New Functional Characterizations and Optimal Structural Results for Assemble-to-Order M-Systems

Emre Nadar  
*Carnegie Mellon University*

Mustafa Akan  
*Carnegie Mellon University, akan@cmu.edu*

Alan Scheller-Wolf  
*Carnegie Mellon University, awolf@andrew.cmu.edu*

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New Functional Characterizations and Optimal Structural Results for Assemble-to-Order $M$-Systems

Emre Nadar, Mustafa Akan and Alan Scheller-Wolf
Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, {enadar, akan, awolf}@andrew.cmu.edu

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We consider an assemble-to-order $M$-system with multiple components, multiple products, batch ordering of components, random lead times, and lost sales. We model the system as an infinite-horizon Markov decision process and seek an optimal control policy: a control policy specifies when a batch of components should be produced, and whether an arriving demand for each product should be satisfied. We introduce new functional characterizations for submodularity and supermodularity restricted to certain subspaces. These enable us to characterize optimal inventory replenishment and allocation policies under a mild condition on component batch sizes via a new type of policy: lattice-dependent base-stock production and lattice-dependent rationing.

Key words: assemble-to-order systems; Markov decision processes; optimal control; inventory rationing

1. Introduction

Assemble-to-order (ATO) production is a popular strategy for manufacturing firms. ATO allows companies to reduce their response window by stocking components, but gives them the flexibility of postponing final assembly until demand is realized (Benjaafar and ElHafsi 2006). Many high-tech firms, facing shrinking product life cycles and increasing demand for product varieties, use ATO to broaden customized product offerings, lower inventory cost, and mitigate the effect of product obsolescence. Besides manufacturing, ATO systems can be observed in cases where customer orders may include several items in different quantities (Song 2000). However, despite its popularity, little is known about the forms of optimal policies for ATO systems. Much of this is attributable to the fact that there is considerable difficulty in identifying optimal policies, as ATO systems build upon the features of both assembly and distribution systems (Song and Zipkin 2003). (An assembly system has only one product and aims to optimally coordinate components. A distribution system has only one component and seeks to optimally allocate the component among different products.) Hence, one needs to address both coordination and allocation issues in an ATO system, making them notoriously difficult to analyze.

ATO systems can be categorized based on their product structures (Lu et al. 2010). Figure 1 depicts the four specific types of ATO product structures: (a) An $N$-system, the simplest of the
Specific types of ATO product structures: (a) $N$-system, (b) $M$-system, (c) $W$-system, and (d) Nested system with three products.

ATO product structures, has two components and two products. One product uses both components while the other product uses only one component. (b) An $M$-system has two components and three products. One product uses both components while the other two products use different components. (c) A $W$-system has three components and two products. Each product is assembled from one product-specific component and one common component. (d) A nested system has multiple components and products, where the set of components required by one product is a subset of the set of components required by the next larger product. Figure 1(d) depicts a nested system with three components. There are papers characterizing the optimal policies for ATO systems with product structures (a), (c), or (d); for instance, see Dogru et al. (2010) for a $W$-system; Lu et al. (2010) for an $N$-system, and a $W$-system and its generalizations; and ElHafsi et al. (2008) for a nested system.

In this paper, we consider the inventory control of a continuous time ATO system with multiple products and components structured according to a generalized version of the $M$-system. Specifically, the system involves a single product which uses multiple units from each component, and
multiple products each of which consumes multiple units from a different component. Our product structure takes the form of $M$-system when there are three products, cf. Figure 1(b).

We formulate the problem as an infinite-horizon Markov decision process (MDP) under the total expected discounted cost criterion. We assume each component is produced in batches of a fixed size in a make-to-stock fashion; production times are independent and exponentially distributed. Demand for each end-product arrives as an independent Poisson process. If not satisfied immediately upon arrival, these demands are lost. A control policy specifies when to produce a batch of any component and, upon arrival of a demand, whether or not to satisfy it from inventory if sufficient inventory exists.

A standard approach for the analysis of optimal policies of MDPs is to explore the first- and/or second-order properties of the optimal cost function (see Koole 2006). In the literature, optimal cost functions are typically shown to be convex (or concave). However, the existence of counter-examples proves that convexity need not hold for our model (see Nadar et al. 2011). Taking an alternative approach we define new functional characterizations for submodularity and supermodularity, restricted to certain subspaces of the state space.

With these new definitions, we characterize the forms of optimal inventory replenishment and allocation policies under a mild condition: If the batch size for any component equals the number of units of that component needed to make one unit of the product using that component only (Assumption 1), the optimal inventory replenishment is a lattice-dependent base-stock production policy and the optimal inventory allocation is a lattice-dependent rationing policy (cf. Theorem 1). This implies that the state space of the problem can be partitioned into disjoint lattices such that, on each lattice, (a) it is optimal to produce a batch of a particular component if and only if the state vector is less than the base-stock level associated with that component, and (b) it is optimal to fulfill a demand of a particular product if and only if the state vector is greater than or equal to the rationing level associated with that product. Furthermore, as the system moves to a different lattice upon replenishment of a particular component, (i) the base-stock level of any other component increases, (ii) the rationing level for any product not using that component increases, and (iii) the rationing level for the product using all components decreases, in a non-strict sense.

Literature on ATO systems is extensive, although the optimal policy is still unknown for the general ATO system. Song and Zipkin (2003) provide a comprehensive survey of this literature. The paper that is most closely related to ours is Benjaafar and ElHafsi (2006). They consider an
ATO system with a single end-product and multiple components. One unit of each component is assembled into the end-product, which is demanded by multiple customer classes. They show that, under Markovian assumptions on production and demand, the optimal replenishment policy is a state-dependent base-stock policy, and the optimal allocation policy is a state-dependent rationing policy. We extend the model of Benjaafar and ElHafsi (2006) in several directions: (i) Each of our components is used by two products, (ii) each of our components may be used by the two products in different quantities, and (iii) the product requiring all components may use different quantities of different components. Furthermore, the state-dependent base-stock and state-dependent rationing (SBSR) policy in Benjaafar and ElHafsi (2006) is a special case of our lattice-dependent base-stock and lattice-dependent rationing (LBLR) policy, implying that LBLR is analytically no worse than SBSR for general ATO systems.

Our contributions in the ATO research stream are as follows: First, to our knowledge, our study is the first attempt to characterize the forms of optimal replenishment and allocation policies for the $M$-system and its generalizations. Second, unlike previous research dealing with the optimal policy characterization for ATO systems under stochastic lead times, we are the first to allow different products to use the same component in different quantities. Third, we define new functional characterizations for submodularity and supermodularity, restricted to certain subspaces of the state space. Fourth, we introduce the notion of a lattice-dependent policy, which represents a significant step towards understanding the problem and may enable researchers to develop near-optimal heuristic solutions for general ATO systems.

The rest of this paper is organized as follows: In Section 2, we formulate our model under the discounted cost criterion. In Section 3, we introduce the new functional characterizations, establish the optimal replenishment and allocation policies, and extend our structural results to the average cost case. In Section 4, we offer some extensions and concluding remarks.

2. Problem Formulation

We consider an ATO system with $n$ components ($j = 1, 2, \ldots, n$) and $n + 1$ products ($i = 1, 2, \ldots, n + 1$), where each component $j$ is consumed by products $i = j$ and $i = n + 1$ only. Notice that the ATO system we consider reduces to an “$M$-system” when $n = 2$, cf. Figure 1(b). Define $a = (a_1, a_2, \ldots, a_n)$ as the vector of component requirements for product $n + 1$; $a_j$ is the number of component $j$ needed to assemble one unit of product $n + 1$. Define $b = (b_1, b_2, \ldots, b_n)$ as the vector of component requirements for all the other products; $b_j$ is the number of component $j$ required to make one
unit of product \(i = j\). Each component \(j\) is produced in batches of a fixed size \(q_j\) in a make-to-stock fashion. Define \(q = (q_1, q_2, ..., q_n)\) as the vector of production batch sizes. Production time for component \(j\) is independent of the system state and the number of outstanding orders of any type, and exponentially distributed with finite mean \(1/\mu_j\). Assembly lead times are negligible so that assembly operations can be postponed until demand is realized. Demand for each product \(i\) arrives as an independent Poisson process with finite rate \(\lambda_i\). Demand for product \(i\) can be fulfilled only if all the required components are available; otherwise, the demand is lost, incurring a unit lost sale cost \(c_i\). Demand may also be rejected in the presence of all the necessary components, again incurring a unit lost sale cost.

