisting of

$$\frac{\partial^2 C(\phi_k)}{\partial(\phi^n_k)^2} = \frac{2W^n_{-k}E^n}{(E^n - \phi^n_k)^3} \geq 0.$$

Because this matrix is positive definite, cost function C is strictly convex. Hence, when γ is sufficiently small (e.g., $\gamma < 1$), profit function π will be strictly quasi-concave. In particular, the upper contour sets for profit function π of any firm will be convex. When we impose the following mild assumption

$$\lim_{\phi\to\infty} \frac{1}{f(\phi)} D_\phi f < \infty,$$

this will guarantee that as ϕ gets large, profit π becomes negative since cost C is unbounded while revenue $R = \frac{B}{Q} f \leq B$ is bounded. Thus, $D_\phi \pi$ will be asymptotically negative. The restriction itself bounds the production function below something exponential, and is sufficient, though not necessary. In this case, when $D_{\phi_k} f(0)$ and E^n are sufficiently large, we obtain

$$D_{\phi_k}\pi_k(0) = \frac{B}{Q_{-k}} D_{\phi_k} f(0) - \frac{W^n_{-k}}{E^n} > 0. \tag{9}$$

The following lemma shows the conditions under which a firm seeks an interior profit maximum.

Lemma 1 *In sectors with sufficiently small γ and sufficiently large $D_\phi f(0)$ and E^n, firms will seek interior profit maximum.*

3.2 Existence of Equilibrium

In this section, we present the main existence theorem.

Theorem 1 *Suppose that there exist at least two firms that use all input goods and satisfy condition (9). Then there exists an equilibrium in which all households make positive bids and some firms make positive bids (i.e., produce).*

Proof. We will prove the existence of equilibrium by using Kakutani's fixed-point theorem. We first define a best-response mapping from a set into itself. When households make positive bids on output goods of two firms that use all inputs, the best response for these two firms, say $\widehat{k}_{\widehat{j}}$ and $\widetilde{k}_{\widetilde{j}}$, is to make positive production, and hence make positive bids on input goods. Similarly, when these two firms make positive bids on input goods, the best response for households is to make positive bids on output goods of these two firms. When households and these two firms start with positive bids, the best response for households and these two firms is to continue to make positive bids. Then, there exists sufficiently small $\epsilon > 0$ such that the vector of household h's bids on output goods $b_h \geq \epsilon \cdot \iota$ for all $h \in \{1, \ldots, H\}$ and the vector of firm k_j's bids on input goods $w_{k_j} \geq \epsilon \cdot \iota$ for all $k_j \in \{\widehat{k}_{\widehat{j}}, \widetilde{k}_{\widetilde{j}}\}$, where ι is the vector of ones. Note that we have a free normalization on all bids because all bids on input and output goods appear in both households' and firms' budget constraints. Thus, we define the vector of all bids β as follows

$$\beta = (b_1, \ldots, b_H, w_{1_1}, \ldots, w_{1_J}, w_{2_1}, \ldots, w_{2_J}, \ldots, w_{K_1}, \ldots, w_{K_J}) \in \Delta_\epsilon^{JH+NK_jJ-1},$$

where b_h is J-dimensional vector for all $h \in \{1, \ldots, H\}$, w_{k_j} is N-dimensional vector for all $k \in \{1, \ldots, K_j\}$ and for all $j \in \{1, \ldots, J\}$, and $\Delta_\epsilon^{JH+NK_jJ-1}$ is ϵ-trimmed unit simplex. We define a mapping $\zeta : \Delta_\epsilon^{JH+NK_jJ-1} \to \Delta_\epsilon^{JH+NK_jJ-1}$, where

$$\zeta(\beta) = \frac{1}{\iota^T \cdot \widehat{\beta}(\beta)} \widehat{\beta}(\beta),$$

ι is the vector of ones, and $\widehat{\beta}\,(\beta)$ is the vector of best responses to β.

