Core Deviation Minimizing Auctions*

Isa E. Hafalir† and Hadi Yektaş‡

February 13, 2012

Abstract

We study dominant strategy implementable direct mechanisms that minimize the expected surplus from core deviations. Using incentive compatibility conditions, we formulate the core deviation miminimization problem as a calculus of variations problem and then numerically solve it for some particular cases.

JEL Classifications Codes: D44, C71

Keywords: Core, Auctions, Mechanism Design

1 Introduction

In an economic environment with complementarities, Vickrey-Clark-Groves (VCG) mechanism may generate low revenues and therefore is vulnerable to collusion (of coalitions that necessarily include the seller). Motivated by this observation, core-selecting auctions (CSA) have been proposed as alternatives to VCG mechanism.[1] In a *complete information* setting, different forms of CSA have been considered and shown to have superior properties in terms of revenue while ensuring noncollusive behavior.[2]

More recently, in a simple stylized environment with *private information*, Goeree and Lien (2009) and Ausubel and Baranov (2010) analyze how VCG and CSA perform in terms of revenue, efficiency, and distance from core allocations. Their results demonstrate that

*We are grateful to Bilkent University for hosting us during the early stages of this research. We also thank Gokhan Apaydin and Mohamad Maleki for discussions about numerical optimization and Ersin Korpeoglu for excellent research assistantship.

†Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA E-mail: isaemin@cmu.edu

‡Department of Economics, Zirve University, Kızılhisar Kampüsü, Gaziantep 27260, Turkey E-mail: hyektas@gmail.com.

[1] See Day and Raghavan (2007) and Day and Milgrom (2008).

[2] There is a small but growing literature on CSA. See, for instance, Day and Crampton (2008), Crampton (2009), Baranov (2010), Erdil and Klemperer (2010), Lamy (2010) and Sano (2011).

VCG and CSA cannot be generally ranked in terms of above measures, and their relative performances depend on specifications of value distributions of the players. The stylized model for CSA considers two goods, two local bidders, and a global bidder. In this model, the global bidder views the two goods as perfect complements in the sense that he/she has positive valuation for the bundle, but zero valuation for a single good. Local bidder i, on the other hand, has positive value for object i and zero value for object j ($\{i, j\} = \{1, 2\}$).

In the same stylized model, we ask a mechanism design problem: Among dominant strategy implementable direct mechanisms, which mechanism achieves an outcome that is closest to the core? When answering this problem, we measure the distance from the core as *expected surplus from core deviations*.[3] Appealing to envelope theorem, we write the transfer function in terms of the allocation function. We then apply some calculus operations and write the mechanism design problem as a standard calculus of variations problem. We prove that the optimal mechanism should "favor the global bidder" in the sense that if global bidder's value is greater than the sum of local bidders' values, then global bidder is always awarded both items.

While an analytical closed-form solution to the calculus of variations problem turns out to be difficult to obtain, we numerically solve it for two interesting cases. If bidders' value distributions are all uniform; interestingly, Vickrey auction turns out to be the optimal mechanism. On the other hand, for another case, we show that the optimal mechanism performs significantly better as compared to the Vickrey auction.

2 Model

There are two goods, goods 1 and 2, that are to be allocated among two local buyers, buyers 1 and 2, and one global buyer, buyer 3. Local buyer i earns a positive value only from good i and global buyer earns a positive value only if he obtains both goods. Local buyer i's value v_i is distributed over $[0, 1]$ according to a distribution function F and global buyer's value for the bundle v_3 is distributed over $[0, 2]$ according to a distribution function G. We assume that distributions F and G are atomless, continuous and differentiable.[4] Also define $\mathbf{v} \equiv (v_1, v_2, v_3)$. In this environment, we consider the set of dominant strategy incentive compatible direct mechanisms that minimize the expected surplus from core deviations. Note that a deviating group has to include the seller, therefore it is composed either of the

[3] Goeree and Lien (2009) already established that there exists no (Bayesian) incentive compatible direct mechanism for which the value of the objective function is zero. We look for a second best, that is, given incentive constraints, how close can a mechanism get to the core according to this measure?

[4] Ausubel and Baranov (2010) also considered the case in which local bidders' valuations may be correlated. We do not analyze that specification here.

seller and the two local buyers or of the seller and the global buyer.