The state of the system at time \(t\) is the vector \(X(t) = (X_1(t), ..., X_n(t))\), where \(X_j(t)\) is a nonnegative integer denoting the on-hand inventory for component \(j\) at time \(t\). Each component held in stock has a holding cost per unit time which is strictly increasing convex in the number of available units of that component. Denote by \(h(X(t)) = \sum_j h_j(X_j(t))\) the inventory holding cost rate at state \(X(t)\). Since all inter-event times are exponentially distributed, the system retains no memory, and decision epochs can be restricted to times when the state changes. Using the memoryless property, we can formulate the problem as an MDP and concentrate on only Markovian policies for which actions at each decision epoch depend solely on the current state. A control policy \(\pi\) specifies for each state \(x = (x_1, ..., x_n)\), the action \(u^\pi(x) = (y_1, ..., y_n, z_1, ..., z_{n+1})\), \(y_j, z_i \in \{0, 1\}, \forall i, j\), where \(y_j = 1\) means produce component \(j\), \(y_j = 0\) means do not produce component \(j\), \(z_i = 1\) means satisfy demand for product \(i\), and \(z_i = 0\) means reject demand for product \(i\). Thus there is at most one outstanding order for each component \(j\) at any time. As component orders are not part of our system state, these can in effect be cancelled upon transition to a new state. Both of these assumptions are standard in the literature (see Ha 1997, Benjaafar and ElHafsi 2006, and ElHafsi et al. 2008).

Define \(0 < \alpha < 1\) as the discount rate. For a given policy \(\pi\) and a starting state \(x \in \mathbb{N}_0^n\) (where \(\mathbb{N}_0\) is the set of nonnegative integers and \(\mathbb{N}_0^n\) is its \(n\)-dimensional cross product), the expected discounted cost over an infinite planning horizon \(v^\pi(x)\) can be written as

\[
v^\pi(x) = \mathbb{E}_x \left[ \sum_{j=1}^n \int_0^\infty e^{-\alpha t} h_j(X_j(t)) dt + \sum_{i=1}^{n+1} \int_0^\infty e^{-\alpha t} c_i dN_i(t) \right]
\]

where \(N_i(t)\) is the number of demands for product \(i\) that have not been fulfilled from on-hand inventory up to time \(t\). Letting \(\beta\) denote the upper bound on transition rates for all system states.
We formulate the optimality equation that holds for the optimal cost function $v^* = v^*$ (see Chapter 5 in Bertsekas 2007 for an explanation of how the continuous-time control problem can be transformed into an equivalent discrete-time control problem):

$$v^*(x) = \frac{1}{\alpha + \beta} \left( h(x) + \sum_j \mu_j T^j T^* (x) + \sum_i \lambda_i T^i T^* (x) \right),$$

where the operators $T^j$ and $T^i$ are defined as

$$T^j v(x) = \min \{ v(x + q_j e_j), v(x) \}, \quad \forall j,$n

and

$$T^i v(x) = \begin{cases} \min \{ v(x) + c_i, v(x - b_i e_i) \} & \text{if } x_i \geq b_i, \\ v(x) + c_i & \text{otherwise,} \end{cases} \quad \forall i \leq n,$n

$T^{n+1} v(x) = \begin{cases} \min \{ v(x) + c_{n+1}, v(x - a) \} & \text{if } x \geq a, \\ v(x) + c_{n+1} & \text{otherwise,} \end{cases} \quad \forall x \geq a.$

where $e_j$ is the $j$th unit vector of dimension $n$. For a given state $x$, the operator $T^j$ specifies whether or not to produce a batch of component $j$; and the operator $T^i$ specifies, upon arrival of a demand for product $i$, whether or not to fulfill it from inventory if sufficient inventory exists. In the optimality equation, as it is always possible to redefine the time scale, without loss of generality we assume $\alpha + \beta = 1$.

### 3. Characterization of the Optimal Policy

In this section we first define new second-order functional characterizations, and show how these properties propagate through our optimal cost function. We then use these propagation results to establish the optimality of lattice-dependent base-stock and rationing policies under a mild condition on component batch sizes.

#### 3.1. Functional Characterizations

Define $f$ as the class of real-valued functions on the $n$-dimensional nonnegative orthant, and let $\Delta_p f = f(x + p) - f(x)$ where $p = (p_1, p_2, \ldots, p_n)$ is a vector of nonnegative integers.

We introduce the notion of “submodularity with step size $p$” for $p \in \mathbb{N}_0^n$ to describe the class of functions $f$ for which $\Delta_{p_j} f$ is nonincreasing with an increase of $p_k$ in the $k$th dimension, $\forall j \neq k$. We denote this class of functions by $Sub(p)$.

We also define the concept of “supermodularity with step sizes $r$ and $p$” for $r, p \in \mathbb{N}_0^n$ to describe the class of functions $f$ with $\Delta_{p_j} f$ nondecreasing with an increase of $r$ in the domain, $\forall j$. We denote this class of functions by $Super(r, p)$. 
Lastly, we define the notion of “$n$-dimensional supermodularity with step sizes $r$ and $p$” for $r, p \in \mathbb{N}_0^n$ to describe the class of functions $f$ with $\Delta_pf$ nondecreasing with an increase of $r$ in the domain, and denote it by $n\text{Super}(r, p)$. Note that both $\text{Super}(1, 1)$ and $n\text{Super}(1, 1)$ are the class of convex functions of one dimension.

**Definition 1** (Second-Order Properties). Let $f$ be a real-valued function defined on $\mathbb{N}_0^n$. Also let $r, p \in \mathbb{N}_0^n$.

(a) $f \in \text{Sub}(p)$, if $f(x + p_j e_j) - f(x) \geq f(x + p_j e_j + p_k e_k) - f(x + p_k e_k), \forall x \in \mathbb{N}_0^n, \forall j$ and $\forall k \neq j$.

(b) $f \in \text{Super}(r, p)$, if $f(x + p_j e_j + r) - f(x + r) \geq f(x + p_j e_j) - f(x), \forall x \in \mathbb{N}_0^n$ and $\forall j$.

(c) $f \in n\text{Super}(r, p)$, if $f(x + p + r) - f(x + r) \geq f(x + p) - f(x), \forall x \in \mathbb{N}_0^n$.

The following lemma shows that the class of $\text{Super}(r, p)$ is a subset of that of $n\text{Super}(r, p)$:

**Lemma 1.** $\text{Super}(r, p) \subseteq n\text{Super}(r, p), \forall r, p \in \mathbb{N}_0^n$.

The proofs of Lemma 1 and all other subsequent results appear in the online appendix.

### 3.2. Propagation Results

We now proceed to the analysis of our optimal cost function based on the functional characterizations of Section 3.1. First notice that our optimal cost function (1) is a linear function of replenishment control operators (i.e., $T^{(i)}, \forall j$), allocation control operators (i.e., $T_i, \forall i$), and holding cost rates (i.e., $h_j, \forall j$). The lemma below shows that (a) each of our replenishment control operators preserves both “submodularity with step size $q$” and “supermodularity with step sizes $a$ and $q$”; (b) each of our allocation control operators preserves both “submodularity with step size $b$” and “supermodularity with step sizes $a$ and $b$”; and (c) our holding cost rate satisfies all these properties:

**Lemma 2.** (a) $T^{(j)} : \text{Sub}(q) \cap \text{Super}(a, q) \rightarrow \text{Sub}(q) \cap \text{Super}(a, q), \forall j$.

(b) $T_i : \text{Sub}(b) \cap \text{Super}(a, b) \rightarrow \text{Sub}(b) \cap \text{Super}(a, b), \forall i$, and

(c) $h \in \text{Sub}(q) \cap \text{Super}(a, q) \cap \text{Sub}(b) \cap \text{Super}(a, b)$.

While our second-order properties are preserved by linear transformations, the second-order properties shown to propagate through our replenishment and allocation control operators above differ in their parameters (i.e., $q$ vs. $b$), and thus need not hold for our optimal cost function: Only for cases with equal parameters ($q = b$) are we able to characterize the structure of our cost function. Therefore, we assume the production batch size for each component $j$ equals the number of units of component $j$ required by one unit of product $i = j$: 
ASSUMPTION 1. $q_j = b_j$, $\forall j$.

Although we make the above assumption for analytical tractability, this corresponds to systems with component batch sizes which are, reasonably, determined by the individual product sizes. This assumption is consistent with previous treatments of Markovian inventory systems (see, for example, Ha 1997, Benjaafar and ElHafsi 2006, and ElHafsi et al. 2008).

We now define $V^*$ as the set of real-valued functions satisfying the properties of $Sub(b)$, $Super(a,b)$, and $nSuper(a,b)$. Then, under Assumption 1, the lemma below follows from Lemmas 1 and 2, and Propositions 3.1.5 and 3.1.6 in Bertsekas (2007):

**Lemma 3.** Under Assumption 1, if $v \in V^*$, then $Tv \in V^*$, where $Tv = h(x) + \sum_j \mu_j T^{(j)} v(x) + \sum_i \lambda_i T^i v(x)$. Furthermore, the optimal cost function $v^*$ is an element of $V^*$.

In the next subsection, we use the second-order properties of our optimal cost function to characterize the forms of optimal inventory replenishment and allocation policies.