We then show that there exists an equilibrium via Kakutani's fixed-point theorem. First, ζ is upper hemicontinuous via the maximum theorem given the standard assumptions on utility and production functions. Moreover, because $\beta \in \Delta_\epsilon^{JH+NK_jJ-1}$, we have $\iota^T \cdot \beta = 1$ and $\iota \geq \beta \geq \epsilon \cdot \iota$. Second, $\Delta_\epsilon^{JH+NK_jJ-1}$ is non-empty because at least $\left(\frac{1}{JH+NK_jJ-1}, \ldots, \frac{1}{JH+NK_jJ-1}\right)$ is in $\Delta_\epsilon^{JH+NK_jJ-1}$. Third, $\Delta_\epsilon^{JH+NK_jJ-1}$ is compact because it is closed and bounded over $[\epsilon, 1]^{JH+NK_jJ-1}$. Finally, $\Delta_\epsilon^{JH+NK_jJ-1}$ is convex as follows. Let $\beta \in \Delta_\epsilon^{JH+NK_jJ-1}$, $\widetilde{\beta} \in \Delta_\epsilon^{JH+NK_jJ-1}$, and $\eta \in [0, 1]$. By definition, we have $\iota^T \cdot \beta = 1$ and $\iota^T \cdot \widetilde{\beta} = 1$ and also $\beta \geq \epsilon \cdot \iota$ and $\widetilde{\beta} \geq \epsilon \cdot \iota$. Then, we obtain $\eta\iota^T \cdot \beta + (1-\eta)\iota^T \cdot \widetilde{\beta} = \eta + (1-\eta) = 1$ and $\eta\beta + (1-\eta)\widetilde{\beta} \geq \eta\epsilon \cdot \iota + (1-\eta)\epsilon \cdot \iota = \epsilon \cdot \iota$. Thus, $\eta\beta + (1-\eta)\widetilde{\beta} \in \Delta_\epsilon^{JH+NK_jJ-1}$. So, $\zeta(.)$ has a fixed point, i.e., there is $\beta \in \Delta_\epsilon^{JH+NK_jJ-1}$ such that $\beta \in \zeta(\beta)$ via Kakutani's fixed-point theorem, and it is a Nash equilibrium of this game. ■

In the proof of Theorem 1, satisfying condition (9) is necessary to guarantee non-negative profits for firms, and the presence of at least two firms that use all inputs is important to open markets for input goods. The proof can be generalized to the case in which any number of firms use subsets of inputs as long as each input is used by at least two firms, and these two firms satisfy condition (9). Besides two firms that use all inputs, other firms can also be active depending on initial endowments.

3.3 Extended Example

In the previous section, we have shown the existence of equilibrium when condition (9) is satisfied for at least two firms. In this section, we will provide more complete characterization under the condition that all firms and households are identical, and under symmetric equilibrium. We will in turn relax the assumptions that production function f is homogeneous of degree γ and that condition (9) is satisfied. Because all firms are identical, there is a single sector in the economy, so we will drop index j. The following arguments can be generalized to multiple sectors but such generalization would complicate the analysis yet would not bring any new insights.

We first introduce best responses in this case. Under identical firms and households, each firm

k's profit maximization problem becomes

$$\max_{w_k^1,\ldots,w_k^N} \frac{B}{Q_{-k}+f\left(\phi_k\right)} f\left(\phi_k\right)-\sum_{n=1}^N w_k^n, \tag{10}$$

and each household h's utility maximization problem becomes

$$\max_{b_h} u_h\left(x_h\right) \tag{11}$$

$$\text{s.t. } b_h \leq \sum_{n=1}^N \frac{W^n}{E^n} e_h^n+\sum_{k=1}^K \theta_h^k\left(\frac{b_h+B_{-h}}{Q_{-k}+f\left(\phi_k\right)} f\left(\phi_k\right)-\sum_{n=1}^N w_k^n\right), \tag{12}$$

where $\phi_k=(w_k^1 \frac{E^1}{W^1}, \ldots, w_k^N \frac{E^N}{W^N})$ and $x_h=b_h \frac{Q}{B}$. The first-order condition of (10) with respect to w_k^n is

$$\frac{B Q_{-k}}{\left(Q_{-k}+f\left(\phi_k\right)\right)^2} \frac{\partial f\left(\phi_k\right)}{\partial \phi_k^n} \frac{E^n W_{-k}^n}{\left(W^n\right)^2}-1, \forall n. \tag{13}$$

Evaluating (13) at symmetric equilibrium $w_k^n=\widehat{w}^n$ for all $n \in\{1, \ldots, N\}$ (noting that $\widehat{W}^n=K \widehat{w}^n$, $Q=K f\left(\phi_k\right)$, and $\phi_k=(w_k^1 \frac{E^1}{W^1}, \ldots, w_k^N \frac{E^N}{W^N}))$ and letting $E=\left(E^1, \ldots, E^N\right)$ gives

$$\begin{aligned}
\frac{B}{f\left(\phi_k\right)} \frac{\partial f\left(\phi_k\right)}{\partial \phi_k^n} \frac{E^n(K-1)^2}{K^4 \widehat{w}^n}-1 &= 0, \forall n \\
\frac{B}{f\left(\frac{E}{K}\right)} \frac{\partial f\left(\frac{E}{K}\right)}{\partial \phi_k^n} \frac{E^n(K-1)^2}{K^4 \widehat{w}^n}-1 &= 0, \forall n .
\end{aligned}$$

Letting λ be the Lagrange multiplier of (12), the first-order conditions of (11) - (12) are (noting that $E^n=H e^n$ because of identical households)

$$u^{\prime}\left(x_h\right) \frac{B_{-h} Q}{B^2}+\lambda\left(\sum_{k=1}^K \frac{\theta_h^k f\left(\phi_k\right)}{Q}-1\right)=0 \tag{14}$$

$$\sum_{n=1}^N \frac{W^n}{H}+\sum_{k=1}^K \theta_h^k\left(\frac{b_h+B_{-h}}{Q} f\left(\phi_k\right)-\sum_{n=1}^N w_k^n\right)-b_h=0. \tag{15}$$

