Carnegie Mellon University

MELLON COLLEGE OF SCIENCE

THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF Doctor of Philosophy

TITLE Dynamics of Phase Separation and Pattern Formation

PRESENTED BY Matteo Rinaldi

ACCEPTED BY THE DEPARTMENT OF Mathematical Sciences

Irene Fonseca and Giovanni Leoni	May 2017
MAJOR PROFESSOR	DATE
Thomas Bohman	May 2017
DEPARTMENT HEAD	DATE

APPROVED BY THE COLLEGE COUNCIL

Rebecca W. Doerge	May 2017
DEAN	DATE

Dynamics of Phase Separation and Pattern Formation

BY

MATTEO RINALDI

DISSERTATION

Submitted in Partial Fulfillment of the Requirements for the Degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

at

CARNEGIE MELLON UNIVERSITY

Department of Mathematical Sciences

Advised by: Prof. Irene Fonseca and Prof. Giovanni Leoni

Approved by:

Prof. Irene Fonseca (co-Chair)

Prof. Giovanni Leoni (co-Chair)

Prof. Nicola Fusco

Prof. Dejan Slepčev

Pittsburgh, PA, April 2017

To my parents

Abstract

The focus of this thesis is the study of the evolution of two models adopted in the context of phase separation and pattern formation, the Cahn-Hilliard model and the Swift-Hohenberg model. In the study of the Cahn-Hilliard model, the PDEs arising as the L^2 and H^{-1} gradient flows in the higher dimensional setting $n > 1$ are studied, and estimates are provided on the evolution of solutions initiated close to configurations that globally or locally minimize the perimeter of the interface are provided. The results rely on a new regularity property of a local version of the well-known isoperimetric function. In the Swift-Hohenberg setting, the one dimensional model is considered, and the slow evolution of a particular class of solutions is established. In this context, existence and regularity of solutions in dimension $n \leq 3$ are provided. In the last part of this thesis, two ongoing project and future research directions are presented.

Acknowledgments

I would like to express my gratitude to my advisors, Irene Fonseca and Giovanni Leoni, for their guidance, patience and support in working on the projects that form my thesis. It has been an honor for me to have had the opportunity to learn from them, and to admire their dedication to education in the broadest sense. Many thanks go to Nicola Fusco and Dejan Slepčev for having taken the time to review this thesis. I have benefited a lot from the enjoyable environment at the Center for Nonlinear Analysis at Carnegie Mellon University where my research was carried out, partially thanks to NSF Grants No. DMS 0905778 and DMS 1412095.

My formation as a mathematician and my growth as a man would not have been the same without all the people I met in the Department of Mathematical Sciences at CMU. Many thanks go to Deborah Brandon, Marco Caroccia, Riccardo Cristoferi, Elisa Davoli, Giovanni Gravina, Adrian Hagerty, Greg Hayrapetyan, Bill Hrusa, David Kinderlehrer, Slav Kirov, Dana Mihai, Léonard Monsaingeon, Marco Morandotti, Ryan Murray, Bob Pego, Daniel Rodríguez, Daniel Schwarz, Dejan Slepčev, Ian Tice, Nicolás García Trillos and all the members of the Energy Minimizers soccer team.

A special thanks goes to the first student I met in the Department of Mathematical Sciences: Laura. To put it in Dante's words: "Galeotto fu 'l libro e chi lo scrisse", that is to say, everything started thanks to a book and its author. In our case it was "Measure Theory and Fine Properties of Functions", by Evans and Gariepy.

I will always be indebted to Stefania Carletti and Dimitri Mugnai. They are not only amazing instructors, but also inspirations, examples and great friends. I am lucky to have met them in my life.

This thesis is dedicated to my parents: they taught me to trust myself, they prepared me to face my fears and showed me how to aim high.

"Perché [...] debbe uno uomo prudente entrare sempre per vie battute da uomini grandi, e quegli che sono stati eccellentissimi imitare, [...] e fare come gli arcieri prudenti, a quali parendo el luogo dove desegnano ferire troppo lontano, [...] pongono la mira assai più alta che il luogo destinato [...] per potere con lo aiuto di sì alta mira pervenire al disegno loro."

"A wise man ought always to follow the paths beaten by great men, and to imitate those who have been more than excellent, [...] and act like the prudent archers who, seeing the target they want to hit too far distant, [...] take aim much higher than the mark, [...] to be able with the aid of so high an aim to fulfill their plan."

N. Machiavelli

Contents

Chapter 1

Introduction

The main area of this thesis is analysis of problems in the context of phase separation and pattern formation. The goal is to use variational methods to study the evolution of solutions and to give estimates on their speed of convergence. The first part of this work deals with the Cahn-Hilliard energy and its gradient flows, the Allen-Cahn and Cahn-Hillard equations. This contribution is mostly contained in [60].

The second part addresses the Swift-Hohenberg equation, seen as a gradient flow of a higher order energy. These results are contained in [46].

Finally, future directions of research are proposed in the final part of this dissertation, where a brief summary on two ongoing projects is given.

1.1 Slow motion

Equations displaying interfacial dynamics have been studied extensively in the last two decades, see, e.g., [2], [3], [4], [5], [17], [16], [19], [39] . Among them, the most well-known are the Allen-Cahn equation

$$u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon - W'(u^\varepsilon) \tag{1.1.1}$$

and the Cahn–Hilliard equation

$$u_t^\varepsilon = -\Delta(\varepsilon^2 \Delta u^\varepsilon - W'(u^\varepsilon)) \tag{1.1.2}$$

where, here and henceforth, we write u^ε to highlight the dependence of solutions on the parameter $\varepsilon > 0$. In the one dimensional setting $n = 1$, it has been shown for the Allen–Cahn equation (see [19] and the references therein) that if $\varepsilon \ll 1$ the evolution from a sufficiently regular initial data occurs in four main stages. In the first stage, the diffusion term $\varepsilon^2 \Delta u^\varepsilon$ can be ignored and the leading order dynamics are driven by the ε independent ordinary differential equation $u_t = -W'(u)$. This is the time-scale in which interfaces develop, i.e., regions in the space domain that separate almost constant solutions corresponding to the stable equilibria of the ordinary differential equation. This stage, referred to as the generation of interface, has been analyzed for the Allen–Cahn equation first in [35], and subsequently in [19], [20], [29], [73].

As the regions separating unequal equilibria decrease in length, the spacial gradient necessarily increases, and after $O(|\ln \varepsilon|)$ time the dynamics are driven by a balance between the two terms on the right–hand side of (1.1.1). In particular, as shown in [19], after $O(\varepsilon^{-1})$ time the solution is exponentially close to the standing–wave profile

$$\Phi(x; p_1, \ldots, p_n) := \pm \prod_n \phi\left(\frac{x - p_i}{\varepsilon}\right), \tag{1.1.3}$$

parametrized by the positions $p_1, \ldots p_n$, where ϕ satisfies

$$\phi'' = W'(\phi), \quad \lim_{z \to \pm\infty} \phi(x) = \pm 1, \quad \phi(0) = 0. \tag{1.1.4}$$

The zeros $p_1(t), \ldots, p_n(t)$ of Φ can be viewed as specifying the location of the interfaces. In particular, the residual $\varepsilon^2 \Phi_{xx} - W'(\Phi)$ is exponentially small and the corresponding third stage of the evolution proceeds on an exponentially slow time scale until two zeros of the solution of (1.1.1) u^ε collide and disappear as part of the fourth stage of the evolution.

The third stage, usually referred to as *slow motion*, has been studied extensively. Some precise interface evolution results for the Allen–Cahn equation can be found in [16], [17], [38], [39]. See also the formal derivation obtained by Neu [61]. To be precise, the zeros of the solution u^ε are approximated by $\{p_i\}$, which at leading order move according to the evolution law

$$p_i' = \varepsilon S\left(\exp\left(-\mu\frac{p_{i+1} - p_i}{\varepsilon}\right) - \exp\left(-\mu\frac{p_i - p_{i-1}}{\varepsilon}\right)\right), \tag{1.1.5}$$

where $\mu = \sqrt{W''(\pm 1)}$, $S > 0$ is a constant depending only on W. The proof of this reduction involves invariant manifold theory and geometric analysis.

A similar approach has been recently adopted by several authors to extend these ideas to a more general setting, by studying the slow manifolds inherent to the dynamics of these equations, see [66] and the references therein.

Subsequently, Bronsard and Kohn [12] introduced a new variational method to study the behavior of solutions of the Allen–Cahn equation (1.1.1). They observed that the motion of solutions of this equation, subject to either Neumann or Dirichlet boundary conditions in an open, bounded interval $\Omega \subset \mathbb{R}$, could be studied by exploiting the gradient flow structure of (1.1.1) (cf. (2.4.3) in Section 2.4). The key tool in their paper is a careful analysis of the asymptotic behavior of the energy

$$G_\varepsilon[u] := \int_\Omega \frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2}|\nabla u|^2 dx, \quad u \in H^1(\Omega), \tag{1.1.6}$$

where W is a double-well potential with $\{W = 0\} = \{a, b\}$ for some $a < b$. See [37] for more details. The L^2–gradient flow of (1.1.6) is precisely (1.1.1). It is well–known (see, e.g., [58], [59], [74]) that if $\{v_\varepsilon\}$ converges in $L^1(\Omega)$ to a function $v \in BV(\Omega;\{a,b\})$ with exactly N jumps, then

$$\liminf_{\varepsilon\to 0} G_\varepsilon[v_\varepsilon] \geq N c_W =: G_0[v], \tag{1.1.7}$$

where

$$c_W := \int_a^b W^{1/2}(s)\,ds.$$

Bronsard and Kohn improved the lower bound (1.1.7) by showing that, for any $k>0$,

$$G_\varepsilon[v_\varepsilon] \geq N c_W - C_1 \varepsilon^k \tag{1.1.8}$$

for ε sufficiently small and some $C_1 > 0$. They then applied (1.1.8) to prove that (cf. Theorem 4.1 in [12]) if the initial data u_0^ε of the equation (1.1.1) converges in $L^1(\Omega)$ to the jump function v, and u_0^ε are energetically "well–prepared", that is,

$$G_\varepsilon[u_0^\varepsilon] \leq N c_W + C_2 \varepsilon^k$$

for some $C_2 > 0$, then for any $M > 0$,

$$\sup_{0\leq t\leq M\varepsilon^{-k}} ||u^\varepsilon(t) - v||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+. \tag{1.1.9}$$

Subsequently, Grant [44] improved the estimate (1.1.8) to

$$G_\varepsilon[v_\varepsilon] \geq N c_W - C_1 e^{-C_2\varepsilon^{-1}} \tag{1.1.10}$$

for ε small, and some $C_1, C_2 > 0$, which in turn gives the more accurate slow motion estimate

$$\sup_{0\leq t\leq Me^{C\varepsilon^{-1}}} ||u^\varepsilon(t) - v||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+ \tag{1.1.11}$$

for some $C > 0$, and his analysis extends to solutions of the Cahn-Hilliard equation (1.1.2). Finally, Bellettini, Nayam and Novaga [8] gave a sharp version of Grant's second–order estimate by proving that

$$\begin{aligned} G_\varepsilon[v_\varepsilon] \geq N c_W &- 2\alpha_+\kappa_+^2 \sum_{k=1}^N e^{-\alpha_+\frac{d_k^\varepsilon}{\varepsilon}} - 2\alpha_-\kappa_-^2 \sum_{k=1}^N e^{-\alpha_-\frac{d_k^\varepsilon}{\varepsilon}} \\ &+ \kappa_+^3\beta_+ \sum_{k=1}^N e^{-\frac{3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}} + \kappa_-^3\beta_- \sum_{k=1}^N e^{-\frac{3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}} \\ &+ o\left(\sum_{k=1}^N e^{-\frac{3\alpha_+}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) + o\left(\sum_{k=1}^N e^{-\frac{3\alpha_-}{2}\frac{d_k^\varepsilon}{\varepsilon}}\right) \end{aligned} \tag{1.1.12}$$

as $\varepsilon \to 0^+$, where $\alpha_\pm, \kappa_\pm, \beta_\pm$ are constants depending on the potential W and d_k^ε is the distance between the k–th and the $(k+1)$–th transitions of v_ε. This last work gives a variational validation of [16], [17]. Indeed, the sharp energy estimate allows the authors to (formally) recover the ODE describing the motion of transition points.

The situation in higher dimensions is more complicated. In particular, we will consider the nonlocal (or mass-preserving) Allen-Cahn equation

$$u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon - W'(u^\varepsilon) + \varepsilon \lambda_\varepsilon, \tag{1.1.13}$$

where λ_ε is the Lagrange multiplier responsible for the preservation of the mass (see Section 1.2 for more details). As in the one–dimensional setting, it is well–known (see, e.g., [14], [71]) that, after rescaling time by ε, (1.1.13) is the L^2–gradient flow of the energy (1.1.6) subject to the mass constraint

$$\int_\Omega u\,dx = m, \tag{1.1.14}$$

where here $\Omega \subset \mathbb{R}^n$, $n \geq 2$. Furthermore, the energy $\mathcal{G}_\varepsilon : L^1(\Omega) \to [0,\infty]$ defined by

$$\mathcal{G}_\varepsilon[u] := \begin{cases} G_\varepsilon[u] & \text{if } u \in H^1(\Omega) \text{ and } \int_\Omega u\,dx = m, \\ \infty & \text{otherwise,} \end{cases} \tag{1.1.15}$$

is known to Γ–converge to $\mathcal{G}_0 : L^1(\Omega) \to [0,\infty]$, where

$$\mathcal{G}_0[u] := \begin{cases} 2c_W P(\{u=a\};\Omega) & \text{if } u \in BV(\Omega;\{a,b\}) \text{ and } \int_\Omega u\,dx = m, \\ \infty & \text{otherwise.} \end{cases} \tag{1.1.16}$$

Here $P(E;\Omega)$ denotes the *relative perimeter of* E *inside* Ω, for any measurable set $E \subset \mathbb{R}^n$ (see Section 2.2). In particular, if

$$u_{E_0} := a\chi_{E_0} + b\chi_{E_0^c} \tag{1.1.17}$$

is a local minimizer of $\mathcal{G}_0$ then ∂E_0 is a surface of constant mean curvature, and the curvatures may affect the slow motion of solutions of (1.2.1). Much of the work in the existing literature in this setting has addressed the motion of phase "bubbles", namely solutions approximating a spherical interface compactly contained in Ω. For example, Bronsard and Kohn [13] utilize variational techniques to analyze radial solutions $u_{\varepsilon,\text{rad}}$ of the Allen–Cahn equation. They prove that $u_{\varepsilon,\text{rad}}$ separates Ω into two regions where $u_{\varepsilon,\text{rad}} \approx +1$ and $u_{\varepsilon,\text{rad}} \approx -1$ and that the interface moves with normal velocity equal to the sum of its principal curvatures. In [32], Ei and Yanagida investigate the dynamics of interfaces for the Allen–Cahn equation, where Ω is a strip–like domain in $\mathbb{R}^2$. They show that the evolution is slower than the mean curvature flow, *but* faster than exponentially slow. This suggests that estimates of the type (1.1.10) cannot be expected to hold in higher dimensions. In the Cahn–Hilliard case, Alikakos, Bronsard and Fusco [3] use energy methods and detailed spectral estimates to show the existence of solutions of (1.1.2) supporting almost spherical interfaces, which evolve by drifting towards the boundary with exponentially small velocity. Other related works include [2], [4] and [5]. Most of these contributions require significant analytical techniques, and often focus only on the existence of slowly moving solutions.

1.2 Allen-Cahn & Cahn-Hilliard

The first part of this thesis addresses the slow motion of phase boundaries for the nonlocal Allen–Cahn equation with Neumann boundary conditions. To be precise, consider

$$\begin{cases} u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon - W'(u^\varepsilon) + \varepsilon\lambda_\varepsilon & \text{in } \Omega \times [0,\infty), \\ \frac{\partial u^\varepsilon}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u^\varepsilon = u_0^\varepsilon & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.2.1}$$

Here $\Omega \subset \mathbb{R}^n$, $1 < n \leq 7$, is an open, bounded, connected set with $\partial\Omega$ regular (see (2.1.1)), $\varepsilon > 0$ is a parameter representing the interaction length, $W : \mathbb{R} \to [0,\infty)$ is a double well-potential with wells at $a < b$, u_0^ε is the initial datum, and λ_ε is a Lagrange multiplier that renders solutions mass–preserving, i.e.,

$$\lambda_\varepsilon = \frac{1}{\varepsilon \mathcal{L}^n(\Omega)} \int_\Omega W'(u^\varepsilon)\, dx.$$

This nonlocal reaction diffusion equation was introduced by Rubinstein and Sternberg [71] to model phase separation after quenching of homogenous binary systems (e.g., glasses or polymers). An important property of this equation is that the total mass $\int_\Omega u^\varepsilon(x,t)\, dx$ is preserved in time. It can be shown that when $\varepsilon \to 0^+$ the domain Ω is divided into regions in which u^ε is close to a and to b, and that the interfaces between these regions as $\varepsilon \to 0^+$ evolve according to a nonlocal volume–preserving mean curvature flow.

A key tool in the analysis of solutions of (1.2.1) in the higher-dimensional setting is the analogue of (1.1.8) that was recently obtained by Leoni and Murray [55]. Their result assumes that the *isoperimetric function*

$$\mathcal{I}_\Omega(r) := \inf\{P(E;\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = r\}, \quad r \in [0, \mathcal{L}^n(\Omega)], \tag{1.2.2}$$

satisfies a Taylor formula of order two at the value

$$r_0 := \frac{b\mathcal{L}^n(\Omega) - m}{b - a}, \tag{1.2.3}$$

where m is the mass constraint given in (1.1.14), and where by a "Taylor formula of order two" we mean that there exists a neighborhood U of r_0 such that

$$\mathcal{I}_\Omega(r) = \mathcal{I}_\Omega(r_0) + \frac{d\mathcal{I}_\Omega}{dr}(r_0)(r - r_0) + O(|r - r_0|^{1+\varsigma}), \tag{1.2.4}$$

for some $\varsigma \in (0,1]$, for all $r \in U$ (see Lemma 3.2.3; see also [7] and [75]).

In certain settings it is known that $\mathcal{I}_\Omega$ is semi–concave (see [7] and [75]), and indeed we will later show that $\mathcal{I}_\Omega$ is semi–concave as long as Ω is $C^{2,\sigma}$ (see Remark 3.2.4). Hence, $\mathcal{I}_\Omega$ satisfies a Taylor formula of order two at $\mathcal{L}^1$–a.e. r or, equivalently, for $\mathcal{L}^1$–a.e. mass m in (1.1.14).

If a set $E_0 \subset \Omega$ satisfies

$$\mathcal{L}^n(E_0) = r_0, \qquad P(E_0; \Omega) = \mathcal{I}_\Omega(r_0), \tag{1.2.5}$$

then we call E_0 a *volume–constrained global perimeter minimizer*. Classical results [45], [56] establish the existence of volume–constrained global perimeter minimizers, and assert that the boundary of any volume–constrained global perimeter minimizer is a surface of (classical) constant mean curvature for $n \leq 7$, provided $\partial\Omega$ is of class $C^{2,\alpha}$ (see Proposition 2.3.7 and Lemma 3.2.1 below).

Under technical hypotheses on Ω, W, m, a simplified version of the main theorem in [55] is the following.

Theorem 1.2.1. *Assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and suppose that $E_0 \subset \Omega$ is a volume–constrained global perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$. Suppose further that $\mathcal{I}_\Omega$ satisfies a Taylor expansion of order two at r_0 (given by* (1.2.3)*) as in* (1.2.4)*. Then given any function $u \in L^1(\Omega)$, the following error bound holds*

$$\mathcal{G}_\varepsilon[u] \geq \mathcal{G}_0[u_{E_0}] - C(\kappa)\varepsilon \tag{1.2.6}$$

for all $\varepsilon > 0$ sufficiently small, where u_{E_0} is the function given in (1.1.17) *and $C(\kappa)$ is a known, sharp constant that depends only upon W, $P(E_0; \Omega)$ and the mean curvature κ of ∂E_0.*

Thanks to the previous energy estimate, we are naturally led to the study of motion of solutions of the initial value problem (1.2.1). We will denote

$$X_1 := \left\{ u \in L^2(\Omega) : \int_\Omega u \, dx = m \right\}.$$

The first main result in the first part of this thesis is the following.

Theorem 1.2.2. *Assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and let E_0 be a volume–constrained global perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$. Furthermore, suppose that $\mathcal{I}_\Omega$ satisfies a Taylor expansion of order two at r_0 as in* (1.2.4)*. Assume that $u_0^\varepsilon \in X_1 \cap L^\infty(\Omega)$ is such that*

$$u_0^\varepsilon \to u_{E_0} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0^+, \tag{1.2.7}$$

and

$$\mathcal{G}_\varepsilon[u_0^\varepsilon] \leq \mathcal{G}_0[u_{E_0}] + C\varepsilon \tag{1.2.8}$$

for some $C > 0$. Let u^ε be a solution to (1.2.1)*. Then, for any $M > 0$*

$$\sup_{0 \leq t \leq M\varepsilon^{-1}} ||u^\varepsilon(t) - u_{E_0}||_{L^2} \to 0 \text{ as } \varepsilon \to 0^+. \tag{1.2.9}$$

Remark 1.2.3. *The assumption $u_0^\varepsilon \in X_1 \cap L^\infty(\Omega)$ is needed in order to ensure regularity of the solutions, see Theorem 2.4.1. In particular,* (2.4.1) *is satisfied thanks to the hypotheses on the potential, see* (2.1.6)*.*

Using Theorem 1.2.1, we can also prove that solutions to the Cahn–Hiliard equation with Neumann boundary conditions

$$\begin{cases} u_t^\varepsilon = -\Delta v_\varepsilon & \text{in } \Omega \times (0,\infty), \\ v_\varepsilon = \varepsilon^2 \Delta u^\varepsilon - W'(u^\varepsilon) & \text{in } \Omega \times [0,\infty), \\ \dfrac{\partial u^\varepsilon}{\partial \nu} = \dfrac{\partial v_\varepsilon}{\partial n} = 0 & \text{on } \partial\Omega \times [0,\infty), \\ u^\varepsilon = u_0^\varepsilon & \text{on } \Omega \times \{0\}. \end{cases} \tag{1.2.10}$$

admit analogous properties. As a matter of fact, it is well–known that the Cahn–Hilliard equation can be seen as the X_2–gradient flow of the energy in (1.1.15), where the space $X_2(\Omega)$ is similar to $H^{-1}(\Omega)$. In particular, following [52], we will formally denote

$$X_2(\Omega) := ((H^1(\Omega))', \langle\ ,\ \rangle_{X_2}),$$

where the inner product will be precisely introduced in Section 2.4. We will prove the following.

Theorem 1.2.4. *Let $n = 2, 3$, assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and let E_0 be a volume–constrained global perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$. Furthermore, suppose that $\mathcal{I}_\Omega$ satisfies a Taylor expansion of order 2 at r_0 as in* (1.2.4)*. Assume that $u_0^\varepsilon \in X_2 \cap L^2(\Omega)$ is such that*

$$u_0^\varepsilon \to u_{E_0} \text{ in } X_2(\Omega) \text{ as } \varepsilon \to 0^+, \tag{1.2.11}$$

and

$$\mathcal{G}_\varepsilon[u_0^\varepsilon] \le \mathcal{G}_0[u_{E_0}] + C\varepsilon \tag{1.2.12}$$

for some $C > 0$. Let u^ε be a solution to (1.2.10)*. Then, for any $M > 0$*

$$\sup_{0 \le t \le M\varepsilon^{-1}} ||u^\varepsilon - u_{E_0}||_{X_2} \to 0 \text{ as } \varepsilon \to 0^+. \tag{1.2.13}$$

Remark 1.2.5. *To the best of our knowledge, regularity results for* (1.2.10) *have not been rigorously derived in the case $n \ge 4$. For this reason, the previous result is stated in a lower dimensional setting and we rely on Theorems 2.4.2 and 2.4.3 for the regularity of solutions. On the other hand, if we assume that solutions $u^\varepsilon(t) \in L^1(\Omega)$ for all $t \ge 0$, then the result holds for any $1 < n \le 7$.*

Next we show that Theorem 1.2.2 continues to hold for certain volume–constrained local perimeter minimizers (for a precise definition see Definition 2.2.6 in Section 2.1). For this purpose, we introduce a local version of the isoperimetric function $\mathcal{I}_\Omega$ defined by

(1.2.2). Given a Borel set $E_0 \subset \Omega$ and $\delta > 0$ we define the *local isoperimetric function* of parameter δ about the set E_0 to be

$$\mathcal{I}_\Omega^{\delta,E_0}(r) := \inf\{P(E,\Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E) = r, \alpha(E_0,E) \leq \delta\}, \tag{1.2.14}$$

where

$$\alpha(E_1,E_2) := \min\{\mathcal{L}^n(E_1 \setminus E_2), \mathcal{L}^n(E_2 \setminus E_1)\} \tag{1.2.15}$$

for all Borel sets $E_1, E_2 \subset \Omega$.