Using revelation principle, we focus on truthful direct mechanisms. A direct mechanism $\langle Q,T\rangle$ is defined by an allocation rule Q and a payment rule T. Here, we focus on deterministic rules. Moreover, we consider monotonic rules in the sense that a bidder's probability of getting the item (or bundle) increases with his own value. Note that the seller would not want to allocate only one object to the global buyer or both objects to one of the local buyers. Therefore, we can write $Q=(Q_l,Q_g)$ where Q_l and Q_g are probabilities of winning for local buyers and global buyer, respectively. Of course, $Q_l+Q_g=1$.

3 Surplus from core deviations

In this setting, the expected surplus from core deviations is given by

$$\int_0^1\int_0^1\int_0^2\left(\begin{array}{c}(\max\{v_3-T_1(\mathbf{v})-T_2(\mathbf{v}),0\})\times I[Q_l(\mathbf{v})=1]\\+(\max\{v_1+v_2-T_3(\mathbf{v}),0\})\times I[Q_g(\mathbf{v})=1]\end{array}\right)dG(v_3)\,dF(v_2)\,dF(v_1) \quad (1)$$

where I is the identity function.

Given the monotonicity assumption, we can analogously define the allocation rule using $\widehat{r}(v_1,v_2)$ such that

$$Q_l(\mathbf{v})=1 \text{ if and only if } v_3<\widehat{r}(v_1,v_2) \quad (2)$$

where $\widehat{r}$ is an increasing function of its arguments. As a simplifying assumption, we further suppose that

$$\widehat{r}(v_1,v_2)=r(v_1+v_2)$$

where $r'>0$, $r(0)=0$ and $r(2)=2$.[5]

Proposition 1 *If $\widehat{r}(v_1,v_2)=r(v_1+v_2)$, then in a dominant strategy implementable mechanism, expected surplus from core deviations can be written as*

$$\min_r \int_0^1\int_0^1\int_0^{r(v_1+v_2)}\max\{v_3-\max\{r^{-1}(v_3)-v_1,0\}-\max\{r^{-1}(v_3)-v_2,0\},0\}dG(v_3)\,dF(v_2)\,dF(v_1)$$
$$+\int_0^1\int_0^1\int_{r(v_1+v_2)}^2\max\{v_1+v_2-r(v_1+v_2),0\}dG(v_3)\,dF(v_2)\,dF(v_1).$$

[5]This assumption is essential for the results we present in the remainder of the paper. Although it is clearly a restricting assumption, it is at the same time quite reasonable. With this assumption, the allocation rule treats local bidders symmetrically and compares the global bidder's value to the sum of the local bidders' values. Also, note that this assumption is satisfied by the VCG mechanism.

Proof. With the specification of $r(v_1+v_2)$, objective function (1) can be rewritten as

$$\int_0^1\int_0^1\int_0^{r(v_1+v_2)} \max\{v_3 - T_1(\mathbf{v}) - T_2(\mathbf{v}), 0\} dG(v_3)\, dF(v_2)\, dF(v_1)$$
$$+\int_0^1\int_0^1\int_{r(v_1+v_2)}^2 \max\{v_1+v_2 - T_3(\mathbf{v}), 0\} dG(v_3)\, dF(v_2)\, dF(v_1) \quad (3)$$

and the incentive compatibility constraints are given by

$$Q_i(v_i, v_{-i})\, v_i - T_i(v_i, v_{-i}) \geq Q_i(v_i', v_{-i})\, v_i - T_i(v_i', v_{-i}) \quad (4)$$

for all $i \in \{1,2,3\}$, v_i, v_i' and v_{-i}.

Define $U_i(v_i, v_{-i}) = Q_i(v_i, v_{-i})\, v_i - T_i(v_i, v_{-i})$. Then, by envelope theorem, the incentive constraints (4) imply that

$$U_i'(v_i, v_{-i}) = Q_i(v_i, v_{-i})$$

and integrating both sides with respect to v_i, we obtain

$$U_i(v_i, v_{-i}) = U_i(0, v_{-i}) + \int_0^{v_i} Q_i(t, v_{-i})\, dt$$

which further implies

$$T_i(v_i, v_{-i}) = Q_i(v_i, v_{-i})\, v_i + T_i(0, v_{-i}) - \int_0^{v_i} Q_i(t, v_{-i})\, dt. \quad (5)$$

