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Bike sharing systems have been installed in many cities around the world and are increasing in popularity. A major operational cost driver in these systems is rebalancing the bikes over time such that the appropriate number of bikes and open docks are available to users. We combine two aspects that have previously been handled separately in the literature: determining service level requirements at each bike sharing station, and designing (near-)optimal vehicle routes to rebalance the inventory. Since finding provably optimal solutions is practically intractable, we propose a new cluster-first route-second heuristic, in which the polynomial-size Clustering Problem simultaneously considers the service level feasibility constraints and approximate routing costs. Extensive computational results on real-world data from Hubway (Boston, MA) and Capital Bikeshare (Washington, DC) are provided, which show that our heuristic outperforms a pure mixed integer programming formulation and a constraint programming approach.

Key words: vehicle routing and scheduling, inventory, queues: applications, programming: integer, programming: constraints, heuristics

1. Introduction

Bike sharing systems are experiencing wide-spread adoption in major cities around the world, with over 300 active systems and more than 200 in planning (Meddin and DeMaio 2012). In these systems, users can pickup and return bikes at designated bike sharing stations with a finite number of docks. Unfortunately, user behavior results in spatial imbalance of the bike inventory over time. The system equilibrium is often characterized by unacceptably low availability of bikes or open docks, for pickups or returns respectively (Fricker et al. 2012, p. 375).

Therefore, operators deploy a fleet of trucks to *rebalance* the bike inventory. We focus on the efficiency of these rebalancing operations, a major cost driver for operators (DeMaio 2009, p. 50). This problem consists of two main components. First, determining the desired inventory level at each bike station, which is typically done by an analysis of historic user data. Second, designing truck routes that will perform the necessary pickups and deliveries in order to reach the target

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inventory levels. In the literature, these two aspects have been mainly considered separately (see Section 2 for a literature review). However, in practice operators often face a tradeoff between user satisfaction and the cost of rebalancing. Moreover, as will be shown in this paper, it may be advantageous to include stations in a route even though their inventory is already satisfactory, in order to satisfy the service level requirements at a nearby location. For these reasons, we propose to consider the inventory balancing problem and the routing problem simultaneously.

We model the stochastic demand by viewing the inventory at each station as a queuing system with finite capacity and derive closed-form *service level requirements* on the transient distribution of the availability of bikes and docks (Section 3). We recognize that service level requirements can be met when the inventory is between a lower and upper bound. We then introduce the notion of *self-sufficient* stations. Self-sufficient stations meet the service level requirements with their starting inventory and hence should *not necessarily* be visited by a vehicle.

Next, we consider the routing of vehicles to achieve the service level requirements. Observe that the pickup and delivery of bikes is unpaired, i.e., our situation corresponds to an extended One-Commodity Pickup-and-Delivery Vehicle Routing Problem (1-PDVRP, see Hernández-Pérez and Salazar-González 2003). However, while preventing unnecessary vehicle visits, the inventory flexibility at each station adds another layer of complexity to the classical Routing Problem (Section 4). Since exact mixed integer programs (MIPs) for vehicle routing problems become intractable for realistic instances, we present a new cluster-first route-second heuristic in Section 5. We incorporate the service level feasibility constraints in a polynomial-size Clustering Problem (modeled as a MIP). The routing costs are incorporated into the clustering problem via a new approximation based on maximum spanning stars. Lastly, we present an improvement scheme based on elimination cuts to mitigate the approximation error.

In addition, we present a Constraint Programming (CP) formulation in Section 6, acting both as an effective (exact) solution method for smaller instances and a benchmark for our dedicated clustered routing heuristics.

Computational experiments (Section 7) using data from Hubway (Boston, MA) and Capital Bikeshare (Washington, DC) show that our heuristics strongly outperform the classical MIP model. Within seconds, we identify a feasible solution with a reasonable optimality gap. In a minute, we find better solutions for almost all instances than the best MIP solution after 2 hours. Moreover, our dedicated Clustered MIP heuristic outperforms Constraint Programming on larger instances.

We summarize the main contributions of this paper as follows. First, we present a novel formulation of the rebalancing problem using dual-bounded service level constraints. Second, we present a new polynomial-size Clustering Problem with service level feasibility constraints. Third, we develop heuristics that are fast and accurate for large instances, and outperform state-of-the-art existing methods.

2. Related Work

The study of bike sharing systems is increasing in popularity. DeMaio (2009) and Shaheen et al. (2010) provide a history of bicycle sharing, starting with the first generation 'white bikes' in Amsterdam as early as 1965. From 1995 onwards, the third generation IT-based systems incorporate 'advanced technologies for bicycle reservations, pickup, drop-off, and information tracking' (Shaheen et al. 2010, p. 7). We identify four research substreams in bike sharing literature: strategic design, demand analysis, service level analysis, and rebalancing operations.

Strategic design. Since most major cities have planned or considered implementation of bike sharing systems, and existing systems are often expanded, several studies present models dedicated to strategic design. Dell'Olio et al. (2011) develop a comprehensive methodology for implementation, from estimating potential demand to optimizing locations. Martinez et al. (2012) and Prem Kumar and Bierlaire (2012) present MIP models for the location problem. Lin and Yang (2011) create a model trading off the interests of both users and investors. We note that none of the studies include a notion of expected inventory imbalance costs.

Demand analysis. The purpose of demand analysis is twofold: forecasting future demand (e.g. for service level requirements) and understanding the explaining factors for managerial decision making. Kaltenbrunner et al. (2010) predict the system inventory state and suggest making such information available to users. Froehlich and Oliver (2008), Borgnat and Abry (2009), Borgnat et al. (2011), and Lathia et al. (2012) identify a temporal demand pattern and forecast the number of rentals. Hampshire and Marla (2012) seek land use and socio-economic factors that explain the use of bike sharing systems. Vogel et al. (2011) construct clusters of stations with similar demand patterns. These studies could provide insights that are helpful in improving the service level requirements developed in this paper.

Service level analysis. Several studies focus on service levels in bike sharing systems. We recall that service level requirements are typically two-sided: for available bikes and docks. Most notably, Nair and Miller-Hooks (2011) and Nair et al. (2013) decompose system-wide reliability into a set of dual-bounded chance constraints for each station. We adapt their dual-bounded service level constraints, but use a more realistic Markov chain to model the station inventory over time, as opposed to observing the total net demand (see Section 3). Raviv et al. (2011a) present a queuing system with finite capacity to model expected user dissatisfaction at each station, similar to our approach. However, we assume time-independent user arrival rates to derive closed-form solutions. Leurent (2012) models bike sharing stations as a dual Markovian waiting system, but contrary to their assumption of waiting customers, we assume immediate lost sales.

Rebalancing operations. Using mean field analysis, Fricker et al. (2012) conclude that the equilibrium system performance collapses under heterogeneity of user behavior and that a pressing need for rebalancing operations exists. Vogel and Mattfeld (2010) motivate rebalancing activities using a system dynamics model. Shu et al. (2010) estimate rebalancing operations can lead to an additional 15-20% of trips supported system-wide.

We identify two modes of rebalancing: providing user incentives and deploying a truck fleet. While Fricker and Gast (2012) and Waserhole and Jost (2012) develop a pricing strategy, and some active systems have already implemented incentive schemes (e.g. *V+* for Vélib', see Fricker and Gast 2012, p. 20), we observe that every bike sharing system still operates a vehicle fleet for rebalancing. Therefore, the underlying vehicle routing problem has received most attention.

Most studies use an exact target inventory for each station, which implies their routing problem is more closely related to the 1-PDVRP than ours, which adds inventory flexibility. Routing costs to attain exact target inventories will always be higher than strictly necessary to maintain appropriate service levels (see Section 3). We note that our models can be parameterized to solve the target inventory problem. Caggiani and Ottomanelli (2012) construct a decision support system for routing. Chemla et al. (2012) present a branch-and-cut algorithm for the *single-vehicle* problem, with results on instances of up to 100 stations. Approximation algorithms for the same problem are given by Benchimol et al. (2011). The work of Contardo et al. (2012) can be considered state of the art for the *multi-vehicle* target inventory problem, using Dantzig-Wolfe and Benders decomposition to derive lower bounds and feasible solutions with low computing times (approximately 5 minutes) for instances of 5 vehicles and up to 100 stations.

Raviv et al. (2011a) present an alternative approach: the expected system-wide user dissatisfaction is minimized subject to a time limit. Their arc-, time-, and sequence-indexed MIP models are intractable for systems of reasonable size, however we rely strongly on these models in Section 4 to present the Bike Sharing Rebalancing Problem in its pure MIP form. As mentioned, the dual-bounded chance constraints from Nair et al. (2013) inspire our service level requirements, but their paper presents a swapping-based rebalancing model and no explicit vehicle routing.

Lin and Chou (2012) propose an optimization method taking road conditions, traffic regulations, and geographical factors into account, rather than simply using Euclidian distance. Their actual distance path calculation is used to implement a heuristic for the VRP. Naturally, using actual distances would lead to decreased costs in practice.

3. Service Level Requirements

In general, inventory rebalancing efforts are made in order to improve customer service. Contrary to traditional inventory theory, where the service level increases as inventory increases, bike sharing systems are subject to a *net demand process*, with an empty station preventing users from picking up bikes *and* a full station preventing returns. For the remainder of the paper, we let $\mathcal{S}$ represent the set of bike sharing stations. Let C_i denote the capacity (i.e. number of docks) of station $i \in \mathcal{S}$.