Under smoothness assumptions on $\mathcal{I}_\Omega^{\delta,E_0}$ and other technical hypotheses on Ω, W, m (see Section 2.1), we will prove the following result.

Theorem 1.2.6. *Assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, let E_0 be a volume–constrained local perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$. Fix $\delta > 0$ and suppose that $\mathcal{I}_\Omega^{\delta,E_0}$ admits a Taylor expansion of order two at r_0 as in* (1.2.4)*. Then for any $u \in L^1(\Omega)$ satisfying*

$$\|u - u_{E_0}\|_{L^1} \leq 2\delta \tag{1.2.16}$$

we have

$$\mathcal{G}_\varepsilon[u] \geq \mathcal{G}_0[u_{E_0}] - C(\kappa)\varepsilon, \tag{1.2.17}$$

for $\varepsilon > 0$ sufficiently small, where $C(\kappa)$ is a known, sharp constant that depends only upon W, $P(E_0;\Omega)$ and the mean curvature κ of ∂E_0.

Remark 1.2.7. *The closeness condition* (1.2.16) *depends on the distance between the wells of W, and it precisely reads as $\|u - u_{E_0}\|_{L^1} \leq (b-a)\delta$. Without loss of generality, we will assume $a = -1 < 1 = b$, see* (2.1.11).

Replacing $\mathcal{I}_\Omega$ with $\mathcal{I}_\Omega^{\delta,E_0}$, we are able to show that Theorem 1.2.1 continues to hold for volume–constrained local perimeter minimizers. In turn, this brings us to the next main result of the first part of this thesis.

Theorem 1.2.8. *Assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and let E_0 be a volume–constrained local perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$. Fix $\delta > 0$, and suppose that $\mathcal{I}_\Omega^{\delta,E_0}$ admits a Taylor expansion of order two at r_0 as in* (1.2.4)*. Assume that $u_0^\varepsilon \in X_1 \cap L^\infty(\Omega)$ is such that*

$$u_0^\varepsilon \to u_{E_0} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0^+, \tag{1.2.18}$$

and

$$\mathcal{G}_\varepsilon[u_0^\varepsilon] \leq \mathcal{G}_0[u_{E_0}] + C\varepsilon \tag{1.2.19}$$

for some $C > 0$. Let u^ε be a solution to (1.2.1)*. Then, for any $M > 0$*

$$\sup_{0 \leq t \leq M\varepsilon^{-1}} ||u^\varepsilon(t) - u_{E_0}||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+.$$

In view of the previous theorem, the regularity of $\mathcal{I}_\Omega^{\delta,E_0}$ at r_0 is of crucial importance. Note that unlike $\mathcal{I}_\Omega$, the function $\mathcal{I}_\Omega^{\delta,E_0}$ depends upon r_0, and thus semi–concavity does not provide enough information. We will focus on the case in which E_0 is either a ball or a set with positive second variation in the sense of (2.3.6). The case in which E_0 is a ball is linked to the case of phase "bubbles", which have been extensively studied in [2], [3], [4], and [5] (see Section 3.2).

Theorem 1.2.9. *Let Ω satisfy* (2.1.1)*, let $E_0 = B_{\rho_0}(x_0) \subset\subset \Omega$ for some $x_0 \in \Omega$ with $\rho_0 = (r_0/\omega_n)^{1/n}$. Then there exist $\delta_0 > 0$ and $0 < r_1 < r_0$ such that*

$$\mathcal{I}_\Omega^{\delta,E_0}(r) = C_n r^{\frac{n-1}{n}}, \tag{1.2.20}$$

for all $r \in [r_0 - r_1, r_0 + r_1]$ and all $0 < \delta \leq \delta_0$, where C_n is a constant depending only on the dimension n. In particular, the map $r \mapsto \mathcal{I}_\Omega^{\delta,E_0}(r)$ admits a Taylor expansion of order two at r_0 as in (1.2.4) *and Theorem 1.2.8 holds for E_0.*

Here $\omega_n := \mathcal{L}^n(B_1(\mathbf{0}))$. Moreover, we are able to prove regularity of $\mathcal{I}_\Omega^{\delta,E_0}$ in the setting of isolated local minimizers with positive second variation in the sense of (2.3.6). The proof of the previous result relies upon the theory of the stability of the perimeter functional developed by Fusco, Maggi and Pratelli [40]. In particular, we use the results obtained by Julin and Pisante [49], who extended the techniques introduced by Acerbi, Fusco and Morini [1].

Theorem 1.2.10. *Suppose that Ω satisfies* (2.1.1)*, and that E_0 is a local volume–constrained perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$ and with positive second variation in the sense of* (2.3.6)*. Then, for sufficiently small δ, $\mathcal{I}_\Omega^{\delta,E_0}$ admits a Taylor expansion of order two at r_0 as in* (1.2.4)*. In particular, Theorem 1.2.8 holds for such E_0.*

1.3 Swift-Hohenberg

The fourth order partial differential equation

$$u_t = ru - (\bar{q}^2 + \nabla^2)^2 u + f(u) \tag{1.3.1}$$

is a generalization of the Swift–Hohenberg equation introduced in 1977 by Swift and Hohenberg [76] as a model for the study of pattern formation, in connection with the Rayleigh–Bénard convection, e.g. see [27],[51]. Among many different applications, the most famous ones in the literature are those in connection to pattern formation in vibrated granular materials [77], buckling of long elastic structures [48], Taylor–Couette flow [47], [69], and the study of lasers [53]. Moreover, in recent years considerable interest has been paid to models of phase transitions in the study of pattern–formation in bilayer membranes, see e.g. [21] where the Swift–Hohenberg equation turns out to be the gradient flow of Ginzburg–Landau type energies with respect to the right inner product structure.

Consider (1.3.1) on a periodic domain with a characteristic size $L = 1/\varepsilon$, where $0 < \varepsilon \ll 1$. Letting W be the primitive of $s \mapsto 2(f(s) + (r - \bar{q}^4)s)$, $q := 2\bar{q}^2$, and rescaling time and space by ε in (1.3.1), we obtain at the rescaled formulation

$$\begin{cases} u_t = -W'(u) - 2\varepsilon^2 q u^\varepsilon_{xx} - 2\varepsilon^4 u^\varepsilon_{xxxx} & x \in \mathbb{T}, t > 0, \\ u^\varepsilon(x, 0) = u^\varepsilon_0(x) & x \in \mathbb{T}, \end{cases} \tag{1.3.2}$$

where $\mathbb{T}$ is a one–dimensional torus, and u^ε_0 is the initial datum. As above, $W : \mathbb{R} \to [0, +\infty)$ is a double–well potential with phases supported at $a < b$, and we study the long–time behavior of solutions when $q > 0$ is sufficiently small. In particular, due to the presence of the small parameter ε in (1.3.2) the solutions are expected to develop interfacial structure driven by the minima of the potential W. Equation (1.3.2) may be viewed as a gradient flow associated to a second order energy functional, and our main result consists of an asymptotic lower bound on the corresponding energy functional and the consequent bounds on the speed of evolution of the developed interfaces, as in the case of the lower order Allen-Cahn equation (1.1.1).

Equations associated to higher–order energy functionals have been studied in the last two decades. For instance, in [50] the authors consider a family of the form

$$\mathcal{H}_\varepsilon(u) := \frac{1}{\varepsilon} \int_I \left(\sum_{k-1}^{n} \frac{\gamma_k \varepsilon^{2k}}{2} |u^{(k)}|^2 + W(u) \right) dx, \tag{1.3.3}$$

where $u^{(k)}$ stands for the k–th spatial derivative of u. Due to difficulties associated with higher order nature of the functional, in particular, the lack of exact solutions of the corresponding Euler–Lagrange equation, sharp bounds analogous to (1.1.12) have not been established. An important condition on $\mathcal{H}_\varepsilon$ in [50] is

(HP) There exist constants $d_0, \eta > 0$ such that for every interval $I \subset \mathbb{R}$ with length $|I| \geq d_0$, and for all $u \in H^n(I) := W^{2,n}(I)$

$$\int_I \left(\sum_{k-1}^{n} \gamma_k |u^{(k)}|^2 \right) dx \geq \eta \int_I \left(|u^{(n)}|^2 + |u'|^2 \right) dx. \tag{1.3.4}$$

Under this hypothesis the authors prove that for any $u \in H^n(I)$ sufficiently close to a step function taking values ± 1 and having exactly N jumps,

$$\mathcal{H}_\varepsilon(u) \geq N m_1 - C \exp\left(-\frac{d\lambda}{3\varepsilon} \right), \tag{1.3.5}$$

where λ is a constant satisfying $\lambda < |\mathrm{Re}(\mu)|$, for all eigenvalues μ of the linearization of

$$\sum_{k=1}^{n} (-1)^k \gamma_k u^{(2k)} + W'(u) = 0 \tag{1.3.6}$$

at $(\pm 1, 0, \dots, 0)$.

The initial value problem (1.3.2) can be seen as the L^2–gradient flow of the second order energy functional

$$E_\varepsilon(u;\mathbb{T}) := \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W(u) - \varepsilon q|u'|^2 + \varepsilon^3|u''|^2\right)dx, \quad u \in H^2(\mathbb{T}), \tag{1.3.7}$$

and our main goals are the extension and the improvement of the bound (1.3.5) for this energy and, in turn, using this to prove the slow motion of solutions of (1.3.2). We note that the functional (1.3.7) does not satisfy (HP) due to the negative term in the energy. We use recently established interpolation inequalities (see [21] and [23]) to overcome this difficulty if $q > 0$ is sufficiently small. Moreover, in the proof of an energy estimate analogous to (1.3.5), see Theorem 1.3.1, we do not assume any closeness condition on the H^2 functions we consider, instead we make an assumptions on the zeros of such functions.

The main result in this framework is the following lower bound on the energy.

Theorem 1.3.1. *Let $\mathbb{T}$ be the one–dimensional unit torus, and let W satisfy the hypotheses* (2.1.3)*,* (2.1.5) *and* (2.1.7)–(2.1.9)*. Let $\alpha_0 > 0$. Then there exist $q_0 > 0$ and $\varepsilon_0 > 0$, possibly dependent on α_0 and q_0, such that if $q < q_0$ and $w \in H^2(\mathbb{T})$ has at least N zeros, $\{x_k\}_{k=1}^N$, satisfying $\min_k |x_{k+1} - x_k| \geq \alpha_0$ then*

$$E_\varepsilon(w;\mathbb{T}) \geq Nm_1 - C\sum_{k=1}^{N} \exp\left(-\frac{d_k\gamma}{\varepsilon}\right), \tag{1.3.8}$$

for every $0 < \varepsilon < \varepsilon_0$, where $d_k = x_{k+1} - x_k$, $\gamma > 0$ is defined in (4.1.45) *and depends only on W, while $C > 0$ is independent of ε.*

We remark that a similar estimate can be obtained when the domain is an interval $I := (a_0, b_0)$, with (1.3.8) replaced with

$$E_\varepsilon(w;I) \geq Nm_1 - C\sum_{k=0}^{N} \exp\left(-\frac{d_k\gamma}{\varepsilon}\right), \tag{1.3.9}$$

where $d_0 := x_1 - a_0$, $d_N := b_0 - x_N$.

Remark 1.3.2. *We highlight the fact that we are not requiring the function w of Theorem 1.3.1 to be L^1–close to a jump function, in contrast with [9], [12], [44], [50]. On the other hand, it is easy to show that if w is L^1–close to a jump function v taking values ± 1, then there exists an $\alpha_0 > 0$ with the property that the zeros of w are at least $\alpha_0 > 0$ apart, as in the statement of Theorem 1.3.1.*

The energy estimate above is a crucial ingredient to prove slow motion of solutions of (1.3.2), when the initial data is close in the L^1 norm to a BV function, as in [12], [44], [50]. In particular, we will consider regular solutions of (1.3.2), whose existence is proved in the Appendix, see Theorem 4.3.1. Our analysis yields the following result.

Theorem 1.3.3. *Let $v \in BV(\mathbb{T};\{\pm 1\})$ be a function with $N(v) \neq 0$ jumps at $x_k(v)$, for $k = 1, \dots, N(v)$, and let $q_0 > 0$ be as in Theorem 1.3.1. Let $d := \min_k |x_{k+1}(v) - x_k(v)|$. Then there exist $\varepsilon_0, \delta_0 > 0$ with $d - 4\delta_0 > 0$ such that, if u^ε is a solution of* (1.3.2) *with $u^\varepsilon \in L^2((0,\infty); H^4(\mathbb{T}))$, $u^\varepsilon_t \in L^2((0,\infty); H^2(\mathbb{T}))$ and initial data $u^\varepsilon_0 \in H^2(\mathbb{T})$ satisfying*

$$||u^\varepsilon_0 - v||_{L^1(\mathbb{T})} \leq \delta \tag{1.3.10}$$

for $0 < \delta < \delta_0$ and

$$E_\varepsilon(u_0;\mathbb{T}) \leq E_0(v;\mathbb{T}) + \frac{1}{h(\varepsilon)}, \tag{1.3.11}$$

for all $0 < \varepsilon < \varepsilon_0$ and for some function $h : (0,\infty) \to (0,\infty)$, then for all $q < q_0$,

$$\lim_{\varepsilon \to 0^+} \left\{ \sup_{0 \leq t \leq T_\varepsilon} \int_{\mathbb{T}} |u^\varepsilon(x,t) - u^\varepsilon_0(x)| dx \right\} = 0, \tag{1.3.12}$$

where

$$T_\varepsilon := \delta^2 \min\{h(\varepsilon), \exp((d - 4\delta)\gamma/\varepsilon)\}.$$

Remark 1.3.4. *If $h(\varepsilon) = \exp(d\gamma/\varepsilon)$, then*

$$T_\varepsilon = \delta \exp((d - 4\delta)\gamma/\varepsilon)$$

which is consistent with the estimates obtained in [44] and [50]. On the other hand, we remark that our Theorem 1.3.3 provides more general results.

Remark 1.3.5. *To the best of our knowledge, only recently some regularity results for the Swift–Hohenberg equation have been proved, see [42]. In the statement of Theorem 1.3.3 we* assume *that the solutions are sufficiently regular. In the Appendix we prove existence of solutions (though with weaker regularity) using De Giorgi's technique of Minimizing Movements (see Theorem 4.3.1).*

Chapter 2

Preliminaries

2.1 Notation and structural assumptions

In the first part of this thesis we consider an open, connected, bounded domain $\Omega \subset \mathbb{R}^n$, with $n \leq 7$, such that

$$\mathcal{L}^n(\Omega) = 1, \quad \partial\Omega \text{ is of class } C^{4,\sigma}, \quad \sigma \in (0,1]. \tag{2.1.1}$$

We use the fact that $\partial\Omega$ is of class $C^{4,\sigma}$ only in the proof of Theorem 1.2.10. All the other results in this thesis continue to hold if the regularity of $\partial\Omega$ is assumed to be $C^{2,\sigma}$. Moreover, following Remark 5.2 in [55], we believe that, for many of our results, assumption (2.1.1) could be weakened to assuming that Ω has a Lipschitz boundary.

We will work with a potential $W : \mathbb{R} \to [0,\infty)$ satisfying:

$$W \text{ is of class } C^2; \tag{2.1.2}$$

$$W \text{ has precisely two zeros at } a < b; \tag{2.1.3}$$

$$W''(a) = W''(b) > 0; \tag{2.1.4}$$

$$W' \text{ has exactly 3 zeros at } a, c, b, \text{ with } a < c < b, \quad W''(c) < 0; \tag{2.1.5}$$

$$\liminf_{|s|\to\infty} |W'(s)| = \infty. \tag{2.1.6}$$

and, only in the second part of the thesis, we will also make use of

$$W \text{ is of class } C^5 \text{ and } W(s) = W(-s), \text{ for all } s \in \mathbb{R}; \tag{2.1.7}$$

$$\text{there exists } 0 < c_W \leq 1 \text{ such that } W(s) \geq c_W|s-a|^2, \text{ for } s \geq 0. \tag{2.1.8}$$

For simplicity, we set

$$a = -1, \quad b = 1, \tag{2.1.9}$$

and a prototype for W is given by

$$W(s) := \frac{1}{4}(s^2-1)^2. \tag{2.1.10}$$

Furthermore, without loss of generality, we will work with masses m (see (1.1.14)) satisfying

$$m \in (-1, 1). \tag{2.1.11}$$

By way of notation, constants C vary from line to line throughout the whole thesis.

2.2 BV functions and perimeter

We recall some definitions and basic results from the theory of functions of bounded variation, see, e.g., [34], [54].

Definition 2.2.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We define the* space of functions of bounded variation $BV(\Omega)$ *as the space of all functions $u \in L^1(\Omega)$ such that for all $i = 1, \dots, n$ there exist finite signed Radon measures $D_i u : \mathcal{B}(\Omega) \to \mathbb{R}$ such that*

$$\int_\Omega u\phi_{x_i} dx = -\int_\Omega \phi dD_i u$$

for all $\phi \in C_0^\infty(\Omega)$. The measure $D_i u$ is called the weak, *or* distributional, partial derivative of u *with respect to x_i. Moreover, if $u \in BV(\Omega)$ then the* total variation measure of Du *is finite, namely*

$$|Du|(\Omega) := \sup \left\{ \sum_{i=1}^{n} \int_\Omega \Phi_i dD_i u : \quad \Phi \in C_0(\Omega; \mathbb{R}^n),\ ||\Phi||_{C_0(\Omega;\mathbb{R}^n)} \leq 1 \right\} < \infty.$$

It is well–known that characteristic functions of smooth sets belong to $BV(\Omega)$. More generally, we have the following.

Definition 2.2.2. *Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $\Omega \subset \mathbb{R}^n$ be an open set. The* perimeter *of E in Ω, denoted $P(E; \Omega)$, is the variation of χ_E in Ω, that is,*

$$P(E; \Omega) := |D\chi_E|(\Omega).$$

The set E is said to have finite perimeter in Ω *if $P(E; \Omega) < \infty$. If $\Omega = \mathbb{R}^n$, we write $P(E) := P(E; \mathbb{R}^n)$.*

Given a set E of finite perimeter, by the Besicovitch derivation theorem (see, e.g., [34]) for $|D\chi_E|$–a.e. $x \in \operatorname{supp}|D\chi_E|$ there exists the derivative of $D\chi_E$ with respect to its total variation $|D\chi_E|$, and it is a vector of length 1. For such points we have

$$\frac{D\chi_E}{|D\chi_E|}(x) = \lim_{r \to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} =: -\nu_E(x) \quad \text{and} \quad |\nu_E(x)| = 1. \tag{2.2.1}$$

Definition 2.2.3. *We denote by $\partial^* E$ the set of all points in supp$(|D\chi_E|)$ where* (2.2.1) *holds. The set $\partial^* E$ is called the* reduced boundary *of E, while the vector $\nu_E(x)$ is the* generalized exterior normal at x.

By the *structure theorem for sets of finite perimeter*, (see, e.g., [34], Theorem 2, (iii), page 205), if E has finite perimeter in $\mathbb{R}^n$ then for any Borel set $F \subset \mathbb{R}^n$

$$P(E; F) = \mathcal{H}^{n-1}(\partial^* E \cap F), \tag{2.2.2}$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$–dimensional Hausdorff measure. A classical result in the theory of sets of finite perimeter is the following *isoperimetric inequality*.

Theorem 2.2.4. *Let $E \subset \mathbb{R}^n$, $n \geq 2$, be a set of finite perimeter. Then either E or $\mathbb{R}^n \setminus E$ has finite Lebesgue measure and*

$$\min\{\mathcal{L}^n(E),\ \mathcal{L}^n(\mathbb{R}^n \setminus E)\}^{\frac{n-1}{n}} \leq \frac{\omega_n^{-1/n}}{n} P(E), \tag{2.2.3}$$

where equality holds if and only if E is a ball.

A version of the isoperimetric inequality also holds in bounded domains (see Corollary 3.2.1 and Lemma 3.2.4 of [57], or [22]).

Proposition 2.2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with Lipschitz boundary. Then there exists $C_\Omega > 0$ such that*

$$\min\{\mathcal{L}^n(E),\ \mathcal{L}^n(\Omega \setminus E)\}^{\frac{n-1}{n}} \leq C_\Omega P(E; \Omega) \tag{2.2.4}$$

for all sets $E \subset \Omega$ of finite perimeter.

Next we give the formal definition of a local volume–constrained perimeter minimizer.

Definition 2.2.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. A measurable set $E_0 \subset \Omega$ is said to be a* volume–constrained local perimeter minimizer *of $P(\cdot, \Omega)$ if there exists $\rho > 0$ such that*

$$P(E_0; \Omega) = \inf \left\{ P(E; \Omega) : E \subset \Omega \text{ Borel}, \mathcal{L}^n(E_0) = \mathcal{L}^n(E),\ \mathcal{L}^n(E_0 \Delta E) < \rho \right\}.$$

The next proposition motivates the definition of local isoperimetric function $\mathcal{I}_\Omega^{\delta, E_0}$ (see (1.2.14)).

Proposition 2.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $E_0 \subset \Omega$ be a Borel set and let $v_{E_0} = -\chi_{E_0} + \chi_{E_0{}^c}$. Then*

$$\alpha(E_0, \{u \leq s\}) \leq \delta \tag{2.2.5}$$

for all $u \in L^1(\Omega)$ such that

$$\|u - v_{E_0}\|_{L^1} \leq 2\delta, \tag{2.2.6}$$

and for every $s \in \mathbb{R}$, where α is the number given in (1.2.15).

Proof. Fix $\delta > 0$ and for $s \in \mathbb{R}$ define $F_s := \{x \in \Omega : u(x) \leq s\}$. If $s \in (-1, 1)$ then, by (2.2.6),

$$\begin{aligned}2\delta &\geq \int_{F_s \setminus E_0} |u - v_{E_0}|\, dx + \int_{E_0 \setminus F_s} |u - v_{E_0}| dx \\ &\geq (1 - s)\mathcal{L}^n(F_s \setminus E_0) + (1 + s)\mathcal{L}^n(E_0 \setminus F_s) \geq 2\alpha(E_0, F_s),\end{aligned}$$

so that (2.2.5) is proved in this case. If $s \geq 1$, again by (2.2.6),

$$2\delta \geq \int_{E_0 \setminus F_s} |u - v_{E_0}| dx \geq (1 + s)\mathcal{L}^n(E_0 \setminus F_s) \geq 2\alpha(E_0, F_s).$$

The case $s \leq -1$ is analogous. □

2.3 First and second variation of perimeter

We recall here the following standard definitions and theorems from Chapter 17 in [56].

Definition 2.3.1. *Let $\Omega \subset \mathbb{R}^n$ be open. A* one-parameter family $\{f(\cdot; t)\}_t$ of diffeomorphisms of $\mathbb{R}^n$ *is a smooth function*

$$(x, t) \in \mathbb{R}^n \times (-\epsilon, \epsilon) \mapsto f(x; t) \in \mathbb{R}^n, \; \epsilon > 0,$$

such that $f(\cdot; t) : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism of $\mathbb{R}^n$ for each fixed $|t| < \epsilon$. In particular, we say that $\{f(\cdot; t)\}_{|t|<\epsilon}$ is a local variation in Ω *if it defines a one-parameter family of diffeomorphisms such that*

$$\begin{aligned} f(x; 0) = x \quad &\textit{for all } x \in \mathbb{R}^n, \\ \{x \in \mathbb{R}^n : f(x; t) \neq x\} \subset\subset \Omega \quad &\textit{for all } 0 < |t| < \epsilon. \end{aligned}$$

It follows from the previous definition that given a local variation $\{f(\cdot; t)\}_{|t|<\epsilon}$ in Ω, then

$$E \Delta f(E; t) \subset\subset \Omega \quad \text{ for all } E \subset \mathbb{R}^n.$$

Moreover, one can show that there exists a compactly supported smooth vector field $T \in C_c^\infty(\Omega; \mathbb{R}^n)$ such that the following expansions hold in $\mathbb{R}^n$,

$$f(x; t) = x + T(x) + O(t^2), \quad \nabla f(x; t) = \text{Id} + t\nabla T(x) + O(t^2), \tag{2.3.1}$$

where ∇ denotes the derivative with respect to x, and T satisfies

$$T(x) = f_t(x; 0) \quad x \in \mathbb{R}^n,$$

with $f_t(x; 0)$ standing for the derivative of f with respect to t evaluated at $(x; 0)$.