Given r, we have

$$Q_i(v_i, v_{-i}) = \begin{cases} 1 & \text{if } v_i > \max\{r^{-1}(v_3) - v_j, 0\} \\ 0 & \text{if } v_i < \max\{r^{-1}(v_3) - v_j, 0\} \end{cases} \quad (6)$$

for $i, j = 1, 2$ and $j \neq i$ and

$$Q_3(v_3, v_1, v_2) = \begin{cases} 1 & \text{if } v_3 > r(v_1+v_2) \\ 0 & \text{if } v_3 < r(v_1+v_2). \end{cases} \quad (7)$$

Substituting (6) and (7) into (5) yields

$$T_i(v_i, v_{-i}) = \begin{cases} T_i(0, v_{-i}) & \text{if } v_i < \max\{r^{-1}(v_3) - v_j, 0\} \\ \begin{array}{l} v_i + T_i(0, v_{-i}) - (v_i - \max\{r^{-1}(v_3) - v_j, 0\}) \\ \quad = T_i(0, v_{-i}) + \max\{r^{-1}(v_3) - v_j, 0\} \end{array} & \text{if } v_i > \max\{r^{-1}(v_3) - v_j, 0\} \end{cases} \tag{8}$$

$$T_3(v_3, v_1, v_2) = \begin{cases} T_3(0, v_1, v_2) & \text{if } v_3 < r(v_1 + v_2) \\ \begin{array}{l} v_3 + T_3(0, v_1, v_2) - (v_3 - r(v_1 + v_2)) \\ \quad = T_3(0, v_1, v_2) + r(v_1 + v_2) \end{array} & \text{if } v_3 > r(v_1 + v_2) \end{cases} \tag{9}$$

where $i, j = 1, 2$ and $j \neq i$. Note that $r^{-1}(v_3) - v_j$ could be greater than 1.

We can also argue that at the optimum, $T_3(0, v_1, v_2) = T_1(0, v_{-1}) = T_2(0, v_{-2}) = 0$.[6] Then, we can summarize the problem as

$$\begin{aligned} O \equiv \min_{r(\cdot)} \int_0^1 \int_0^1 \int_0^{r(v_1+v_2)} & \max\left\{ \begin{pmatrix} v_3 - \max\{r^{-1}(v_3) - v_1, 0\} \\ - \max\{r^{-1}(v_3) - v_2, 0\} \end{pmatrix}, 0 \right\} dG(v_3)\, dF(v_2)\, dF(v_1) \\ + \int_0^1 \int_0^1 \int_{r(v_1+v_2)}^2 & \max\{v_1 + v_2 - r(v_1 + v_2), 0\} dG(v_3)\, dF(v_2)\, dF(v_1). \end{aligned} \tag{10}$$

where r is a continuos and strictly increasing function defined over $[0, 2]$ with $r(0) = 0$ and $r(2) = 2$.

This problem is equivalent to the original mechanism design problem of finding optimal dominant strategy implementable mechanism where the objective is to minimize the surplus from core deviations. This is true due to standard mechanism design technique: by envelope conditions, the incentive constraints are satisfied and built in to objective function, and by $T_3(0, v_1, v_2) = T_1(0, v_{-1}) = T_2(0, v_{-2}) = 0$, individual rationality conditions are also satisfied. ■

4 Calculus of variations problem

We now establish that the mechanism design problem can be expressed as a calculus of variations problem.

Proposition 2 *The core deviation minimization problem can be written as a calculus of*

[6] Note that $T_i(0, v_{-i})$ cannot be positive because otherwise individual rationality constraint of bidder i with type 0 is violated. If, on the other hand, $T_i(0, v_{-i})$ is strictly negative, then we can subtract that number from the payments of bidder i, which will not change i's incentive constraints, but decrease the value of the objective function.

variations problem.