Net demand implies that after a pickup and a return occur, the inventory is unchanged. Previous studies (Nair et al. 2013) observe the *total* net demand (pickups minus returns during observation period) at each station $i \in \mathcal{S}$. However, we note that while observing a net demand of zero, we often still need bikes *and* docks. We view the net demand as a stochastic process which needs to be satisfied during the entire observation period, not only at the end. In Appendix A we motivate why observing the total net demand is insufficient.

3.1. Service Level Definition

Operators can benefit greatly from measuring their service level (Gunasekaran et al. 2001). But due to censoring, it is nontrivial to observe lost sales in bike sharing systems. Therefore, operators commonly measure the (fraction of) time that their stations are full or empty. Some operators are even penalized by the local government in proportion to such a time measure (e.g. Vélib' in Paris).

In the next section, we show that the number of dissatisfied customers is proportional to this time measure under the assumption of Poisson demand. Therefore, we can implement a measurable *type 2* service level: the *fraction of demands satisfied directly* should be larger than β_i^- for pickups and larger than β_i^+ for returns. We assume no backorders, i.e., the effect of waiting customers is negligible. Customers often choose alternative stations in case of stock outs, we assume this behavior is implicit in the (independent) demand processes.

Definition 1. The service level requirements at station $i \in \mathcal{S}$ are

$$\frac{\text{E[Satisfied bike pickup demands]}}{\text{E[Total bike pickup demands]}} \geq \beta_i^-$$

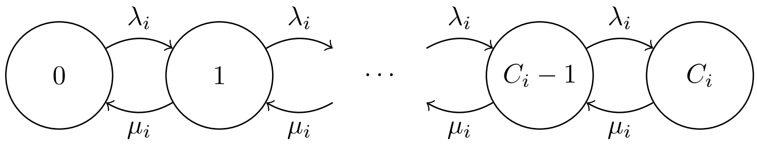
$$\frac{\text{E[Satisfied bike return demands]}}{\text{E[Total bike return demands]}} \geq \beta_i^+$$

for given $\beta_i^-, \beta_i^+ \in [0,1]$.

3.2. Markov Chain Formulation

The inventory at station $i \in \mathcal{S}$ can be modeled as an $M/M/1/K$ queuing system (Kendall 1953), with the number of customers in the queue representing the inventory. This implies that the customer inter-arrival times (for bike returns) and service times (i.e. inter-arrival times for bike pickups) are exponentially distributed with rates λ_i and μ_i, respectively, at each station $i \in \mathcal{S}$. We implicitly assume that user behavior during the observation period is stationary. While some users (e.g. tourists) arrive simultaneously (compound Poisson process), we assume this effect is negligible. There is 1 server and there are $K = C_i$ waiting spaces in the system for station $i \in \mathcal{S}$. The $M/M/1/K$ queue is well-studied (Morse 1958) and closed-form expressions for the transient probabilities given a starting state are available.

Figure 1 Markov chain for the inventory $S_i(t)$ at station $i \in \mathcal{S}$.



Note. Users arrive with rate λ_i to return bikes and with rate μ_i to pickup bikes.

Denote by $\{S_i(t) : t \geq 0\}$ the stochastic process on state space $\{0, \ldots, C_i\}$ representing the inventory of station $i \in \mathcal{S}$ at time $t \geq 0$. Define $p_i(s, \sigma, t) \equiv \Pr(S_i(t) = \sigma \mid S_i(0) = s)$, the transient probability that the inventory at station $i \in \mathcal{S}$ equals $\sigma \in \{0, \ldots, C_i\}$ at time $t \geq 0$ given starting inventory $s \in \{0, \ldots, C_i\}$. Define $g_i(s, \sigma) \equiv \frac{1}{T} \int_0^T p_i(s, \sigma, t)\, \mathrm{d}t$ as the expected fraction of the observation period $[0, T]$ for which the inventory at station $i \in \mathcal{S}$ equals σ given starting inventory s.

Then, in order to meet both the pickup and return service level from Definition 1 during the observation period $[0, T]$, we calculate the expected values:

$$\begin{aligned}
\frac{\mathrm{E}[\text{Satisfied bike pickup demands}]}{\mathrm{E}[\text{Total bike pickup demands}]} &= \frac{\int_0^T \mu_i \,(1 - p_i(s, 0, t))\, \mathrm{d}t}{\mu_i T} \\
&= 1 - \frac{\int_0^T p_i(s, 0, t)\, \mathrm{d}t}{T} \\
&= 1 - g_i(s, 0)
\end{aligned}$$

and similarly

$$\frac{\mathrm{E}[\text{Satisfied bike return demands}]}{\mathrm{E}[\text{Total bike return demands}]} = 1 - g_i(s, C_i).$$

Thus, inventory level s satisfies the service level requirements at station $i \in \mathcal{S}$ when $1 - g_i(s, 0) \geq \beta_i^-$ and $1 - g_i(s, C_i) \geq \beta_i^+$.

LEMMA 1. *A closed-form expression for $g_i(s, \sigma)$ exists.*

Proof of Lemma 1. Morse (1958, p. 64) presents a transient solution for the $M/M/1/N$ queue (note that $N = C_i$):

$$p_i(s, \sigma, t) = \pi_i(\sigma) + \frac{2\rho_i^{\frac{1}{2}(\sigma - s)}}{C_i + 1} \sum_{m=1}^{C_i} K_{i,m} e^{-k_{i,m} t}$$

with

$$\rho_i = \frac{\lambda_i}{\mu_i}$$
$$\pi_i(\sigma) = \begin{cases} \frac{1}{C_i+1} & \text{if } \rho_i = 1 \\ \frac{1-\rho_i}{1-\rho_i^{C_i+1}}\rho_i^\sigma & \text{otherwise} \end{cases}$$
$$K_{i,m} = \left(\frac{\mu_i}{k_{i,m}}\right)\left(\sin\frac{ms\pi}{C_i+1} - \sqrt{\rho_i}\sin\frac{m(s+1)\pi}{C_i+1}\right)\left(\sin\frac{m\sigma\pi}{C_i+1} - \sqrt{\rho_i}\sin\frac{m(\sigma+1)\pi}{C_i+1}\right)$$
$$k_{i,m} = \lambda_i + \mu_i - 2\sqrt{\lambda_i\mu_i}\cos\left(\frac{m\pi}{C_i+1}\right)$$

The antiderivative $P_{s,\sigma}(t)$ follows naturally:

$$\begin{aligned} P_i(s,\sigma,t) &= \int p_i(s,\sigma,t)\,\mathrm{d}t \\ &= \pi_i(\sigma)t - \frac{2\rho_i^{\frac{1}{2}(\sigma-s)}}{C_i+1}\sum_{m=1}^{C_i} K_{i,m}\frac{e^{-k_{i,m}t}}{k_{i,m}}. \end{aligned}$$

Thus, $g_i(s,\sigma) = \frac{1}{T}\int_0^T p_i(s,\sigma,t)\,\mathrm{d}t = \frac{1}{T}\left(P_i(s,\sigma,T) - P_i(s,\sigma,0)\right)$. □

Intuitively, as the starting inventory increases, the bike pickup service level increases and the bike return service level decreases. We note that indeed

$$g_i(s+1,0) - g_i(s,0) \le 0 \text{ and}$$
$$g_i(s+1,C_i) - g_i(s,C_i) \ge 0$$

for $s \in \{0,\ldots,C_i-1\}$, which gives us Lemma 2.

LEMMA 2. *To meet the service level requirements from Definition 1 at station $i \in \mathcal{S}$, it must hold that $s_i \ge s_i^{\min}$ and $s_i \le s_i^{\max}$ with*

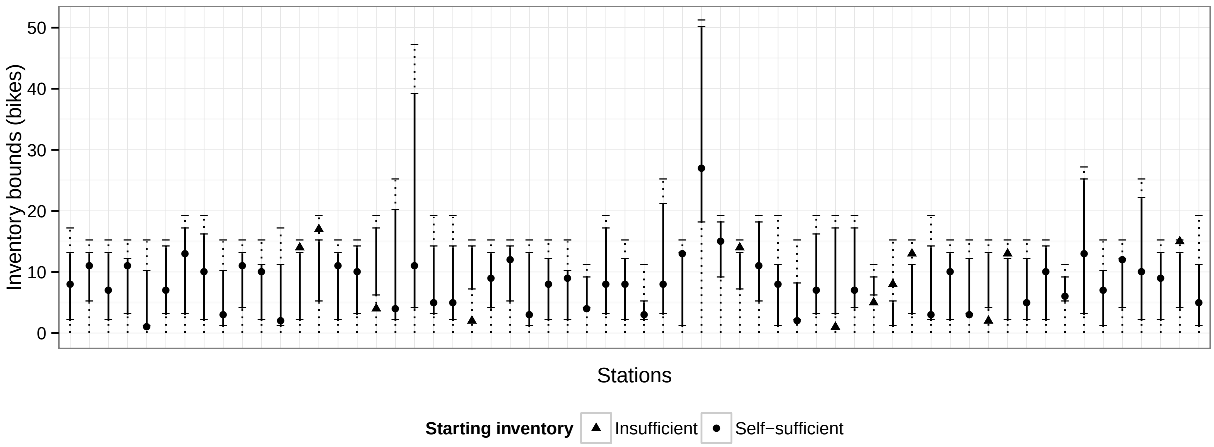
$$s_i^{\min} = \min\left\{s \in \{0,\ldots,C_i\} : 1 - g_i(s,0) \ge \beta_i^-\right\}$$
$$s_i^{\max} = \max\left\{s \in \{0,\ldots,C_i\} : 1 - g_i(s,C_i) \ge \beta_i^+\right\}$$

and $g_i(s,\sigma) = \frac{1}{T}\int_0^T p_i(s,\sigma,t)\,\mathrm{d}t$.