Definition 2.3.2. *The smooth vector field T in* (2.3.1) *is called the* initial velocity *of* $\{f(\cdot; t)\}_{|t|<\epsilon}$.

The following result gives an explicit expression for the *first variation of the perimeter* of a set E, relative to Ω, with respect to local variations $\{f(\cdot;t)\}_{|t|<\epsilon}$ in Ω, that is, a formula for

$$\frac{d}{dt}\Big|_{t=0} P(f(E;t);\Omega) \quad \text{for } T \in C_c^\infty(\Omega;\mathbb{R}^n) \text{ given.}$$

Theorem 2.3.3 (First variation of perimeter)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, let E be a set of locally finite perimeter, and let $\{f(\cdot;t)\}_{|t|<\epsilon}$ be a local variation in Ω. Then*

$$P(f(E;t);\Omega) = P(E;\Omega) + t\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} + O(t^2), \tag{2.3.2}$$

where T is the initial velocity of $\{f(\cdot;t)\}_{|t|<\epsilon}$ and $\mathrm{div}_E T : \partial^* E \to \mathbb{R}$, defined by

$$\mathrm{div}_E T(x) := \mathrm{div} T - \nu_E(x) \cdot \nabla T(x) \nabla_E(x),\ x \in \partial^* E, \tag{2.3.3}$$

is a Borel function called the boundary divergence of T on E.

In the case of volume–constrained perimeter minimizers, the following holds.

Theorem 2.3.4 (Constant Mean Curvature)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $E_0 \subset \Omega$ be a volume–constrained perimeter minimizer in the open set Ω. Then there exists $\lambda_0 \in \mathbb{R}$ such that*

$$\int_{\partial^* E} \mathrm{div}_E T \, d\mathcal{H}^{n-1} = \lambda_0 \int_{\partial^* E} (T \cdot \nu_E) \, d\mathcal{H}^{n-1} \quad \textit{for all } T \in C_c^\infty(\Omega;\mathbb{R}^n).$$

In particular, E_0 has distributional mean curvature in Ω constantly equal to λ_0, and we denote $\kappa_{E_0} := \lambda_0$.

In order to characterize the second variation for the perimeter of open, regular sets, we need to introduce some preliminary tools.

Proposition 2.3.5. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $E \subset \Omega$ be an open set such that $\partial E \cap \Omega$ is C^2. Then there exists an open set Ω' with $\Omega \cap \partial E \subset \Omega' \subset \Omega$ such that the signed distance function $s_E : \mathbb{R}^n \to \mathbb{R}$ of E,*

$$s_E(x) := \begin{cases} \mathrm{dist}(x, \partial E) & \textit{if } x \in \mathbb{R}^n \setminus E, \\ -\mathrm{dist}(x, \partial E) & \textit{if } x \in E, \end{cases}$$

satisfies $s_E \in C^2(\Omega')$.

The previous result allows us to define a vector field $N_E \in C^1(\Omega';\mathbb{R}^n)$ and a tensor field $A_E \in C^0(\Omega';\mathbf{Sym}(n))$ via

$$N_E := \nabla s_E, \quad A_E := \nabla^2 s_E \ \text{ on } \Omega'. \tag{2.3.4}$$

In particular, one can show that for every $x \in \Omega \cap \partial E$ there exist $r > 0$, vector fields $\{\tau_h\}_{h=1}^{n-1} \subset C^1(B_r(x); S^{n-1})$, and functions $\{\kappa_h\}_{h=1}^{n-1} \subset C^0(B_r(x))$, such that $\{\tau_h\}_{h=1}^{n-1}$ is an orthonormal basis of $T_y\partial E$ for every $y \in B_r(x) \cap \partial E$, $\{\tau_h\}_{h=1}^{n-1} \cup \{N_E(y)\}$ is an orthonormal basis of $\mathbb{R}^n$ for every $y \in B_r(x)$, and

$$A_E(y) = \sum_{h=1}^{n-1} \kappa_h(y)\tau_h(y) \otimes \tau_h(y) \text{ for all } y \in B_r(x).$$

Definition 2.3.6. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $E \subset \Omega$ be an open set such that $\partial E \cap \Omega$ is C^2. For any $y \in B_r(x) \cap \partial E$, $A_E(y)$ (seen as symmetric tensor on $T_y\partial E \otimes T_y\partial E$) is called the* second fundamental form of ∂E at y, *while $\{\tau_h\}_{h=1}^{n-1} \subset S^{n-1} \cap T_y\partial E$ and $\{\kappa_h\}_{h=1}^{n-1}$ are denoted the* principal directions *and the* principal curvatures *of ∂E at y.*

We recall that for any matrix $\mathfrak{M}$ the *Frobenius norm*, which we will write $|\mathfrak{M}|$, is given by

$$|\mathfrak{M}| := \sqrt{\sum_i \sum_j |\mathfrak{M}_{ij}|^2} \tag{2.3.5}$$

Proposition 2.3.7. *Let $\Omega \subset \mathbb{R}^n$ be open, and let $E \subset \Omega$ be an open set such that $\partial E \cap \Omega$ is C^2. The* scalar mean curvature *κ_E of the C^2–hypersurface $\Omega \cap \partial E$ is locally representable as*

$$\kappa_E(y) = \sum_{h=1}^{n-1} \kappa_h(y) \textit{ for all } y \in B_r(x) \cap \partial E,$$

while the second fundamental form satisfies

$$|A_E(y)|^2 = \sum_{h=1}^{n-1} (\kappa_h(y))^2 \textit{ for all } y \in B_r(x) \cap \partial E.$$

We now state the following theorem.

Theorem 2.3.8 (Second variation of perimeter)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, let $E \subset \Omega$ be an open set such that $\partial E \cap \Omega$ is C^2, $\zeta \in C_c^\infty(\Omega)$, and let $\{f(\cdot;t)\}_{|t|<\epsilon}$ be a local variation associated with the normal vector field $T = \zeta N_E \in C_c^1(\Omega;\mathbb{R}^n)$. Then*

$$\frac{d^2}{dt^2}\Big|_{t=0} P(f(E;t);\Omega) = \int_{\partial E} |\nabla_E \zeta|^2 + \left(\kappa_E^2 - |A_E|^2\right)\zeta^2 d\mathcal{H}^{n-1},$$

where $\nabla_E\zeta := \nabla\zeta - (\nu_E \cdot \nabla\zeta)\nu_E$ denotes the tangential gradient of ζ with respect to the boundary of E.

We will say that E has *positive second variation* if

$$\frac{d^2}{dt^2}\Big|_{t=0} P(f(E;t);\Omega) > 0 \tag{2.3.6}$$

for every local variation $\{f(\cdot;t)\}_{|t|<\epsilon}$.

We conclude this section with the following version of the divergence theorem, see, e.g., [56], Theorem 11.8 and equation 11.14.

Theorem 2.3.9. *Let $M \subset \mathbb{R}^n$ be a C^2–hypersurface with boundary Γ. Then there exists a normal vector field $\mathbf{H}_M \in C(M;\mathbb{R}^n)$ to M and a normal vector field $\nu_\Gamma^M \in C^1(\Gamma; S^{n-1})$ to Γ such that for every $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$*

$$\int_M \mathrm{div}_M T d\mathcal{H}^{n-1} = \int_M T \cdot \mathbf{H}_M d\mathcal{H}^{n-1} + \int_\Gamma (T \cdot \nu_\Gamma^M) d\mathcal{H}^{n-2},$$

where $\mathbf{H}_M$ is the mean curvature vector *to M and $\mathrm{div}_M T$ is the* tangential divergence *of T on M, defined by*

$$\mathrm{div}_M T := \mathrm{div} T - (\nabla T \nu_M) \cdot \nu_M = \mathrm{trace}(\nabla^M T), \tag{2.3.7}$$

with $\nu_M : M \to S^{n-1}$ being any unit normal vector field to M.

2.4 Regularity of solutions and gradient flows

We start by recalling results about the regularity of solutions of (1.2.1) and (1.2.10), respectively. In the case of the nonlocal Allen–Cahn equation, we follow [62]: assume that Ω and W satisfy (2.1.1)–(2.1.6) and let $s_1 < s_2$ be two arbitrarily chosen constants such that

$$W'(s_2) < W'(s) < W'(s_1),$$

for all $s \in (s_1, s_2)$. Furthermore, assume that the initial data u_0^ε in (1.2.1) satisfy

$$u_0^\varepsilon \in L^2(\Omega) \text{ and } s_1 \le u_0^\varepsilon \le s_2 \text{ a.e. in } \Omega, \tag{2.4.1}$$

and set

$$\Omega_T := \Omega \times (0, T).$$

Then the following holds.

Theorem 2.4.1 ([62], Theorem 1.1.1)**.** *Fix $\varepsilon > 0$, let Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, $n \ge 2$ and assume that* (2.4.1) *holds. Then the problem* (1.2.1) *admits a solution $u^\varepsilon \in C([0,\infty); L^2(\Omega))$ such that, for every $T > 0$,*

$$u^\varepsilon \in L^\infty(\Omega_T) \cap L^2(0;T;H^1(\Omega)) \textit{ and } u_t^\varepsilon \in L^2(0,T;(H^1(\Omega))').$$

Moreover, $u^\varepsilon \in C^\infty(\overline{\Omega} \times (0,\infty))$,

$$s_1 \le u^\varepsilon(x,t) \le s_2 \textit{ for all } x \in \overline{\Omega} \textit{ and all } t > 0.$$

The variational approach we will follow throughout this thesis relies on the concept of gradient flow of a given energy. In the case of the nonlocal Allen–Cahn equation, we notice that integrating (1.2.1) with respect to x gives

$$\begin{aligned} 0 &= \frac{d}{dt}\int_\Omega u^\varepsilon \, dx - \int_\Omega \left(-\varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} W'(u^\varepsilon) - \lambda_\varepsilon \right) dx \\ &= \frac{d}{dt}\int_\Omega u^\varepsilon \, dx - \int_\Omega \left(\frac{1}{\varepsilon} W'(u^\varepsilon) - \lambda_\varepsilon \right) dx = \frac{d}{dt}\int_\Omega u^\varepsilon \, dx, \end{aligned} \tag{2.4.2}$$

where we have used the Neumann boundary conditions, see (1.2.1). In other words, (2.4.2) is highlighting the fact that solutions of the nonlocal Allen–Cahn equation preserve the volume, thanks to the presence of the Lagrange multiplier λ_ε. Moreover, the regularity results of Theorem 2.4.1 allow us to remark that multiplying the nonlocal Allen–Cahn equation by u_t and integrating by parts, using boundary conditions and the volume preserving condition (2.4.2), gives

$$\mathcal{G}_\varepsilon[u^\varepsilon](0) - \mathcal{G}_\varepsilon[u^\varepsilon](T) = \varepsilon^{-1} \int_0^T ||u_t^\varepsilon(s)||_{L^2}^2 \, ds, \tag{2.4.3}$$

for any $T > 0$, which is precisely what we mean when we say that (1.2.1) has a *gradient flow structure*. It is very important to recall that our energy (1.1.15) is slightly different from the *unconstrained* version of the energy that is used in [12], [8], as those works consider the classical Allen–Cahn equation (1.1.1).

In the case of the Cahn–Hilliard equation, we define the space

$$H_N^2(\Omega) := \{w \in H^2(\Omega) : \nu_{\partial\Omega} \cdot \nabla v = 0 \text{ on } \partial\Omega\},$$

where $\nu_{\partial\Omega}$ denotes the exterior normal to the boundary of Ω. The following regularity result was proved in [33]. See also [63] for a refined version, and Chapter 4 in [28].

Theorem 2.4.2. *Fix $\varepsilon > 0$, let Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, for $n \leq 2$, and assume that $u_0^\varepsilon \in H_N^2(\Omega)$. Then for any $T > 0$ there exists a unique global solution u^ε of* (1.2.10) *such that*

$$u^\varepsilon \in H^{4,1}(\Omega_T).$$

Theorem 2.4.3. *Fix $\varepsilon > 0$, let Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, for $n \leq 3$, and assume that $u_0^\varepsilon \in L^2(\Omega)$. Then there exists a unique solution u^ε to* (1.2.10) *such that for all $T > 0$*

$$u^\varepsilon \in C([0;T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap L^4(0,T; L^4(\Omega)).$$

Moreover, if $u_0^\varepsilon \in H_N^2(\Omega)$, then for all $T > 0$,

$$u^\varepsilon \in C([0,T]; H_N^2(\Omega)) \cap L^2(0,T; \mathcal{D}(\mathcal{A}^2)),$$

where $\mathcal{D}(\cdot)$ stands for the domain of a given operator, while $\mathcal{A}$ is the Laplacian with Neumann boundary conditions.

Furthermore, (1.1.2) can be seen as the gradient flow with respect to a variant of $(H^1(\Omega))'$ of the energy $\mathcal{G}_\varepsilon$. To be precise, the following approach is standard in studying the Cahn–Hilliard equation (see, e.g., [52]): let $\langle \ , \ \rangle$ denote the dual pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$, and recall that for every $f \in (H^1(\Omega))'$ there is a $g \in H^1(\Omega)$ such that

$$\langle f, \varphi \rangle = \int_\Omega \nabla g \cdot \nabla \varphi \, dx \text{ for all } \varphi \in H^1(\Omega).$$

As the function g is unique, up to an additive constant, we denote by $-\Delta_{X_2}^{-1} f$ the function g with 0 mean over Ω. We then define the inner product

$$\langle u, v\rangle_{X_2} := \int_\Omega \nabla(\Delta_{X_2}^{-1} u) \cdot \nabla(\Delta_{X_2}^{-1} v)\, dx \qquad \text{for } u, v \in (H^1(\Omega))',$$

so that $X_2 := ((H^1(\Omega))', \langle\ ,\ \rangle_{X_2})$ is a Hilbert space. After rescaling time by ε, we get

$$u_t^\varepsilon = -\nabla_{X_2} \mathcal{G}_\varepsilon(u^\varepsilon),$$

where

$$\nabla_{X_2} \mathcal{G}_\varepsilon(u) = -\Delta(-\varepsilon^2 \Delta u + W'(u)).$$

In particular, in this case we have

$$\mathcal{G}_\varepsilon[u^\varepsilon](0) - \mathcal{G}_\varepsilon[u^\varepsilon](T) = \varepsilon^{-1} \int_0^T ||u_t^\varepsilon(s)||_{X_2}^2\, ds. \tag{2.4.4}$$

2.5 Γ–convergence and interpolation inequalities

In this section we recall some properties of the energy

$$E_\varepsilon(u; \Omega) := \int_\Omega \left(\frac{1}{\varepsilon} W(u) - \varepsilon q |\nabla u|^2 + \varepsilon^3 |\nabla^2 u|^2 \right) dx, \tag{2.5.1}$$

in the more general setting where Ω is a bounded open set of $\mathbb{R}^n$ with C^1 boundary, $q > 0$ is a small parameter, and W is a double–well potential, as in (2.1.10). In [21] Chermisi, Dal Maso, Fonseca and Leoni proved that the sequence of functionals $\mathscr{E}_\varepsilon : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\mathscr{E}_\varepsilon(u) := \begin{cases} E_\varepsilon(u; \Omega) & \text{if } u \in H^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus H^2(\Omega), \end{cases}$$

Γ–converges as $\varepsilon \to 0^+$ to the functional $\mathscr{E}_0 : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathscr{E}_0(u) := \begin{cases} m_n P(\{u = 1\}; \Omega) & \text{if } u \in BV(\Omega; \{-1, +1\}), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega; \{-1, +1\}), \end{cases}$$

where

$$m_n := \inf \{E_\varepsilon(u; Q) : 0 < \varepsilon \le 1, u \in \mathcal{A}_n\},$$

$Q := \left(-\frac{1}{2}, \frac{1}{2}\right)^n$, and

$$\begin{aligned} \mathcal{A}_n := \Big\{ u \in & H^2_{\text{loc}}(\mathbb{R}^n),\ u(x) = -1 \text{ near } x \cdot e_n = -\frac{1}{2}, \\ & u(x) = 1 \text{ near } x \cdot e_n = \frac{1}{2}, \\ & u(x) = u(x + e_i) \text{ for all } x \in \mathbb{R}^n, i = 1, \dots, n-1 \Big\}. \end{aligned}$$

We define the one–dimensional rescaled energy

$$E(v;A) := \int_A \left(W(v) - q|v'|^2 + |v''|^2\right) dx, \tag{2.5.2}$$

and we introduce the set of admissible functions

$$\mathcal{A} := \left\{v \in H^2_{\text{loc}}(\mathbb{R}) : v(x) = -1 \text{ near } x = a, v(x) = 1 \text{ near } x = b\right\}. \tag{2.5.3}$$

It was proved in [21], Section 5.1, that

$$m_1 = \inf\left\{E(v;\mathbb{R}) : v \in H^2_{\text{loc}}(\mathbb{R}), \lim_{x\to\pm\infty} v(x) = \pm 1\right\}, \tag{2.5.4}$$

so that in dimension $n = 1$ we have

$$\mathscr{E}_0(u) = \begin{cases} Nm_1 & \text{if } u \in BV((a,b);\{-1,+1\}), \\ +\infty & \text{if } u \in L^2((a,b)) \setminus BV((a,b);\{-1,+1\}), \end{cases}$$

where N is the number of jumps of the function u. We further define

$$\begin{aligned} m_\pm &:= \inf\left\{E(u;\mathbb{R}^+) : u \in H^2_{\text{loc}}(\mathbb{R}^+), \lim_{x\to\infty} u(x) = \pm 1, \quad u(0) = 0\right\} \\ &= \inf\left\{E(u;\mathbb{R}^-) : u \in H^2_{\text{loc}}(\mathbb{R}^-), \lim_{x\to-\infty} u(x) = \pm 1, \quad u(0) = 0\right\} \end{aligned} \tag{2.5.5}$$

and remark that in our case of a symmetric potential W, $m_+ = m_- = m_1/2$. One of the key tools to prove the Γ–convergence result is the following nonlinear interpolation inequality, see e.g. Theorem 3.4 in [21].

Lemma 2.5.1. *Let Ω be a bounded open set of $\mathbb{R}^n$ with C^1 boundary, and assume that W satisfies* (2.1.3)*,* (2.1.5) *and* (2.1.7)–(2.1.9)*. Then there exists a constant $q^* > 0$, independent of Ω, such that for every $-\infty < q < q^*/N$ there exists $\varepsilon_0 = \varepsilon_0(\Omega, q) > 0$ such that*

$$q\varepsilon^2 \int_\Omega |\nabla u|^2 dx \le \int_\Omega W(u)dx + \varepsilon^4 \int_\Omega |\nabla^2 u|^2 dx$$

for every $\varepsilon \in (0, \varepsilon_0)$ and every $u \in H^2(\Omega)$.

In particular, in the one dimensional setting we will often use the following nonrescaled version of the previous result, see Lemma 3.1 in [23].

Lemma 2.5.2. *Let W be a continuous potential satisfying* (2.1.3)*,* (2.1.5) *and* (2.1.8)*. Let $I \subset \mathbb{R}$ be an open, bounded interval. Then there exists a constant $q^* > 0$ such that*

$$q^* \int_I (u')^2 dx \le \frac{1}{\mathcal{L}^1(I)^2} \int_I W(u)dx + \mathcal{L}^1(I)^2 \int_I (u'')^2 dx$$

for every $u \in H^2(I)$.

2.6 Smooth linearization near the hyperbolic fixed point

In the proof of Lemma 4.1.4 we use the fact that in a sufficiently small neighborhood of the fixed point x_0 of the system (4.1.19), F admits a C^1 linearization. This variant of the classical Hartman–Grobman Theorem is based on the concept of Q–smoothness of the Jacobian matrix $DF(x_0)$ introduced in [72]. Following [72], we define

$$\gamma(\lambda; m) := \lambda - \sum_{i=1}^{4} m_i r_i, \text{ for } \lambda \in \mathbb{C},\ m_i \in \mathbb{N}_0, \tag{2.6.1}$$

where r_i are the eigenvalues in (4.1.46).

Definition 2.6.1. *A matrix A is said to satisfy the* Sternberg condition of order N, $N \geq 2$*, if*

$$\gamma(\lambda; m) \neq 0, \textit{ for all } \lambda \in \Sigma(A), \textit{and for all } m \textit{ such that } 2 \leq |m| \leq N, \tag{2.6.2}$$

where $|m| := \sum m_i$. We will say that A satisfies the strong Sternberg condition of order N*, if A satisfies* (2.6.2) *and*

$$\mathrm{Re}\gamma(\lambda; m) \neq 0, \tag{2.6.3}$$

for all $\lambda \in \Sigma(A)$ and all m such that $|m| = N$.

Definition 2.6.2. *Let $\Sigma^+(A)$ and $\Sigma^-(A)$ be the set of eigenvalues of A having positive and negative real part respectively. A is said to be* strictly hyperbolic *if*

$$\Sigma^+(A) \neq \emptyset, \quad \Sigma^-(A) \neq \emptyset.$$

The spectral spread *of A is defined by*

$$\rho^j := \frac{\max\{|\mathrm{Re}\lambda| : \lambda \in \Sigma^j(A)\}}{\min\{|\mathrm{Re}\lambda| : \lambda \in \Sigma^j(A)\}},$$

for $j = \pm$.

Definition 2.6.3. *Let $Q \in \mathbb{N}$ and A be hyperbolic. The Q–*smoothness *of A is the largest integer $K \geq 0$ such that*

(i) $Q - K\rho^- \geq 0$, if $\Sigma^+(A) = \emptyset$;

(ii) $Q - K\rho^+ \geq 0$, if $\Sigma^-(A) = \emptyset$;

(iii) there exist $M, N \in \mathbb{N}$ with $Q = M + N$ and $M - K\rho^+ \geq 0$, $N - K\rho^- \geq 0$, when A is strictly hyperbolic.

The following theorem is proved in [72] (Theorem 1, page 4).

Theorem 2.6.4. *Let X be a finite dimensional Banach space. Let $Q \geq 2$ be an integer. Assume that G is of class C^{3Q} on $U \subset X$ with $0 \in U$, where $D^p G(0) = 0$ for $p = 0, 1$. Let A be strictly hyperbolic and assume it satisfies the strong Sternberg condition of order Q. Then*

$$x' = Ax + G(x) \tag{2.6.4}$$

admits a C^K–linearization, where K is the Q–smoothness of A. In other words, there exists a C^K–diffeomorphism between solutions of (2.6.4) *and solutions of its linear part.*

In fact, as remarked in [72], in the case of A strictly hyperbolic it suffices to assume that G is of class $C^{Q+\max(M,N)+K}$. In the remainder, we show that under the assumptions of Lemma 4.1.4, the matrix $DF(0)$ satisfies the strong Sternberg condition of order $N = 2$ and the 2-smoothness of $DF(0)$ is $K = 1$.

Lemma 2.6.1. *Consider the ordinary differential equation*

$$x' = F(x), \tag{2.6.5}$$

where F is a C^4 mapping $\mathbb{R}^4 \to \mathbb{R}^4$ satisfying $F(0) = 0$. Assume the linearization $DF(0)$ has four eigenvalues $\pm\gamma \pm \delta i$, where $\gamma \geq \lambda > 0$. Then, the matrix $DF(0)$ satisfies the strong Sternberg condition of order $N = 2$. Moreover, the Q–smoothness of $DF(0)$ is $K = 1$, and (4.1.19) *admits a C^1–linearization around the hyperbolic fixed point* 0.

Proof. We write (2.6.5) as

$$x' = DF(0)x + G, \tag{2.6.6}$$

where $G(x) := F(x) - DF(0)x$ is of class C^4, $G(0) = F(0) = 0$, $DG(0) = DF(0) - DF(0) = 0$ and show that (2.6.2) and (2.6.3) hold for $N = 2$. Recalling (2.6.1), we have

$$\gamma(r_1; m) = (1 - m_1)r_1 - m_2 r_2 - m_3 r_3 - m_4 r_4, \tag{2.6.7}$$

where $|m| = \sum_{i=1}^{4} m_i = 2$ and $r_1 := \gamma + \delta i, r_2 := \gamma - \delta i, r_3 := -\gamma + \delta i, r_4 := -\gamma - \delta i$ are the eigenvalues of $DF(0)$. Assume, for the sake of contradiction, that $\mathrm{Re}\gamma(r_1; m) = 0$ with $|m| = 2$. Setting the real part of (2.6.7) to 0 and recalling that $|m| = 2$, we have

$$\begin{cases} 1 - m_1 - m_2 + m_3 + m_4 = 0, \\ m_1 + m_2 + m_3 + m_4 = 2, \end{cases} \tag{2.6.8}$$

Adding the two equations and dividing by two, one has

$$m_3 + m_4 = 1/2, \tag{2.6.9}$$

a contradiction since m_3 and m_4 are integers. A similar argument for any $\lambda \in \Sigma(Df(0))$ shows that (2.6.3) and (2.6.2) hold for the matrix $Df(0)$, and $N = 2$.