Proof. First of all, by changing the orders of integrations, we can rewrite (10) as

$$\int_0^2 \left(\begin{array}{c} \int_{\max\{r^{-1}(v_3)-1,0\}}^1 \int_{\max\{r^{-1}(v_3)-v_1,0\}}^1 \max\left\{ \left(\begin{array}{c} v_3 - \max\{r^{-1}(v_3) - v_1, 0\} \\ - \max\{r^{-1}(v_3) - v_2, 0\} \end{array} \right), 0 \right\} dF(v_2)\, dF(v_1) \\ + \int_0^{\min\{r^{-1}(v_3),1\}} \int_0^{\min\{r^{-1}(v_3)-v_1,1\}} \max\{(v_1 + v_2 - r(v_1 + v_2)), 0\} dF(v_2)\, dF(v_1) \end{array} \right) dG(v_3)$$

By changing the variable $r^{-1}(v_3)$ to v, we obtain

$$\min_r \int_0^2 \left(\begin{array}{c} \int_{\max\{v-1,0\}}^1 \int_{\max\{v-v_1,0\}}^1 \left(\max\left\{ \left(\begin{array}{c} r(v) - \max\{v - v_1, 0\} \\ - \max\{v - v_2, 0\} \end{array} \right), 0 \right\} \right) dF(v_2)\, dF(v_1) \\ + \int_0^{\min\{v,1\}} \int_0^{\min\{v-v_1,1\}} \max\{(v_1 + v_2 - r(v_1 + v_2)), 0\} dF(v_2)\, dF(v_1) \end{array} \right) g(r(v))\, r'(v)\, dv \tag{11}$$

In calculus of variations problems, the objective is to optimize an integral by choosing a function, where the integrand involves the function and its derivative. Above minimization is of this sort. ■

Moreover, it can be shown that the optimal mechanism should "favor the global bidder" in the sense that $r(v) \leq v$.

Proposition 3 *In the optimal mechanism,* $r(v) \leq v$.

Proof. Suppose $r(v) > v$ on some interval (c, d), then consider the function $\widetilde{r}$ which satisfies $\widetilde{r}(v) = v$ for $v \in (c, d)$, and $\widetilde{r}(v) = r(v)$ for $v \notin (c, d)$. We argue that $\widetilde{r}$ will achieve a smaller value than r at the objective function. If $v_1 + v_2 \in (c, d)$, then $\max\{(v_1 + v_2 - r(v_1 + v_2)), 0\}$ is zero, and decreasing r in this interval up to $r(v) = v$, does not change the value of this term. Moreover, decreasing r in this interval makes $\max\{r(v) - \max\{v - v_1, 0\} - \max\{v - v_2, 0\}, 0\}$ smaller. Hence this decrease has no cost, but only benefit. Hence r has to satisfy $r(v) \leq v$ at the optimal solution. ■

Hence, we can write the objective function as

$$\min_r \int_0^2 \left(\begin{array}{c} \int_{\max\{v-1,0\}}^1 \int_{\max\{v-v_1,0\}}^1 \max\{ \left(\begin{array}{c} r(v) - \max\{v - v_1, 0\} \\ - \max\{v - v_2, 0\} \end{array} \right), 0\} dF(v_2)\, dF(v_1) \\ + \int_0^{\min\{v,1\}} \int_0^{\min\{v-v_1,1\}} (v_1 + v_2 - r(v_1 + v_2)) dF(v_2)\, dF(v_1) \end{array} \right) g(r(v))\, r'(v)\, dv$$

subject to

$$r(0) = 0$$
$$r(2) = 2$$
$$r'(v) > 0$$
$$r(v) \leq v$$

We now show that one can eliminate the integrals in the integrands of Equation (11).

Proposition 4 *The core deviation minimization problem can be written as a standard calculus of variations problem of the form*

$$\int_0^2 A(r(v), r'(v), v)\, dv$$

Proof. Denote

$$\int_0^2 \left(\int_0^{\min\{v,1\}} \int_0^{\min\{v-v_1,1\}} (v_1 + v_2 - r(v_1 + v_2)) dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv$$

by O_1 and

$$\int_0^2 \left(\int_{\max\{v-1,0\}}^1 \int_{\max\{v-v_1,0\}}^1 \max\{r(v) - \max\{v - v_1, 0\} - \max\{v - v_2, 0\}, 0\} dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv$$

by O_2.