### 3.3. Optimal Inventory Replenishment and Allocation

We introduce the notation $\mathbb{L}(p,r) = \{p + kr : k \in \mathbb{N}_0\}$ to denote an $n$-dimensional lattice with initial vector $p \in \mathbb{N}_0^n$ and common difference $r \in \mathbb{N}_0^n$, where $\exists j$ such that $p_j < r_j$. With this we are now ready to state the main result of this paper:

**Theorem 1.** Under Assumption 1, there exists an optimal stationary policy that can be specified as follows.

1. The optimal inventory replenishment policy for each component $j$ is a lattice-dependent base-stock policy with lattice-dependent base-stock levels $S^*_j(p) \in \mathbb{L}(p,a)$, $\forall p$: It is optimal to produce a batch of component $j$ if and only if $x \in \mathbb{L}(p,a)$ is less than $S^*_j(p)$.

2. The optimal inventory allocation policy for each product $i \leq n$ is a lattice-dependent rationing policy with lattice-dependent rationing levels $R^*_i(p) \in \mathbb{L}(p,a)$, $\forall p$: It is optimal to fulfill a demand for product $i \leq n$ if and only if $x \in \mathbb{L}(p,a)$ is greater than or equal to $R^*_i(p)$.

3. The optimal inventory allocation policy for product $n+1$ is a lattice-dependent rationing policy with lattice-dependent rationing levels $R^*_{n+1}(p) \in \mathbb{L}(p,b)$, $\forall p$: It is optimal to fulfill a demand for product $n+1$ if and only if $x \in \mathbb{L}(p,b)$ is greater than or equal to $R^*_{n+1}(p)$.

The optimal policy has the following additional properties:
i. As the system moves to a difference lattice with an increment of \( b_k \) in the inventory level of component \( k \), both the optimal base-stock level of component \( j \neq k \) and the optimal rationing level for (small) product \( i \notin \{k, n+1\} \) increase in a non-strict sense, \( \forall k \).

ii. As the system moves to a difference lattice with an increment of \( b_k \) in the inventory level of component \( k \), the optimal rationing level for product \( n+1 \) decreases in a non-strict sense, \( \forall k \).

iii. It is optimal to fulfill a demand of product \( n+1 \) if \( x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor \), \( \forall j \).

Theorem 1 builds upon the properties of \( \text{Super}(a, b) \), \( n\text{Super}(a, b) \), and \( \text{Sub}(b) \): \( \text{Super}(a, b) \) implies that, as the system moves to a higher inventory level on the lattice \( L(p, a) \), the desirability of producing a batch of any component decreases in a non-strict sense (i.e., optimality of base-stock policies, point 1), and the desirability of satisfying a demand for any product \( i \leq n \) increases in a non-strict sense (i.e., optimality of rationing policies for each product \( i \leq n \), point 2). \( n\text{Super}(a, b) \) implies that, as the system moves to a higher inventory level on the lattice \( L(p, b) \), the incentive to fulfill a demand for product \( n+1 \) increases in a non-strict sense (i.e., optimality of a rationing policy for product \( n+1 \), point 3).

Notice that the rationing policy for each product \( i \leq n \) in point 2 is defined over lattices with common difference \( a \), while the rationing policy for product \( n+1 \) in point 3 is defined over lattices with common difference \( b \). The intuition behind these results is as follows: Demands of each product \( i \leq n \) compete with those of product \( n+1 \) for the same component. For a given product \( i \leq n \), an increment of \( a \) in the inventory level increases the total demand for its competitor product that can be satisfied, thereby mitigating the competition. Hence, the incentive to fulfill a demand of product \( i \leq n \) increases in a non-strict sense (point 2). Likewise, for product \( n+1 \), an increment of \( b \) in the inventory level mitigates the competition as the total demand for each of its competitors that can be satisfied increases. Hence, the incentive to fulfill a demand of product \( n+1 \) increases in a non-strict sense (point 3). Note that under the rationing policy described in Theorem 1, for a given product, an increment in the inventory level that does not increase the total demand for any of its competitors that can be satisfied, may reduce the incentive to fulfill a demand of this product (in a non-strict sense).

Theorem 1, using the properties of \( \text{Sub}(b) \) and \( \text{Super}(a, b) \), proves the following additional properties of the optimal policy: Point (i) says that, based on the property of \( \text{Sub}(b) \), upon replenishment of a batch of a component \( k \), the desirability of producing a batch of component \( j \neq k \) increases while the desirability of satisfying a demand for product \( i \notin \{k, n+1\} \) decreases, in a
non-strict sense. Therefore, both the base-stock level of component \( j \neq k \) and the rationing level for product \( i \notin \{k, n+1\} \) increase in a non-strict sense. The intuition is that the presence of product \( n+1 \) requires us to coordinate inventory replenishment and fulfillment decisions across components; it is less beneficial to produce or hold a batch of one component when the inventory level of any other component is significantly smaller. Point (ii) states that, based on the property of \( \text{Super}(a, b) \), upon replenishment of a batch of any component, the incentive to fulfill a demand for product \( n+1 \) increases in a non-strict sense since the total demand for one of its competitors that can be satisfied increases. Lastly, point (iii) shows that it is optimal to fulfill a demand of product \( n+1 \) as long as the total demand for any other product that can be satisfied stays the same.

To our knowledge, we are the first to introduce the notion of a lattice-dependent base-stock and rationing (LBLR) policy. Such a policy differs from state-dependent base-stock and rationing (SBSR) policies shown to be optimal in a single-product ATO system by Benjaafar and ElHafsi (2006) in the following ways: There are inventory levels \( x_1 \in L(p_1, a) \) and \( x_2 \in L(p_2, a), x_1 \geq x_2, p_1 \neq p_2 \), such that an LBLR policy allows a particular component to be produced at \( x_1 \) even if it is not produced at \( x_2 \), but an SBSR policy does not. Likewise, there are inventory levels \( x_1 \in L(p_1, b) \) and \( x_2 \in L(p_2, b), x_1 \geq x_2, p_1 \neq p_2 \), such that an LBLR policy allows a demand for product \( n+1 \) to be rejected at \( x_1 \) even if it is satisfied at \( x_2 \), but again an SBSR policy does not. Conversely, if \( a \neq \sum_j z e_j \) for \( z \in \mathbb{N}_0 \), then there also may exist inventory levels \( x_1 \geq x_2 \), such that an SBSR policy allows a particular component to be produced at \( x_1 \) even if it is not produced at \( x_2 \), but an LBLR policy does not. But if \( a \) is chosen optimally, then it can be shown that an SBSR policy is a subclass of LBLR policies.

### 3.4. The Case of Average Cost

In this subsection, as our optimization criterion, we take the average cost per unit time over an infinite planning horizon. Given a policy \( \pi \), the average cost rate is given by

\[
v^\pi(x) = \lim_{T \to \infty} \frac{1}{T} \left\{ \sum_{j=1}^{n} \int_{0}^{T} h_j(X_j(t)) dt + \sum_{i=1}^{n+1} \int_{0}^{T} c_i dN_i(t) \right\}.
\]

The objective is to identify a policy \( \pi^* \) that yields \( v^*(x) = \inf_\pi v^\pi(x) \) for all states \( x \). The following proposition shows that our structural results carry over to the average cost case:

**Proposition 1.** Suppose that Assumption 1 holds and the Markov chain governing the system is irreducible. There exists a stationary policy that is optimal under the average cost criterion.
The policy retains all the properties of the optimal policy under the discounted cost criterion, as introduced in Theorem 1. Also, the optimal average cost is finite and independent of the initial state; there exists a finite constant \( v^* \) such that \( v^*(x) = v^* \) for all \( x \).

4. Extensions and Concluding Remarks

We have studied the optimal inventory replenishment and allocation policies in an ATO production system with generalized \( M \)-system product structure. We extend the existing literature by characterizing the optimal policy while allowing different products to use different quantities of the same component. Assuming component batch sizes are determined by the individual product sizes, we establish the optimality of a lattice-dependent base-stock and lattice-dependent rationing policy for both the discounted cost and average cost cases. We discuss below two extensions to our analysis and several concluding remarks.

First, our analysis can be extended to systems where a nonempty subset of the components is required only by product \( n+1 \). These systems take the form of \( N \)-system if there are two components. Define \( A_1 \) as the set of components used by product \( n+1 \) only, and \( A_2 \) as the set of components \( j \) used by products \( i = j \) and \( i = n+1 \) (i.e., \( A_1 = \{1, 2, \ldots, n\} \setminus A_2 \)). Such systems are a special case of our model in which the demand rate for each product \( i \in A_1 \) is zero, and therefore an LBLR policy is optimal for these systems. Notice that Assumption 1 is no longer required in setting the batch sizes for components \( i \in A_1 \). Since the demand rate for each product \( i \in A_1 \) is zero, one can choose \( b_i \) to be the ideal batch size for component \( j = i \), \( \forall i \in A_1 \).