It remains to prove that the 2-smoothness of $DF(0)$ is $K = 1$. Since $|\mathrm{Re}\lambda| = \gamma$, for all $\lambda \in \Sigma(Df(0))$, then the spectral radius of $Df(0)$ is $\rho^i = 1$, for $i = \pm$. Being $Df(0)$

strictly hyperbolic, we are in case (iii) of Definition 2.6.3 and $Q = 2$ implies $M = N = 1$. In turn, the largest integer K that satisfies

$$\begin{cases} M - K\rho^{+} = 1 - K \geq 0, \\ N - K\rho^{-} = 1 - K \geq 0, \end{cases} \tag{2.6.10}$$

is $K = 1$, which is then the 2–smoothness of $Df(0)$. We now apply Theorem 2.6.4 with $Q = 2$ and $A = DF(0)$ to conclude that (2.6.6) admits a C^1–linearization. $\square$

Chapter 3

Allen-Cahn & Cahn-Hilliard

3.1 Energy estimates and slow motion

This section is devoted to the study of the motion of solutions for both the nonlocal Allen–Cahn equation (1.2.1) and the Cahn–Hilliard equation (1.2.10). We start by proving Theorem 1.2.2 and Theorem 1.2.4, and subsequently we study solutions of the nonlocal Allen–Cahn equation whose initial data is close to a configuration that locally minimizes the perimeter of the interface, by proving Theorem 1.2.8. In the latter, we make use of a new local version of the well–known isoperimetric function, whose regularity properties will be investigated in Section 3.2.

3.1.1 Slow motion near global perimeter minimizers

Due to the fact that the same strategy of proof holds for both Theorem 1.2.2 and Theorem 1.2.4, we will follow the convention that $||\cdot||_X$ stands for the L^2 norm in the case of the nonlocal Allen–Cahn equation, Theorem 1.2.2 (so that in this case $X = L^2$), while $X = X_2$ in the case of Theorem 1.2.4.

Proof of Theorem 1.2.2 and Theorem 1.2.4. Fix $\varepsilon > 0$, let $M > 0$ and let $t \in [0, \varepsilon^{-1}M]$. By properties of the Bochner integral (see, e.g., [11], [30]) and Hölder's inequality

$$\begin{aligned} ||u^\varepsilon(t) - u_{E_0}||_X &\leq ||u^\varepsilon(t) - u_0^\varepsilon||_X + ||u_0^\varepsilon - u_{E_0}||_X \\ &\leq \int_0^t ||u_s^\varepsilon(s)||_X \, ds + ||u_0^\varepsilon - u_{E_0}||_X \\ &\leq t^{1/2} \left(\int_0^t ||u_s^\varepsilon(s)||_X^2 \, ds \right)^{1/2} + ||u_0^\varepsilon - u_{E_0}||_X. \end{aligned} \tag{3.1.1}$$

Since $u^\varepsilon(t) \in L^1(\Omega)$ for all $t \geq 0$ (see Theorems 2.4.1, 2.4.2, 2.4.3), we apply Theorem

1.2.1 and use the gradient flow structure (2.4.3) and (2.4.4) to obtain

$$\begin{aligned}\int_0^t ||u_s^\varepsilon(s)||_X^2\, ds &= \varepsilon\mathcal{G}_\varepsilon[u^\varepsilon](0) - \varepsilon\mathcal{G}_\varepsilon[u^\varepsilon](t) \\ &\leq \varepsilon\mathcal{G}_0[u_{E_0}] + C\varepsilon^2 - \varepsilon\mathcal{G}_0[u_{E_0}] + C(\kappa)\varepsilon^2 \leq C\varepsilon^2,\end{aligned} \tag{3.1.2}$$

where we have used (1.2.8) and (1.2.12). In turn, by (3.1.1) and (3.1.2)

$$\begin{aligned}||u^\varepsilon(t) - u_{E_0}||_X &\leq C\varepsilon t^{1/2} + ||u_0^\varepsilon - u_{E_0}||_X \\ &\leq C\varepsilon^{1/2} + ||u_0^\varepsilon - u_{E_0}||_X.\end{aligned}$$

Taking the supremum over all $t \in [0, \varepsilon^{-1}M]$ on both sides, followed by a limit as $\varepsilon \to 0^+$, and using (1.2.7) in the Allen–Cahn case, or (1.2.11) in the Cahn–Hilliard one, gives the desired result. □

Remark 3.1.1. *It follows from the previous proof that*

$$\lim_{\varepsilon\to 0^+} \sup_{0<t\leq g(\varepsilon)} ||u^\varepsilon(t) - u_{E_0}||_X = 0$$

for every decreasing function $g : (0,\infty) \to (0,\infty)$ *with*

$$\lim_{s\to 0^+} s^2 g(s) = 0.$$

In particular, we can take $g(s) := s^{\delta-2}$*, where* $\delta > 0$*. This is also true when we later study slow motion near local perimeter minimizers.*

3.1.2 Slow motion near local perimeter minimizers

Here we prove Theorem 1.2.8. In order to do so, we need to introduce some tools and prove the key energy estimate Theorem 1.2.6. Throughout this section we will assume that $\Omega \subset \mathbb{R}^n$ is as in Section 2.1 (see (2.1.1)) and that E_0 is a volume–constrained local perimeter minimizer with $\mathcal{L}^n(E_0) = r_0$, see Definition 2.2.6. Moreover, we will assume that $\mathcal{I}_\Omega^{\delta,E_0}$ admits a Taylor expantion of order 2 as in (1.2.4), at r_0, for some $\delta > 0$. We remark that Theorem 1.2.9 and Theorem 1.2.10 are two cases where we will prove the validity of the last assumption, as long as δ is sufficiently small (see Section 2.4). For simplicity, we write $\mathcal{I}^\delta$ in place of $\mathcal{I}_\Omega^{\delta,E_0}$.

By Proposition 3.1 in [55], we may select a function $\mathcal{I}^* \in C^{1,\varsigma}_{\text{loc}}((0,1))$ satisfying

$$\mathcal{I}^*(r_0) = \mathcal{I}^\delta(r_0), \tag{3.1.3}$$

$$0 \leq \mathcal{I}^*(r) \leq \mathcal{I}^\delta(r) \text{ for all } r \in (0,1), \tag{3.1.4}$$

$$\mathcal{I}^*(r) \geq \min\left\{Cr^{\frac{n-1}{n}}, C(1-r)^{\frac{n-1}{n}}\right\} \text{ for all } r \in (0,1), \tag{3.1.5}$$

for some $C > 0$, where ς is given in (1.2.4).

After extending $\mathcal{I}^*$ to be zero outside of $(0,1)$, we define the function V_Ω via the initial value problem

$$\begin{cases} \dfrac{d}{ds} V_\Omega(s) = \mathcal{I}^*(V_\Omega(s)), \\ V_\Omega(0) = \dfrac{1}{2}. \end{cases}$$

Remark 3.1.2. *Using* (3.1.5)*, and as* $0 < \frac{n-1}{n} < 1$*, a straightforward argument gives that there exist* $S_1, S_2 > 0$ *finite, such that* $V_\Omega(s) \in (0,1)$ *for all* $s \in (-S_1, S_2)$ *and* $V_\Omega(s) \notin (0,1)$ *otherwise.*

Definition 3.1.3. *Let* $u \in L^1(\Omega)$*. For* $s \in \mathbb{R}$ *we denote*

$$\eta(s) := \mathcal{I}^*(V_\Omega(s)), \quad \varrho(s) := \mathcal{L}^n(\{u < s\}),$$

and define the increasing rearrangement *of* u *by*

$$f_u(s) := \sup\{z : \varrho(z) < V_\Omega(s)\}.$$

We remark that our definitions of ϱ and f_u differ from [55], and from other standard sources on rearrangements, in the direction of our inequalities. In particular, we are choosing to construct an increasing rearrangement, as opposed to a decreasing one. In the case where η is symmetric there is no difference between using an increasing or decreasing rearrangement (see [55] Remark 3.11). Since $\mathcal{I}_\Omega^{\delta,E_0}$ is not symmetric in general, in our case η may not be symmetric either. However, the arguments for the increasing rearrangement do not differ from the decreasing one in our case (see Remark 3.11 in [55]).

Definition 3.1.4. *Let* $I \subset \mathbb{R}$ *be an open, bounded interval and consider the function* η *in Definition 3.1.3. We denote the weighted spaces with weight* η *as*

$$L^1_\eta(I) := L^1(I;\eta), \quad H^1_\eta(I) := H^1(I;\eta),$$

endowed with the norms

$$\|u\|_{L^1_\eta} = \int_I |u(s)|\eta(s)\,ds, \quad \|u\|_{H^1_\eta} = \left(\int_I u(s)^2\eta(s)\,ds\right)^{1/2} + \left(\int_I u'(s)^2\eta(s)\,ds\right)^{1/2},$$

respectively.

We give the following auxiliary result.

Lemma 3.1.5. *Assume that* Ω, W, m *satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and let* $E_0 \subset \Omega$ *be a volume–constrained local perimeter minimizer. Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a Borel function,* $u \in L^1(\Omega)$*,* S_1, S_2 *be as in Remark 3.1.2. Fix* $\delta > 0$ *and suppose that* $\mathcal{I}_\Omega^{\delta,E_0}$ *admits a Taylor expansion of order two at* r_0 *as in* (1.2.4)*. Then*

$$\int_\Omega \psi(u(x))\,dx = \int_{-S_1}^{S_2} \psi(f_u(s))\eta(s)\,ds, \tag{3.1.6}$$

provided the integral on the right hand side of (3.1.6) *is well–defined. Moreover,*

$$\int_\Omega |u(x) - w(x)|\,dx \geq \int_{-S_1}^{S_2} |f_u(s) - f_w(s)|\eta(s)\,ds \tag{3.1.7}$$

for all $w \in L^1(\Omega)$. *Furthermore, if* $u \in W^{1,p}(\Omega)$ *for some* $1 \leq p < \infty$ *and* $\|u - u_{E_0}\|_{L^1} \leq 2\delta$*, then*

$$\int_\Omega |\nabla u(x)|^p\,dx \geq \int_{-S_1}^{S_2} |f_u'(s)|^p \eta(s)\,ds \tag{3.1.8}$$

In particular, it follows that if $\|u - u_{E_0}\|_{L^1} \leq 2\delta$ *then*

$$\mathcal{G}_\varepsilon[u] \geq \int_{-S_1}^{S_2} (\varepsilon^{-1} W(f_u(s)) + \varepsilon\,(f_u'(s))^2)\eta(s)\,ds. \tag{3.1.9}$$

Proof. We will only show (3.1.8), since (3.1.6) and (3.1.7) follow from Lemma 3.3 and Proposition 3.4 in [55] (see also [26]), and (3.1.9) is a consequence of (3.1.6) and (3.1.8). By Proposition 2.2.7, for any $u \in L^1(\Omega)$ satisfying $\|u - u_{E_0}\|_{L^1} \leq 2\delta$, we have

$$\alpha(E_0, \{u \leq s\}) \leq \delta$$

for every $s \in \mathbb{R}$, (see (2.2.6)). In turn, by definition of $\mathcal{I}^\delta$ (see (1.2.14)) we get

$$\mathcal{I}^\delta(\mathcal{L}^n(\{u \leq s\})) \leq P(\{u \leq s\}; \Omega) \quad \text{for } \mathcal{L}^1\text{–a.e. } s \in \mathbb{R}. \tag{3.1.10}$$

In particular, since Ω has finite measure, (3.1.10) holds true for any function in $W^{1,p}(\Omega)$. Since the proofs of Lemma 3.3, Proposition 3.4 and Theorem 3.10 in [55] only rely on properties (3.1.4)–(3.1.5) and (3.1.10), which are shared by $\mathcal{I}_\Omega$ and $\mathcal{I}^\delta$, the same results hold true if we replace $\mathcal{I}_\Omega$ with $\mathcal{I}^\delta$. We omit the details. □

We consider the functional $\mathcal{F}_\varepsilon : L^1_\eta((-S_1, S_2)) \to [0, \infty]$ defined by

$$\mathcal{F}_\varepsilon[f] := \begin{cases} \int_{-S_1}^{S_2} (\varepsilon^{-1}(W \circ f) + \varepsilon (f')^2)\eta\,ds & \text{if } f \in \hat{H}^1_\eta((-S_1, S_2)), \\ \infty & \text{otherwise} \end{cases}$$

where $\hat{H}^1_\eta((-S_1, S_2))$ is the space of functions $f \in H^1_\eta((-S_1, S_2))$ with the property that $\int_{-S_1}^{S_2} (f - f_{u_{E_0}})\eta\,ds = 0$. The following theorem is a simplified version of Theorem 4.20 from [55].

Theorem 3.1.6. *Assume that* Ω, W, m *satisfy hypotheses* (2.1.1)–(2.1.6), (2.1.9), (2.1.11), *and let* E_0 *be a volume–constrained local perimeter minimizer with* $\mathcal{L}^n(E_0) = r_0$. *Fix* $\delta > 0$ *and suppose that* $\mathcal{I}_\Omega^{\delta, E_0}$ *admits a Taylor expansion of order two at* r_0 *as in* (1.2.4), *and let* $f_0 := f_{u_{E_0}}$ *be such that*

$$f_\varepsilon \to f_0 \text{ in } L^1_\eta((-S_1, S_2)) \text{ as } \varepsilon \to 0^+.$$

Then

$$\mathcal{F}_\varepsilon[f_\varepsilon] \geq \mathcal{G}_0[u_{E_0}] - C(\kappa)\varepsilon,$$

for ε sufficiently small, where $C(\kappa)$ is a positive constant depending only on the curvature of ∂E_0.

We now prove our main energy estimate, Theorem 1.2.6.

Proof of Theorem 1.2.6. Thanks to (3.1.7), if $u^\varepsilon \to u_{E_0}$ in $L^1(\Omega)$, then $f_{u^\varepsilon} \to f_0$ in L^1_η and in light of (3.1.9),

$$\mathcal{G}_\varepsilon[u^\varepsilon] \geq \mathcal{F}_\varepsilon[f_{u^\varepsilon}],$$

for all ε sufficiently small. This, combined with Theorem 3.1.6, gives the desired result. □

The techniques we use in the remainder of this section are very similar to those found in [12] and [44]. We begin with the following auxiliary result.

Proposition 3.1.7. *Assume that Ω, W, m satisfy hypotheses* (2.1.1)–(2.1.6)*,* (2.1.9)*,* (2.1.11)*, and let E_0 be a volume–constrained local perimeter minimizer. Suppose further that $\mathcal{I}_\Omega^{\delta,E_0}$ admits a Taylor expansion of order two at r_0 as in* (1.2.4)*, for some $\delta > 0$. Assume that $u_0^\varepsilon \in X_1$ satisfy*

$$u_0^\varepsilon \to u_{E_0} \text{ in } L^1(\Omega) \text{ as } \varepsilon \to 0^+ \tag{3.1.11}$$

and

$$\mathcal{G}_\varepsilon[u_0^\varepsilon] \leq \mathcal{G}_0[u_{E_0}] + C\varepsilon \tag{3.1.12}$$

for some $C > 0$. Then there exist two positive constants k_1 and k_2, not depending on ε, such that

$$\int_0^{k_1\varepsilon^{-2}} ||u_t^\varepsilon(t)||_{L^2}^2 \, dt \leq k_2\varepsilon^2, \tag{3.1.13}$$

where u^ε is the solution of (1.2.1)*.*

Proof. By the gradient flow structure (2.4.3), for any $T > 0$ we have

$$\mathcal{G}_\varepsilon[u_0^\varepsilon] - \mathcal{G}_\varepsilon[u^\varepsilon](T) = \varepsilon^{-1} \int_0^T ||u_t^\varepsilon(s)||_{L^2}^2 \, ds, \tag{3.1.14}$$

which shows that $t \mapsto \mathcal{G}_\varepsilon(u^\varepsilon)(t)$ is decreasing and $||u_t^\varepsilon||_{L^2}^2$ is integrable. Given δ as in the assumptions, then by (3.1.11),

$$||u_0^\varepsilon - u_{E_0}||_{L^1} \leq \delta$$

for ε sufficiently small. Now suppose that there exists $T_\varepsilon > 0$ small enough that

$$\int_0^{T_\varepsilon} ||u_t^\varepsilon(t)||_{L^1} \, dt \leq \delta. \tag{3.1.15}$$

Then,
$$\delta \geq \int_0^{T_\varepsilon} ||u_t^\varepsilon(t)||_{L^1}\, dt \geq \left|\left|\int_0^{T_\varepsilon} u_t^\varepsilon(t)\, dt\right|\right|_{L^1} = ||u^\varepsilon(T_\varepsilon) - u_0^\varepsilon||_{L^1},$$
so that
$$||u^\varepsilon(T_\varepsilon) - u_{E_0}||_{L^1} \leq ||u^\varepsilon(T_\varepsilon) - u_0^\varepsilon||_{L^1} + ||u_0^\varepsilon - u_{E_0}||_{L^1} \leq 2\delta \tag{3.1.16}$$
and, in particular, by Theorem 1.2.6,
$$\mathcal{G}_\varepsilon[u^\varepsilon](T_\varepsilon) \geq \mathcal{G}_0[u_{E_0}] - C(\kappa)\varepsilon. \tag{3.1.17}$$
By (3.1.12) and (3.1.17) together with (3.1.14),
$$\begin{aligned}\int_0^{T_\varepsilon} ||u_t^\varepsilon(s)||_{L^2}^2\, ds &= \varepsilon\mathcal{G}_\varepsilon[u_0^\varepsilon] - \varepsilon\mathcal{G}_\varepsilon[u^\varepsilon](T_\varepsilon) \\ &\leq \varepsilon\mathcal{G}_0[u_{E_0}] + C\varepsilon^2 - \varepsilon\mathcal{G}_0[u_{E_0}] \leq C\varepsilon^2.\end{aligned} \tag{3.1.18}$$
In turn, by Hölder's inequality we get
$$\left(\int_0^{T_\varepsilon} ||u_t^\varepsilon(t)||_{L^1}\, dt\right)^2 \leq CT_\varepsilon\varepsilon^2,$$
so that
$$T_\varepsilon \geq \frac{1}{C\varepsilon^2}\left(\int_0^{T_\varepsilon} ||u_t^\varepsilon(t)||_{L^1}\, dt\right)^2. \tag{3.1.19}$$
In order to conclude the proof, we need to make sure that it is always possible to choose T_ε as in (3.1.15) and that $T_\varepsilon \geq k_1\varepsilon^{-2}$ for some $k_1 > 0$. We argue as follows: suppose first that
$$\int_0^\infty ||u_t^\varepsilon(t)||_{L^1}\, dt > \delta.$$
Then by continuity we can choose $T_\varepsilon > 0$ such that
$$\int_0^{T_\varepsilon} ||u_t^\varepsilon(t)||_{L^1}\, dt = \delta,$$
and for such a choice of T_ε, (3.1.19) gives
$$T_\varepsilon \geq \frac{\delta^2}{C\varepsilon^2}.$$
Thus, by (3.1.18),
$$\int_0^{k_1\varepsilon^{-2}} ||u_t^\varepsilon(s)||_{L^2}^2\, ds \leq C\varepsilon^2 =: k_2\varepsilon^2, \tag{3.1.20}$$

for

$$k_1 := \frac{\delta^2}{C}.$$

On the other hand, if

$$\int_0^\infty ||u_t^\varepsilon(t)||_{L^1}\,dt \le \delta,$$

then (3.1.18) must hold for all $T_\varepsilon > 0$, and (3.1.20) holds true in this case as well. □

We are now ready to prove the main result.

Proof of Theorem 1.2.8. Let k_1, k_2 be as in Proposition 3.1.7, and rescale u^ε by setting $\tilde{u}^\varepsilon(x,t) = u^\varepsilon(x, \varepsilon^{-1}t)$. Proposition 3.1.7 applied to $\tilde{u}^\varepsilon$ reads

$$\int_0^{k_1\varepsilon^{-1}} ||\tilde{u}_t^\varepsilon(t)||_{L^2}^2\,dt \le k_2\varepsilon,$$

and, in turn, by Hölder's inequality, for $0 < M < k_1\varepsilon^{-1}$,

$$\int_0^M ||\tilde{u}_t^\varepsilon(t)||_{L^1}\,dt \le M^{1/2}(k_2\varepsilon)^{1/2}. \tag{3.1.21}$$

For any $0 < s < M$, by the properties of the Bochner integral we have

$$\begin{aligned} ||\tilde{u}^\varepsilon(s) - u_0^\varepsilon||_{L^1} = \left|\left|\int_0^s \tilde{u}_t^\varepsilon(t)\,dt\right|\right|_{L^1} &\le \int_0^s ||\tilde{u}_t^\varepsilon(t)||_{L^1}\,dt \\ &\le \int_0^M ||\tilde{u}_t^\varepsilon(t)||_{L^1}\,dt, \end{aligned}$$

and thus

$$\sup_{0\le s\le M} ||\tilde{u}^\varepsilon(s) - u_0^\varepsilon||_{L^1} \le \int_0^M ||\tilde{u}_t^\varepsilon(t)||_{L^1}\,dt. \tag{3.1.22}$$

On the other hand, by (3.1.11),

$$||\tilde{u}_0^\varepsilon - u_{E_0}||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+. \tag{3.1.23}$$

Putting together (3.1.21), (3.1.22) and (3.1.23) leads to

$$\sup_{0\le s\le M} ||\tilde{u}^\varepsilon(t) - u_{E_0}||_{L^1} \to 0 \text{ as } \varepsilon \to 0^+,$$

which implies the desired results (1.2.9) and (1.2.13). □

3.2 The local isoperimetric Function $\mathcal{I}_\Omega^{\delta,E_0}$

As discussed in the introduction, our analysis heavily depends on the regularity of the local isoperimetric function $r \mapsto \mathcal{I}_\Omega^{\delta,E_0}(r)$ in a neighborhood of $r_0 := \mathcal{L}^n(E_0)$, where E_0 is a mass–constrained local perimeter minimizer (see Definition 2.2.6). This is due to the fact that Theorem 1.2.6 assumes that the function $\mathcal{I}_\Omega^{\delta,E_0}$ satisfies a Taylor expansion of order 2 at r_0 (see (1.2.4)). As previously stated, we will write $\mathcal{I}^\delta$ instead of $\mathcal{I}_\Omega^{\delta,E_0}$ when the set E_0 is clear from the context.

3.2.1 Regularity in the case $E_0 = B_\rho(0)$

Here we prove Theorem 1.2.9, namely that $\mathcal{I}^\delta$ is smooth near r_0 when E_0 is a ball. This particular choice of E_0 corresponds to the case of "bubbles", which has been widely studied in the last two decades (see e.g. [3], [4]). Our approach is rooted in the recent rigorous study of isoperimetric problems, and thus draws on ideas from geometric measure theory. This offers transparent, quantitative tools that permit a variational approach to the problem that, to our knowledge, is novel. We believe that these techniques may also prove to be useful in the study of other similar PDE problems.

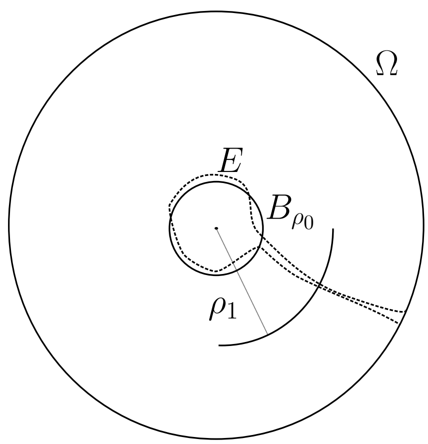

Figure 3.2.1: Finding a good "slice" ρ_1.

In what follows, we denote by B_ρ the ball centered at $\mathbf{0}$ and radius ρ.

Proof of Theorem 1.2.9. **Step 1.** We start by assuming that $\Omega = B_1$ and that $E_0 = B_{\rho_0}$, with $\rho_0 < 1$.

Given $\gamma > 0$ (which we will fix later), choose $0 < c_1 < \gamma/4$ and $0 < 2\delta < \gamma$. Fix a Borel set $E \subset B_1$ with $\mathcal{L}^n(E) = r$, admissible in the definition of $\mathcal{I}_{B_1}^\delta(r)$, with $|r - r_0| < c_1$, and satisfying $P(E; B_1) = \mathcal{I}_{B_1}^\delta(r)$.