Denote the expectation of F by L,

$$L(v) = \int_0^v v' f(v')\, dv'. \tag{12}$$

Moreover, denote $v_1 + v_2$ by t and the distribution of t by H. That is

$$H(t) = \int_0^{\min\{t,1\}} \int_0^{\min\{t-v_1,1\}} dF(v_2)\, dF(v_1). \tag{13}$$

Also denote its density by h, $h(t) = H'(t)$, and expectation by K,

$$K(t) = \int_0^t t' h(t')\, dt'. \tag{14}$$

Lastly, denote $v_1 + v_2$ conditional on $v_1, v_2 < v$ by t_v and its distribution by H_v. That is,

$$H_v(t_v) = \int_0^{\min\{t_v, v\}} \int_0^{\min\{t - v_1, v\}} dF(v_2)\, dF(v_1) \tag{15}$$

Also denote its density by h_v, $h_v(t) = \frac{d}{dt} H_v(t)$, and expectation by K_v,

$$K_v(t) = \int_0^t t' h_v(t')\, dt'. \tag{16}$$

Then, we can write

$$\begin{aligned}
O_1 &= \int_0^2 \left(\int_0^v (t - r(t)) dH(t) \right) g(r(v))\, r'(v)\, dv \\
&= \int_0^2 \left(K(v) - \int_0^v r(t)\, h(t)\, dt \right) g(r(v))\, r'(v)\, dv \\
&= \int_0^2 K(v)\, g(r(v))\, r'(v)\, dv - \int_0^2 \left(\int_0^v r(t)\, h(t)\, dt \right) dG(r(v)) \\
&= \int_0^2 K(v)\, g(r(v))\, r'(v)\, dv - \int_0^2 \left(\int_t^2 dG(r(v)) \right) r(t)\, h(t)\, dt \\
&= \int_0^2 K(v)\, g(r(v))\, r'(v)\, dv - \int_0^2 (1 - G(r(t)))\, r(t)\, h(t)\, dt \\
&= \int_0^2 \left(K(v)\, g(r(v))\, r'(v) - (1 - G(r(v)))\, r(v)\, h(v) \right) dv
\end{aligned}$$

where, to achieve fourth line, we change order of integration.

On the other hand,

$$\begin{aligned}
O_2 = &\int_0^1 \left(\int_v^1 \int_v^1 r(v)\, dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv \\
&+ \int_0^1 \left(\int_v^1 \int_0^v \max\{r(v) - v + v_2, 0\} dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv \\
&+ \int_0^1 \left(\int_0^v \int_v^1 \max\{r(v) - v + v_1, 0\} dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv \\
&+ \int_0^1 \left(\int_0^v \int_{v - v_1}^v \max\{r(v) - 2v + v_1 + v_2, 0\} dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv \\
&+ \int_1^2 \left(\int_{v-1}^1 \int_{v - v_1}^1 \max\{r(v) - 2v + v_1 + v_2, 0\} dF(v_2)\, dF(v_1) \right) g(r(v))\, r'(v)\, dv
\end{aligned}$$

Let us denote the first, second, third, fourth and fifth summands by $O_{2,1}$, $O_{2,2}$, $O_{2,3}$, $O_{2,4}$ and $O_{2,5}$ respectively.

We can write

$$O_{2,1} = \int_0^1 r(v)(1-F(v))^2 g(r(v))\, r'(v)\, dv$$

and

$$\begin{aligned}
O_{2,2} &= \int_0^1 \left(\int_v^1 \int_{v-r(v)}^{v} (r(v)-v+v_2)\, dF(v_2)\, dF(v_1)\right) g(r(v))\, r'(v)\, dv \\
&= \int_0^1 \left(\int_v^1 ((r(v)-v)(F(v)-F(v-r(v))) + L(v) - L(v-r(v)))\, dF(v_1)\right) g(r(v))\, r'(v)\, dv \\
&= \int_0^1 ((r(v)-v)(F(v)-F(v-r(v))) + L(v) - L(v-r(v)))(1-F(v))\, g(r(v))\, r'(v)\, dv
\end{aligned}$$

and

$$\begin{aligned}
O_{2,3} &= \int_0^1 \left(\int_0^v \max\{r(v)-v+v_1, 0\}(1-F(v))\, dF(v_1)\right) g(r(v))\, r'(v)\, dv \\
&= \int_0^1 \left(\int_{v-r(v)}^{v} (r(v)-v+v_1)\, dF(v_1)\right)(1-F(v))\, g(r(v))\, r'(v)\, dv \\
&= \int_0^1 ((r(v)-v)(F(v)-F(v-r(v))) + L(v) - L(v-r(v)))(1-F(v))\, g(r(v))\, r'(v)\, dv \\
&= O_{2,2}
\end{aligned}$$