Hence, vehicles should rebalance the starting inventory s_i^0 such that $s_i^{\min} \le s_i \le s_i^{\max}$ for each station $i \in \mathcal{S}$, in order to meet the service levels β_i^-, β_i^+. Denote by $\mathcal{S}_0 = \{i \in \mathcal{S} : s_i^{\min} \le s_i^0 \le s_i^{\max}\}$ the set of *self-sufficient* stations which satisfy the service level requirements with their starting inventory.

Note that the service level requirements could theoretically be *infeasible*, e.g. during peak hours. We identify three types of infeasibility:

Figure 2 Example of Service Level Requirements for Hubway (Boston, MA).



Note. Observation period 8–9AM on weekdays with $\beta_i^- = \beta_i^+ = 95\%$. The range $[s_i^{\min}, s_i^{\max}]$ is displayed solid with $[0, C_i]$ displayed dotted. For reference, we show the starting inventory s_i^0 for each station as a solid marker, based on a snapshot of the system taken at 8AM on Friday June 1st 2012.

- $s_i^{\max} < s_i^{\min}$ requires the operator to prioritize the service level requirement for either pickups or returns at station $i \in \mathcal{S}$, or choose a weighted average. We observe that most operators prioritize returns, to prevent users incurring unintended late-return fees.
- $1 - g_i(C_i, 0) < \beta_i^-$ or $1 - g_i(0, C_i) < \beta_i^+$ implies the inventory always violates (one of) the service level requirements. This requires the operator to choose the best possible inventory bounds.
- $\sum_{i \in \mathcal{S}} s_i^0 < \sum_{i \in \mathcal{S}} s_i^{\min}$ or $\sum_{i \in \mathcal{S}} s_i^0 > \sum_{i \in \mathcal{S}} s_i^{\max}$ implies a system-wide shortage or excess (ignoring vehicle inventory and capacity for the sake of simplicity).

All types of infeasibility would require the operator to take alternative measures in case service level problems persist, e.g., increase station capacity, influence user behavior, or introduce or remove bikes from the system. However, we encountered no infeasibilities in processing any of our data.

Using Lemma's 1 and 2, we are able to efficiently calculate the service level requirements at each station using closed-form expressions.

Example 1. For this example, we use trip data provided by Hubway (Boston, MA) for the 60 stations that were active between November 1st 2011 and May 31st 2012. In Appendix B, we give a detailed overview of the data sources used for our examples and computational results. The observation period is 8–9AM ($T = 1$) on weekdays, for which we have 82 observations. We estimate λ and μ by the mean (Maximum Likelihood Estimator for Poisson variables) number of returns and pickups, respectively, per observation period.

By requiring a $\beta_i^- = \beta_i^+ = 95\%$ service level at each station $i \in \mathcal{S}$, we calculate $s_i^{\min}$ and $s_i^{\max}$ using Lemma 2. The service level requirements for each station are depicted as bounds in Figure 2. We observe that there is a lot of flexibility in the target inventory, with most stations having relatively wide service level bounds. For one of the stations, $s_i^{\min}$ is relatively high, but this is paired with a higher than average $s_i^{\max}$, in anticipation of bike pickups clearing sufficient docks.

4. Routing Problem

Recall that vehicles should rebalance the starting inventory s_i^0 such that $s_i^{\min} \leq s_i \leq s_i^{\max}$ for each station $i \in \mathcal{S}$, in order to meet the service level requirements. We define the Routing Problem, a pure MIP approach to the Bike Sharing Rebalancing Problem, inspired by Raviv et al. (2011a). The bike sharing system is represented as a complete directed graph with vertex set $\mathcal{S}$ and distances $d_{i,j}$ for all $i, j \in \mathcal{S}$. We observe that using a model that is both arc- and sequence-indexed increases the size, but yields much stronger relaxations than the sequence-indexed model.

Several objectives could be applied to this routing problem, e.g. minimizing the total distance. However, to maximize user satisfaction it is usually desired to finish the rebalancing operations as soon as possible. For this reason, we minimize the maximum tour length of the vehicles, i.e. the *makespan* of the schedule. Denote by h_v the routing costs (distance traveled) of vehicle $v \in \mathcal{V}$. Then, our objective is to minimize the makespan $H = \max_{v\in\mathcal{V}} h_v$ of rebalancing the bike inventory such that service level requirements at each station are met. We assume user activity during the rebalancing operations is negligible. We allow arbitrary route start and end points (no closed tour) to implement the model on a rolling horizon.

We use a set $\mathcal{T} = \{1, \ldots, T\}$ for sequence indexing. We introduce binary decision variables $x_{i,j,t,v}$ to indicate whether vehicle $v \in \mathcal{V}$ traverses arc (i, j) in time step $t \in \mathcal{T}$. Decision variables $y^-_{i,t,v}, y^+_{i,t,v}$ indicate bike pickup or delivery, respectively, by vehicle $v \in \mathcal{V}$. We use q_v^0 and Q_v for the initial inventory and capacity of vehicle $v \in \mathcal{V}$, respectively. The formulation then becomes:

$$\text{minimize } H \tag{P1}$$

s.t.

$$s_i^0 + \sum_{t\in\mathcal{T}} \sum_{v\in\mathcal{V}} (y^+_{i,t,v} - y^-_{i,t,v}) \geq s_i^{\min} \qquad \forall i \in \mathcal{S} \tag{1}$$

$$s_i^0 + \sum_{t\in\mathcal{T}} \sum_{v\in\mathcal{V}} (y^+_{i,t,v} - y^-_{i,t,v}) \leq s_i^{\max} \qquad \forall i \in \mathcal{S} \tag{2}$$

$$\sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{N}} x_{i,j,1,v} \leq 1 \qquad \forall v \in \mathcal{V} \tag{3}$$

$$\sum_{j\in\mathcal{N}} x_{i,j,t,v} \leq \sum_{j\in\mathcal{S}} x_{j,i,t-1,v} \qquad \forall i \in \mathcal{S}, t \in \mathcal{T} \setminus \{1\}, v \in \mathcal{V} \tag{4}$$

$$\sum_{i\in\mathcal{S}} \sum_{t\in\mathcal{T}} \sum_{v\in\mathcal{V}} x_{i,i,t,v} = 0 \tag{5}$$

$$y^-_{i,t,v} \le Q_v \sum_{j\in\mathcal{N}} x_{i,j,t,v} \qquad \forall i\in\mathcal{S}, t\in\mathcal{T}, v\in\mathcal{V} \qquad (6)$$

$$y^+_{i,t,v} \le Q_v \sum_{j\in\mathcal{S}} x_{j,i,t,v} \qquad \forall i\in\mathcal{S}, t\in\mathcal{T}, v\in\mathcal{V} \qquad (7)$$

$$\sum_{t\in\mathcal{T}}\sum_{v\in\mathcal{V}} y^-_{i,t,v} \le s^0_i \qquad \forall i\in\mathcal{S} \qquad (8)$$

$$\sum_{t\in\mathcal{T}}\sum_{v\in\mathcal{V}} y^+_{i,t,v} \le C_i - s^0_i \qquad \forall i\in\mathcal{S} \qquad (9)$$

$$q^0_v + \sum_{i\in\mathcal{S}}\sum_{\bar{t}\in\mathcal{T}:\bar{t}\le t} (y^-_{i,t,v} - y^+_{i,t,v}) \ge 0 \qquad \forall t\in\mathcal{T}, v\in\mathcal{V} \qquad (10)$$

$$q^0_v + \sum_{i\in\mathcal{S}}\sum_{\bar{t}\in\mathcal{T}:\bar{t}\le t} (y^-_{i,t,v} - y^+_{i,t,v}) \le Q_v \qquad \forall t\in\mathcal{T}, v\in\mathcal{V} \qquad (11)$$

$$h_v = \sum_{i,j\in\mathcal{S}}\sum_{t\in\mathcal{T}} d_{i,j} x_{i,j,t,v} \qquad \forall v\in\mathcal{V} \qquad (12)$$

$$H \ge h_v \qquad \forall v\in\mathcal{V} \qquad (13)$$

$$x_{i,j,t,v} \in \{0,1\} \qquad \forall i\in\mathcal{S}, j\in\mathcal{N}, t\in\mathcal{T}, v\in\mathcal{V}$$

$$y^-_{i,t,v}, y^+_{i,t,v} \in \mathbb{N}_0 \qquad \forall i\in\mathcal{S}, t\in\mathcal{T}, v\in\mathcal{V}$$

$$H, h_v \ge 0 \qquad \forall v\in\mathcal{V}$$

Constraints (1)-(2) impose the service level requirements from Lemma 2. Constraints (3)-(7) take care of vehicle routing. Constraints (3) imply that each vehicle starts at most one route. Constraints (4) take care of flow conservation. Constraints (5) prevent dwelling. Constraints (6)-(7) ensure pickup or delivery can only take place when leaving from or arriving at a station, respectively. Note that $\mathcal{N} \equiv \mathcal{S}\cup\{0\}$ extends the station set with an artificial vertex 0 to allow picking up bikes at the final stop without incurring routing costs. Constraints (8)-(9) limit the amount of transshipments, because of the model's inability to track station inventory over time (cf. Raviv et al. 2011a, p. 10). Constraints (10)-(11) ensure that the vehicle inventory remains non-negative and within vehicle capacity at all times. Constraints (12) define the routing costs for each vehicle. Finally, constraints (13) linearize the objective $H = \max_{v\in\mathcal{V}} h_v$. Note that the integrality constraints on $y^-_{i,t,v}, y^+_{i,t,v}$ can be relaxed without implications.