Define

$$V_1(\rho) := \int_\rho^1 \mathcal{H}^{N-1}(E \cap \partial B_s)\, ds = \mathcal{L}^n(E \cap (B_1 \setminus \overline{B}_\rho)), \quad \rho \in [0,1], \tag{3.2.1}$$

where we have used spherical coordinates. In particular, we have

$$\mathcal{L}^n(E \setminus B_{\rho_0}) = V_1(\rho_0). \tag{3.2.2}$$

We claim that for γ chosen appropriately we must have that $V_1(\rho) \equiv 0$ in a left neighborhood of $\rho = 1$.

We assume, to obtain a contradiction, that $V_1(\rho) > 0$ for all $\rho < 1$. Our goal will be to find an appropriate radius ρ_1 at which to "slice" our set (see Figure 3.2.1). We will then estimate the perimeter of the set inside and outside of the slice to demonstrate that a ball with the same mass decreases the perimeter.

We begin by studying $\alpha(B_{\rho_0}, E)$. Notice that if $r = r_0$, then

$$\mathcal{L}^n(E \cap B_{\rho_0}) + \mathcal{L}^n(E \setminus B_{\rho_0}) = \mathcal{L}^n(B_{\rho_0}) = \mathcal{L}^n(B_{\rho_0} \cap E) + \mathcal{L}^n(B_{\rho_0} \setminus E), \tag{3.2.3}$$

and, in turn,

$$\mathcal{L}^n(E \setminus B_{\rho_0}) = \mathcal{L}^n(B_{\rho_0} \setminus E).$$

In particular, by (3.2.2) this implies that

$$\alpha(B_{\rho_0}, E) = \mathcal{L}^n(B_{\rho_0} \setminus E) = \mathcal{L}^n(E \setminus B_{\rho_0}) = V_1(\rho_0).$$

Next, if $r_0 - r =: \xi_r > 0$ we find that

$$\mathcal{L}^n(B_{\rho_0} \setminus E) - \xi_r = \mathcal{L}^n(E \setminus B_{\rho_0}),$$

and thus by (3.2.2),

$$\alpha(B_{\rho_0}, E) = \mathcal{L}^n(E \setminus B_{\rho_0}) = V_1(\rho_0),$$

while if $r_0 - r =: \xi_r < 0$, then

$$\mathcal{L}^n(B_{\rho_0} \setminus E) + \xi_r = \mathcal{L}^n(E \setminus B_{\rho_0}) = V_1(\rho_0),$$

which gives

$$\alpha(B_{\rho_0}, E) = \mathcal{L}^n(B_{\rho_0} \setminus E) = V_1(\rho_0) - \xi_r.$$

Summarizing, we obtain

$$\alpha(B_{\rho_0}, E) = \begin{cases} V_1(\rho_0) & \text{if } r \in (r_0 - c_1, r_0], \\ V_1(\rho_0) - \xi_r & \text{if } r \in (r_0, r_0 + c_1). \end{cases} \tag{3.2.4}$$

By definition of $\mathcal{I}^\delta$, see (1.2.14), we know that

$$\alpha(B_{\rho_0}, E) \le \delta. \tag{3.2.5}$$

Thus we find that

$$V_1(\rho_0) \le \delta + |\xi_r| \le \frac{\gamma}{2} + \frac{\gamma}{4} = \frac{3}{4}\gamma, \tag{3.2.6}$$

where we used the fact that $|r - r_0| < c_1 < \gamma/4$.

We claim that for any $C^* > 0$, if $\gamma > 0$ (to be fixed later) is so small that

$$C^* > \frac{\gamma^{\frac{1}{n}}}{n(1-\rho_0)}, \tag{3.2.7}$$

then there exists a measurable set $F \subset [\rho_0, 1]$ with $\mathcal{L}^1(F) > 0$ such that

$$-C^* \left(V_1(\rho)\right)^{\frac{n-1}{n}} \leq \frac{dV_1}{d\rho}(\rho), \tag{3.2.8}$$

for all $\rho \in F$.

In order to prove (3.2.8), we argue by contradiction and suppose that

$$-C^* \left(V_1(\rho)\right)^{\frac{n-1}{n}} > \frac{dV_1}{d\rho}(\rho)$$

for a.e. $\rho \in [\rho_0, 1]$. Then, since $V_1 > 0$ in $[0, 1)$, for all $\rho \geq \rho_0$,

$$-C^* > \frac{1}{n}\frac{d}{d\rho}\left(V_1(\rho)\right)^{\frac{1}{n}},$$

and, in turn, by the fundamental theorem of calculus, we have

$$\left(V_1(\rho)\right)^{\frac{1}{n}} = \left(V_1(1)\right)^{\frac{1}{n}} - \int_\rho^1 \frac{d}{ds}\left(V_1(s)^{\frac{1}{n}}\right)\, ds > nC^*(1-\rho),$$

which, using (3.2.6), implies that

$$\gamma > V_1(\rho_0) > (nC^*)^n(1-\rho_0)^n,$$

a contradiction with (3.2.7). Hence (3.2.7) holds on a set of positive measure.

Next we note that for a.e. $\rho \in [\rho_0, 1]$ we have that

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B_\rho) = 0. \tag{3.2.9}$$

Thanks to (3.2.8), we can now choose $\rho_1 \in F$ such that the condition in (3.2.9) is satisfied. We define

$$E_1 := E \cap (B_1 \setminus \overline{B}_{\rho_1}), \quad E_2 := E \cap \overline{B}_{\rho_1}. \tag{3.2.10}$$

Since

$$\mathcal{L}^n(E_1) = V_1(\rho_1) \leq V_1(\rho_0) < \frac{3\gamma}{4} \tag{3.2.11}$$

by (3.2.1), (3.2.2) and (3.2.6), taking $\gamma < r_{B_1}$, where r_{B_1} is the constant given in (2.2.4) with $\Omega = B_1$, we have that

$$P(E_1; B_1) \geq C_{B_1}(\mathcal{L}^n(E_1))^{\frac{n-1}{n}} = C_{B_1}(V_1(\rho_1))^{\frac{n-1}{n}}. \tag{3.2.12}$$

On the other hand, in view of (3.2.9),

$$\begin{aligned} P(E_2; B_1) &= P(E \cap B_{\rho_1}; B_1) = P(E \cap B_{\rho_1}) \\ &\geq n\omega_n^{1/n}(\mathcal{L}^n(E \cap B_{\rho_1}))^{\frac{n-1}{n}} = n\omega_n^{1/n}(r - V_1(\rho_1))^{\frac{n-1}{n}}, \end{aligned} \tag{3.2.13}$$

where we have used the isoperimetric inequality in $\mathbb{R}^n$ (2.2.3), and (3.2.1). Using the inequality

$$(1-s)^{\frac{n-1}{n}} \geq 1 - \frac{n-1}{n}\frac{s}{(1-s)^{\frac{1}{n}}} \geq 1 - \frac{n-1}{n}2^{\frac{1}{n}}s \tag{3.2.14}$$

for all $0 < s < \frac{1}{2}$, we can bound from below the right hand side of (3.2.13) by

$$n\omega_n^{\frac{1}{n}}r^{\frac{n-1}{n}} - \omega_n^{\frac{1}{n}}(n-1)2^{\frac{1}{n}}r^{-\frac{1}{n}}V_1(\rho_1), \tag{3.2.15}$$

provided $\gamma < \frac{r}{2}$ (see (3.2.11)).

We notice that

$$\begin{aligned} \partial^* E_1 &\subset (\partial^* E \cap (B_1 \setminus B_{\rho_1})) \cup (E \cap \partial B_{\rho_1}), \\ \partial^* E_2 &\subset (\partial^* E \cap B_{\rho_1}) \cup (E \cap \partial B_{\rho_1}). \end{aligned} \tag{3.2.16}$$

Since E_1 is a set of finite perimeter, using the structure theorem for sets of finite perimeter (2.2.2), (3.2.16) implies

$$\mathcal{H}^{n-1}(\partial^* E \cap (B_1 \setminus B_{\rho_1})) \geq P(E_1; B_1) - \mathcal{H}^{n-1}(E \cap \partial B_{\rho_1})$$

and similarly for E_2,

$$\mathcal{H}^{n-1}(\partial^* E \cap B_{\rho_1}) \geq P(E_2; B_1) - \mathcal{H}^{n-1}(E \cap \partial B_{\rho_1}).$$

In turn,

$$\begin{aligned} P(E; B_1) &= \mathcal{H}^{n-1}(\partial^* E \cap (B_1 \setminus B_{\rho_1})) + \mathcal{H}^{n-1}(\partial^* E \cap B_{\rho_1})) \\ &\geq P(E_1; B_1) + P(E_2; B_1) - 2\mathcal{H}^{n-1}(E \cap \partial B_{\rho_1}) \\ &\geq C_{B_1}(V_1(\rho_1))^{\frac{n-1}{n}} + n\omega_n^{\frac{1}{n}}r^{\frac{n-1}{n}} - \omega_n^{\frac{1}{n}}(n-1)2^{\frac{1}{n}}r^{-\frac{1}{n}}V_1(\rho_1) - 2\mathcal{H}^{n-1}(E \cap B_{\rho_1}), \end{aligned} \tag{3.2.17}$$

where the first inequality holds in view of (3.2.9), and where we have used (3.2.12), (3.2.13) and (3.2.14).

Using the fundamental theorem of calculus in (3.2.1) we have that $\frac{dV_1(\rho)}{d\rho} = -\mathcal{H}^{n-1}(E\cap \partial B_\rho)$ for all $0 < \rho < 1$, and so also by (3.2.8) the right-hand side of (3.2.17) can be bounded from below by

$$(C_{B_1} - 2C^*)(V_1(\rho))^{\frac{n-1}{n}} + n\omega_n^{\frac{1}{n}}r^{\frac{n-1}{n}} - \omega_n^{\frac{1}{n}}(n-1)2^{\frac{1}{n}}r^{-\frac{1}{n}}V_1(\rho). \tag{3.2.18}$$

Fix $C^* := \frac{1}{4}C_{B_1}$. By taking γ so small that

$$(C_{B_1} - 2C^*) - \omega_n^{\frac{1}{n}}(n-1)2^{\frac{1}{n}}r^{-\frac{1}{n}}\gamma^{\frac{1}{n}} > 0,$$

by (3.2.11) we have that
$$P(E;B_1) > n\omega^{1/n} r^{\frac{n-1}{n}}.$$
Let $\rho_r := (\frac{r}{\omega_n})^{\frac{1}{n}}$ so that $\mathcal{L}^n(B_{\rho_r}) = r$. Then

$$\mathcal{H}^{n-1}(\partial B_{\rho_r}) = n\omega_n \rho_r^{n-1} = n\omega_n^{\frac{1}{n}} r^{\frac{n-1}{n}},$$

and so $P(E;B_1) > P(B_{\rho_r};B_1)$. On the other hand,

$$\alpha(B_{\rho_r}, B_{\rho_0}) = 0 \leq \delta,$$

and we have reached a contradiction (see(1.2.14)). It follows that $V_1(\rho) = 0$ for all ρ close to 1. This shows that $E \subset B_\rho$ for some $\rho < 1$. In turn, $P(E;B_1) = P(E)$. Hence we can use the isoperimetric inequality in $\mathbb{R}^n$ (see (2.2.3)), to conclude that E is in fact a ball of radius ρ_r. This proves (1.2.20).

Step 2. Now suppose that $E_0 = B_{r_0}(x) \subset\subset \Omega$, for an arbitrary Ω satisfying (2.1.1), for some $x \in \Omega$. Again, given a $\gamma > 0$ (which we will fix later), we choose $0 < c_1 < \gamma/4$ and $0 < 2\delta < \gamma$. Let $R > r_0$ be such that $B_R(x) \subset\subset \Omega$. Fix E as in step 1, so that $\mathcal{L}^n(E) = r$, $|r - r_0| < c_1$ and $P(E;\Omega) = \mathcal{I}^\delta(r)$.

We define
$$E_1 := E \cap (\Omega \setminus B_R(x)), \quad E_2 := E \cap B_R(x)$$
and estimate
$$P(E;\Omega) \geq P(E_1; \Omega \setminus B_R(x)) + P(E_2; B_R(x)).$$
Following the same reasoning in the derivation of equation (3.2.6) in Step 1, we have that $\mathcal{L}^n(E_1) \leq \frac{3\gamma}{4}$, and thus (2.2.4) implies that

$$P(E_1; \Omega \setminus B_R(x)) \geq C_{\Omega \setminus B_R(x)} \mathcal{L}^n(E_1)^{\frac{n-1}{n}}$$

as long as γ is small enough. It is clear that $\alpha(E_2, E_0) \leq \alpha(E, E_0)$, and that $|\mathcal{L}^n(E_2) - r_0| \leq \mathcal{L}^n(E_1) + |r - r_0| \leq \gamma$. By the results of Step 1 we know that for γ small enough

$$P(E_2; B_R(x)) \geq \mathcal{I}^\delta_{B_R(x)}(\mathcal{L}^n(E_2)) = n\omega_n^{\frac{1}{n}} (\mathcal{L}^n(E_2))^{\frac{n-1}{n}} = n\omega_n^{\frac{1}{n}} (r - \mathcal{L}^n(E_1))^{\frac{n-1}{n}}.$$

As in Step 1, defining $\rho_r := (\frac{r}{\omega_n})^{\frac{1}{n}}$, if $\mathcal{L}^n(E_1) > 0$ this implies that

$$P(E;\Omega) > P(B_{\rho_r}(x)) = P(B_{\rho_r}(x);\Omega)$$

while $\alpha(B_{\rho_r}(x), E_0) \leq \delta$, which is a contradiction. Again, as in Step 1, the classical isoperimetric inequality (2.2.3) then implies that E must be a ball, which concludes the proof. $\square$

3.2.2 Regularity in the case of positive second variation

Here we prove Theorem 1.2.10. We begin by stating the following lemma, which summarizes a number of classical results (see e.g. [43], [45], [55], [56], [75]), see Lemma 5.4 in [55] for details.

Lemma 3.2.1. *Let Ω satisfy the assumptions in Section 2.1 (see* (2.1.1)*), and let $E_0 \subset \Omega$ be a volume-constrained local perimeter minimizer in Ω. Then ∂E_0 is a surface of constant mean curvature κ_{E_0}, which intersects the boundary of Ω orthogonally. Moreover, there exists a neighborhood I of r_0 and a family of sets $\{V_r\}_r$ constructed via a normal perturbation of E_0 (see Theorem 2.3.8), satisfying*

$$\mathcal{L}^n(V_r) = r, \qquad \lim_{r \to r_0} |V_r \Delta E_0| = 0, \tag{3.2.19}$$

and such that the function

$$r \mapsto \phi(r) := P(V_r; \Omega), \quad \textit{for } r \in I,$$

is smooth. Moreover, the function ϕ satisfies

$$\phi(r_0) = P(E_0; \Omega), \quad \frac{d\phi(r)}{dr}\Big|_{r=r_0} = \kappa_{E_0}(n-1), \tag{3.2.20}$$

and

$$\frac{d^2\phi(r)}{dr^2}\Big|_{r=r_0} = -\frac{\int_{\partial E_0} |A_{E_0}|^2 \, d\mathcal{H}^{n-1} + \int_{\partial E_0 \cap \partial\Omega} \nu_{\partial E_0} \cdot A_\Omega \nu_{\partial E_0} \, d\mathcal{H}^{n-2}}{P(E_0; \Omega)^2},$$

where A_{E_0} and A_Ω are the second fundamental forms, see Definition 2.3.6.

Remark 3.2.2. *Recalling the definition of $\mathcal{I}_\Omega^{\delta, E_0}$, if follows from* (3.2.19) *and* (3.2.20) *that $\mathcal{I}_\Omega^{\delta, E_0}$ is upper semi-continuous at r_0.*

We start by proving the following.

Lemma 3.2.3. *Let Ω satisfy the assumptions in Section 2.1 (see* (2.1.1)*), and let E_0 be a volume–constrained local perimeter minimizer with $r_0 := \mathcal{L}^n(E_0)$. Let $\delta > 0$, and let $I_{r_0} \subset\subset [0, \mathcal{L}^n(\Omega]$ be an open interval containing r_0. Suppose that for every $r \in I_{r_0}$ at least one minimizer E_r of the problem*

$$\min\{P(E; \Omega) : \mathcal{L}^n(E) = r, \ \alpha(E, E_0) \le \delta\}$$

satisfies

$$\alpha(E_r, E_0) < \delta. \tag{3.2.21}$$

Then the local isoperimetric function $\mathcal{I}_\Omega^{\delta, E_0}$ is semi–concave in I_{r_0}, that is, there exists a constant $C > 0$ such that

$$r \mapsto \mathcal{I}_\Omega^{\delta, E_0}(r) - Cr^2 \tag{3.2.22}$$

is a concave function in I_{r_0}.

Remark 3.2.4. *By setting δ large enough this establishes that the isoperimetric function $\mathcal{I}_\Omega$ is semi–concave on any interval $[a,b] \subset [0, \mathcal{L}^n(\Omega)] = [0,1]$.*

Proof. By lower semicontinuity of the perimeter and BV compactness, it follows that $\mathcal{I}_\Omega^{\delta,E_0}$ is lower semicontinuous. By (3.2.21) we have that E_r must be a local volume-constrained perimeter minimizer. Thus by Lemma 3.2.1 applied to E_r, for any $r \in I_{r_0}$ there exists a smooth function ϕ_r and a constant $\delta_r > 0$ depending on r such that

$$\phi_r(s) \geq \mathcal{I}_\Omega^{\delta,E_0}(s) \text{ for all } s \in (r-\delta_r, r+\delta_r), \quad \phi_r(r) = P(E_r;\Omega) = \mathcal{I}_\Omega^{\delta,E_0}(r), \tag{3.2.23}$$

and

$$\frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} = -\frac{\int_{\partial E_r} |A_{E_r}|^2 \, d\mathcal{H}^{n-1} + \int_{\partial E_r \cap \partial\Omega} \nu_{E_r} \cdot A_\Omega \nu_{E_r} \, d\mathcal{H}^{n-2}}{P(E_r;\Omega)^2}, \tag{3.2.24}$$

where we recall that $|A_{E_r}|$ is the Frobenius norm, see equation (2.3.5). Furthermore, we notice that the lower semicontinuity of $\mathcal{I}_\Omega^{\delta,E_0}$, together with (3.2.23), implies that $\mathcal{I}_\Omega^{\delta,E_0}$ is continuous on I_{r_0}.

Let $C_\Omega := \max_{x\in\partial\Omega} |A_\Omega(x)|$. Then we have

$$\left| \int_{\partial E_r \cap \partial\Omega} \nu_{E_r} \cdot A_\Omega \nu_{E_r} \, d\mathcal{H}^{n-2} \right| \leq C_\Omega \int_{\partial E_r \cap \partial\Omega} \nu_\Omega \cdot \nu_\Omega \, d\mathcal{H}^{n-2}. \tag{3.2.25}$$

Since Ω is of class $C^{2,\alpha}$, we can locally express $\partial\Omega$ as the graph of a function of class $C^{2,\alpha}$ and, in turn, we can locally extend the normal to the boundary ν_Ω to a $C^{1,\alpha}$ vector field. Thus, using a partition of unity, we may extend the vector field $C_\Omega \nu_\Omega$ to a vector field $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ satisfying

$$\|T\|_\infty \leq C, \qquad \|\nabla T\|_\infty \leq C \tag{3.2.26}$$

for some constant $C > 0$. We then apply the divergence theorem (see Theorem 2.3.9) with $M = \overline{(\partial E_r) \cap \Omega}$ and $\Gamma = \partial E_r \cap \partial\Omega$ to find that

$$\begin{aligned} C_\Omega \int_{\partial E_r \cap \partial\Omega} \nu_\Omega \cdot \nu_\Omega \, d\mathcal{H}^{n-2} &= \int_{\partial E_r} \mathrm{div}_{E_r} T \, d\mathcal{H}^{n-1} - \int_{\partial E_r} T \cdot \kappa_{E_r} \nu_\Omega \, d\mathcal{H}^{n-1} \\ &\leq C P(E_r;\Omega) + C \int_{\partial E_r} |\kappa_{E_r}| \, d\mathcal{H}^{n-1}, \end{aligned} \tag{3.2.27}$$

where in the last inequality we have used (2.3.3) and (3.2.26). Moreover, we recall that (see Proposition 2.3.7) for every $x \in \Omega \cap \partial E_r$,

$$|A_{E_r}(y)|^2 = \sum_{h=1}^{n-1} \kappa_{h,E_r}(y)^2, \qquad \kappa_{E_r}(y) = \sum_{h=1}^{n-1} \kappa_{h,E_r}(y) \text{ for all } y \in B_r(x) \cap \partial E_r \tag{3.2.28}$$

where κ_{h,E_r} are the principal curvatures of E_r. Thus, using (3.2.28), if we consider the principal curvatures κ_{h,E_r} as a vector in $\mathbb{R}^{n-1}$ then we have that

$$C|\kappa_{E_r}| \leq \sqrt{n-1}C|A_{E_r}| \leq \max\{(n-1)C^2, |A_{E_r}|^2\}. \tag{3.2.29}$$

In turn, putting together (3.2.24), (3.2.25), (3.2.27) and (3.2.29), we get

$$\begin{aligned}\frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} &\leq \frac{-\int_{\partial E_r}|A_{E_r}|^2\,d\mathcal{H}^{n-1} + CP(E_r;\Omega) + \int_{\partial E_r}\max\{(n-1)C^2, |A_{E_r}|^2\,\mathcal{H}^{n-1}}{P(E_r;\Omega)^2}\\ &\leq \frac{CP(E_r;\Omega) + (n-1)C^2P(E_r;\Omega)}{P(E_r;\Omega)^2}.\end{aligned}$$

Denote

$$m_1 := \min_{s\in\overline{I_{r_0}}} \mathcal{I}_\Omega^{\delta,E_0}(s),\ m_2 := C + (n-1)C^2 < \infty,$$

and notice that

$$\min_{s\in\overline{I_{r_0}}} \mathcal{I}_\Omega^{\delta,E_0}(s) \geq \min_{s\in\overline{I_{r_0}}} \mathcal{I}_\Omega(s) > 0$$

where the last inequality follows from Proposition 2.2.5 (see also Lemma 3.2.4 in [57]). From (3.2.24) we have that

$$\frac{d^2\phi_r(s)}{ds^2}\Big|_{s=r} \leq \frac{m_2}{m_1}. \tag{3.2.30}$$

Thus by (3.2.23) for any r we can find a $\delta_r > 0$ so that for $s \in (r-\delta_r, r+\delta_r)$,

$$\begin{aligned}\mathcal{I}_\Omega^{\delta,E_0}(s) - \frac{m_2}{m_1}s^2 &\leq \phi_r(s) - \frac{m_2}{m_1}s^2\\ &= \phi_r(s) - \frac{m_2}{m_1}((s-r)^2 + 2sr - r^2)\\ &=: \psi(s) - \frac{m_2}{m_1}(2sr - r^2),\end{aligned} \tag{3.2.31}$$

where $\psi(s) = \phi_r(s) - \frac{m_1}{m_2}(s-r)^2$ is a concave function on $(r-\delta_r, r+\delta_r)$ by (3.2.30). The estimate (3.2.31) allows us to apply Lemma 2.7 in [75] and conclude that $\mathcal{I}_\Omega^{\delta,E_0}(s) - \frac{m_2}{m_1}s^2$ is a concave function on I_{r_0}. In turn, $\mathcal{I}_\Omega^{\delta,E_0}$ is semi–concave on I_{r_0}. □

Corollary 3.2.5. *Under the assumptions of Lemma 3.2.3, the local isoperimetric function $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}. Furthermore, for all $J_{r_0} \subset\subset I_{r_0}$, for all $r \in J_{r_0}$, the values $\kappa_{E_r}(n-1)$ belong to the supergradient of $\mathcal{I}_\Omega^{\delta,E_0}$, and hence*

$$|\kappa_{E_r}| \leq L, \tag{3.2.32}$$

where L is the Lipschitz constant of $\mathcal{I}_\Omega^{\delta,E_0}$ in J_{r_0}.

Proof. Thanks to (3.2.21) in Lemma 4.3, for any $r \in I_{r_0}$ there exists a volume–constrained local perimeter minimizer E_r such that

$$\mathcal{I}_\Omega^{\delta,E_0}(r) = P(E_r;\Omega),\ \mathcal{L}^n(E_r) = r,\ \alpha(E_r, E_0) < \delta.$$

By Lemma 3.2.1 applied to E_r, in particular from (3.2.20), we have that $\kappa_{E_r}(n-1)$ belongs to the supergradient of $\mathcal{I}_\Omega^{\delta,E_0}$. From (3.2.22) we know that the mapping $r \mapsto \mathcal{I}_\Omega^{\delta,E_0}(r) - Cr^2$ is concave, and hence locally Lipschitz. In turn, $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}. Finally, as $\kappa_{E_r}(n-1)$ is in the supergradient of a locally Lipschitz function, there exists a constant $L > 0$ so that (3.2.32) holds on J_{r_0} (see Theorem 9.13 in [70]). □

Recently stability estimates have been proved for a nonlocal version of the perimeter functional by Acerbi, Fusco and Morini [1]. We recall the generalization of their result obtained by Julin and Pisante (see Theorem 1.1 in [49]), which will turn out to be a key tool for our analysis.