Second, our model can be extended to allow each product to be requested by multiple demand classes with different lost sale costs. Suppose that there are \( D^i \) different demand classes for product \( i \), and let \( d^i = 1, 2, \ldots, D^i \). A demand for one unit of product \( i \) from class \( d^i \) arrives as an independent Poisson process with rate \( \lambda_{i,d^i} \) and has a lost sale cost \( c_{i,d^i} \), \( \forall i \). Without loss of generality, we assume \( c_{i,1} \geq c_{i,2} \geq \cdots \geq c_{i,D^i} \), \( \forall i \). We can revise our optimal cost function by augmenting the allocation control operator \( T_i \) to include the index of demand class \( d^i \), \( \forall i \). We can then prove the optimality of LBLR under the following modifications: (i) The optimal inventory allocation for demand class \( d^i \) of each product \( i \leq n \) is a lattice-dependent rationing policy with rationing levels \( R^*_{i,d^i}(p) \in \mathbb{L}(p, a) \), \( \forall p \), (ii) the optimal inventory allocation for demand class \( d^{n+1} \) of product \( n+1 \) is a lattice-dependent rationing policy with rationing levels \( R^*_{n+1,d^{n+1}}(p) \in \mathbb{L}(p, b) \), \( \forall p \), and (iii) it is optimal to fulfill a demand of product \( n+1 \) from class 1 as long as the total demand for any
other product that can be satisfied stays the same. Furthermore, \( R_{i,1}(p) \leq R_{i,2}(p) \leq \cdots \leq R_{i,D}(p) \), \( \forall p, \forall i \).

In a related work (Nadar et al. 2011), we conduct numerical experiments to evaluate the use of an LBLR policy as a heuristic for general ATO systems, comparing it with two other heuristics: a state-dependent base-stock and rationing policy (SBSR), and a fixed base-stock and rationing policy (FBFR), both adapted from Benjaafar and ElHafsi (2006). We numerically show, in the average cost case, that LBLR always yields the optimal solution in over one thousand examples, while SBSR (or FBFR) provides solutions within 2.4% (or 4.8%) of the optimal cost. We are also able to analytically show that LBLR outperforms the other heuristics. Based on these results, future research could investigate whether an LBLR policy is indeed optimal for general ATO systems, and if so, how the state space should be partitioned into disjoint lattices. However, one may need to develop a different methodology to prove the optimality of LBLR, because in Nadar et al. (2011) we also provide counter-examples showing that the second-order properties of our optimal cost function, which are sufficient to ensure the optimality of LBLR, may fail to hold for general ATO systems.

Future extensions of the current paper could also consider ATO systems with backordering. In this case, we would need to include the number of backordered demands for each product in the state space, and investigate the optimal backorder clearing mechanism upon replenishment of any component. Another direction for future research is to extend our model to phase-type or even general component production and demand interarrival times. Also, it would be more realistic to allow for dependent demand across products and over time. Lastly, extending our model to include nonzero assembly times is an interesting problem to pursue. However, with today’s manufacturing technology, assembly times are usually small and our model would likely provide a good approximation.

References


Online Technical Appendix

EC.1. Proofs of the Results in Section 3.1

Proof of Lemma 1. \( f \in \text{Super}(r, p) \) implies the following inequalities:

\[
\begin{align*}
f(x + p_1 e_1 + r) - f(x + p_1 e_1) &\geq f(x + r) - f(x), \\
f(x + p_1 e_1 + p_2 e_2 + r) - f(x + p_1 e_1 + p_2 e_2) &\geq f(x + p_1 e_1 + r) - f(x + p_1 e_1), \\
\vdots \\
f(x + \sum_{j \leq n} p_j e_j + r) - f(x + \sum_{j \leq n} p_j e_j) &\geq f(x + \sum_{j < n} p_j e_j + r) - f(x + \sum_{j < n} p_j e_j)
\end{align*}
\]

Summation of the inequalities above implies \( f(x + p + r) - f(x + p) \geq f(x + r) - f(x) \), and therefore \( f \in n\text{Super}(r, p) \).

\[\Box\]

EC.2. Proofs of the Results in Section 3.2

Proof of Lemma 2. Recall that \( T^{(j)}v(x) = \min\{v(x + q_j e_j), v(x)\}, T_i v(x) = \min\{v(x) + c_i, v(x - b_i e_i)\} \) if \( x_i \geq b_i \), and \( T_i v(x) = v(x) + c_i \) otherwise, for \( i \leq n \); and \( T_{n+1} v(x) = \min\{v(x) + c_{n+1}, v(x - a)\} \) if \( x_j \geq a_j \) for all \( j \), and \( T_{n+1} v(x) = v(x) + c_{n+1} \) otherwise.

(a) Assume that \( v \in \text{Sub}(q) \cap \text{Super}(a, q) \). We will show \( T^{(j)}v \in \text{Sub}(q) \cap \text{Super}(a, q) \).

- First we show \( T^{(j)}v \in \text{Sub}(q) \), i.e., \( T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \geq T^{(j)}v(x + q_i e_i + q_k e_k) - T^{(j)}v(x + q_k e_k), \forall k \neq i \). Pick arbitrary \( k \in \{1, 2, \ldots, n\} \). There are four different scenarios we need to consider depending on the optimal actions at \( T^{(j)}v(x + q_i e_i) \) and \( T^{(j)}v(x + q_k e_k) \) (if this inequality holds under suboptimal actions of \( T^{(j)}v(x) \) and/or \( T^{(j)}v(x + q_i e_i + q_k e_k) \), it also holds under optimal actions of these operators, and thus we do not enforce the optimal actions at these operators). These four scenarios are as follows:

1. Suppose that \( T^{(j)}v(x + q_i e_i) = v(x + q_i e_i) < v(x + q_j e_j + q_k e_k) \) and \( T^{(j)}v(x + q_k e_k) = v(x + q_k e_k) < v(x + q_j e_j + q_k e_k) \). As we assume \( v \in \text{Sub}(q) \), the following inequalities hold:

\[
T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \geq v(x + q_i e_i) - v(x)
\]

\[
\geq v(x + q_i e_i + q_k e_k) - v(x + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_i e_i + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]

2. Suppose that \( T^{(j)}v(x + q_i e_i) = v(x + q_j e_j + q_k e_k) \) and \( T^{(j)}v(x + q_k e_k) = v(x + q_k e_k) < v(x + q_j e_j + q_k e_k) \). As we assume \( v \in \text{Sub}(q) \), the following inequalities hold:

\[
T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \geq v(x + q_j e_j + q_i e_i) - v(x)
\]

\[
\geq v(x + q_j e_j + q_k e_k) - v(x + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_j e_j + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]
Next we show that \( T^{(j)}v(x + q_i e_i) = v(x + q_i e_i) < v(x + q_j e_j + q_i e_i) \) and \( T^{(j)}v(x + q_k e_k) = v(x + q_j e_j + q_k e_k) < v(x + q_k e_k) \) if \( j \neq i \). If \( j = i \), then it is easy to verify that

\[
T^{(j)}v(x + q_j e_j) - T^{(j)}v(x) \geq v(x + q_j e_j) - v(x + q_j e_j)
\]

\[
= v(x + q_j e_j + q_k e_k) - v(x + q_j e_j + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_j e_j + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]

(3) Suppose that \( T^{(j)}v(x + q_i e_i) = v(x + q_i e_i) < v(x + q_j e_j + q_i e_i) \) and \( T^{(j)}v(x + q_k e_k) = v(x + q_j e_j + q_k e_k) < v(x + q_k e_k) \). If \( j = i \), then it is easy to verify that

\[
T^{(j)}v(x + q_j e_j) - T^{(j)}v(x) \geq v(x + q_j e_j) - v(x + q_j e_j)
\]

\[
= v(x + q_j e_j + q_k e_k) - v(x + q_j e_j + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_j e_j + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]

If \( j \neq i \), as we assume \( v \in \text{Sub}(q) \), the following inequalities hold:

\[
T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \geq v(x + q_i e_i) - v(x)
\]

\[
\geq v(x + q_j e_j + q_i e_i) - v(x + q_j e_j)
\]

\[
\geq v(x + q_j e_j + q_i e_i + q_k e_k) - v(x + q_j e_j + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_j e_j + q_i e_i + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]

(4) Suppose that \( T^{(j)}v(x + q_i e_i) = v(x + q_i e_i) < v(x + q_j e_j + q_i e_i) \) and \( T^{(j)}v(x + q_k e_k) = v(x + q_j e_j + q_k e_k) < v(x + q_k e_k) \). As we assume \( v \in \text{Sub}(q) \), the following inequalities hold:

\[
T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \geq v(x + q_i e_i) - v(x)
\]

\[
\geq v(x + q_j e_j + q_i e_i) - v(x + q_j e_j)
\]

\[
\geq v(x + q_j e_j + q_i e_i + q_k e_k) - v(x + q_j e_j + q_k e_k)
\]

\[
\geq T^{(j)}v(x + q_j e_j + q_i e_i + q_k e_k) - T^{(j)}v(x + q_k e_k)
\]

Hence our inequality holds in each of the possible scenarios. Therefore, \( T^{(j)}v \in \text{Sub}(q) \).