and

$$\begin{aligned}
O_{2,4} &= \int_0^1 \left(\int_v^{2v} \max\{r(v)-2v+t_v, 0\} dH_v(t_v)\right) g(r(v))\, r'(v)\, dv \\
&= \int_0^1 \left(\int_{2v-r(v)}^{2v} (r(v)-2v+t_v)\, dH_v(t_v)\right) g(r(v))\, r'(v)\, dv \\
&= \int_0^1 ((r(v)-2v)(H_v(2v)-H_v(2v-r(v))) + K_v(2v) - K_v(2v-r(v)))\, g(r(v))\, r'(v)\, dv
\end{aligned}$$

and lastly,

$$
\begin{aligned}
O_{2,5} &= \int_1^2 \left(\int_v^2 \max\{r(v) - 2v + t, 0\} dH(t) \right) g(r(v)) r'(v) dv \\
&= \int_1^2 \left(\int_{2v-r(v)}^2 (r(v) - 2v + t) dH(t) \right) g(r(v)) r'(v) dv \\
&= \int_1^2 ((r(v) - 2v)(1 - H(2v - r(v))) + K(2) - K(2v - r(v))) g(r(v)) r'(v) dv
\end{aligned}
$$

Adding all the terms up, we obtain

$$
\begin{aligned}
O = &\int_0^1 (K(v) g(r(v)) r'(v) - (1 - G(r(v))) r(v) h(v)) dv \qquad (17) \\
&+ \int_0^1 r(v)(1 - F(v))^2 g(r(v)) r'(v) dv \\
&+ \int_0^1 2((r(v) - v)(F(v) - F(v - r(v))) + L(v) - L(v - r(v)))(1 - F(v)) g(r(v)) r'(v) dv \\
&+ \int_0^1 ((r(v) - 2v)(H_v(2v) - H_v(2v - r(v))) + K_v(2v) - K_v(2v - r(v))) g(r(v)) r'(v) dv \\
&+ \int_1^2 (K(v) g(r(v)) r'(v) - (1 - G(r(v))) r(v) h(v)) dv \\
&+ \int_1^2 ((r(v) - 2v)(1 - H(2v - r(v))) + K(2) - K(2v - r(v))) g(r(v)) r'(v) dv
\end{aligned}
$$

This problem is now of the form

$$
\int_0^2 A(r(v), r'(v), v) dv
$$

■

In the following two subsections, we numerically solve above calculus of variations problem for two pairs of distribution functions that have been considered in Ausubel and Baranov (2010). We solve a discrete version of the problem using the Newton-Cotes formulas to evaluate integrals numerically. We divide the interval of integration to n pieces, and convert $r(v)$ function to a vector of $n+1$ elements (this corresponds to $n+1$ decision variables). We use the following approximation formula

$$
\int_a^b f(x) dx \approx \sum\nolimits_{i=0}^n f\left(a + i\frac{b-a}{n}\right) \frac{b-a}{n}
$$

to evaluate the integrals in the objective function. We then use MATLAB's fmincon function to optimize the provided objective function.

4.1 Uniform case

Suppose now that both F and G are uniform: $F(v) = v$, $G(v) = \frac{v}{2}$. By using the formulas (12), (13), (14), (15), and (16), we obtain

$$L(v) = \frac{v^2}{2}, \qquad H(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \in [0,1] \\ 2t - 1 - \frac{t^2}{2} & \text{if } t \in [1,2] \end{cases},$$

$$K(t) = \begin{cases} \frac{t^3}{3} & \text{if } t \in [0,1] \\ t^2 - \frac{t^3}{3} - \frac{1}{3} & \text{if } t \in [1,2] \end{cases}, \quad h(t) = \begin{cases} t & \text{if } t \in [0,1] \\ 2 - t & \text{if } t \in [1,2] \end{cases}.$$

Moreover, for $v \in [0,1]$,

$$H_v(t) = \begin{cases} \frac{v^2}{2} & \text{if } t \in [0,v] \\ 2vt - v^2 - \frac{t^2}{2} & \text{if } t \in [v,2v] \end{cases}, \quad h_v(t) = \begin{cases} v & \text{if } t \in [0,v] \\ 2v - t & \text{if } t \in [v,2v] \end{cases},$$

$$K_v(t) = \begin{cases} \frac{t^3}{3} & \text{if } t \in [0,v] \\ vt^2 - \frac{t^3}{3} - \frac{v^3}{3} & \text{if } t \in [v,2v] \end{cases}.$$