Theorem 3.2.6. *Suppose that Ω satisfies* (2.1.1) *and that E_0 is a mass–constrained local perimeter minimizer with strictly positive second variation in the sense of* (2.3.6)*. Then E_0 is a strict local minimum for $P(\cdot;\Omega)$ in the L^1 sense, and there exist $c > 0$ and $\delta_0 > 0$ such that*

$$P(E;\Omega) \geq P(E_0;\Omega) + c\mathcal{L}^n(E\Delta E_0)^2 \tag{3.2.33}$$

for every set E of finite perimeter in Ω satisfying $\mathcal{L}^n(E) = \mathcal{L}^n(E_0)$ and $\mathcal{L}^n(E\Delta E_0) < \delta_0$.

Remark 3.2.7. *The original version of Theorem 1.1 in [49] requires the set E_0 in the statement to be a "regular critical" set of the perimeter functional (see Definition 2.1 in [49]). In essence, they require the set E_0 to be such that the first variation of $P(\cdot,\Omega)$ is zero in the direction of every admissible vector field of class C^1. We notice that this condition is always satisfied when E_0 is a mass–constrained local perimeter minimizer.*

We are now ready to give the proof of Theorem 1.2.10.

Proof of Theorem 1.2.10. The proof will be divided into several steps, and we will invoke the previous results and the stability estimate (3.2.33) proved by Julin and Pisante [49]. By Theorem 3.2.6 we know that E_0 is an isolated local volume-constrained perimeter minimizer, and hence the unique minimizer of the problem

$$\min\left\{P(E;\Omega) :\ E \subset \Omega \text{ Borel, } \mathcal{L}^n(E) = r,\ \alpha(E, E_0) \leq \delta\right\}, \tag{3.2.34}$$

for $r = r_0$ and for some fixed $0 < \delta < \delta_0$ small enough, where δ_0 is given in (3.2.33).

Let I be a neighborhood of r_0 (to be fixed later) and consider a sequence $\{r_k\}$ satisfying $r_k \to r_0$ as $k \to \infty$. Let E_{r_k} be a minimizer of the problem (3.2.34) for $r = r_k$.

Step 1. By considering level sets of the signed distance function (see, e.g. Lemma 5.4 in [55] or [65]), and recalling the definition of $\mathcal{I}_\Omega^{\delta,E_0}$, it is straightforward to show that

$$\mathcal{I}_\Omega^{\delta,E_0} \leq C \tag{3.2.35}$$

for some $C > 0$ and, in turn, by BV compactness, there exists a subsequence of $\{E_{r_k}\}$ (not relabeled) such that

$$E_{r_k} \to E^* \text{ in } L^1(\Omega) \text{ as } k \to \infty, \tag{3.2.36}$$

for some measurable set E^* such that $\chi_{E^*} \in BV(\Omega)$ and $\mathcal{L}^n(E^*) = r_0$.

We notice that since $\alpha(E^*, E_0) \leq \delta$ and $\mathcal{L}^n(E^*) = r_0$, by lower semi-continuity of the perimeter (see [34]), and Remark 3.2.2, we have that

$$\begin{aligned} P(E^*;\Omega) &\leq \liminf_{k\to\infty} P(E_{r_k};\Omega) = \liminf_{k\to\infty} \mathcal{I}_\Omega^{\delta,E_0}(r_k) \leq \limsup_{k\to\infty} \mathcal{I}_\Omega^{\delta,E_0}(r_k) \\ &\leq \mathcal{I}_\Omega^{\delta,E_0}(r_0) = P(E_0;\Omega) \leq P(E^*;\Omega). \end{aligned}$$

By uniqueness of (3.2.34) for $r = r_0$, $E^* = E_0$, and so (3.2.36) reads

$$E_{r_k} \to E_0 \text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{3.2.37}$$

Thanks to (3.2.37), we obtain

$$\alpha(E_{r_k}, E_0) < \delta,$$

for k big enough. In turn, this implies that there exists an open neighborhood I_{r_0} of r_0 as in Lemma 3.2.3. By Corollary 3.2.5, we have that $\mathcal{I}_\Omega^{\delta,E_0}$ is locally Lipschitz in I_{r_0}.

Step 2. Fix an open neighborhood $J_{r_0} := (r_0 - R, r_0 + R) \subset\subset I_{r_0}$ of r_0, and let L be the associated Lipschitz constant of $\mathcal{I}_\Omega^{\delta,E_0}$ in J_{r_0} (see Corollary 3.2.5). Let k be large enough so that $r_k \in J_{r_0}$. Let $x_0 \in \Omega$, $\rho_0 > 0$. We claim that E_{r_k} is a (Λ, ρ_0)–perimeter minimizer (see e.g. [56]), that is

$$P(E_{r_k}; B_\rho(x_0)) \leq P(E; B_\rho(x_0)) + \Lambda\mathcal{L}^n(E_{r_k}\Delta E), \tag{3.2.38}$$

for all $\rho < \rho_0$ and all measurable E satisfying

$$E_{r_k}\Delta E \subset\subset B_\rho(x_0), \tag{3.2.39}$$

and with

$$\Lambda = \max\left\{L, \frac{2C}{\delta}, \frac{2C}{R}\right\},$$

where $C > 0$ is as in Step 1. Because of (3.2.39), we know that $P(E_{r_k}; B_\rho(x_0)) - P(E; B_\rho(x_0)) = P(E_{r_k};\Omega) - P(E;\Omega)$, and thus it suffices to prove that

$$P(E_{r_k};\Omega) \leq P(E;\Omega) + \Lambda\mathcal{L}^n(E_{r_k}\Delta E). \tag{3.2.40}$$

We divide the proof of (3.2.40) into three cases. If

$$\alpha(E_0, E) \leq \delta \text{ and } \mathcal{L}^n(E) \in J_{r_0},$$

then by our choice of L (see Corollary 3.2.5), we have

$$\begin{aligned}P(E_{r_k};\Omega) &= \mathcal{I}_\Omega^{\delta,E_0}(E_{r_k}) \le \mathcal{I}_\Omega^{\delta,E_0}(\mathcal{L}^n(E)) + L\,|\mathcal{L}^n(E_{r_k}) - \mathcal{L}^n(E))| \\ &\le P(E;\Omega) + L\,|\mathcal{L}^n(E_{r_k}) - \mathcal{L}^n(E))| \\ &\le P(E;\Omega) + L\mathcal{L}^n(E_{r_k}\Delta E),\end{aligned}$$

and (3.2.40) is proved in this case.

If instead E is such that

$$\alpha(E_0, E) > \delta,$$

then by (3.2.37),

$$\mathcal{L}^n(E_{r_k}\Delta E) \ge \mathcal{L}^n(E_0\Delta E) - \mathcal{L}^n(E_{r_k}\Delta E_0) \ge \frac{\delta}{2}, \tag{3.2.41}$$

for k sufficiently large. Moreover, by (3.2.35) and (3.2.41),

$$P(E_{r_k};\Omega) \le C \le \frac{2C}{\delta}\mathcal{L}^n(E_{r_k}\Delta E) \le \frac{2C}{\delta}\mathcal{L}^n(E_{r_k}\Delta E) + P(E;\Omega), \tag{3.2.42}$$

so that (3.2.40) follows from our choice of Λ.

Finally, if

$$\mathcal{L}^n(E) \notin J_{r_0},$$

then for $r_k \in (r_0 - R/2, r_0 + R/2)$ we have that

$$\mathcal{L}^n(E_{r_k}\Delta E) \ge \frac{R}{2},$$

and so (3.2.40) follows as in the previous case.

Step 3. Fix $\boldsymbol{z}_0 \in \Omega \cap \partial E_0$, and choose $\mathfrak{r} > 0$ such that $B_{\mathfrak{r}}(\boldsymbol{z}_0) \subset\subset \Omega$ and

$$\partial E_0 \cap B_{\mathfrak{r}}(\boldsymbol{z}_0) = \mathrm{graph}(u_0),$$

for some regular function u_0. By the theory of (Λ, ρ_0) minimizers (see Theorem 26.6 in [56]), choosing ρ_0 smaller if needed, it follows that for any sequence of points $\boldsymbol{z}_k \in \partial E_{r_k}$ such that $\boldsymbol{z}_k \to \boldsymbol{z}_0 \in \Omega \cap \partial E_0$, then for k large enough $\boldsymbol{z}_k \in \Omega \cap \partial^* E_{r_k}$ and

$$\lim_{k\to\infty} \nu_{E_{r_k}}(\boldsymbol{z}_k) = \nu_{E_0}(\boldsymbol{z}_0), \tag{3.2.43}$$

uniformly on $B_{\mathfrak{r}}(\boldsymbol{z}_0)$. In turn, by (3.2.37), for k big enough

$$\partial E_{r_k} \cap B_{\mathfrak{r}}(\boldsymbol{z}_0) = \text{graph}(u_k), \tag{3.2.44}$$

for some functions u_k. In particular, by equation (26.52) in [56], we obtain

$$\nabla u_k \to \nabla u_0, \text{ in } C^{0,\gamma}(\Omega), \tag{3.2.45}$$

for all $\gamma \in (0, 1/2)$.

Step 4. Since ∂E_{r_k} is a surface of constant mean curvature, u_k solves

$$\operatorname{div}\left(\frac{\nabla u_k}{\sqrt{1+|\nabla u_k|^2}}\right) = \kappa_k \text{ in } B_{\mathfrak{r}}(\boldsymbol{z}_0),$$

where κ_k is the mean curvature of ∂E_{r_k}. By standard Schauder estimates (see e.g. [41]) and (3.2.43), it follows that

$$||u_k||_{C^{2,\gamma}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} \leq c_1|\kappa_k| \leq C, \tag{3.2.46}$$

where $B'_{\mathfrak{r}/2}(\boldsymbol{z}_0)$ is the $(n-1)$–dimensional ball and the uniform bound on the curvatures comes from Corollary 3.2.5.

Step 5. By Rellich–Kondrachov compactness theorem and by a bootstrapping argument on (3.2.46), we deduce that there exists a subsequence of $\{r_k\}$, not relabeled, and $\tilde{u} \in W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))$ such that

$$u_{r_j} \to \tilde{u} \text{ in } W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0)) \tag{3.2.47}$$

for all $m > 0$. It follows from (3.2.37), that necessarily $\tilde{u} = u_0$.

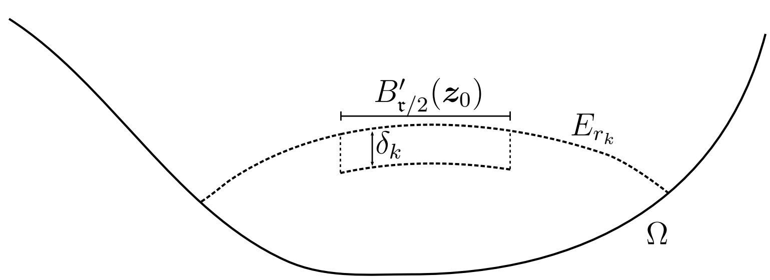

Figure 3.2.2: Mass fixing perturbation of E_{r_k} from Step 6.

Step 6. Define

$$\delta_k := (r_0 - r_k)\left(\frac{\mathfrak{r}}{2}\right)^{1-n}\omega_{n-1}^{-1}, \tag{3.2.48}$$

and let

$$\tilde{u}_{r_k} = \begin{cases} u_{r_k} + \delta_k & \text{on } B'_{\mathfrak{r}/2}(\boldsymbol{z}_0) \\ u_{r_k} & \text{on } B'_{\mathfrak{r}}(\boldsymbol{z}_0) \setminus B'_{\mathfrak{r}/2}(\boldsymbol{z}_0). \end{cases}$$

Let $\tilde{E}_{r_k}$ be the subgraph of u_{r_k} (inside a cylinder with base $B'_{\mathfrak{r}}(\boldsymbol{z}_0)$, and equal to E_{r_k} otherwise), and notice that $\mathcal{L}^n(\tilde{E}_{r_k}) = \mathcal{L}^n(E_0)$ by our choice of δ_k. Moreover, we have that

$$P(\tilde{E}_{r_k};\Omega) = P(E_{r_k};\Omega) + c_n\left(\frac{\mathfrak{r}}{2}\right)^{n-2}\delta_k = P(E_{r_k};\Omega) + O(|r_k - r_0|), \tag{3.2.49}$$

where c_n is the surface area of the $n-1$ dimensional unit ball, and where we have used (3.2.48). Furthermore, it follows from Corollary 3.2.5 that

$$P(E_{r_k};\Omega) = P(E_0;\Omega) + O(|r_k - r_0|). \tag{3.2.50}$$

By (3.2.33), together with (3.2.49), (3.2.50), we infer that

$$\mathcal{L}^n(\tilde{E}_{r_k}\Delta E_0) \le \sqrt{P(\tilde{E}_{r_k};\Omega) - P(E_0;\Omega)} \le O(|r_k - r_0|^{1/2}).$$

Moreover, by the triangle inequality, we have

$$\mathcal{L}^n(E_{r_k}\Delta E_0) \le \mathcal{L}^n(E_0\Delta\tilde{E}_{r_k}) + \mathcal{L}^n(E_{r_k}\Delta\tilde{E}_{r_k}) \le O(|r_k - r_0|^{1/2}) + O(|r_k - r_0|),$$

where the first term is estimated above while the second one follows by the construction of the $\tilde{E}_{r_k}$. In turn,

$$\mathcal{L}^n(E_{r_k}\Delta E_0) \le O(|r_k - r_0|^{1/2}) \tag{3.2.51}$$

and

$$\begin{aligned} |\kappa_{r_k} - \kappa_0| &\le C||D^2u_{r_k} - D^2u_0||_{L^2(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} \\ &\le C||u_{r_k} - u_0||^{1-\beta}_{L^1(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))}||u - u_{r_k}||^{\beta}_{W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} + C||u_{r_k} - u_0||_{L^1(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} \\ &= C\mathcal{L}^n(E_{r_k}\Delta E_0)^{1-\beta}||u - u_{r_k}||^{\beta}_{W^{m,2}(B'_{\mathfrak{r}/2}(\boldsymbol{z}_0))} + C\mathcal{L}^n(E_{r_k}\Delta E_0), \end{aligned} \tag{3.2.52}$$

for $m > 2$ and for some $\beta \in (0,1)$, where we have used the fact that ∂E_{r_k} are surfaces of constant mean curvature, Nirenberg's interpolation inequality (see [64], p. 125-126) and (3.2.44).

Hence, (3.2.47), (3.2.51) and (3.2.52) imply that

$$|\kappa_{r_k} - \kappa_0| = O(|r_k - r_0|^{(1-\beta)/2}). \tag{3.2.53}$$

Since $(n-1)\kappa_{E_r}$ belongs to the supergradient of $\mathcal{I}^{\delta,E_0}_{\Omega}$ at $r \in I_{r_0}$ (see Lemma 3.2.5), at any point s where $\mathcal{I}^{\delta,E_0}_{\Omega}$ is differentiable we have that

$$\frac{d\mathcal{I}^{\delta,E_0}_{\Omega}(r)}{dr}\Big|_{r=s} = (n-1)\kappa_{E_s}.$$

Since $\mathcal{I}^{\delta,E_0}_{\Omega}$ is locally Lipschitz in I_{r_0} (see Step 1), we apply the fundamental theorem of calculus for $r \ge r_0$, $r \in I_{r_0}$ to obtain

$$\begin{aligned} \left|\mathcal{I}^{\delta,E_0}_{\Omega}(r) - \mathcal{I}^{\delta,E_0}_{\Omega}(r_0) - (r - r_0)\kappa_{r_0}(n-1)\right| &\le (n-1)\int_{r_0}^{r}|\kappa_s - \kappa_{r_0}|ds \\ &\le C\int_{r_0}^{r}|s - r_0|^{(1-\beta)/2}ds \end{aligned}$$

where in the last inequality we have used (3.2.53), and (1.2.4) follows. The case $r \leq r_0$ is analogous. $\square$

Chapter 4

Swift-Hohenberg

4.1 Qualitative properties of minimizers

Corollary 4.1.1. *Let W and q^* be as in Lemma 2.5.2. Then there exist $\sigma > 0$ such that for every open interval I, every $0 < \varepsilon \leq \mathcal{L}^1(I)$, and every $-\infty < q \leq q^*/4$,*

$$q\varepsilon^2 \int_I (u')^2 dx \leq \int_I \left(W(u) + \varepsilon^4 (u'')^2\right) dx, \tag{4.1.1}$$

and

$$E_\varepsilon(u; I) \geq \sigma \int_I \left(W(u) + \varepsilon^2 (u')^2 + \varepsilon^4 (u'')^2\right) dx. \tag{4.1.2}$$

for all $u \in H^2_{loc}(I)$.

Proof. Let $I = (a, b)$ and $u \in H^2((a, b))$. We change variables $v(y) := u(\varepsilon x)$, subdivide the resulting rescaled domain $I_\varepsilon = (a/\varepsilon, b/\varepsilon)$ into $\left[\frac{b-a}{\varepsilon}\right]$ subintervals, I^k_ε, of length between $1/2$ and 2 (since $0 < \varepsilon \leq b - a$) and use Lemma 2.5.2 to obtain

$$\begin{aligned}
\frac{q^*}{4} \int_a^b (u')^2 dx &= \frac{q^*}{4\varepsilon} \int_{a/\varepsilon}^{b/\varepsilon} (v')^2 dy = \frac{1}{4\varepsilon} \sum_k q^* \int_{I^k_\varepsilon} (v')^2 dy \\
&\leq \frac{1}{4\varepsilon} \sum_k \int_{I^k_\varepsilon} \left(4W(v) + 4(v'')^2\right) dy \\
&= \frac{1}{\varepsilon} \int_{a/\varepsilon}^{b/\varepsilon} \left(W(v) + (v'')^2\right) dy \\
&= \int_a^b (W(u) + \varepsilon^3 (u'')^2) dx.
\end{aligned} \tag{4.1.3}$$

Since $q \leq q^*/4$, (4.1.1) easily follows. To prove (4.1.2) we follow closely the strategy used in the proof of Theorem 1.1 of [21] and proceed as follows. Fix $\sigma \in (0, 1)$ sufficiently small so that $(q+\sigma)/(1-\sigma) < q^*/4$. Then,

$$\begin{aligned}\int_a^b (W(u) - q\varepsilon^2(u')^2 + \varepsilon^4(u'')^2)dx& \\ &= (1-\sigma)\int_a^b \left(W(u) - \frac{q+\sigma}{1-\sigma}\varepsilon^2(u')^2 + \varepsilon^4(u'')^2 \right) dx \\ &+ \sigma \int_a^b \left(W(u) + \varepsilon^2(u')^2 + \varepsilon^4(u'')^2 \right) dx,\end{aligned} \tag{4.1.4}$$

and (4.1.2) follows since by (4.1.3) the first term on the right-hand side of (4.1.4) is non-negative. □

The following lemmas established for a generalization of the Modica–Mortola Functional in [50] will be useful to prove our main result. While our energy does not satisfy the assumptions of [50], their argument is easily extended to our case with the help of the interpolation inequality (4.1.2). In particular, Lemma 4.1.2, shows that an H^2 function with a uniformly bounded energy, necessarily takes values close to $\{\pm 1\}$ and has small derivatives, except on a set of measure $O(\varepsilon)$ and Lemma 4.1.3 gives a characterization of the global minimizers for the energy $E(\cdot,\cdot)$, defined in (2.5.2), subject to small boundary conditions.

Lemma 4.1.2. *Let I be an open interval, $M > 0$ and $0 < \delta < 1$. Then there exists a constant $C_1 > 0$ such that for any $0 < \varepsilon \leq \mathcal{L}^1(I)$ and every $u \in H^2(I)$ with $E_\varepsilon(u; I) \leq M$ the following property holds: there is a measurable set $J \subset I$ with $\mathcal{L}^1(J) \leq C_1\varepsilon$ such that*

$$\operatorname{dist}(u(x), \{\pm 1\}) < \delta \quad \textit{and} \quad |\varepsilon u'(x)| < \delta \quad \textit{and}$$

hold for all $x \in I \setminus J$, where dist *denotes the usual distance between a point and a set.*

Proof. By (4.1.2), for every $0 < \varepsilon \leq \mathcal{L}^1(I)$ and $u \in H^2(I)$,

$$\begin{aligned}\int_I \left(W(u) - q\varepsilon^2(u')^2 + \varepsilon^4(u'')^2 \right) dx &\geq \sigma \int_I \left(W(u) + \varepsilon^2(u')^2 + \varepsilon^4(u'')^2 \right) dx \\ &\geq \sigma \int_I W(u)dx.\end{aligned} \tag{4.1.5}$$

We now let $J_0 := \{x \in I : \text{dist}(u(x), \{\pm 1\}) \geq \delta\}$ and from the definition of W we have $c := \inf\{W(s) : \text{dist}(s, \{\pm 1\}) \geq \delta\} > 0$. Then (4.1.5) implies

$$M \geq E_\varepsilon(u; I) \geq \frac{\sigma}{\varepsilon}\int_I W(u)dx \geq \frac{c\sigma}{\varepsilon}\mathcal{L}^1(J_0),$$

and therefore

$$\mathcal{L}^1(J_0) \leq \frac{M\varepsilon}{c\sigma}.$$

Similarly, setting $J_1 := \{x \in I : |\varepsilon u'(x)| \geq \delta\}$, (4.1.5) yields the estimates

$$M \geq E_\varepsilon(u; I) \geq \frac{\sigma}{\varepsilon} \int_I \varepsilon^2 (u')^2 dx \geq \frac{\sigma \delta^2}{\varepsilon} \mathcal{L}^1(J_1) \tag{4.1.6}$$

and consequently

$$\mathcal{L}^1(J_1) \leq \frac{M\varepsilon}{\sigma \delta^2}.$$

Setting $J := J_0 \cup J_1$ yields the desired result. $\square$

Lemma 4.1.3. *Let $I := (a, b)$ be an open interval and $W \in C^2$ satisfy* (2.1.3)*,* (2.1.5) *and* (2.1.8)*. Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ define*

$$\mathcal{M}^{\pm}_{\alpha,\beta} := \{v \in H^2(I) : v(a) = \pm 1 + \alpha_0,\ v'(a) = \alpha_1,\ v(b) = \pm 1 + \beta_0,\ v'(b) = \beta_1\}. \tag{4.1.7}$$

Then there exist constants $\delta_0, C > 0$ such that the following holds. If $\mathcal{L}^1(I) > 1$ and $||\alpha||, ||\beta|| \leq \delta < \delta_0$ then the functional $E(\cdot; I)$ defined in (2.5.2) *has a global minimizer $v_\pm$ on $\mathcal{M}^{\pm}_{\alpha,\beta}$. This minimizer $v_\pm$ solves the Euler–Lagrange equation, and satisfies the estimates*

$$||v_\pm \pm 1||_{L^\infty(I)} \leq C\delta, \tag{4.1.8}$$

$$||v^{(k)}_\pm|_{L^2(I)} \leq C\delta \textit{ for } k = 1, \ldots, 4. \tag{4.1.9}$$

$$||v^{(k)}_\pm||_{L^\infty(I)} \leq C\delta \textit{ for } k = 1, \ldots, 3. \tag{4.1.10}$$

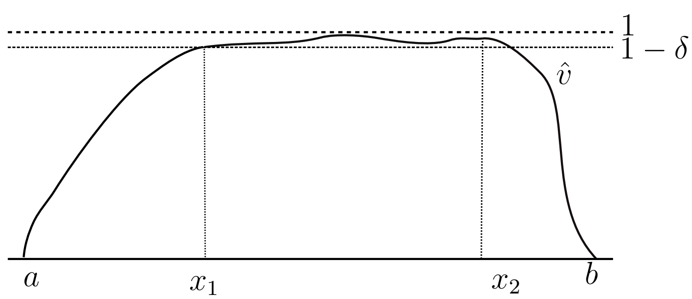

Figure 4.1.1: If $\hat{v}$ is close to 1 at x_1 and x_2, then it stays close in between.

Proof. We prove the proposition when $s = -1$, the $s = 1$ case being identical. We divide the proof into several steps. Moreover, we simplify the notation used for the L^p norms when the domain of integration will be clear from the context.