- Next we show \( T^{(j)}v \in \text{Super}(a, q) \), i.e., \( T^{(j)}v(x + q_i e_i + a) - T^{(j)}v(x + a) \geq T^{(j)}v(x + q_i e_i) - T^{(j)}v(x) \), \( \forall i \). Again, there are four different scenarios depending on the optimal actions at \( T^{(j)}v(x + q_i e_i + a) \) and \( T^{(j)}v(x) \):

1. Suppose that \( T^{(j)}v(x + q_i e_i + a) = v(x + q_i e_i + a) < v(x + q_j e_j + q_i e_i + a) \) and \( T^{(j)}v(x) = v(x) < v(x + q_j e_j) \). As we assume \( v \in \text{Super}(a, q) \), the following inequalities hold:

\[
T^{(j)}v(x + q_i e_i + a) - T^{(j)}v(x + a) \geq v(x + q_i e_i + a) - v(x + a)
\]

\[
\geq v(x + q_i e_i) - v(x)
\]

\[
\geq T^{(j)}v(x + q_i e_i) - T^{(j)}v(x)
\]
(2) Suppose that $T^{(j)}v(x + q_ie_i + a) = v(x + q_je_j + q_ie_i + a) < v(x + q_ie_i + a)$ and $T^{(j)}v(x) = v(x) < v(x + q_je_j)$. As we assume $v \in Super(a, q)$, the following inequalities hold:

\[
T^{(j)}v(x + q_ie_i + a) - T^{(j)}v(x + a) \geq v(x + q_je_j + q_ie_i + a) - v(x + a) \\
\geq v(x + q_je_j + q_ie_i) + v(x + q_je_j + a) \\
- v(x + q_je_j) - v(x + a) \\
\geq v(x + q_je_j + q_ie_i) - v(x) \\
\geq T^{(j)}v(x + q_ie_i) - T^{(j)}v(x)
\]

(3) Suppose that $T^{(j)}v(x + q_ie_i + a) = v(x + q_i e_i + a) < v(x + q_j e_j + q_ie_i + a)$ and $T^{(j)}v(x) = v(x + q_j e_j) < v(x)$. If $j = i$, then it is easy to verify that

\[
T^{(j)}v(x + q_j e_j + a) - T^{(j)}v(x + a) \geq v(x + q_j e_j + a) - v(x + q_j e_j) \\
= v(x + q_j e_j) - v(x + q_j e_j) \\
\geq T^{(j)}v(x + q_j e_j) - T^{(j)}v(x)
\]

If $j \neq i$, as we assume $v \in Super(a, q)$ and $v \in Sub(q)$, the following inequalities hold:

\[
T^{(j)}v(x + q_ie_i + a) - T^{(j)}v(x + a) \geq v(x + q_i e_i + a) - v(x + a) \\
\geq v(x + q_i e_i) - v(x) \\
\geq v(x + q_j e_j + q_i e_i) - v(x + q_j e_j) \\
\geq T^{(j)}v(x + q_ie_i) - T^{(j)}v(x)
\]

(4) Suppose that $T^{(j)}v(x + q_ie_i + a) = v(x + q_j e_j + q_i e_i + a) < v(x + q_i e_i + a)$ and $T^{(j)}v(x) = v(x + q_j e_j) < v(x)$. As we assume $v \in Super(a, q)$, the following inequalities hold:

\[
T^{(j)}v(x + q_ie_i + a) - T^{(j)}v(x + a) \geq v(x + q_j e_j + q_i e_i + a) - v(x + q_j e_j + a) \\
\geq v(x + q_j e_j + q_i e_i) - v(x + q_j e_j) \\
\geq T^{(j)}v(x + q_ie_i) - T^{(j)}v(x)
\]

Hence our inequality holds in all the possible scenarios. Therefore, $T^{(j)}v \in Super(a, q)$.

(b) Assume that $v \in Sub(b) \cap Super(a, b)$. We will show $T_i v \in Sub(b) \cap Super(a, b)$.

**Case I:** Suppose that $i \leq n.$
• First we show $T_i v \in \text{Sub}(b)$, i.e., $T_i v(x + b_j e_j) - T_i v(x) \geq T_i v(x + b_j e_j + b_k e_k) - T_i v(x + b_k e_k)$, $\forall k \neq j$. Pick arbitrary $k \in \{1, 2, \ldots, n\}$. There are four different scenarios we need to consider depending on the optimal actions at $T_i v(x + b_j e_j)$ and $T_i v(x + b_k e_k)$ (if this inequality holds under suboptimal actions of $T_i v(x)$ and/or $T_i v(x + b_j e_j + b_k e_k)$, it also holds under optimal actions of these operators, and thus we do not enforce the optimal actions at these operators).

These four scenarios are as follows:

1. Suppose that $T_i v(x + b_j e_j) = v(x + b_j e_j) + c_i$ and $T_i v(x + b_k e_k) = v(x + b_k e_k) + c_i$. As we assume $v \in \text{Sub}(b)$, the following inequalities hold:

   $T_i v(x + b_j e_j) - T_i v(x) \geq v(x + b_j e_j) + c_i - v(x) - c_i$
   $\geq v(x + b_j e_j + b_k e_k) + c_i - v(x + b_k e_k) - c_i$
   $\geq T_i v(x + b_j e_j + b_k e_k) - T_i v(x + b_k e_k)$

2. Suppose that $x_i \geq b_i$, $T_i v(x + b_j e_j) = v(x + b_j e_j) + c_i$ and $T_i v(x + b_k e_k) = v(x + b_k e_k - b_i e_i)$.

   As we assume $v \in \text{Sub}(b)$, the following inequalities hold:

   $T_i v(x + b_j e_j) - T_i v(x) \geq v(x + b_j e_j) + c_i - v(x - b_i e_i)$
   $\geq v(x) - v(x + b_k e_k)$
   $\geq v(x + b_j e_j + b_k e_k) + c_i - v(x - b_i e_i)$
   $\geq v(x + b_j e_j + b_k e_k) + c_i - v(x + b_k e_k - b_i e_i)$
   $\geq T_i v(x + b_j e_j + b_k e_k) - T_i v(x + b_k e_k)$

3. Suppose that $x_i \geq b_i$ if $i \neq j$, $T_i v(x + b_j e_j) = v(x + b_j e_j - b_i e_i)$ and $T_i v(x + b_k e_k) = v(x + b_k e_k) + c_i$. If $i = j$, then it is easy to verify that

   $T_i v(x + b_i e_i) - T_i v(x) \geq v(x) - v(x) - c_i$
   $= v(x + b_k e_k) - v(x + b_k e_k) - c_i$
   $\geq T_i v(x + b_i e_i + b_k e_k) - T_i v(x + b_k e_k)$

If $i \neq j$, as we assume $v \in \text{Sub}(b)$, the following inequalities hold:

   $T_i v(x + b_j e_j) - T_i v(x) \geq v(x + b_j e_j - b_i e_i) - v(x - b_i e_i)$
   $\geq v(x + b_j e_j) - v(x)$
   $\geq v(x + b_j e_j + b_k e_k) + c_i - v(x + b_k e_k) - c_i$
   $\geq T_i v(x + b_j e_j + b_k e_k) - T_i v(x + b_k e_k)$
(4) Suppose that \( x_i \geq b_i, T_i v(x + b_i e_i) = v(x + b_i e_j - b_i e_i) \) and \( T_i v(x + b_i e_k) = v(x + b_k e_k - b_i e_i) \).

As we assume \( v \in \text{Sub}(b) \), the following inequalities hold:

\[
T_i v(x + b_i e_j) - T_i v(x) \geq v(x + b_i e_j - b_i e_i) - v(x - b_i e_i)
\]
\[
\geq v(x + b_i e_j + b_k e_k - b_i e_i) - v(x + b_k e_k - b_i e_i)
\]
\[
\geq T_i v(x + b_i e_j + b_k e_k) - T_i v(x + b_k e_k)
\]

Hence our inequality holds in all the possible scenarios. Therefore, \( T_i v \in \text{Sub}(b) \).

• Next we show \( T_i v \in \text{Super}(a, b) \), i.e., \( T_i v(x + b_i e_j + a) - T_i v(x + a) \geq T_i v(x + b_i e_j) - T_i v(x), \forall j \).

Again, there are four different scenarios depending on the optimal actions at \( T_i v(x + b_i e_j + a) \) and \( T_i v(x) \):

(1) Suppose that \( x_i \geq b_i, T_i v(x + b_i e_j + a) = v(x + b_i e_j + a - b_i e_i) \) and \( T_i v(x) = v(x - b_i e_i) \).