Denote by $H^*(\mathcal{S},\mathcal{V})$ the optimal solution obtained by solving (P1) for station set $\mathcal{S}$ and vehicle set $\mathcal{V}$. Solving (P1) gives an optimal solution for the Bike Sharing Rebalancing Problem (apart from the limited transshipments). However, we observe that(P1) is practically intractable for realistic station sets with $|\mathcal{S}| \ge 50$ and vehicle fleets with $|\mathcal{V}| \ge 3$. Therefore, we introduce Clustered Routing heuristics in Section 5 and a Constraint Programming heuristic in Section 6 as alternatives to find high quality solutions in a short amount of time.

5. Clustered Routing

We formulate a Clustering Problem to decompose the Routing Problem (P1) into separate *single-vehicle* Routing Problems, thereby reducing combinatorial complexity.

A feasible clustering solution assigns disjoint clusters of stations $\mathcal{S}_v \subseteq \mathcal{S}$ to vehicles $v \in \mathcal{V}$ such that the service level requirements can be satisfied using *only* within-cluster vehicle routing. Implementing these feasibility constraints in existing clustering algorithms, e.g. the Fisher-Jaikumar algorithm (Fisher and Jaikumar 1981), is non-trivial (see Section 5.3). Therefore, we propose to formulate the Clustering Problem as an extended Set Partitioning Problem. The core of the model is a set of binary decision variables:

$$z_{i,v} = \begin{cases} 1 & \text{if station } i \in \mathcal{S} \text{ is assigned to vehicle } v \in \mathcal{V}, \\ 0 & \text{otherwise.} \end{cases}$$

These variables assign a cluster of stations $\mathcal{S}_v = \{i \in \mathcal{S} : z_{i,v} = 1\}$ to each vehicle $v \in \mathcal{V}$.

The objective of the Clustering Problem is to find a feasible solution while minimizing makespan $H = \max_{v \in \mathcal{V}} h_v$, i.e., rebalance the system as soon as possible and divide the workload between vehicles. Optimally, stations are clustered with known exact routing costs for any (feasible) combination of stations $\mathcal{S}_v$. However, the computational complexity (see Section 4) requires us to use approximations to estimate the routing costs. Therefore, we are interested in non-algorithmic routing costs approximations that correlate highly with the exact vehicle routing costs within a cluster, with a *consistent* over- or underestimation.

5.1. Routing Costs Approximation

It is not straightforward to approximate the optimal routing costs for a cluster, because feasibility constraints on the vehicle route may require (many) revisits. However, assuming that all stations in $\mathcal{S}_v$ need to be visited, the within-cluster routing costs $H^*(\mathcal{S}_v, \{v\})$ are bounded from below by the optimal solution of a Traveling Salesman Problem with an added artificial zero-distance depot (to model the open tour for rolling horizon implementation). If we impose the triangle inequality, then this lower bound equals the length of the shortest Hamiltonian path over $\mathcal{S}_v$.

TSP approximations are widely studied, see e.g. Laporte (1992). However, exponential approximations do not scale well in MIPs and, as mentioned, we refrain from algorithmic approximations because of the feasibility constraints. For example, the MIP formulation of the Held-Karp relaxation (Held and Karp 1970, Charikar et al. 2004) would require constraints for each non-empty subset of $\mathcal{S}$. Furthermore, the polynomial-size Assignment Problem relaxation (Dantzig et al. 1954) has limited applicability, because it does not satisfy *monotonicity*, i.e., adding stations to a cluster may actually decrease the routing costs approximation. Since sub-tours are not eliminated, our experimentation with the Assignment Problem resulted in the undesirable assignment of geographically separated groups of stations to the same cluster.

Instead, we introduce the Maximum Spanning Star approximation (we note that Wu et al. (1998, p. 2) introduce the algorithmic minimum k-star approximation). Denote by $\mathrm{SPS}_i(\mathcal{S}_v) = \sum_{j\in\mathcal{S}_v} d_{i,j}$ the cost of the spanning star (spanning tree with depth one) of $\mathcal{S}_v$ rooted at station $i\in\mathcal{S}_v$. The routing costs are approximated by the maximum-cost spanning star $\max_{i\in\mathcal{S}_v}\mathrm{SPS}_i(\mathcal{S}_v)$. In Section 5.3 we show that the Maximum Spanning Star can be implemented using a polynomial number of binary assignment variables and constraints. Moreover, the Maximum Spanning Star is an upper bound on the shortest Hamiltonian path over the cluster. Most importantly, the Maximum Spanning Star satisfies monotonicity, such that stations are only assigned to a cluster if this is necessary for feasibility of the service level requirements.

5.2. Properties Maximum Spanning Star

Next, we prove the monotonicity and upper bound of the Maximum Spanning Star approximation, to motivate our intuition that $\mathrm{MAXSPS}(\mathcal{S}_v)$ and $H^*(\mathcal{S}_v,\{v\})$ correlate. Namely, both $H^*(\mathcal{S}_v,\{v\})$ and $\mathrm{MAXSPS}(\mathcal{S}_v)$ are bounded from below by the shortest Hamiltonian path over $\mathcal{S}_v$, given that the Maximum Spanning Star satisfies monotonicity (which ensures all stations are visited).

Lemma 3. *The Maximum Spanning Star approximation satisfies monotonicity:*

$$\mathrm{MAXSPS}(\mathcal{S}_v\cup\{i\})\geq\mathrm{MAXSPS}(\mathcal{S}_v).$$

Proof of Lemma 3. Assume without loss of generality that $\mathrm{MAXSPS}(\mathcal{S}_v)$ is rooted at station $j\in\mathcal{S}_v$. Then, $\mathrm{MAXSPS}(\mathcal{S}_v)=\mathrm{SPS}_j(\mathcal{S}_v)\leq\mathrm{SPS}_j(\mathcal{S}_v)+d_{j,i}\leq\mathrm{MAXSPS}(\mathcal{S}_v\cup\{i\})$. □

Denote by $\mathrm{TSP}_0^*(\mathcal{S}_v)$ the length of the shortest Hamiltonian path over $\mathcal{S}_v$, which, under the triangle inequality, equals the optimal solution of a Traveling Salesman Problem with an artificial zero-distance depot.

Lemma 4. *The Maximum Spanning Star approximation is an upper bound on the length of the shortest Hamiltonian path:*

$$\mathrm{MAXSPS}(\mathcal{S}_v)\geq\mathrm{TSP}_0^*(\mathcal{S}_v).$$

Proof of Lemma 4. Denote by $C(\mathcal{S}_v)=\sum_{i,j\in\mathcal{S}_v} d_{i,j}$ the cost of the directed clique on $\mathcal{S}_v$. Denote by $n=|\mathcal{S}_v|$ the number of stations in a candidate cluster. Then:

$$\sum_{i\in\mathcal{S}_v}\mathrm{SPS}_i(\mathcal{S}_v)=C(\mathcal{S}_v),\text{ which yields}$$

$$\mathrm{MAXSPS}(\mathcal{S}_v)\geq\frac{C(\mathcal{S}_v)}{n}.$$

Akiyama et al. (2004, p. 40) present the *Walecki decomposition* of a *complete undirected* graph K_n with odd $n \geq 3$ into $(n-1)/2$ Hamiltonian cycles. It follows that K_n with even $n \geq 2$ can be decomposed into $n/2$ Hamiltonian paths. This result gives us:

Case 1: $n \geq 2$ is even. The complete *directed* graph on $\mathcal{S}_v$ can be decomposed into n directed Hamiltonian paths. Hence,

$$\mathrm{TSP}_0^*(\mathcal{S}_v) \leq \frac{C(\mathcal{S}_v)}{n}.$$

Case 2: $n \geq 3$ is odd. The complete *directed* graph on $\mathcal{S}_v$ can be decomposed into $n-1$ directed Hamiltonian cycles. We can remove one of the n edges in any of these $n-1$ Hamiltonian cycles to obtain a Hamiltonian path. Thereby,

$$\mathrm{TSP}_0^*(\mathcal{S}_v) \leq \frac{n-1}{n}\frac{C(\mathcal{S}_v)}{n-1} = \frac{C(\mathcal{S}_v)}{n}.$$

Thus, for both even and odd $n \geq 2$ we have

$$\mathrm{TSP}_0^*(\mathcal{S}_v) \leq \frac{C(\mathcal{S}_v)}{n} \leq \mathrm{MAXSPS}(\mathcal{S}_v). \quad \square$$

We report on the approximation performance of the Maximum Spanning Star in Section 7.1. In particular, we show that the Maximum Spanning Star approximation is highly correlated with the actual routing distance for our data sets (correlation of more than 85%).