Step 1. Fix $\delta > 0$. We claim that there exists $C_1 > 0$ such that if $||\alpha||, ||\beta|| \leq \delta$, then

$$\inf_{\mathcal{M}^-_{\alpha,\beta}} E(\cdot; I) \leq C_1 \delta^2. \tag{4.1.11}$$

To show this we note that, if $\varphi_0, \varphi_1 \in C^\infty(\mathbb{R})$ satisfy $\varphi_i(x) = 0$ for all $x \geq 1/2$, with $\varphi_0(0) = 1, \varphi_0'(0) = 0, \varphi_1(0) = 0$, and $\varphi_1'(0) = 1$, then the function

$$\phi(x) := -1+\alpha_0\varphi_0(x-a)+\alpha_1\varphi_1(x-a)+\beta_0\varphi_0(b-x)-\beta_1\varphi_1(b-x), \; x \in (a,b), \tag{4.1.12}$$

belongs to $\mathcal{M}^-_{\alpha,\beta}$. Using ϕ as a test function, (4.1.11) follows from Taylor's formula for W and the facts that $W(\pm 1) = W'(\pm 1) = 0$ and $W \in C^2(\mathbb{R})$.

Step 2. Fix $0 < \delta < 1$. We will show that there exists $C_2 > 0$ such that for every $v \in \mathcal{M}^-_{\alpha,\beta}$, with $v \leq 0$ on I and $||\alpha||, ||\beta|| \leq \delta$ we have

$$E(v; I) \geq C_2 ||v+1||^2_{L^\infty}. \tag{4.1.13}$$

Suppose that $|v(x) + 1| \geq ||v+1||_\infty/2$ for all $x \in I$. Using (2.1.8) and (4.1.2) with $\varepsilon = 1$ we have,

$$E(v; I) \geq \sigma \int_I W(v)dx \geq \sigma c_W \int_I |v+1|^2 \geq \mathcal{L}^1(I)\frac{\sigma}{4}c_W ||v+1||^2_{L^\infty} \tag{4.1.14}$$

Otherwise, there are points $x_0, x_1 \in \bar{I}$ satisfying

$$|v(x_0) + 1| = \frac{||v+1||_\infty}{2} \;\text{ and }\; |v(x_1)+1| = ||v+1||_\infty,$$

in which case, again by (2.1.8), (4.1.2) and Young's Inequality

$$\begin{aligned} E(v; I) &\geq \sigma \int_I \left(W(v) + |v'|^2\right) dx \geq 2\sigma \int_I \sqrt{W(v)}|v'| \\ &\geq 2c_W\sigma \left| \int_{x_0}^{x_1} |v+1| v' dx \right| \\ &= c_W\sigma \left((v+1)^2(x_1) - (v+1)^2(x_0)\right) = \frac{\sigma}{2} c_W ||v+1||_{L^\infty} \end{aligned}$$

and this proves (4.1.13).

Step 3. We claim that there exists $\delta_0 > 0$ and $C_3 = C_3(\delta_0) > 0$ such that if $||\alpha||, ||\beta|| \leq \delta < \delta_0$ and $v \in \mathcal{M}^-_{\alpha,\beta}$, with $E(v; I) \leq 2 \inf_{\mathcal{M}^-_{\alpha,\beta}} E$, then

$$||v+1||_{L^\infty} \leq C_3\delta. \tag{4.1.15}$$

By taking $0 < \delta < 1$ sufficiently small, we may assume that $v \leq 0$ on I. Indeed, since $v(a) = -1 + \alpha_0 \leq -1 + \delta < 0$, if $v(x) > 0$ for some x, then necessarily there exists x_1 such that $v(x_1) = 0$, and so by (2.1.8),

$$\begin{aligned} E(v; I) &\geq \sigma \int_I \left(W(v) + |v'|^2\right) dx \geq 2\sigma \int_a^{x_1} \sqrt{W(v)}|v'| \\ &\geq 2C_W\sigma \left| \int_a^{x_1} |v+1| v' dx \right| \\ &= \sigma C_W \left((v+1)^2(x_1) - (v+1)^2(a)\right) \\ &\geq \sigma(1 - |\alpha_0|^2), \end{aligned}$$

which contradicts Step 1 for δ sufficiently small. Hence, Steps 1 and 2 imply (4.1.15).

Step 4. Finally, (4.1.2) with $\varepsilon = 1$ and standard compactness and lower semicontinuity arguments imply the existence of minimizer v_- of $E(\cdot; I)$ and since by previous step $v_- \leq 0$ for $\delta < \delta_0$ and

$$||v_- + 1||_{L^2}^2 \leq \mathcal{L}^1(I)||v_- + 1||_{L^\infty}^2 \leq C\delta^2, \tag{4.1.16}$$

for some $C > 0$, again using (4.1.2) along with (4.1.11) yields

$$||v_-^{(k)}||_{L^2} \leq C\delta, \text{ for } k = 1, 2.$$

Furthermore, since W is C^2, from (4.1.16) and the Mean Value Theorem we have

$$W'(v_-) = W'(v_-) - W'(-1) \leq \max_{0 \leq \xi \leq 2} W''(\xi)(v_- + 1). \tag{4.1.17}$$

The Euler–Lagrange equation

$$2v_-^{(iv)} + 2qv_-'' + W'(v_-) = 0,$$

the L^∞ bound from Step 3 and (4.1.17) imply

$$||v_-^{(iv)}||_{L^2} \leq |q|||v_-''||_{L^2} + \frac{1}{2}||W'(v_-)||_{L^2} \leq |q|||v_-''||_{L^2} + C||v_- + 1||_{L^2} \leq C\delta \tag{4.1.18}$$

for some $C > 0$.

The energy bound (4.1.11) and standard interpolation inequalities (e.g., see Theorem 6.4 in [36]) imply (4.1.8), (4.1.9), (4.1.10). $\square$

4.1.1 The Euler–Lagrange equation

Here we further analyze the behavior of the minimizers of the energy E_ε with the aid of the corresponding Euler-Lagrange equation, and we prove our main result, Theorem 1.3.1.

Lemma 4.1.4. *Consider the ordinary differential equation*

$$x' = F(x), \tag{4.1.19}$$

where $F : \mathbb{R}^4 \to \mathbb{R}^4$ is a C^4 mapping satisfying $F(x_0) = 0$ for some $x_0 \in \mathbb{R}^4$. Assume $DF(x_0)$ has four eigenvalues $\pm\gamma \pm \delta i$, where $\gamma > 0$ and $\delta \in \mathbb{R}$. Then for $0 < \lambda \leq \gamma$ there exist a constant $C(\gamma, \delta) > 0$, $T_0(\gamma, \delta) > 0$ and $R > 0$ such that for all $T > T_0$, if $x : [0, T] \to B(x_0, R)$ is a solution of (4.1.19)*, then the inequality*

$$|x(t) - x_0| \leq C(\gamma, \delta) \exp(-\lambda T/2) \tag{4.1.20}$$

holds for all $t \in \left[\frac{\lambda T}{2\gamma}, T - \frac{\lambda T}{2\gamma}\right]$. In particular, if $\gamma = \lambda$,

$$|x(T/2) - x_0| \leq C(\gamma, \delta) \exp(-\gamma T/2). \tag{4.1.21}$$

Proof. Changing variables if necessary, we may assume, without loss of generality, that $x_0 = 0$. Let $A := DF(0)$. By an extension of the Hartman–Grobman Theorem (see, e.g. [72] and Lemma 2.6.1), there exist two open neighborhoods of 0, $V_1, V_2 \subset \mathbb{R}^4$, and a diffeomorphism $h : V_1 \to V_2$ of class C^1, with $h(0) = 0$, such that if $x(t) \in V_1$ for all $t \in [0, T]$ then the funciton $y(t) := h(x(t)), t \in [0, T]$ is a solution of the linearized system

$$y' = Ay. \tag{4.1.22}$$

Let $R > 0$ be so small that $\overline{B(0, R)} \subset V_1$, and define $V := h(B(0, R))$. Then V is bounded and since $h(0) = 0$, there exists $L > 0$ such that $V \subset B(0, L)$. Hence if $x(t) \in B(0, R)$ for all $t \in [0, T]$, then $y(t) \in B(0, L)$ for all $t \in [0, T]$.

Since the eigenvalues of A are all distinct, the solution of (4.1.22) has the form

$$\begin{aligned} y(t) = c_1 v_1 \exp\left((-\gamma - \delta i)t\right) &+ c_2 v_2 \exp\left((-\gamma + \delta i)t\right) \\ &+ c_3 v_3 \exp\left((\gamma - \delta i)t\right) + c_4 v_4 \exp\left((-\gamma - \delta i)t\right), \end{aligned}$$

where $c_1, \ldots, c_4$ are complex valued constants and $\{v_i\} \subset \mathbb{C}^4$ is a linearly independent set of eigenvectors of A. Letting $P = [v_1, v_2, v_3, v_4]$ be the matrix of eigenvectors of A, we write the above solution as

$$y(t) = Pw, \tag{4.1.23}$$

where

$$w := \left[c_1 \exp\left((-\gamma - \delta i)t\right), c_2 \exp\left((-\gamma + \delta i)t\right), c_3 \exp\left((\gamma - \delta i)t\right), c_4 \exp\left((\gamma - \delta i)t\right)\right]^{\mathrm{Tr}}$$

and the superscript Tr denotes the transpose of a matrix. Since $y(t) \in B(0, L)$ for all $t \in [0, T]$,

$$|w|^2 \leq ||P^{-1}||^2 |y(t)|^2 \leq L^2 ||P^{-1}||^2,$$

where $||P^{-1}||$ is the operator norm of P^{-1}. In particular,

$$|c_1|^2 \leq L^2 ||P^{-1}||^2 \exp(2\gamma t), \quad |c_2|^2 \leq L^2 ||P^{-1}||^2 \exp(2\gamma t), \tag{4.1.24}$$

$$|c_3|^2 \leq L^2 ||P^{-1}||^2 \exp(-2\gamma t), \quad |c_4|^2 \leq L^2 ||P^{-1}||^2 \exp(-2\gamma t), \tag{4.1.25}$$

for all $t \in [0, T]$. Setting $t = 0$ and $t = T$ in the first and second row respectively we obtain bounds on the constants $c_1, ..., c_4$,

$$|c_1| \leq L||P^{-1}||, \quad |c_2| \leq L||P^{-1}||, \tag{4.1.26}$$

$$|c_3| \leq L||P^{-1}|| \exp(-\gamma T), \quad |c_4| \leq L||P^{-1}|| \exp(-\gamma T). \tag{4.1.27}$$

Using the resulting bounds in (4.1.23) yields

$$\begin{aligned} \exp(\lambda T) |y(t)|^2 &\leq \exp(\lambda T) ||P||^2 \big(|c_1|^2 \exp(-2\gamma t) + |c_2|^2 \exp(-2\gamma t) \\ &\qquad + |c_3|^2 \exp(2\gamma t) + |c_4|^2 \exp(2\gamma t)\big) \\ &\leq 4L^2 ||P||^2 ||P^{-1}||^2, \end{aligned}$$

provided

$$\lambda T - 2\gamma t \leq 0 \text{ and } \lambda T - 2\gamma T + 2\gamma t \leq 0.$$

Both of these conditions are satisfied as long as

$$t \in \left[\frac{\lambda T}{2\gamma}, T - \frac{\lambda T}{2\gamma}\right] =: [t_1, t_2].$$

Hence for $t \in [t_1, t_2]$,

$$|y(t)|^2 \leq 4L^2||P||^2||P^{-1}||^2 \exp\left(-\lambda T\right).$$

In particular, if T is sufficiently large (depending only on γ, δ, and V_2), there exists a compact set E such that $y(t) \in E \subset V_2$ for all $t \in [t_1, t_2]$. Since h^{-1} is C^1 and $h(0) = 0$, by the Mean Value Theorem,

$$|x(t)| = |h^{-1}(y(t))| \leq \sup_{s \in E} |\nabla h^{-1}(s)||y(t)| \leq C_{\gamma,\delta} \exp\left(-\lambda T/2\right) \tag{4.1.28}$$

for all $t \in [t_1, t_2]$, where $C_{\gamma,\delta} := L \sup_{s \in E} |\nabla h^{-1}(s)|||P||||P^{-1}||$. $\square$

For a given open interval I and a subinterval $(y_1, y_2) \subset I$ we define

$$\mathcal{M} := \left\{w \in H^2((y_1, y_2)) : w(y_1) = 0,\ w(y_2) = 0\right\}. \tag{4.1.29}$$

Proposition 4.1.1. *Let $\varepsilon_0 > 0$ and let $\hat{w}_\varepsilon$ be a global minimizer of $E_\varepsilon(\cdot; (y_1, y_2))$ on $\mathcal{M}$ satisfying*

$$E_\varepsilon(\hat{w}_\varepsilon; (y_1, y_2)) \leq M, \tag{4.1.30}$$

for all $\varepsilon < \varepsilon_0$. Then $\hat{w}_\varepsilon$ solves the Euler–Lagrange equation

$$2\varepsilon^4 \hat{w}_\varepsilon^{(iv)} + 2q\varepsilon^2 \hat{w}_\varepsilon'' + W'(\hat{w}_\varepsilon) = 0, \tag{4.1.31}$$

with additional natural boundary conditions $\hat{w}_\varepsilon''(y_1) = \hat{w}_\varepsilon''(y_2) = 0$, and for all $\varepsilon < \varepsilon_0$ satisfies the estimates

$$\operatorname{dist}(\hat{w}_\varepsilon((y_1 + y_2)/2), \{\pm 1\}) \leq C_M \exp\left(-\frac{d\gamma}{2\varepsilon}\right), \tag{4.1.32}$$

$$|\hat{w}_\varepsilon^{(m)}((y_1 + y_2)/2)| \leq C_M \exp\left(-\frac{d\gamma}{2\varepsilon}\right), \quad m = 1, \ldots 3, \tag{4.1.33}$$

where $d := y_2 - y_1$ and $C_M > 0$ is a positive constant dependent only on M, q and the potential W.

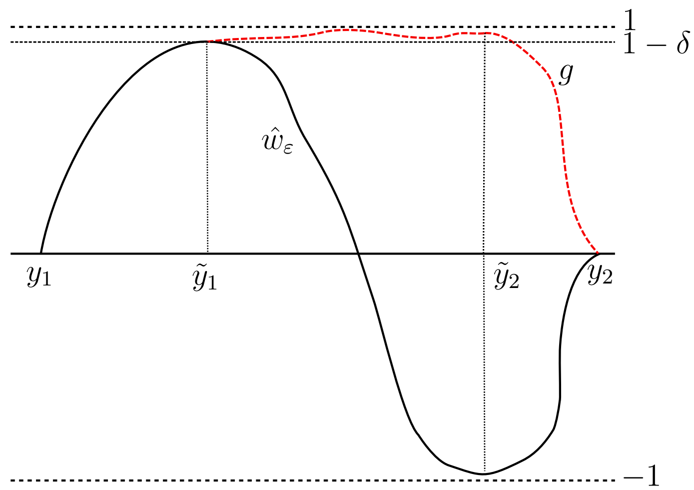

Figure 4.1.2: The contradiction argument.

Proof. Fix $\delta > 0$ to be chosen later. We first observe that, due to the upper bound (4.1.30) and Lemma 4.1.2, there exists $c = c(\delta, M) > 0$ and points $\tilde{y}_1 \in (y_1, y_1 + c\varepsilon)$ and $\tilde{y}_2 \in (y_2 - c\varepsilon, y_2)$ such that

$$\operatorname{dist}(\hat{w}_\varepsilon(\tilde{y}_1), \{\pm 1\}) < \delta, \quad |\varepsilon \hat{w}_\varepsilon'(\tilde{y}_1)| < \delta, \tag{4.1.34}$$

$$\operatorname{dist}(\hat{w}_\varepsilon(\tilde{y}_2), \{\pm 1\}) < \delta, \quad |\varepsilon \hat{w}_\varepsilon'(\tilde{y}_2)| < \delta. \tag{4.1.35}$$

In addition, we claim that since $\hat{w}_\varepsilon$ is a minimizer, at $\tilde{y}_1$ and $\tilde{y}_2$ its value is near the same well of W, i.e., we may assume without loss of generality that

$$|\hat{w}_\varepsilon(\tilde{y}_1) - 1| < \delta, \quad |\hat{w}_\varepsilon(\tilde{y}_2) - 1| < \delta. \tag{4.1.36}$$

As a matter of fact, if this was not the case and for example

$$|\hat{w}_\varepsilon(\tilde{y}_1) - 1| < \delta, \quad |\hat{w}_\varepsilon(\tilde{y}_2) + 1| < \delta. \tag{4.1.37}$$

then consider

$$g(x) := \begin{cases} \hat{w}_\varepsilon(x), & y_1 \le x \le \tilde{y}_1, \\ \phi(x), & \tilde{y}_1 \le x \le \tilde{y}_2, \\ -\hat{w}_\varepsilon(x), & \tilde{y}_2 \le x \le y_2, \end{cases}$$

where

$$\begin{aligned} \phi(x) := 1 &+ (\hat{w}_\varepsilon(\tilde{y}_1) - 1)\varphi_0(x - \tilde{y}_1) + \hat{w}_\varepsilon'(\tilde{y}_1)\varphi_1(x - \tilde{y}_1) \\ &+ (-\hat{w}_\varepsilon(\tilde{y}_2) - 1)\varphi_0(\tilde{y}_2 - x) + \hat{w}_\varepsilon'(\tilde{y}_2)\varphi_1(\tilde{y}_2 - x), \end{aligned} \tag{4.1.38}$$

and φ_0, φ_1 satisfy

$$\varphi_j \in C^\infty(\mathbb{R}), \ \varphi_j(x) = 0 \text{ for all } x \ge (y_2 - y_1)/2,$$

$$\varphi_0(0) = 1,\ \varphi_0'(0) = 0,\ \varphi_1(0) = 0,\ \varphi_1'(0) = 1.$$

It is easy to see that

$$\phi(\tilde{y}_1) = \hat{w}_\varepsilon(\tilde{y}_1), \quad \phi'(\tilde{y}_1) = \hat{w}_\varepsilon'(\tilde{y}_1),$$
$$\phi(\tilde{y}_2) = -\hat{w}_\varepsilon(\tilde{y}_2), \quad \phi'(\tilde{y}_2) = -\hat{w}_\varepsilon'(\tilde{y}_2),$$

and consequently $g \in H^2((y_1, y_2))$. Obtaining ϕ' from (4.1.38) and using (4.1.37), we get

$$||\phi'||^2_{L^\infty(\tilde{y}_1,\tilde{y}_2)} \leq c(||\varphi_0'||^2_{L^\infty(\mathbb{R})} + ||\varphi_1'||^2_{L^\infty(\mathbb{R})})\delta^2,$$

where $c > 0$ is a constant and we notice that

$$\int_{\tilde{y}_1}^{\tilde{y}_2} |\phi'|^2 dx \leq c(y_2 - y_1)(||\varphi_0'||^2_{L^\infty(\mathbb{R})} + ||\varphi_1'||^2_{L^\infty(\mathbb{R})})\delta^2.$$

Similarly, an analogous bound for ϕ'' can be derived. Additionally, using Taylor's formula for W and the facts that $W(\pm 1) = W'(\pm 1) = 0$ and $W \in C^2(\mathbb{R})$, it follows that

$$E_\varepsilon(\phi; (\tilde{y}_1, \tilde{y}_2)) \leq \xi_1 \delta^2, \tag{4.1.39}$$

where ξ_1 only depends on y_1 and y_2, which do not depend on δ, while interpolation inequality of Corollary 4.1.1 yields for δ sufficiently small

$$\begin{aligned}
E_\varepsilon(\hat{w}_\varepsilon; (\tilde{y}_1, \tilde{y}_2)) &= \int_{\tilde{y}_1}^{\tilde{y}_2} \left(\frac{1}{\varepsilon} W(\hat{w}_\varepsilon) - q\varepsilon |\hat{w}_\varepsilon'|^2 + \varepsilon^3 |\hat{w}_\varepsilon''|^2 \right) dx \\
&\geq \sigma \int_{\tilde{y}_1}^{\tilde{y}_2} \left(\frac{1}{\varepsilon} W(\hat{w}_\varepsilon) + \varepsilon |\hat{w}_\varepsilon'|^2 \right) dx \\
&\geq \sigma \int_{\tilde{y}_1}^{\tilde{y}_2} \sqrt{W(\hat{w}_\varepsilon)} \hat{w}_\varepsilon' dx = \sigma \int_{\hat{w}_\varepsilon(\tilde{y}_1)}^{\hat{w}_\varepsilon(\tilde{y}_2)} \sqrt{W(s)} ds \\
&\geq \sigma \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{W(s)} ds =: \sigma \xi_2 > 0.
\end{aligned}$$

In turn, from (4.1.39), possibly choosing δ even smaller we get a contradiction with the fact that $\hat{w}_\varepsilon$ is a minimizer.

Since $\hat{w}_\varepsilon$ is a minimizer of $E_\varepsilon(\cdot; (y_1, y_2))$, it follows from standard arguments that it satisfies the Euler–Lagrange equation (4.1.31). We change variables $z = \frac{x - y_1}{\varepsilon}$ and define $\hat{v}(z) := \hat{w}_\varepsilon(x)$. Observe that

$$E_\varepsilon(\hat{w}_\varepsilon; (y_1, y_2)) = E(\hat{v}; (0, d/\varepsilon)) \tag{4.1.40}$$

and the rescaled minimizer $\hat{v}$ satisfies the Euler–Lagrange equation

$$2\hat{v}^{(iv)} + 2q\hat{v}'' + W'(\hat{v}) = 0, \quad \hat{v}''(0) = \hat{v}''(d/\varepsilon) = 0. \tag{4.1.41}$$

We now apply Lemma 4.1.3 on the interval $\left(\frac{\tilde{y}_1-y_1}{\varepsilon}, \frac{\tilde{y}_2-y_1}{\varepsilon}\right)$ with

$$\alpha_0 := \hat{w}_\varepsilon(\tilde{y}_1) = \hat{v}\left(\frac{\tilde{y}_1 - y_1}{\varepsilon}\right), \quad \alpha_1 := \varepsilon \hat{w}'_\varepsilon(\tilde{y}_1) = \hat{v}'\left(\frac{\tilde{y}_1 - y_1}{\varepsilon}\right), \tag{4.1.42}$$

$$\beta_0 := \hat{w}_\varepsilon(\tilde{y}_2) = \hat{v}\left(\frac{\tilde{y}_2 - y_1}{\varepsilon}\right), \quad \beta_1 := \varepsilon \hat{w}'_\varepsilon(\tilde{y}_2) = \hat{v}'\left(\frac{\tilde{y}_2 - y_1}{\varepsilon}\right). \tag{4.1.43}$$

The resulting minimizer agrees with $\hat{v}$ on this interval and given $R > 0$, for δ sufficiently small the bounds (4.1.34) and (4.1.35) imply that

$$\chi := [\hat{v} - 1, \hat{v}', \hat{v}'', \hat{v}'''] \in B(0, R).$$

Using the notation $\chi = [\chi_1, \chi_2, \chi_3, \chi_4]$, we rewrite (4.1.41) in the system form

$$\chi' = F(\chi) \tag{4.1.44}$$

where

$$F(\chi) = \begin{bmatrix} \chi_2 \\ \chi_3 \\ \chi_4 \\ -\frac{1}{2}W'(\chi_1) - q\chi_2 \end{bmatrix}$$

and the Jacobian of F at 0 is given by

$$DF(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{2}W''(1) & 0 & -q & 0 \end{bmatrix}.$$

The eigenvalues of $DF(0)$ are the roots of the characteristic polynomial

$$2r^4 + 2qr^2 + W''(1) = 0.$$

In particular,

$$r^2 = \frac{-2q \pm \sqrt{4q^2 - 8W''(1)}}{4},$$

and since $q > 0$ is small, the expression under the square root is negative. We write

$$\begin{cases} r^2 = \dfrac{-2q + \sqrt{4q^2 - 8W''(1)}}{4}, \\ r^2 = \dfrac{-2q - \sqrt{4q^2 - 8W''(1)}}{4} \end{cases}$$

and let r_1, r_2 be the roots of the first equation, r_3, r_4 those of the second one. We recall that

$$\sqrt{a + ib} = \pm(\gamma + i\delta),$$

for

$$\gamma = \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}, \quad \delta = \operatorname{sgn}(b)\sqrt{\frac{-a+\sqrt{a^2+b^2}}{2}}.$$