As we assume \( v \in \text{Super}(a, b) \), the following inequalities hold:

\[
T_i v(x + b_i e_j + a) - T_i v(x + a) \geq v(x + b_i e_j + a - b_i e_i) - v(x + a - b_i e_i)
\]
\[
\geq v(x + b_i e_j - b_i e_i) - v(x - b_i e_i)
\]
\[
\geq T_i v(x + b_i e_j) - T_i v(x)
\]

(2) Suppose that \( x_i \geq b_i, T_i v(x + b_i e_j + a) = v(x + b_i e_j + a) + c_i \) and \( T_i v(x) = v(x - b_i e_i) \). As we assume \( v \in \text{Super}(a, b) \), the following inequalities hold:

\[
T_i v(x + b_i e_j + a) - T_i v(x + a) \geq v(x + b_i e_j + a) + c_i - v(x + a - b_i e_i)
\]
\[
\geq v(x + b_i e_j) + v(x + a)
\]
\[
-v(x) + c_i - v(x + a - b_i e_i)
\]
\[
\geq v(x + b_i e_j) + c_i - v(x - b_i e_i)
\]
\[
\geq T_i v(x + b_i e_j) - T_i v(x)
\]

(3) Suppose that \( x_i + a_i \geq b_i \) if \( i \neq j \), \( T_i v(x + b_i e_j + a) = v(x + b_j e_j + a - b_i e_i) \) and \( T_i v(x) = v(x) + c_i \). If \( i = j \), it is easy to verify that

\[
T_i v(x + b_i e_i + a) - T_i v(x + a) \geq v(x + a) - v(x + a) - c_i
\]
\[
= v(x) - v(x) - c_i
\]
\[
\geq T_i v(x + b_i e_i) - T_i v(x)
\]
If \( i \neq j \), as we assume \( v \in \text{Sub}(b) \) and \( v \in \text{Super}(a,b) \), the following inequalities hold:

\[
T_i v(x + bj_j + a) - T_i v(x + a) \geq v(x + bj_j + a - bj_i) - v(x + a - bj_i)
\]

\[
\geq v(x + bj_j + a) - v(x + a)
\]

\[
\geq v(x + bj_j) + bj_i - v(x) - c_i
\]

\[
\geq T_i v(x + bj_j) - T_i v(x)
\]

(4) Suppose that \( T_i v(x + bj_j + a) = v(x + bj_j + a) + bj_i \) and \( T_i v(x) = v(x) + c_i \). As we assume \( v \in \text{Super}(a,b) \), the following inequalities hold:

\[
T_i v(x + bj_j + a) - T_i v(x + a) \geq v(x + bj_j + a) + bj_i - v(x + a) - c_i
\]

\[
\geq v(x + bj_j) + bj_i - v(x) - c_i
\]

\[
\geq T_i v(x + bj_j) - T_i v(x)
\]

Hence our inequality holds in all the possible scenarios. Therefore, \( T_i v \in \text{Super}(a,b) \).

**Case II:** Suppose that \( i = n + 1 \).

- First we show \( T_{n+1} v \in \text{Sub}(b) \), i.e., \( T_{n+1} v(x + bj_j) - T_{n+1} v(x) \geq T_{n+1} v(x + bj_j + bk ek), \forall k \neq j \). Pick arbitrary \( k,j \in \{1, 2, ..., n\} \). There are four possible scenarios depending on the optimal actions at \( T_{n+1} v(x + bj_j) \) and \( T_{n+1} v(x + bk ek) \):

  (1) Suppose that \( T_{n+1} v(x + bj_j) = v(x + bj_j) + c_{n+1} \) and \( T_{n+1} v(x + bk ek) = v(x + bk ek) + c_{n+1} \). As we assume \( v \in \text{Sub}(b) \), the following inequalities hold:

\[
T_{n+1} v(x + bj_j) - T_{n+1} v(x) \geq v(x + bj_j) + c_{n+1} - v(x) - c_{n+1}
\]

\[
\geq v(x + bj_j + bk ek) + c_{n+1} - v(x + bk ek) - c_{n+1}
\]

\[
\geq T_{n+1} v(x + bj_j + bk ek) - T_{n+1} v(x + bk ek)
\]

(2) Suppose that \( x + bk ek \geq a \), \( T_{n+1} v(x + bj_j) = v(x + bj_j) + c_{n+1} \) and \( T_{n+1} v(x + bk ek) = v(x + bk ek - a) \). As we assume \( v \in \text{Sub}(b) \) and \( v \in \text{Super}(a,b) \), the following inequalities hold:

\[
T_{n+1} v(x + bj_j) - T_{n+1} v(x) \geq v(x + bj_j) + c_{n+1} - v(x) - c_{n+1}
\]

\[
\geq v(x + bj_j + bk ek) - v(x + bk ek)
\]

\[
\geq v(x + bj_j + bk ek - a) - v(x + bk ek - a)
\]

\[
\geq T_{n+1} v(x + bj_j + bk ek) - T_{n+1} v(x + bk ek)
\]
(3) Suppose that $x + b_j e_j \geq a$, $T_{n+1}v(x + b_j e_j) = v(x + b_j e_j - a)$ and $T_{n+1}v(x + b_k e_k) = v(x + b_k e_k) + c_{n+1}$. As we assume $v \in Super(a,b)$ and $v \in Sub(b)$, the following inequalities hold:

$$T_{n+1}v(x + b_j e_j) - T_{n+1}v(x) \geq v(x + b_j e_j - a) - v(x) - c_{n+1}$$

$$\geq v(x + b_j e_j) - v(x + b_j e_j + b_k e_k)$$

$$+ v(x + b_j e_j + b_k e_k - a) - v(x) - c_{n+1}$$

$$\geq v(x + b_j e_j + b_k e_k - a) - v(x + b_k e_k) - c_{n+1}$$

$$\geq T_{n+1}v(x + b_j e_j + b_k e_k) - T_{n+1}v(x + b_k e_k)$$

(4) Suppose that $x + b_j e_j \geq a$, $x + b_k e_k \geq a$, $T_{n+1}v(x + b_j e_j) = v(x + b_j e_j - a)$ and $T_{n+1}v(x + b_k e_k) = v(x + b_k e_k - a)$. Notice that, for $j \neq k$, $x + b_j e_j \geq a$ and $x + b_k e_k \geq a$, imply, respectively, $x_t \geq a_t$ for all $t \neq j$ and $x_t \geq a_t$ for all $t \neq k$, and therefore $x \geq a$. As we assume $v \in Sub(b)$, the following inequalities hold:

$$T_{n+1}v(x + b_j e_j) - T_{n+1}v(x) \geq v(x + b_j e_j - a) - v(x - a)$$

$$\geq v(x + b_j e_j + b_k e_k - a) - v(x + b_k e_k - a)$$

$$\geq T_{n+1}v(x + b_j e_j + b_k e_k) - T_{n+1}v(x + b_k e_k)$$

Hence our inequality holds in all the possible scenarios. Therefore, $T_{n+1}v \in Sub(b)$.

- Next we show $T_{n+1}v \in Super(a,b)$, i.e., $T_{n+1}v(x + b_j e_j + a) - T_{n+1}v(x + a) \geq T_{n+1}v(x + b_j e_j) - T_{n+1}v(x)$, $\forall j$. Again, there are four different scenarios depending on the optimal actions at $T_{n+1}v(x + b_j e_j + a)$ and $T_{n+1}v(x)$:

(1) Suppose that $x \geq a$, $T_{n+1}v(x + b_j e_j + a) = v(x + b_j e_j)$ and $T_{n+1}v(x) = v(x - a)$. As we assume $v \in Super(a,b)$, the following inequalities hold:

$$T_{n+1}v(x + b_j e_j + a) - T_{n+1}v(x + a) \geq v(x + b_j e_j) - v(x)$$

$$\geq v(x + b_j e_j - a) - v(x - a)$$

$$\geq T_{n+1}v(x + b_j e_j) - T_{n+1}v(x)$$

(2) Suppose that $x \geq a$, $T_{n+1}v(x + b_j e_j + a) = v(x + b_j e_j + a) + c_{n+1}$ and $T_{n+1}v(x) = v(x - a)$. As we assume $v \in Super(a,b)$, the following inequalities hold:

$$T_{n+1}v(x + b_j e_j + a) - T_{n+1}v(x + a) \geq v(x + b_j e_j + a) + c_{n+1} - v(x + a) - c_{n+1}$$
\[ \geq v(x + b_j e_j) - v(x) \]
\[ \geq v(x + b_j e_j - a) - v(x - a) \]
\[ \geq T_{n+1} v(x + b_j e_j) - T_{n+1} v(x) \]

(3) Suppose that \( T_{n+1} v(x + b_j e_j + a) = v(x + b_j e_j) \) and \( T_{n+1} v(x) = v(x) + c_{n+1} \). Then it is easy to verify that

\[
T_{n+1} v(x + b_j e_j + a) - T_{n+1} v(x + a) \geq v(x + b_j e_j) - v(x) \\
= v(x + b_j e_j) + c_{n+1} - v(x) - c_{n+1} \\
\geq T_{n+1} v(x + b_j e_j) - T_{n+1} v(x)
\]

(4) Suppose that \( T_{n+1} v(x + b_j e_j + a) = v(x + b_j e_j + a) + c_{n+1} \) and \( T_{n+1} v(x) = v(x) + c_{n+1} \). As we assume \( v \in Super(a, b) \), the following inequalities hold:

\[
T_{n+1} v(x + b_j e_j + a) - T_{n+1} v(x + a) \geq v(x + b_j e_j + a) + c_{n+1} - v(x + a) - c_{n+1} \\
\geq v(x + b_j e_j) + c_{n+1} - v(x) - c_{n+1} \\
\geq T_{n+1} v(x + b_j e_j) - T_{n+1} v(x)
\]

Hence our inequality holds in all the possible scenarios. Therefore, \( T_{n+1} v \in Super(a, b) \).