Evaluating (14) and (15) at symmetric equilibrium $b_h=\widehat{b}$, $\theta_h^k=\frac{1}{H}$, $w_k^n=\widehat{w}^n$ (noting that $\widehat{B}=H \widehat{b}$, $Q=K f\left(\phi_k\right)$, $\widehat{W}^n=K \widehat{w}^n$, and $x_h=b_h \frac{Q}{B}$), we obtain

$$\begin{aligned}
u^{\prime}\left(\frac{Q}{H}\right) \frac{(H-1) Q}{H^2 \widehat{b}}+\lambda\left(\frac{1}{H}-1\right) &= 0 \\
\sum_{n=1}^N \frac{K \widehat{w}^n}{H}+\frac{K}{H}\left(\frac{H \widehat{b}}{K}-\sum_{n=1}^N \widehat{w}^n\right) &= \widehat{b} .
\end{aligned} \tag{16}$$

After simplifications, budget constraint (16) yields

$$\widehat{b} = \sum_{n=1}^{N} \frac{K\widehat{w}^n}{H} + \left(\widehat{b} - \frac{K}{H} \sum_{n=1}^{N} \widehat{w}^n \right) = \widehat{b}.$$

Thus, (16) yields an identity. Then, the following system of equations is necessary for equilibrium

$$\frac{H\widehat{b}}{f\left(\frac{E}{K}\right)} \frac{\partial f\left(\frac{E}{K}\right)}{\partial \phi_k^n} \frac{E^n(K-1)^2}{K^4} = \widehat{w}^n, \forall n \tag{17}$$

$$u'\left(\frac{Kf\left(\frac{E}{K}\right)}{H}\right) \frac{Kf\left(\frac{E}{K}\right)}{H\widehat{b}} = \lambda. \tag{18}$$

Finally, we need to check whether firms make nonnegative profits under any solution to (17) - (18) because only such a solution can be equilibrium. The profit of each firm under solutions to (17) - (18) is

$$\pi = \frac{H\,\widehat{b}}{K} - \sum_{n=1}^{N} \widehat{w}^n.$$

Substituting (17) gives

$$\begin{aligned} \pi &= \frac{H\,\widehat{b}}{K} \left[1 - \left(\frac{K-1}{K} \right)^2 \sum_{n=1}^{N} \frac{\partial f\left(\frac{E}{K}\right)}{\partial \phi_k^n} \frac{\frac{E^n}{K}}{f\left(\frac{E}{K}\right)} \right] \\ &= \frac{H\,\widehat{b}}{K} \left[1 - \left(\frac{K-1}{K} \right)^2 \sum_{n=1}^{N} \mu^n(\phi_k) \right], \end{aligned}$$

where $\mu^n(\phi_k) = \frac{\partial \log(f(\phi_k))}{\partial \log(\phi_k^n)}$. In fact, $\mu^n(\phi_k)$ is the n^{th} input elasticity of production. Then, firms make nonnegative profits if and only if

$$\sum_{n=1}^{N} \mu^n\left(\frac{E}{K}\right) \leq \left(1 + \frac{1}{K-1}\right)^2.$$

As a result, we reach the following proposition.

Proposition 1 *Given the number of firms K, input vector $E = (E^1, \ldots, E^N)$, and production function f, a symmetric equilibrium exists if and only if*

$$\sum_{n=1}^{N} \mu^n\left(\frac{E}{K}\right) \leq \left(1 + \frac{1}{K-1}\right)^2, \tag{19}$$

where $\mu^n(\phi_k) = \frac{\partial \log(f(\phi_k))}{\partial \log(\phi_k^n)}$ for all $n \in \{1, ..., N\}$, and $\phi_k = (\phi_k^1, \ldots, \phi_k^N)$.

Corollary 1 shows the relation between the input elasticity of production, total endowments, and the number of firms.

Corollary 1 *Only a few firms can be active in equilibrium when i) the input elasticity of production is constant, or ii) the input elasticity of production is decreasing and total endowments are limited.*

Example 1 *Suppose that each firm uses input good ϕ, and its technology is specified by a Cobb-Douglas production function, i.e., $f(\phi) = \alpha(\phi)^c$, where $\alpha > 0$ and $c > 0$. The input elasticity of production is c, which is a constant. Then, (19) implies that the number of firms $\sqrt{c} \leq \left|1 + \frac{1}{K-1}\right|$. For example, if $c = 1.44$, the number of firms $K \leq 6$.*

Note that in Example 1, the input elasticity of production c is in fact returns to scale. Thus, when $c > 1$, technology exhibits increasing returns to scale. Therefore, in the presence of increasing returns, only a few firms can be active in equilibrium.

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