5.3. Clustering Problem

Next, we implement the Maximum Spanning Star approximation in our Clustering Problem. The objective is to minimize the estimated makespan $\hat{H}$ such that service level requirements can be satisfied using only within-cluster vehicle routing.

$$\text{minimize } \hat{H} \tag{P2}$$

s.t.

$$\sum_{v \in \mathcal{V}} z_{i,v} = 1 \qquad \forall i \in \mathcal{S} \setminus \mathcal{S}_0 \tag{14}$$

$$\sum_{v \in \mathcal{V}} z_{i,v} \leq 1 \qquad \forall i \in \mathcal{S}_0 \tag{15}$$

$$q_v^0 + \sum_{i \in \mathcal{S}} s_i^0 z_{i,v} \geq \sum_{i \in \mathcal{S}} s_i^{\min} z_{i,v} \qquad \forall v \in \mathcal{V} \tag{16}$$

$$-(Q_v - q_v^0) + \sum_{i \in \mathcal{S}} s_i^0 z_{i,v} \leq \sum_{i \in \mathcal{S}} s_i^{\max} z_{i,v} \qquad \forall v \in \mathcal{V} \tag{17}$$

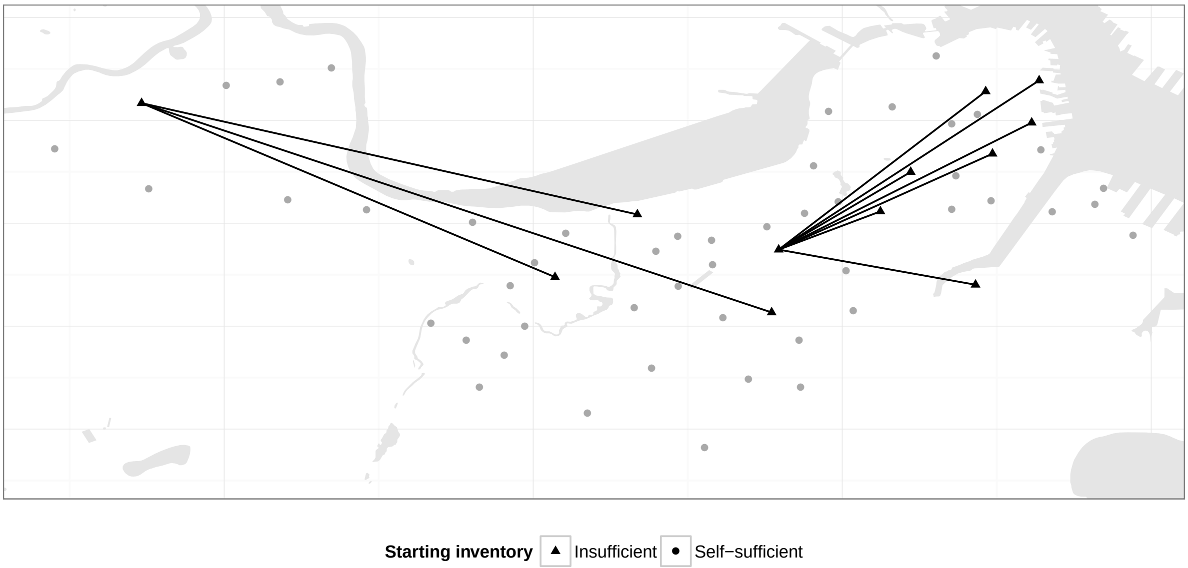
$$\hat{h}_v \geq \sum_{j \in \mathcal{S}} d_{i,j}(z_{i,v} + z_{j,v} - 1) \qquad \forall i \in \mathcal{S}, v \in \mathcal{V} \tag{18}$$

$$\hat{H} \geq \hat{h}_v \qquad \forall v \in \mathcal{V} \tag{19}$$

$$z_{i,v} \in \{0,1\} \qquad \forall i \in \mathcal{S}, v \in \mathcal{V}$$

$$\hat{H}, \hat{h}_v \geq 0 \qquad \forall v \in \mathcal{V}$$

Figure 3 Example of a solution of the Clustering Problem (P2) for Hubway (Boston, MA).



Constraints (14)-(17) ensure feasibility of the clustering solution. Insufficient stations *must* be visited by a vehicle (14) and self-sufficient stations *can* be visited (15). A vehicle cluster must contain enough bikes, possibly using the vehicle starting inventory q_v^0, such that service level requirements can be met through within-cluster repositioning (16). Constraints (17) are similar but then for the maximum inventory in the cluster, possibly using vehicle surplus capacity $Q_v - q_v^0$.

Constraints (18) impose $\mathrm{SPS}_i(\mathcal{S}_v)$ as a lower bound on the estimated routing costs $\hat{h}_v$ if station $i \in \mathcal{S}$ is assigned to vehicle $v \in \mathcal{V}$. Since $\hat{h}_v$ is indirectly minimized, $\hat{h}_v \geq \mathrm{MAXSPS}(\mathcal{S}_v)$. The estimated makespan $\hat{H} = \max_{v \in \mathcal{V}} \hat{h}_v$ is linearized through constraints (19).

Note that we have formulated a compact clustering model with routing costs approximation, which guarantees that the service level requirements can be satisfied at each station while using only within-cluster vehicle routing.

Example 2. Figure 3 shows the assignment of stations to the two Hubway vehicles obtained by solving the Clustering Problem (P2) for the service level requirements from Example 1. We use real inventory data, based on a snapshot of the system taken at 8AM on Friday June 1st 2012.

5.4. Heuristics

Having presented MIP formulations for the Clustering Problem and the Routing Problem, we now formally introduce our Clustered Routing heuristics.

Heuristic 1 (Clustered MIP). Subsequently:

1. Solve the Clustering Problem (P2).

2. For each $v \in \mathcal{V}$ solve the Routing Problem (P1) with $\mathcal{S} = \mathcal{S}_v$ and $\mathcal{V} = \{v\}$ to obtain $H^*(\mathcal{S}_v, \{v\})$.
3. $H^* = \max_{v \in \mathcal{V}} H^*(\mathcal{S}_v, \{v\})$.

Since the routing costs approximation is imperfect and may therefore lead to sub-optimal clusters, the Clustered MIP heuristic has an optimality loss. However, we can use the information obtained from routing the individual clusters to add *cuts* to the Clustering Problem.

Assume that an unknown optimal solution exists with makespan H^{OPT} strictly less than our best found solution H^*. This implies that all clusters $\mathcal{S}_v$ with known optimal routing costs $H^*(\mathcal{S}_v, \{v\}) \geq H^*$ are *not* part of the optimal solution. Thereby, these clusters can be eliminated from the solution space of the Clustering Problem. Note that, contrary to problems in which total routing costs are minimized, these cuts do not need to be removed later.

Heuristic 2 (Clustered MIP with Cuts). Subsequently:

1. Define a cut set $\mathcal{C} \subseteq 2^{\mathcal{S}}$ of subsets of $\mathcal{S}$. Initialize $\mathcal{C} = \emptyset$.
2. Solve the Clustering Problem (P2) with additional constraints:

$$\sum_{i \in c} z_{i,v} - \sum_{i \in \mathcal{S} \setminus c} z_{i,v} \leq |c| - 1 \qquad \forall c \in \mathcal{C}, v \in \mathcal{V} \tag{20}$$

3. For each $v \in \mathcal{V}$ solve the Routing Problem (P1) with $\mathcal{S} = \mathcal{S}_v$ and $\mathcal{V} = \{v\}$ to obtain $H^*(\mathcal{S}_v, \{v\})$.
   (a) If $\max_{v \in \mathcal{V}} H^*(\mathcal{S}_v, \{v\}) < H^*$ or $\mathcal{C} = \emptyset$ then (re)define H^* and store the routing solution.
   (b) For each $v \in \mathcal{V}$ with $H^*(\mathcal{S}_v, \{v\}) \geq H^*$ redefine $\mathcal{C} = \mathcal{C} \cup \{\mathcal{S}_v\}$
4. Go to step 2.

Note that H^* is only redefined if an improvement is found (3a), in which case at least one cut is added per iteration (we have $H^*(\mathcal{S}_v, v) = H^*$ for at least one $v \in \mathcal{V}$). If no improvement is found, then H^* is not redefined and at least one cluster has actual costs strictly larger than the best found solution H^*. We add a cut for all non-improving clusters (3b), and continue. Note that any *strict* subset or superset of a cut $c \in \mathcal{C}$ is *not* eliminated by constraints (20). The cuts force the Clustering Problem to iteratively adjust the assignment of stations to vehicles, thereby mitigating the approximation error. After finitely, but possibly exponentially many steps, the heuristic identifies the optimal solution.

Unfortunately, to our knowledge no stronger terminating condition for Clustered MIP with Cuts heuristic than an exhaustive search of the Clustering Problem solution space exists. However, Table 1 in Section 7.2 shows how quickly improvements over the Clustered MIP heuristic are attained.

6. Constraint Programming

In this section we present our Constraint Programming (CP) model for the Bike Sharing Rebalancing Problem. Constraint Programming is among the state of the art for solving complex routing and scheduling problems, even though it applies a generic modeling and solving approach. In particular, CP has been applied before to constrained routing problems (Kilby and Shaw 2006). Most industrial CP solvers combine constraint propagation with large neighborhood search for solving routing problems (Shaw 1998).