(c) We below show \( h \in Sub(p) \cap Super(r, p) \), for any \( r \) and \( p \).

- First we prove \( h \in Sub(p) \) (i.e., \( h(x + p_j e_j) - h(x) \geq h(x + p_j e_j + p_k e_k) - h(x + p_k e_k), \forall k \neq j \)):
  \[
h(x + p_j e_j) - h(x) = \sum_{i \neq j} h_i(x_i) + h_j(x_j + p_j) - \sum_{i \neq j} h_i(x_i) - h_j(x_j) = h_j(x_j + p_j) - h_j(x_j) = \sum_{i \neq j, k} h_i(x_i) + h_j(x_j + p_j) + h_k(x_k + p_k) - \sum_{i \neq j, k} h_i(x_i) - h_j(x_j) - h_k(x_k + p_k) = h(x + p_j e_j + p_k e_k) - h(x + p_k e_k), \forall k \neq j.
  \]

- Second we prove \( h \in Super(r, p) \) (i.e., \( h(x + p_j e_j + r) - h(x + r) \geq h(x + p_j e_j) - h(x), \forall j \)):
  \[
h(x + p_j e_j + r) - h(x + r) = \sum_{i \neq j} h_i(x_i + r_i) + h_j(x_j + p_j + r_j) - \sum_{i \neq j} h_i(x_i + r_i) - h_j(x_j + r_j) = h_j(x_j + p_j + r_j) - h_j(x_j + r_j) \geq h_j(x_j + p_j) - h_j(x_j) = \sum_{i \neq j} h_i(x_i) + h_j(x_j + p_j) - \sum_{i \neq j} h_i(x_i) - h_j(x_j) = h(x + p_j e_j) - h(x), \forall j.\]

The inequality above follows from the assumption that \( h_j \) is a convex function, \( \forall j \).

Since \( h \in Sub(p) \cap Super(r, p) \), for any \( r \) and \( p \), we have \( h \in Sub(q) \cap Super(a, q) \cap Sub(b) \cap Super(a, b) \).
Proof of Lemma 3. Define $V^*$ as the set of functions satisfying the properties of $\text{Sub}(b)$, $\text{Super}(a, b)$, and $n\text{Super}(a, b)$. Also, define the operator $T$ on the set of real-valued functions $v$: $Tv(x) = h(x) + \sum_j \mu_j T^{(j)} v(x) + \sum_i \lambda_i T_i v(x)$. First we show $T : V^* \rightarrow V^*$. By Lemma 1, $\text{Super}(r, p) \subseteq n\text{Super}(r, p)$, and therefore $\text{Sub}(p) \cap \text{Super}(r, p) \subseteq n\text{Super}(r, p)$. This, combined with Lemma 2, yields $T^{(j)} : \text{Sub}(q) \cap \text{Super}(a, q) \cap n\text{Super}(a, q) \rightarrow \text{Sub}(q) \cap \text{Super}(a, q) \cap n\text{Super}(a, q)$, and $T_i : \text{Sub}(b) \cap \text{Super}(a, b) \cap n\text{Super}(a, b) \rightarrow \text{Sub}(b) \cap \text{Super}(a, b) \cap n\text{Super}(a, b)$. By Assumption 1, $q = b$, and therefore $T^{(j)}, T_i : \text{Sub}(b) \cap \text{Super}(a, b) \cap n\text{Super}(a, b) \rightarrow \text{Sub}(b) \cap \text{Super}(a, b) \cap n\text{Super}(a, b)$. That is, $T^{(j)} : V^* \rightarrow V^*$ and $T_i : V^* \rightarrow V^*$. By Lemmas 1 and 2, we also know $h \in V^*$. Now let $v \in V^*$. Since $T^{(j)} v \in V^*$, $T_i v \in V^*$, and $h \in V^*$, and our second-order properties are preserved by linear transformations, $Tv \in V^*$. Hence, $T : V^* \rightarrow V^*$. Following Propositions 3.1.5 and 3.1.6 in Bertsekas (2007), we verify that $\lim_{k \rightarrow \infty} (T^k v_0)(x) = v^*(x)$ where $v_0$ is the zero function, $v^*$ is the optimal cost function, and $T^k$ refers to $k$ compositions of operator $T$. Since $v_0 \in V^*$ and $T : V^* \rightarrow V^*$, we have $T^k v_0 \in V^*$, and therefore $v^* \in V^*$.

□

EC.3. Proofs of the Results in Section 3.3

Proof of Theorem 1. By Lemma 3, we know $v^* \in V^*$. Define, for $v^* \in V^*$,

\[ S_j^*(p) = \min\{p + za : v^*(p + za + q_je_j) - v^*(p + za) > 0, \ z \in \mathbb{N}_0\}, \ \forall j, \]

\[ R_i^*(p) = \min\{p + za : v^*(p + za) - v^*(p + za - bi_e_i) > -c_i, \ z \in \mathbb{N}_0, \ and \ p_i + za_i \geq b_i\}, \ \forall i \leq n, \]

\[ R_{n+1}^*(p) = \min\{p + zb : v^*(p + zb) - v^*(p + zb - a) > -c_{n+1}, \ z \in \mathbb{N}_0, \ and \ p + zb \geq a\}. \]

(1) Since $v^* \in \text{Super}(a, b)$ and $q = b$, $v^*(p + za + q_je_j) - v^*(p + za)$ is increasing in $z$. As $z$ increases, since the holding cost rate $h$ is strictly increasing, this difference will eventually cross 0. Therefore, the lattice-dependent base-stock policy is optimal.

(2) Since $v^* \in \text{Super}(a, b)$, $v^*(p + za) - v^*(p + za - bi_e_i)$ is increasing in $z$. We know that, as $z$ increases, this difference will eventually cross 0. Therefore, as $z$ increases, this difference should also cross $-c_i$. Hence, the lattice-dependent rationing policy is optimal.

(3) Since $v^* \in n\text{Super}(a, b)$, $v^*(p + zb) - v^*(p + zb - a)$ is increasing in $z$. As $z$ increases, since the holding cost rate $h$ is strictly increasing, this difference will eventually cross $-c_{n+1}$. Therefore, the lattice-dependent rationing policy is optimal.

Next we will prove properties (i)-(iii):
i. To prove property (i), first, we show that the optimal base-stock levels for each component \( j \) obey \( S_j^*(p + b_k e_k) \geq S_j^*(p) + b_k e_k \), \( \forall k \neq j \). Let \( S_j^*(p) = p + z_j a \) and \( S_j^*(p + b_k e_k) = p + b_k e_k + z_2 a \). Then, it is not optimal to produce a batch of component \( j \) at \( x = p + z_j a \) and \( x = p + b_k e_k + z_2 a \).

Since \( v^* \in Sub(b) \), it is not optimal to produce a batch of component \( j \) at \( x = p + z_j a \) and \( x = p + b_k e_k + z_2 a \). Therefore, we must have \( S_j^*(p + b_k e_k) \geq S_j^*(p) + b_k e_k \).