After substituting above functions in (17), we solve the numerical calculus of variations problem. Interestingly, the numeric solution for this case (which is the case that Goeree and Lien (2009) analyzed) is virtually the Vickrey Auction with $r(v) = v$:

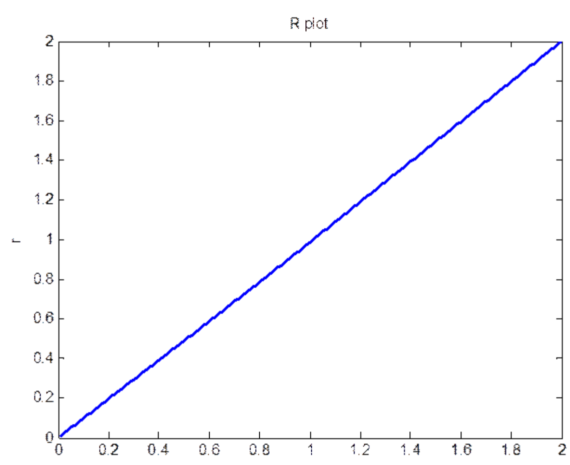

Figure 1: Optimal r for uniform case.

Optimized value is 0.1262.

4.2 Power case

Suppose now that both G is uniform, but we have $F(v) = v^2$. Again, by using the formulas (12), (13), (14), (15), and (16) we obtain

$$L(v) = \frac{2}{3}v^3, \qquad H(t) = \begin{cases} \frac{1}{6}t^4 & \text{if } t \in [0,1] \\ 1 - \frac{8}{3}t + 2t^2 - \frac{1}{6}t^4 & \text{if } t \in [1,2] \end{cases},$$

$$K(t) = \begin{cases} \frac{2}{15}t^5 & \text{if } t \in [0,1] \\ -\frac{2}{15}t^5 + \frac{4}{3}t^3 - \frac{4}{3}t^2 + \frac{4}{15} & \text{if } t \in [1,2] \end{cases}, \quad h(t) = \begin{cases} \frac{2}{3}t^3 & \text{if } t \in [0,1] \\ -\frac{8}{3} + 4t - \frac{2}{3}t^3 & \text{if } t \in [1,2] \end{cases}.$$

Moreover, for $v \in [0,1]$,

$$H_v(t) = \begin{cases} \frac{1}{6}t^4 & \text{if } t \in [0,v] \\ v^4 - \frac{8}{3}tv^3 + 2t^2v^2 - \frac{1}{6}t^4 & \text{if } t \in [v,2v] \end{cases}, \quad h_v(t) = \begin{cases} \frac{2}{3}t^3 & \text{if } t \in [0,v] \\ -\frac{8}{3}v^3 + 4tv^2 - \frac{2}{3}t^3 & \text{if } t \in [v,2v] \end{cases},$$

$$K_v(t) = \begin{cases} \frac{2}{15}t^5 & \text{if } t \in [0,v] \\ -\frac{2}{15}t^5 + \frac{4}{3}t^3v^2 - \frac{4}{3}t^2v^3 + \frac{4}{3}v^3 - \frac{4}{3}v^2 + \frac{4}{15} & \text{if } t \in [v,2v] \end{cases}.$$

After substituting above functions in (17), we solve the numerical calculus of variations problem. For this case, the optimal $r(v)$ is given by the below graph:

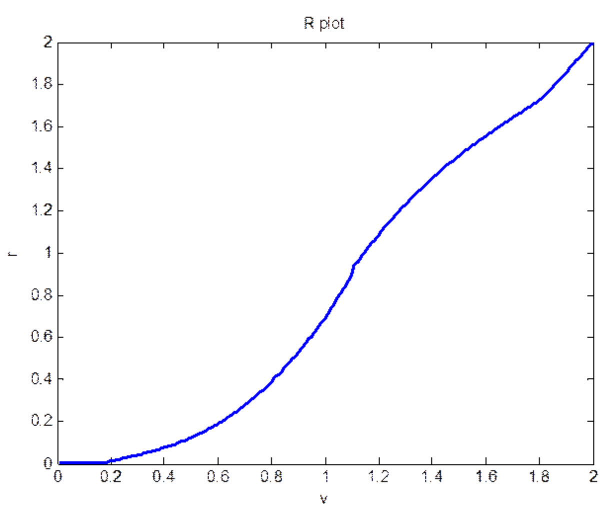

Figure 2: Optimal r for $F(v) = v^2$, $G(v) = \frac{v}{2}$

with optimized value 0.1754. For this case objective function value for Vickrey auction, $r(v) = v$ is only 0.2777. Hence, the optimal solution as compared to Vickrey auction improves the objective approximately by 37%.