In order to take advantage of the strengths of CP, it is common to represent routing problems as scheduling problems, by representing the visit of a location as an *activity*. An activity is a high-level CP modeling structure that implicitly defines integer variables for its start time, duration, and end time, and a Boolean variable for its presence. Activities for which the presence is not fixed to true are called *optional activities*. Traveling the distance between two locations is represented by sequence-dependent setup times between the respective activities (for each pair of locations), i.e., if station j is visited directly after station i, we need to respect the distance $d_{i,j}$ as 'setup' time. In CP, activities impact *resources* which, in case of routing problems, correspond to the vehicles. For example, for each vehicle, we must ensure that no two activities overlap.

We next specify the details of our CP model, following the AIMMS notation for activities and resources (Roelofs and Bisschop 2012). In particular, each activity `A` induces the variables `A.Start`, end time `A.End`, and presence `A.Present`, as explained above. For each station $i \in \mathcal{S}$ and vehicle $v \in \mathcal{V}$, we define optional activities `Pickup[i,v]` and `Delivery[i,v]` with duration 0. That is, each station may be visited by any of the vehicles. Variables $y^+_{i,v}$ and $y^-_{i,v}$ represent the pickup, respectively delivery, amount for vehicle v at station i, as in our models above. Our CP model then becomes:

$$\text{minimize} \max \left\{ \max_{\substack{i\in\mathcal{S}\\ v\in\mathcal{V}}} \{\texttt{Pickup}[\texttt{i},\texttt{v}].\texttt{End}\}, \max_{\substack{i\in\mathcal{S}\\ v\in\mathcal{V}}} \{\texttt{Delivery}[\texttt{i},\texttt{v}].\texttt{End}\} \right\} \tag{P3}$$

s.t.

$$s_i^0 + \sum_{v\in\mathcal{V}} y^+_{i,v} - y^-_{i,v} \geq s_i^{\min} \qquad \forall i \in \mathcal{S} \tag{21}$$

$$s_i^0 + \sum_{v\in\mathcal{V}} y^+_{i,v} - y^-_{i,v} \leq s_i^{\max} \qquad \forall i \in \mathcal{S} \tag{22}$$

$$\sum_{v\in\mathcal{V}} \texttt{Pickup}[\texttt{i},\texttt{v}].\texttt{Present} + \texttt{Delivery}[\texttt{i},\texttt{v}].\texttt{Present} \leq 1 \qquad \forall i \in \mathcal{S} \tag{23}$$

$$\texttt{Pickup}[\texttt{i},\texttt{v}].\texttt{Present} = 1 \Leftrightarrow y^-_{i,v} \geq 1 \qquad \forall i \in \mathcal{S}, v \in \mathcal{V} \tag{24}$$

$$\texttt{Pickup}[\texttt{i},\texttt{v}].\texttt{Present} = 0 \Leftrightarrow y^-_{i,v} = 0 \qquad \forall i \in \mathcal{S}, v \in \mathcal{V} \tag{25}$$

$$\texttt{Delivery}[\texttt{i},\texttt{v}].\texttt{Present} = 1 \Leftrightarrow y^+_{i,v} \geq 1 \qquad \forall i \in \mathcal{S}, v \in \mathcal{V} \tag{26}$$

$$\texttt{Delivery[i, v].Present} = 0 \Leftrightarrow y^+_{i,v} = 0 \qquad \forall i \in \mathcal{S}, v \in \mathcal{V} \qquad (27)$$

$$y^-_{i,v}, y^+_{i,v} \in \{0, \ldots, Q_v\} \qquad \forall i \in \mathcal{S}, v \in \mathcal{V}$$

with resources

`Sequential resource VehicleTime[v](`

`Schedule domain:` $\{0, ..., \texttt{MaxTime}\}$

`Activities: Pickup[i,v], Delivery[i,v]`

`Transition:` $d_{i,j}$

`)`

`Parallel resource VehicleInventory[v](`

`Activities: Pickup[i,v], Delivery[i,v]`

`Level range:` $\{0, \ldots, Q_v\}$

`Initial value:` q^0_v

`Begin change: Delivery[i,v]:` $-y^+_{i,v}$

`End change: Pickup[i,v]:` $y^-_{i,v}$

`)`

`Parallel resource StationInventory[i](`

`Activities: Pickup[i,v], Delivery[i,v]`

`Level range:` $\{0, \ldots, C_i\}$

`Initial value:` s^0_i

`Begin change: Delivery[i,v]:` $y^+_{i,v}$

`End change: Pickup[i,v]:` $-y^-_{i,v}$

`)`

As before, constraints (21)-(22) impose the service level requirements from Lemma 2. Constraints (23) are so-called *alternative resource* constraints, which limit the number of visits to one per station. Adding these constraints can greatly improve the performance of the constraint propagation, but the optimal solution may be eliminated. Note that it may not always be possible to impose the alternative resource constraints, for example due to limited vehicle capacity. In such cases, the model can trivially be extended to allow multiple visits by increasing the right-hand side. We can index the activities and variables $y^-_{i,v}, y^+_{i,v}$ correspondingly. However, our preliminary experimentation showed strongly decreasing computational performance if we allowed multiple visits per location. Therefore, we imposed the alternative resource constraints in our experiments. Constraints (24)-(27) link the vehicle presence constraints with performing a pickup or delivery. Note that the *if and only if* constraints enhance propagation.

For each vehicle we introduce two types of resources. The first represent the no-overlap conditions with respect to the vehicle time, using a `Sequential resource` named `VehicleTime[v]`. For each such resource, we identify the discrete time horizon as its `Schedule domain`, while the keyword `Activities` specifies which activities impact the resource. The arc-dependent transition times model the travel distances via `Transition`.

The second resource associated with a vehicle is its inventory, modeled as a `Parallel resource` named `VehicleInventory[v]`. In addition to specifying the set of activities in its scope, we define its `Level range` to be $\{0,\ldots,Q_v\}$, which is initialized at q_v^0. Furthermore, we specify for each activity in its scope how it impacts the level. For `Delivery[i,v]`, the level is changed at the start of the activity, with amount $-y_{i,v}^+$. Likewise, for `Pickup[i,v]`, the level is changed at the end of the activity with amount $y_{i,v}^-$.

Lastly, for each station we define a `Parallel resource` representing the station inventory, named `StationInventory[i]`. The range of this inventory is $\{0,\ldots,C_i\}$ with initial value s_i^0. Level changes for pickups and deliveries are exactly opposite to the vehicle inventory changes.

7. Computational Results

In this section we report on the performance of our routing costs approximation and heuristics. Recall that we give a detailed overview of our data sources in Appendix B.

7.1. Approximation performance

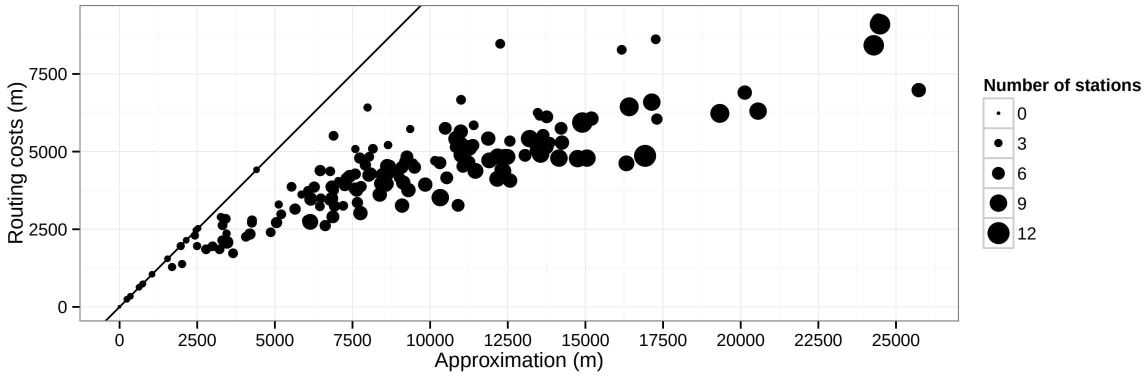
In Section 5.2 we proved the monotonicity and upper bound $\text{MAXSPS}(\mathcal{S}_v) \geq \text{TSP}_0^*(\mathcal{S}_v)$ of the Maximum Spanning Star to motivate our intuition that $\text{MAXSPS}(\mathcal{S}_v)$ approximates $H^*(\mathcal{S}_v,\{v\}) \geq \text{TSP}_0^*(\mathcal{S}_v)$. Figure 4 visualizes this relationship with a scatterplot, in which the line $H^*(\mathcal{S}_v,\{v\}) = \text{MAXSPS}(\mathcal{S}_v)$ is shown for reference.

While we established that $H^*(\mathcal{S}_v,\{v\}) \geq \text{TSP}_0^*(\mathcal{S}_v)$, we note that the additional feasibility constraints on the vehicle route imposed in (P1) do *not* lead to routing costs higher than the MAXSPS approximation for any instance. Rather, the Maximum Spanning Star consistently overestimates the routing costs, which is reflected in the high correlation of 87.99%. A linear regression on $H^*(\mathcal{S}_v,\{v\})$ with an imposed zero intercept estimates a .4238 coefficient for $\text{MAXSPS}(\mathcal{S}_v)$ yielding $R^2 = 94.28\%$. Nonetheless, a small approximation error is present. Next, we show how the improvement scheme with elimination cuts overcomes this limitation of the Clustered MIP heuristic.