Second, we show that the optimal rationing levels for each product \( i \leq n \) obey \( R_i^*(p + b_k e_k) \geq R_i^*(p) + b_k e_k \), \( \forall k \neq i \). Let \( R_i^*(p) = p + z_i b \) and \( R_i^*(p + b_k e_k) = p + b_k e_k + z_2 b \). Then, it is optimal to fulfill a demand for product \( i \) at \( x = p + z_i b \) and \( x = p + b_k e_k + z_2 b \). Since \( v^* \in Sub(b) \), it is also optimal to fulfill a demand for product \( i \) at \( x = p + z_i b + b_k e_k \), implying \( z_2 \geq z_1 \). Therefore, we must have \( R_i^*(p + b_k e_k) \geq R_i^*(p) + b_k e_k \).

ii. To prove (ii), we will show that the optimal rationing levels for product \( n + 1 \) obey \( R_{n+1}^*(p + b_k e_k) \leq R_{n+1}^*(p) + b_k e_k \), \( \forall k \). Let \( R_{n+1}^*(p) = p + z_i b \) and \( R_{n+1}^*(p + b_k e_k) = p + b_k e_k + z_2 b \). Then, it is optimal to fulfill a demand for product \( n + 1 \) at \( x = p + z_i b \) and \( x = p + b_k e_k + z_2 b \). Since \( v^* \in Super(a, b) \), it is also optimal to fulfill a demand for product \( n + 1 \) at \( x = p + z_i b + b_k e_k \), implying \( z_1 \geq z_2 \). Therefore, we must have \( R_{n+1}^*(p + b_k e_k) \geq R_{n+1}^*(p) + b_k e_k \).

iii. Lastly, we will prove that it is optimal to fulfill a demand of product \( n + 1 \) if \( x_j \geq a_j + b_j \left\lfloor \frac{z_j}{b_j} \right\rfloor \), \( \forall j \). Define \( \tilde{V} \) as the set of real-valued functions \( f \) defined on \( \mathbb{N}_0^* \) such that \( f(x) - f(x - a) + c_{n+1} \geq 0 \), for \( x_j \geq a_j + b_j \left\lfloor \frac{z_j}{b_j} \right\rfloor \), \( \forall j \). Recall \( T v(x) = h(x) + \sum_j \mu_j T^{(j)} v(x) + \sum_i \lambda_i T_i v(x) \). We show below \( T : \tilde{V} \rightarrow \tilde{V} \).

Assume that \( v \in \tilde{V} \). We want to prove \( T v \in \tilde{V} \). Since \( h \) is an increasing convex function and \( \sum_j \mu_j + \sum_i \lambda_i \leq 1 \), the following inequality holds:

\[
\begin{align*}
Tv(x) - Tv(x - a) &+ c_{n+1} \\
= h(x) - h(x - a) + \sum_j \mu_j (T^{(j)} v(x) - T^{(j)} v(x - a)) + \sum_i \lambda_i (T_i v(x) - T_i v(x - a)) + c_{n+1} \\
\geq \sum_j \mu_j (T^{(j)} v(x) - T^{(j)} v(x - a) + c_{n+1}) + \sum_i \lambda_i (T_i v(x) - T_i v(x - a) + c_{n+1}) \\
\end{align*}
\]

To prove \( T v \in \tilde{V} \), it suffices to show \( T^{(j)} v(x) - T^{(j)} v(x - a) + c_{n+1} \geq 0 \), \( \forall j \), and \( T_i v(x) - T_i v(x - a) + c_{n+1} \geq 0 \), \( \forall i \), where \( x_k \geq a_k + b_k \left\lfloor \frac{z_k}{b_k} \right\rfloor \), \( \forall k \). We prove these inequalities as follows:

- First we show \( T^{(j)} v(x) - T^{(j)} v(x - a) + c_{n+1} \geq 0 \). There are two possible scenarios depending on the optimal action at \( T^{(j)} v(x) \):

  1. Suppose that \( T^{(j)} v(x) = v(x + q_j e_j) < v(x) \): \( T^{(j)} v(x) - T^{(j)} v(x - a) + c_{n+1} \geq v(x + q_j e_j) - v(x + q_j e_j - a) + c_{n+1} \geq 0 \). The second inequality follows from the fact that \( v \in \tilde{V} \) and \( x_j + q_j \geq a_j + b_j \left\lfloor \frac{z_j}{b_j} \right\rfloor + q_j = a_j + b_j \left\lfloor \frac{z_j + q_j}{b_j} \right\rfloor \). (By Assumption 1, \( q_j = b_j \)).
(2) Suppose that \( T^{(j)}v(x) = v(x) \leq v(x + q_j e_j) \): \( T^{(j)}v(x) - T^{(j)}v(x - a) + c_{n+1} \geq v(x) - v(x - a) + c_{n+1} \geq 0 \). The second inequality follows from the assumption of \( v \in \tilde{V} \).

- Second we show \( T_iv(x) - T_iv(x - a) + c_{n+1} \geq 0 \), for \( i \leq n \). There are two possible scenarios depending on the optimal action at \( T_iv(x) \):

1. Suppose that \( T_iv(x) = v(x) + c_i : T_iv(x) - T_iv(x - a) + c_{n+1} \geq v(x) + c_i - v(x - a) - c_i + c_{n+1} \geq 0 \). The second inequality follows from the assumption of \( v \in \tilde{V} \).

2. Suppose that \( x_i \geq b_i \) and \( T_iv(x) = v(x - b_i e_i) \): \( T_iv(x) - T_iv(x - a) + c_{n+1} \geq v(x - b_i e_i) - v(x - a - b_i e_i) + c_{n+1} \geq 0 \). The second inequality follows from the fact that \( v \in \tilde{V} \) and \( x_i - b_i \geq a_i + b_i \left\lfloor \frac{x_i}{b_i} \right\rfloor - b_i = a_i + b_i \left\lfloor \frac{x_i - b_i}{b_i} \right\rfloor \). Here notice that, as we assume \( x_i \geq a_i + b_i \), we should have \( x_i \geq a_i + b_i \), implying \( x \geq a + b_i e_i \).

- Lastly we show \( T_{n+1}v(x) - T_{n+1}v(x - a) + c_{n+1} \geq 0 \). There are two possible scenarios depending on the optimal action at \( T_{n+1}v(x) \):

1. Suppose that \( T_{n+1}v(x) = v(x) + c_n < v(x - a) : T_{n+1}v(x) - T_{n+1}v(x - a) + c_{n+1} \geq v(x) + c_n - v(x - a) - c_n + c_{n+1} \geq 0 \). The second inequality follows from the assumption of \( v \in \tilde{V} \).

2. Suppose that \( T_{n+1}v(x) = v(x - a) \leq v(x) + c_n : T_{n+1}v(x) - T_{n+1}v(x - a) + c_{n+1} \geq v(x - a) - v(x - a) - c_n + c_{n+1} = 0 \).

Since \( \sum_j \mu_j(T^{(j)}v(x) - T^{(j)}v(x - a) + c_{n+1}) + \sum_i \lambda_i(T_iv(x) - T_iv(x - a) + c_{n+1}) \geq 0 \), we have \( Tv(x) - Tv(x - a) + c_j \geq 0 \). Hence, \( T : \tilde{V} \rightarrow \tilde{V} \). Following Propositions 3.1.5 and 3.1.6 in Bertsekas (2007), we verify that \( \lim_{k \rightarrow \infty}(T^kv_0)(x) = v^*(x) \) where \( v_0 \) is the zero function, \( v^* \) is the optimal cost function, and \( T^k \) refers to \( k \) compositions of operator \( T \). Since \( v_0 \in \tilde{V} \) and \( T : \tilde{V} \rightarrow \tilde{V} \), we have \( T^kv_0 \in \tilde{V} \), and therefore \( v^* \in \tilde{V} \). Since \( v^*(x) - v^*(x - a) + c_{n+1} \geq 0 \), for \( x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor , \forall j \), it is optimal to fulfill a demand of product \( n + 1 \) if \( x_j \geq a_j + b_j \left\lfloor \frac{x_j}{b_j} \right\rfloor , \forall j \).

\( \square \)

EC.4. Proofs of the Results in Section 3.4

**Proof of Proposition 1.** We first prove the following conditions: (i) There exists a stationary policy \( \pi \) that induces an irreducible positive recurrent Markov chain with finite average cost \( v^* \), and (ii) the number of states for which \( h(x) \leq v^* \) is finite. To prove condition (i), consider a policy where the production of each component is controlled by a base-stock policy with an independent and fixed critical level, and inventory allocation follows a first-come-first-served policy. Notice that we have a finite-state Markov chain under this policy. Hence, this policy yields a finite average.
cost. It is easy to prove condition (ii) as the inventory holding cost rate for each component is increasing convex in its inventory level. Thus, for any positive value $\gamma$, the number of states for which $h(x) \leq \gamma$ is always finite. Under conditions (i) and (ii), there exists a constant $v^*$ and a function $f(x)$ such that $f(x) + v^* = \inf \{ h(x) + \sum_j \mu_j T^{(j)} f(x) + \sum_i \lambda_i T_i f(x) \}$ (Weber and Stidham 1987). The stationary policy that minimizes the righthand side of the above equation for each state $x$ is an optimal policy for the average cost criterion and yields a constant average cost $v^*$. Hence, properties of the optimal policy for the average cost are determined through the function $f(x)$. Recall that properties of the optimal policy for the discounted costs are determined through $v^*(x)$. Since the same event operators are applied to $f(x)$, the optimal policy for the average cost retains the same structure as in the discounted cost case.

\[ \square \]

References