5 Conclusion and Discussion

In a stylized model with complementarities, we ask how close can we get to the core among the strategy proof mechanisms. If the allocation function of the mechanism compares the sum of local bidders' values and global bidder's value, it turns out that this problem can be reduced to a calculus of variations problem which can be solved at least numerically. We consider two specific examples. When both F and G are uniform, it turns out that Vickrey auction is the core deviation minimizing auction, whereas if we consider $F(v) = v^2$ and $G(v) = \frac{v}{2}$, the optimal solution improves the objective by approximately 37% as compared to Vickrey auction.

Although we worked on the stylized model, core deviation minimizing auction problem can be defined for more general settings. Consider an auctioneer who has k different items to sell and n bidders ($\{1, ..., n\} = N$) who have combinatorial valuations over these items. More specifically, for a vector $\mathbf{q} = \left(q^1, ..., q^k\right) \in \{0,1\}^k$, let $v_i(\mathbf{q})$ represent the value of player i when he/she wins the set of items as defined by $\mathbf{q}$ (that is, when she is awarded item j's with $q^j = 1$.) Consider a deterministic direct mechanism, which as a function of announced valuations, allocates $\mathbf{q}_i$ to bidder i, and charges him/her t_i. Let us consider a deviation by the seller and subset M of agents. Denote the aggregate value generated by the most efficient allocation of k items among bidders in M by V_M^d. That is

$$\begin{aligned} V_M^d &= \max \sum\nolimits_{i \in M} v_i(\widetilde{\mathbf{q}}_i) \\ \text{subject to } & \sum\nolimits_{i \in M} \widetilde{\mathbf{q}}_i = \mathbf{I} \end{aligned}$$

where $\mathbf{I}$ is the unit vector. Moreover, denote the sum of utilities of the seller and the players in set M given by the mechanism by V_M. That is,

$$V_M = \sum\nolimits_{i \in N} t_i + \sum\nolimits_{i \in M} (v_i(\mathbf{q}_i) - t_i)$$

The surplus from a core deviation that involves the seller and subset M of agents is then given by

$$V_M^s \equiv \max\{0, V_M^d - V_M\}$$

Thus, the highest surplus from any deviations is given by

$$\max_{M \subset N} V_M^s.$$

Hence, for the general case, the core deviation minimizing auction minimizes the expected value of $\max_{M \subset N} V_M^s$ subject to the standard dominant strategy incentive compatibility and individual rationality constraints.

References

[1] Ausubel, L.M. and O.V. Baranov (2010). "Core-selecting auctions with incomplete information." Working paper, University of Maryland

[2] Baranov, O.V. (2010). "First-price package auctions with incomplete information." Working paper, University of Maryland.

[3] Cramton, P. (2009). "Spectrum auction design." Working paper, University of Maryland.

[4] Day, R.W. and P. Cramton (2008). "The quadratic core-selecting payment rule for combinatorial auctions." *Operations Research*, forthcoming.

[5] Day, R.W. and S. Raghavan (2007). "Fair payments for efficient allocations in public sector combinatorial auctions." *Management Science* 53, 1389-1406.

[6] Day, R.W. and P.R. Milgrom (2008). "Core-selecting package auctions." *International Journal of Game Theory,* Volume 36, 393-407.

[7] Erdil, A. and P. Klemperer (2010). "A new payment rule for core-selecting package auctions." *Journal of the European Economic Association*, Volume 8, 537-547.

[8] Goeree, J. K. and Y. Lien (2009). "On the impossibility of core-selecting auctions. " Working paper, Caltech.

[9] Lamy, L. "Core-selecting package auctions: a comment on revenue-monotonicity." *International Journal of Game Theory,* Volume 39, 503-510.

[10] Sano, R. (2010). "Incentives in core-selecting auctions with single-minded bidders." *Games and Economic Behavior*, Volume 72, 602-606.