7.2. Heuristics performance

We report computational results comparing the exact MIP (P1), Clustered MIP (Heuristic 1), Clustered MIP with Cuts (Heuristic 2), and CP (P3) approaches to the Bike Sharing Rebalancing

Figure 4 Computational results for the Maximum Spanning Star routing costs approximation.



Note. Routing costs $H^*(\mathcal{S}, \{v\})$ are plotted against the approximation MAXSPS$(\mathcal{S}_v)$ for all 82 two-vehicle instances of Hubway (Boston, MA). Each data point corresponds to a cluster. The correlation equals 87.99%.

Problem. We consider multiple families of instances based on real trip and inventory data provided by Hubway (Boston, MA) and Capital Bikeshare (Washington, DC).

All experiments were performed on an Intel Xeon X3323 @ 2.50GHz with 4GB of memory, using AIMMS 3.13 FR1 modeling software with MIP solver GUROBI 5.0 (which outperformed CPLEX 12.4 on test instances) and CP solver IBM ILOG CP Optimizer 12.4.

Parameters. A family of instances is defined by a market (Hubway or Capital Bikeshare), observation period (8–9AM or 4–5PM), service level (consistently 95%) and number of vehicles. We observe that the morning and afternoon commute are the most challenging rebalancing problems. Furthermore, different service levels (90%, 99%) did not substantially impact our findings. Subsequently, for each family we generate multiple instances by using different *inventory snapshots* containing the starting inventory s_i^0 for each station $i \in \mathcal{S}$ at the beginning of the observation period.

We calculate Euclidian distances $d_{i,j} = d_{j,i}$ in meters based on the latitude and longitude of stations. We assume $q_v = 0$ to refrain from random data generation. For MIP we set $|\mathcal{T}| = |\mathcal{S} \setminus \mathcal{S}_0|$ (this ensures feasibility for our instances) and for the Clustered MIP heuristics we set $|\mathcal{T}| = |\mathcal{S}_v| + 1$ (this allows a revisit).

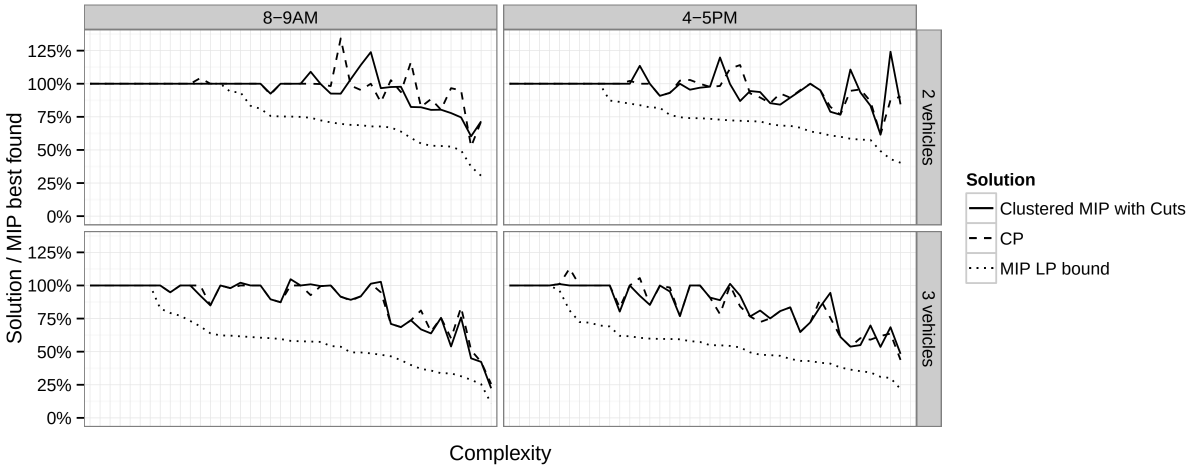
Hubway (Boston, MA). We restrict to $|\mathcal{S}| = 60$ stations to obtain sufficient trip observations to calibrate the service level requirements on 82 weekdays between November 1st 2011 and May 31st 2012. We use 41 inventory snapshots on weekdays between June 1st and July 29th 2012. Hubway currently operates $|\mathcal{V}| = 3$ vehicles with capacity $Q_v = 22$ since opening an additional 40 stations in Summer 2012. However, during our observations Hubway only operated two vehicles. We investigate both truck fleet sizes to reveal the possible implications for performance.

Table 1 Computational results for Hubway (Boston, MA).

		MIP			*Clustered MIP*		*Clustered MIP with Cuts*			*CP*	
Family	$\lvert\mathcal{V}\rvert$	LP bound	Best found	Time	Solution	Time	Solution	Iterations	Time	Solution	Time
8–9AM	2	3228.18	4625	5220.51	4495	0.66	4229	82	60.05	4310	58.56
8–9AM	3	1660.29	3758	6139.66	3097	0.55	2669	57	60.09	2727	60.08
4–5PM	2	3347.55	4674	5787.08	4656	0.88	4429	76	60.03	4393	55.69
4–5PM	3	1674.90	3399	6558.48	3285	0.53	2699	51	60.14	2673	60.07

These results are averaged over the 41 instances of each instance family. The mean number of insufficient stations was equal to $|\mathcal{S} \setminus \mathcal{S}_0| = 10$. The MIP solver was unable to find a solution of the full model for three instances. These are excluded from this summary and shown in Appendix C.

Figure 5 Computational results for Hubway (Boston, MA).



Note. Instances are sorted with decreasing MIP LP bound to MIP best found solution ratio, to approximate complexity. Here, the MIP best found solution is shown as 100%. If the LP bound equals the best found solution (100%), then the solver was able to solve the exact MIP to optimality.

After extensive experiments we set the computational cut-offs (if unsolved): MIP after 7200 seconds; Clustered MIP after 20 seconds for both (P2) and (P1); Clustered MIP with Cuts after 60 seconds in total with 20 seconds for (P2) and (P1); CP after 60 seconds. For CP we set the schedule domain with `MaxTime` $= 50000$, which proved necessary to quickly identify a feasible (but low-quality) solution.

Table 1 summarizes our computational results for Hubway. We show the average solutions and computation times per instance family for the exact MIP and our heuristics. For our improvement scheme (Clustered MIP with Cuts), we show the number of iterations.

We observe that in 1 minute, our Clustered MIP with Cuts and CP heuristic outperform the best found solution of the MIP after 2 hours with 5–10% on average for the two-vehicle families. For the three-vehicle families, the improvement is 15–25%. Note that our Clustered MIP heuristic can, on average, even generate better solutions than the MIP model *within 1 second*. This allows more

Table 2 Computational results for Capital (Washington, DC).

			Clustered MIP		*Clustered MIP with Cuts*			*CP*	
Family	$\|\mathcal{V}\|$	$\|\mathcal{S} \setminus \mathcal{S}_0\|$	Solution	Time	Solution	Iterations	Time	Solution	Time
8–9AM	5	25	7573	32.61	6594	3	120.20	12123	120.39
4–5PM	5	11	2970	11.64	2548	13	120.18	2736	120.36

The $|\mathcal{S} \setminus \mathcal{S}_0|$ column shows the mean number of insufficient stations per instance family.

than 60 iterations of the Clustered MIP with Cuts heuristic in 1 minute, yielding an improvement of approximately 10% over the Clustered MIP heuristic.

In Figure 5 we present a more detailed comparison of our dedicated Clustered MIP with Cuts heuristic with the CP heuristic for the Hubway instance families. We show how they perform in comparison to the best found MIP solution after two hours (depicted as 100%). We note that the performance of the full MIP model decreases strongly for the three vehicle families, with our 1 minute heuristic solutions up to 75% better than the best found solutions after two hours.

CP performs very well for the Hubway instances. We believe that due to the low number of *insufficient* stations, on average 10 out of 60, CP is able to quickly identify a feasible solution which is subsequently improved. We observe that the Routing Problem (P1), when applied to the individual clusters, becomes intractable when there are more than 15 stations in a cluster. For instances on which CP outperformed the Clustered MIP heuristics, indeed one vehicle was assigned more than 15 stations. In these situations, we suggest a hybrid cluster-first, CP-second approach would leverage both strengths, because CP performed well on single-vehicle test instances.

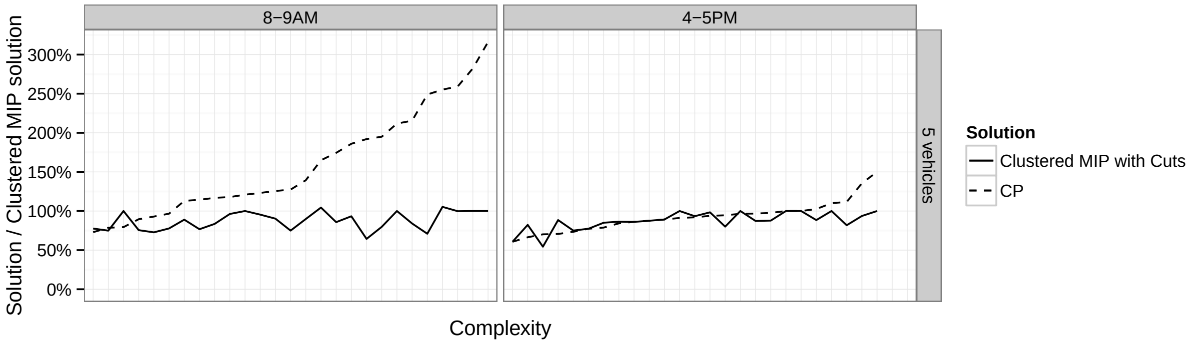
Capital Bikeshare (Washington, DC). We restrict to $|\mathcal{S}| = 135$ stations to obtain 130 trip observations on weekdays between January 1st and June 30th 2012. We use 27 8AM and 25 4PM inventory snapshots of weekdays obtained between December 17th 2012 and January 25th 2013. Capital Bikeshare currently operates $|\mathcal{V}| = 5$ vehicles with $Q_v = 25$.

We increase the computational cut-offs of Clustered MIP with Cuts and CP to 120 seconds to accommodate the increased complexity. For CP we increase `MaxTime` to 75000. Other cut-offs are identical to those used for Hubway.

With 135 stations and 5 vehicles, the Capital Bikeshare instances were too complex to derive feasible solutions or even useful LP bounds from the full MIP (P1) within a reasonable amount of time. Therefore, we report only on the performance of our heuristics.

Table 2 summarizes our computational results for Capital. The 8–9AM instances are more complex than the 4–5PM instances, with on average 25 insufficient stations per instance instead of 11 (see column $|\mathcal{S} \setminus \mathcal{S}_0|$. The Clustered MIP with Cuts heuristic is 45% better than CP for the 8–9AM family of instances. This highlights exactly where we believe our cluster-first route-second

Figure 6 Computational results for Capital Bikeshare (Washington, DC).



Note. Instances are sorted with increasing CP solution to Clustered MIP solution ratio, to approximate complexity. The Clustered MIP solution is shown as 100%.

heuristic excels. The polynomial-size Clustering Problem (P2) allows rapid decomposition of the multi-vehicle problem into reasonably good single-vehicle clusters. Then, the Routing Problem (P1) can be solved to optimality for individual clusters in under a second. The improvement cuts mitigate both the approximation error and (possibly) sub-optimality incurred from cutting off the Clustering Problem before the solver is finished.

In Figure 6 we present a comparison of the heuristics, with the Clustered MIP heuristic shown as 100%. For some instances, even the simple Clustered MIP heuristic is up to 65% better than CP. Figure 6 also shows clearly how adding cuts in the Clustered MIP with Cuts heuristic can yield improvements of up to 40% over the simple Clustered MIP heuristic.

The results for Capital Bikeshare show that our dedicated Clustered MIP (with Cuts) heuristic performs better than CP, especially for instances with a large vehicle fleet and a low number of stations per vehicle. This implies the polynomial-size Clustering Problem can handle large sets of insufficient stations, given that enough vehicles are available to divide the workload. When an instance is more similar to a scheduling problem (i.e., a lower number of vehicles and longer routes, like for some Hubway instances), the techniques embedded in CP show their strength.

8. Conclusions

This paper is the first to unify dual-bounded service level constraints, which add inventory flexibility, and vehicle routing in bike sharing systems. We represent the inventory at each station as a finite-buffer single-server queuing system and use closed-form analysis of the transient probabilities to calculate service level requirements. We introduce the notion of self-sufficient stations, which fulfill these requirements with their starting inventory. Hence, self-sufficient stations do not necessarily need to be visited by a vehicle, but may act as source or sink nodes.

We present a mixed integer programming based Clustering Problem that decomposes the multi-vehicle rebalancing problem into separate single-vehicle problems, while taking into account service level feasibility constraints (a cluster-first route-second approach). We introduce a novel polynomial-size Maximum Spanning Star routing costs approximation for the Clustering Problem to achieve high computational performance. We develop an improvement scheme based on elimination cuts to mitigate the approximation error. Furthermore, we provide the first constraint programming formulation of the bike sharing rebalancing problem.

Using empirical data from two bike sharing systems, we extensively test the Clustered MIP heuristics against the classical full MIP model and the constraint programming approach. Our Clustered MIP with Cuts heuristic outperforms the constraint programming formulation as the number of vehicles (and correspondingly, the number of stations) grows. Constraint programming performs well when the number of vehicles is low and the number of stations per vehicle is high. Both the Clustered MIP and constraint programming approaches identify better solutions within one or two minutes, than the often-used full MIP after two hours. We thus believe that our approach is suitable for practical implementation in bike sharing systems.

The novel heuristics and our approximation may be applicable to other constrained routing problems, specifically to (extended) One-Commodity Pickup-and-Delivery VRPs like the empty freight container rebalancing problem, as well as to other sharing systems.

Appendix A: Net demand process

Figure 7 motivates why we adapt a *process* view to net demand for bikes, instead of observing total net demand.

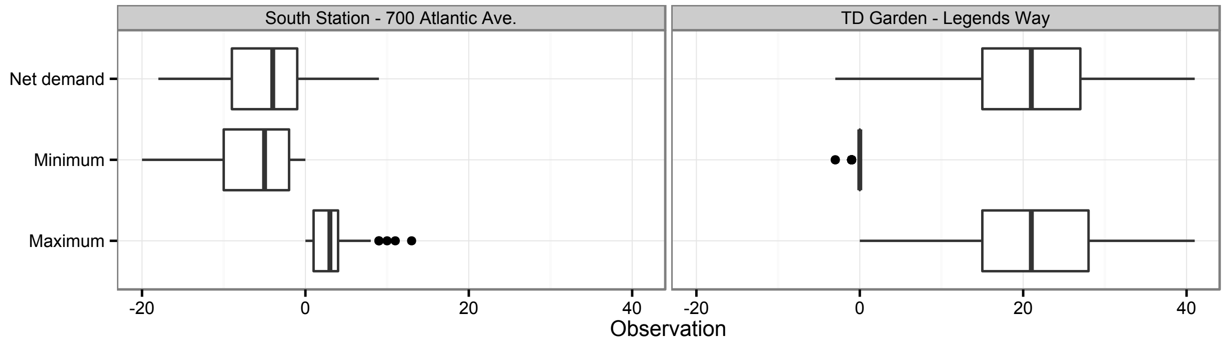
Appendix B: Data sources

In order to produce the examples and computational results, we processed two data sets from Hubway (Boston, MA) and Capital Bikeshare (Washington, DC), which have identical formatting. These data sets were made available through their websites `thehubway.com` and `capitalbikeshare.com`. Each data set consists of three tables: `Stations`, `Trips` and `Snapshots`.

The `Stations` table contains the following fields for each station: `id`, `name`, `lat`, `lng`, `installed`, `locked` and `temporary`. We use the `id` field to create the relationship with the `Trips` and `Inventory` tables. We use the `lat` and `lng` fields to calculate the Euclidean distance matrix d in meters with the `spDists` function of the `sp` package in R.

The `Trips` table contains the following fields for each trip: `id`, `start_date`, `end_date`, `start_station`, `end_station`, `bike_name`, `bike` and `member_type`. We calculate the number of pickups at a station during the observation period using `start_date` and `start_station`, and the number of returns using `end_date` and `end_station`, to synchronize these events. As mentioned

Figure 7 Net demand during observation period 8–9AM for two popular stations of Hubway (Boston, MA).



Note. We show both the *total* net demand at the end of the observation period $[0,T]$ and the minimum and maximum value of the net demand *process*. For the TD Garden station, we observe that the total net demand and the process extremes are almost identical, in accordance with previous studies (Nair et al. 2013). However, for South Station, the negative total net demand might suggest that we do not need bikes (i.e. positive net demand). Yet, the maximum implies that at some moment during most observation periods, the net demand is positive, which requires a positive starting inventory.

in Section 7.2, we select a subset of the stations for calculating the service level requirements, to ensure that sufficient trip observations are available.

The `Snapshots` table contains the following fields for each station/date combination: `id`, `bikes`, `docks` and `date`. We calculate `capacity` = `bikes` + `docks` for each snapshot, since reparations or extensions may lead to changes in the station capacity over time. We set C_i equal to the maximum `capacity` of station $i \in \mathcal{S}$ to prevent infeasibilities. We web scraped the `Snapshots` table for Capital Bikeshare from `http://capitalbikeshare.com/data/stations/bikeStations.xml`.

Appendix C: Instances without MIP solution

Table 3 shows the three Hubway instances for which the solver was unable to identify a feasible solution of the full MIP within two hours.

Table 3 Hubway (Boston, MA) instances without a MIP solution.

			MIP		*Clustered MIP*		*Clustered MIP with Cuts*			*CP*	
Family	$\lvert\mathcal{V}\rvert$	$\lvert\mathcal{S}\setminus\mathcal{S}_0\rvert$	LP bound	Time	Solution	Time	Solution	Iterations	Time	Solution	Time
8–9AM	2	17	4599.00	7201.91	9251	20.62	8553	5	60.47	8413	60.06
4–5PM	3	14	1778.00	7202.25	3873	0.94	3873	41	60.00	3873	60.08
4–5PM	2	17	3632.00	7201.71	5937	5.54	4980	4	60.00	5862	60.05

The $\lvert\mathcal{S}\setminus\mathcal{S}_0\rvert$ column shows the number of insufficient stations of the instance.

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