In the case of r_1, we write

$$r_1 = \left(-\frac{q}{2} + i\frac{\sqrt{2W''(1)-q^2}}{2}\right)^{1/2},$$

and a simple calculation shows that

$$\gamma = \frac{1}{2}\left(-q+\sqrt{2W''(1)}\right)^{1/2}, \quad \delta = \frac{1}{2}\left(q+\sqrt{2W''(1)}\right)^{1/2}. \tag{4.1.45}$$

Similarly, one can show that

$$\begin{cases} r_1 = \gamma + i\delta, \\ r_2 = -r_1, \\ r_3 = \gamma - i\delta, \\ r_4 = -r_3, \end{cases} \tag{4.1.46}$$

Applying Lemma 4.1.4 on the interval $\left(c, \frac{y_2-y_1-c\varepsilon}{\varepsilon}\right) \subset \left(\frac{\tilde{y}_1-y_1}{\varepsilon}, \frac{\tilde{y}_2-y_1}{\varepsilon}\right)$ yields

$$\left|\varphi\left(\frac{y_2-y_1}{2\varepsilon}\right)\right| \leq C(\gamma,\delta)\exp\left(-\gamma\frac{y_2-y_1-2c\varepsilon}{2\varepsilon}\right) \leq C(\gamma,\delta)\exp\left(\gamma c - \gamma\frac{d}{2\varepsilon}\right) \tag{4.1.47}$$

and (4.1.32), (4.1.33) follow from definition of φ and the fact that

$$\hat{w}\left(\frac{y_1+y_2}{2}\right) = \hat{v}\left(\frac{y_2-y_1}{2\varepsilon}\right). \tag{4.1.48}$$

□

Proof of Theorem 1.3.1. Without loss of generality we can assume that $N(v) \geq 2$ and define

$$\mathcal{M}_k := \{w \in H^2((x_k, x_{k+1})) : w(x_k) = 0, \ w(x_{k+1}) = 0\}. \tag{4.1.49}$$

We define $\hat{w}_k \in H^2((x_k, x_{k+1}))$, for $1 \leq k \leq N$, to be the minimizer of $E_\varepsilon(\cdot, (x_k, x_{k+1}))$ over $\mathcal{M}_k$. We also let $\hat{w}_0 := \hat{w}_N$. In turn, $\hat{w}_k$ solves the Euler–Lagrange equation (4.1.31) with

$$\hat{w}_k(x_k) = \hat{w}_k(x_{k+1}) = 0.$$

Define $d_k := x_{k+1} - x_k$ for $k = 1, \ldots, N(v)$ and

$$I_k^-(x_k) := \left(x_k - \frac{d_{k-1}}{2}, x_k\right) \quad \text{and} \quad I_k^+(x_k) := \left(x_k, x_k + \frac{d_k}{2}\right).$$

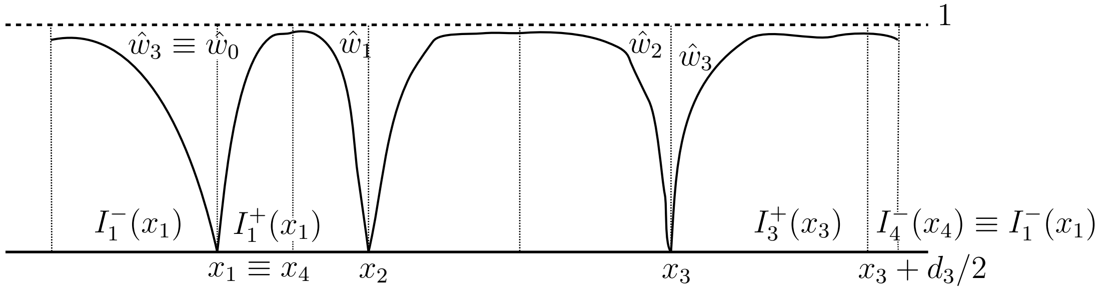

Figure 4.1.3: $\hat{w}_k$ and $I_k^\pm$

From the minimality of $\hat{w}_k$, we have

$$\begin{aligned} E_\varepsilon(w;\mathbb{T}) &= \sum_{k=1}^{N(v)} E_\varepsilon(w;(x_k,x_{k+1})) \geq \sum_{k=1}^{N(v)} E_\varepsilon(\hat{w}_k;(x_k,x_{k+1})) \\ &= \sum_{k=1}^{N(v)} E_\varepsilon(\hat{w}_{k-1};I_k^-(x_k)) + E_\varepsilon(\hat{w}_k;I_k^+(x_k)), \end{aligned} \tag{4.1.50}$$

where in the last equality we have used the fact that $x_{N+1} := x_1$. To complete the proof, it remains to show that

$$E_\varepsilon(\hat{w}_{k-1};I_k^-(x_k)) \geq \frac{m_1}{2} - C\exp\left(-\frac{d_{k-1}\gamma}{\varepsilon}\right) \tag{4.1.51}$$

and

$$E_\varepsilon(\hat{w}_k;I_k^+(x_k)) \geq \frac{m_1}{2} - C\exp\left(-\frac{d_k\gamma}{\varepsilon}\right). \tag{4.1.52}$$

We will only prove (4.1.52), the proof of the first inequality being analogous. Applying the change of variables $z := \frac{x-x_k}{\varepsilon}$ gives

$$\begin{aligned} E_\varepsilon(\hat{w}_k;I_k^+(x_k)) &= \int_{\frac{1}{\varepsilon}I_k^+(0)} \left(W(\hat{w}_k(x_k+\varepsilon z)) - \varepsilon q|\hat{w}_k'(x_k+\varepsilon z)|^2 + \varepsilon^3|\hat{w}_k''(x_k+\varepsilon z)|^2\right)\varepsilon dz \\ &= \int_{\frac{1}{\varepsilon}I_k^+(0)} \left(W(\hat{v}_k(z)) - q|\hat{v}_k'(z)|^2 + |\hat{v}_k''(z)|^2\right)dz = E\left(\hat{v}_k;\frac{1}{\varepsilon}I_k^+(0)\right), \end{aligned}$$

where $E(\cdot;\cdot)$ is the rescaled functional defined in (2.5.2) and

$$\hat{v}_k(z) := \hat{w}_k(x) \text{ on each } \frac{1}{\varepsilon}I_k^+(0).$$

In addition, we notice that $\hat{v}_k(0) = \hat{w}_k(x_k) = 0$ for $1 \leq k \leq N$ and Proposition 4.1.1, together with the change of variables we performed, gives

$$|\hat{v}_k(d_k/2\varepsilon) - s_k| = |\hat{w}_k((x_k+x_{k+1})/2) - s_k| \leq C_f\exp\left(-\frac{d_k\gamma}{2\varepsilon}\right) \tag{4.1.53}$$

and

$$|\hat{v}_k'(d_k/2\varepsilon)| = |\hat{w}_k'((x_k + x_{k+1})/2)| \le C_f \exp\left(-\frac{d_k\gamma}{2\varepsilon}\right), \tag{4.1.54}$$

where s_k is equal to either 1 or -1. We claim that

$$E\left(\hat{v}_k; \frac{1}{\varepsilon} I_k^{\varepsilon,+}(0)\right) \ge \frac{m_1}{2} - E\left(\eta_k; \mathbb{R}^+\right) \tag{4.1.55}$$

where

$$\begin{aligned} \eta_k(x) := {} & s_k + (\hat{v}_k(d_k/2\varepsilon) - s_k) \exp(-\gamma x) \cos(\delta x) \\ & + \frac{\hat{v}_k'(d_k/2\varepsilon) + \gamma(\hat{v}_k(d_k/2\varepsilon) - s_k)}{\delta} \exp(-\gamma x) \sin(\delta x). \end{aligned} \tag{4.1.56}$$

See Remark 4.1.5 for the motivations behind the definition of (4.1.56). Indeed, let $\theta_\varepsilon^+ \in H^2_{loc}(\mathbb{R}^+)$ be the function that coincides with $\hat{v}_k$ on $\frac{1}{\varepsilon} I_k^{\varepsilon,+}(0)$ and $\eta_k^+ := \eta_k(\cdot - d_k/2\varepsilon)$ on $\mathbb{R}^+ \setminus \frac{1}{\varepsilon} I_k^{\varepsilon,+}(0)$. Then,

$$E(\theta_\varepsilon^+; \mathbb{R}^+) \ge m_1/2,$$

and in turn (4.1.55) follows. We now want to find an upper bound for $E(\eta_k; \mathbb{R}^+)$, for ε small enough.

The bounds (4.1.53), (4.1.54) and the definition of η_k imply that there exists a constant $C > 0$ such that

$$|\eta_k(x) - s_+| + |\eta_k'(x)| + |\eta_k''(x)| \le C \exp\left(-\frac{d_k\gamma}{2\varepsilon}\right) \exp(-\gamma x) \quad \text{for all } x > 0 \tag{4.1.57}$$

and consequently

$$\begin{aligned} E(\eta_k; \mathbb{R}^+) &= \int_0^\infty W(\eta_k) - q|\eta_k'|^2 + |\eta_k''|^2 dx \\ &= \int_0^\infty \frac{W''(s_+)}{2} (\eta_k - s_+)^2 - q|\eta_k'|^2 + |\eta_k''|^2 + O((\eta^k - s_+)^3) dx \\ &\le C \exp\left(-\frac{d_k\gamma}{\varepsilon}\right) \int_0^\infty \exp(-2\gamma x) dx \le C \exp\left(-\frac{d_k\gamma}{\varepsilon}\right). \end{aligned}$$

□

Remark 4.1.5. *The construction of a competitor for the minimization of the energy $E(\cdot; \mathbb{R}^+)$ relies on exploiting, heuristically, asymptotic properties of the minimizers. In particular, notice that if $v(x) := u(x) - 1$ solves the Euler–Lagrange equation*

$$2v^{(iv)} + 2qv'' + W''(1)v + g(v) = 0 \quad \textit{on } [0, \infty), \tag{4.1.58}$$

where $g(v) := W'(v+1) - W'(1) - W''(1)v$ then the solution to the linear part of the equation is of the form

$$v_\ell(x) = c_1 e^{-\gamma x} \cos(\delta x) + c_2 e^{-\gamma x} \sin(\delta x) + c_3 e^{\gamma x} \cos(\delta x) + c_4 e^{\gamma x} \sin(\delta x).$$

The structure of the ODE (4.1.58) *allows us to use similar arguments to the ones used in [24] (see Theorem 4.1, page 330) and infer that* $v(x) \to 0$ *and* $v_\ell(x) \to 0$ *as* $x \to \infty$ *under appropriate conditions on the initial data. We then consider the function* $x \mapsto \psi_k(x)$ *defined via*

$$\psi_k(x) := c_1^k e^{-\gamma x}\cos(\delta x) + c_2^k e^{-\gamma x}\sin(\delta x)$$

obtained by setting $c_3 = c_4 = 0$ *in the expression for* v_ℓ*, as* v_ℓ *would not converge to* 0 *as* $x \to \infty$ *otherwise. Now consider the function* $\hat{v}_k$*. Our goal is to glue the functions* $\hat{v}_k$ *and* ψ_k *in a* C^1 *manner and the first step in this program is to find appropriate values for the constants* c_i^k*. In particular, we need to require that*

$$\begin{cases} \psi_k(0) = v_k(d_k/2\varepsilon) - s_k, \\ \psi_k'(0) = v_k'(d_k/2\varepsilon). \end{cases}$$

Straightforward computations show that

$$v_k(d_k/2\varepsilon) - s_k = \psi_k(0) = c_1^k$$

and

$$v_k'(d_k/2\varepsilon) = \psi_k'(0) = -\gamma c_1^k + \delta c_2^k = -\gamma[v_k(d_k/2\varepsilon) - s_k] + \delta c_2^k$$

so that

$$c_2^k = \frac{\gamma[v_k(d_k/2\varepsilon) - s_k] - v_k'(d_k/2\varepsilon)}{\delta}.$$

For the sake of consistency, we introduce the map $x \mapsto \eta_k(x)$ *defined via*

$$\eta_k(x) := \psi_k(x) - s_k.$$

We can finally define a competitor θ_ε^+ *for the minimization of the energy* $E(\cdot;\mathbb{R}^+)$ *as in the final part of the previous proof.*

4.2 Slow motion dynamics: proof of Theorem 1.3.3

Proof of Theorem 1.3.3. Fix $0 < \delta < \min\{1, d/8\}$. We recall that by definition of E_ε

$$E_\varepsilon(u^\varepsilon(\cdot,t);\mathbb{T}) = \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W(u^\varepsilon) - \varepsilon q|u_x^\varepsilon|^2 + \varepsilon^3|u_{xx}^\varepsilon|^2\right)dx.$$

Integrating by parts and using the regularity of the solution u^ε and equation (1.3.2) gives

$$\begin{aligned} \frac{d}{dt}E_\varepsilon(u^\varepsilon(\cdot,t);\mathbb{T}) &= \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W'(u^\varepsilon)u_t^\varepsilon - 2\varepsilon q u_x^\varepsilon u_{xt}^\varepsilon + 2\varepsilon^3 u_{xx}^\varepsilon u_{xxt}^\varepsilon\right)dx \\ &= \int_{\mathbb{T}} \left(\frac{1}{\varepsilon}W'(u^\varepsilon)u_t^\varepsilon + 2\varepsilon q u_{xx} u_t^\varepsilon + 2\varepsilon^3 u_{xxxx}^\varepsilon u_t^\varepsilon\right)dx \\ &= -\int_{\mathbb{T}} |u_t^\varepsilon|^2 dx. \end{aligned}$$

It follows that for every $T > 0$,

$$E_\varepsilon(u_0^\varepsilon; \mathbb{T}) - E_\varepsilon(u^\varepsilon(\cdot, T); \mathbb{T}) = \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{T}} |u_t^\varepsilon|^2 dx dt. \tag{4.2.1}$$

Suppose there exists T_ε such that

$$\int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dx dt \leq \delta \tag{4.2.2}$$

Then,

$$\int_{\mathbb{T}} |u_0^\varepsilon - u^\varepsilon(\cdot, T_\varepsilon)| dx = \int_{\mathbb{T}} \left| \int_0^{T_\varepsilon} u_t^\varepsilon dt \right| dx \leq \int_{\mathbb{T}} \int_0^{T_\varepsilon} |u_t^\varepsilon| dt dx \leq \delta \tag{4.2.3}$$

and using (1.3.10) and the triangle inequality

$$||u^\varepsilon(\cdot, T_\varepsilon) - v||_{L^1(\mathbb{T})} \leq 2\delta. \tag{4.2.4}$$

We claim that $u^\varepsilon(\cdot, T_\varepsilon)$ has at least N_ε zeros, $\{x_k^\varepsilon\}_{k=1}^{N_\varepsilon}$ that satisfy $\min_k |x_{k+1}^\varepsilon - x_k^\varepsilon| \geq d - 4\delta$.

Indeed, consider x_k, the k–th jump point of v. Since the distance between jump points of v is at least d and $\delta \leq d/8$, we know that v is constant on $(x_k - 2\delta, x_k)$ and on $(x_k, x_k + 2\delta)$ and may assume without loss of generality that its value is equal to 1 on $(x_k - 2\delta, x_k)$ and to -1 on $(x_k, x_k + 2\delta)$. It follows from (4.2.4) that $u^\varepsilon(\cdot, T_\varepsilon)$ must take a positive value somewhere on $(x_k - 2\delta, x_k)$ and a negative value on $(x_k, x_k + 2\delta)$. Hence, there exists a zero $x_k^\varepsilon \in (x_k - 2\delta, x_k + 2\delta)$ of $u^\varepsilon(\cdot, T_\varepsilon)$.

Applying Hölder inequality, (1.3.10), (4.2.1), and Theorem 1.3.1 yields

$$\begin{aligned}
\frac{1}{T_\varepsilon} \left(\int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dx dt \right)^2 &\leq \int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon|^2 dx dt \\
&= \varepsilon \left(E_\varepsilon(u_0^\varepsilon; \mathbb{T}) - E_\varepsilon(u^\varepsilon(\cdot, T_\varepsilon); \mathbb{T}) \right) \\
&\leq \varepsilon \left(E_0(v; \mathbb{T}) + \frac{1}{h(\varepsilon)} - m_1 N_\varepsilon + C \sum_{k=1}^{N_\varepsilon} \exp\left(-\frac{(x_{k+1}^\varepsilon - x_k^\varepsilon)\gamma}{\varepsilon} \right) \right) \qquad (4.2.5) \\
&\leq \varepsilon \left(E_0(v; \mathbb{T}) + \frac{1}{h(\varepsilon)} - E_0(v; \mathbb{T}) + C \exp\left(-\frac{(d - 4\delta)\gamma}{\varepsilon} \right) \right) \\
&= \varepsilon \left(\frac{1}{h(\varepsilon)} + C \exp\left(-\frac{(d - 4\delta)\gamma}{\varepsilon} \right) \right)
\end{aligned}$$

and as a consequence,

$$T_\varepsilon \geq \frac{1}{C\varepsilon} \left[\frac{1}{h(\varepsilon)} + \exp\left(-(d - 4\delta)\gamma/\varepsilon \right) \right]^{-1} \left(\int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dx dt \right)^2. \tag{4.2.6}$$

Following the ideas of [44], we prove the existence of T_ε as in (4.2.2) by dividing the analysis into two cases: first assume that

$$\int_0^\infty \int_{\mathbb{T}} |u_t^\varepsilon| dx dt > \delta.$$

Since by (4.2.1) with T replaced by any $S > 0$,

$$\int_0^S \int_{\mathbb{T}} |u_t^\varepsilon|^2 dx dt \leq \varepsilon E_\varepsilon(u_0^\varepsilon; \mathbb{T}) < \infty$$

we can choose T_ε such that

$$\int_0^{T_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dx dt = \delta, \tag{4.2.7}$$

and thanks to (4.2.7), equation (4.2.6) gives

$$T_\varepsilon \geq \frac{\delta^2}{C\varepsilon \left[\frac{1}{h(\varepsilon)} + \exp\left(-(d-4\delta)\gamma/\varepsilon\right)\right]} \geq \frac{\delta^2}{2C\varepsilon} \min\{h(\varepsilon), \exp((d-4\delta)\gamma/\varepsilon)\} =: \Lambda_\varepsilon.$$

In turn, (4.2.2) is satisfied and (4.2.5) yields

$$\int_0^{\Lambda_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon|^2 dx dt \leq C\varepsilon \left[\frac{1}{h(\varepsilon)} + \exp\left(-(d-4\delta)\gamma/\varepsilon\right)\right]. \tag{4.2.8}$$

On the other hand, if

$$\int_0^\infty \int_{\mathbb{T}} |u_t^\varepsilon| dx dt \leq \delta,$$

then (4.2.2) holds true for all $T > 0$ and again (4.2.8) follows. To conclude the proof note that for ε sufficiently small

$$s_\varepsilon := \delta^2 \min\{h(\varepsilon), \exp((d-4\delta)\gamma/\varepsilon)\} \leq \Lambda_\varepsilon$$

and Hölder's inequality together with (4.2.8) yield

$$\begin{aligned}
&\sup_{0 \leq t \leq s_\varepsilon} \int_{\mathbb{T}} |u^\varepsilon(x,t) - u_0^\varepsilon(x)| dx \leq \int_0^{s_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon| dx dt \\
&\leq \left(\min\left\{h(\varepsilon), \exp\left(\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right\} \int_0^{s_\varepsilon} \int_{\mathbb{T}} |u_t^\varepsilon|^2 dx dt\right)^{1/2} \\
&\leq C\left(\min\left\{h(\varepsilon), \exp\left(\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right\} \varepsilon\delta^2 \left[\frac{1}{h(\varepsilon)} + \exp\left(-\frac{(d-4\delta)\gamma}{\varepsilon}\right)\right]\right)^{1/2} \\
&\leq C\sqrt{\varepsilon}\delta.
\end{aligned}$$

Letting $\varepsilon \to 0^+$ gives (1.3.12). □

4.3 Existence of solutions via Minimizing Movements

We now turn to the existence and regularity of solutions for (1.3.2) in the more general case of an open, bounded domain $\Omega \subset \mathbb{R}^d$. We notice that the same proof carries over in the case of the one–dimensional torus $\Omega = \mathbb{T}$, that is, when we deal with periodic Dirichlet boundary conditions, which is the framework in which we have analyzed slow motion of solutions of (1.3.2).

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^d$, $d \leq 3$, be an open bounded set with C^2 boundary, let $u_0 \in H^2(\Omega)$ and the real valued function $z \mapsto W(z)$ be a double–well potential satisfying hypotheses* (2.1.3), (2.1.5) *and* (2.1.7)–(2.1.9). *Then for every $T > 0$ there exists a weak solution $u^\varepsilon \in L^\infty((0,T); H^2(\Omega))$ in the sense of* (4.3.25), *with $u_t^\varepsilon \in L^2((0,T); L^2(\Omega))$ of*

$$\begin{cases} u_t = -\frac{1}{\varepsilon}W'(u) - 2\varepsilon q\Delta u - 2\varepsilon^3\Delta^2 u & \text{in } \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{4.3.1}$$

such that

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx + \int_0^t \int_\Omega \frac{1}{\varepsilon}W'(u(x,s))dxds.$$

Moreover, the following estimates hold

$$\int_0^T \int_\Omega |u_t(x,t)|^2 dxdt \leq M_\varepsilon \sigma^{-1},$$
$$\int_\Omega |\nabla u(x,t)|^2 dx \leq 3M_\varepsilon \sigma^{-1},$$
$$\int_\Omega |\nabla^2 u(x,t)|^2 dx \leq 3M_\varepsilon \sigma^{-1},$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where $\sigma \in (0,1)$ and

$$M_\varepsilon := 2\int_\Omega \left(\frac{1}{\varepsilon}W(u_0) + \varepsilon|\nabla u_0|^2 + \varepsilon^3|\nabla^2 u_0|^2\right) dx. \tag{4.3.2}$$

Proof. **Step 1.** For $\ell \in \mathbb{N}$ we set $\tau := T/\ell$ and subdivide the interval $(0,T)$ into ℓ subintervals of length τ,

$$\tau_0 := 0 < \tau_1 < \ldots < \tau_\ell := T,$$

where $\tau_n := n\tau$ for $n = 1, \ldots, \ell$. For every $n = 1, \ldots, \ell$, we let $u_n \in H^2(\Omega)$ be a solution of the minimization problem

$$\min_{v \in H^2(\Omega)} J_{\varepsilon,n}(v; \Omega),$$

where

$$\begin{aligned} J_{\varepsilon,n}(v;\Omega) &:= \int_\Omega \left(\frac{1}{\varepsilon}W(v) - \varepsilon q|\nabla v|^2 + \varepsilon^3|\nabla^2 v|^2\right) dx + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 dx \\ &= E_\varepsilon(v;\Omega) + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 dx. \end{aligned}$$

In order to prove the existence of u_n, we begin by showing that J_n is non–negative and coercive in $H^2(\Omega)$. We fix $q^* > 0$ such that the interpolation inequality Lemma 2.5.1 holds in Ω, namely

$$k\varepsilon^2 \int_\Omega |\nabla u|^2 dx \leq \int_\Omega \left[W(u) + \varepsilon^4 |\nabla^2 u|^2\right] dx, \ -\infty < k \leq q^*,$$

and we let $\sigma \in (0,1)$ be such that $(q+\sigma)/(1-\sigma) < q^*$, so that we can write

$$\begin{aligned} W(u) - q^2\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2 = (1-\sigma)\left(W(u) - \frac{q+\sigma}{1-\sigma}\varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\right) \\ + \sigma(W(u) + \varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2), \end{aligned} \tag{4.3.3}$$

and in turn $J_{\varepsilon,n}$ is non–negative. Then by (2.1.8), and using the fact that $c_W \leq 1$, we obtain

$$E_\varepsilon(u;\Omega) \geq \sigma c_W \int_\Omega \left((|u|-1)^2 + \varepsilon^2|\nabla u|^2 + \varepsilon^4|\nabla^2 u|^2\right) dx. \tag{4.3.4}$$

The above chain of inequalities implies that

$$J_{\varepsilon,n}(u;\Omega) = E_\varepsilon(u;\Omega) + \frac{1}{2\tau}\int_\Omega (v - u_{n-1})^2 dx \to \infty \quad \text{as } ||u||_{H^2(\Omega)} \to \infty,$$

and hence J_ε is coercive in $H^2(\Omega)$.

We now let $m_n := \inf_{v \in H^2(\Omega)} J_{\varepsilon,n}(v;\Omega)$, and consider a minimizing sequence $\{v_k\} \subset H^2(\Omega)$ satisfying

$$m_n \leq J_{\varepsilon,n}(v_k;\Omega) \leq m_n + \frac{1}{k},$$

so that

$$\lim_{k\to\infty} J_{\varepsilon,n}(v_k;\Omega) = m_n.$$

It follows from (4.3.4) that $\{v_k\}$ is bounded in $H^2(\Omega)$, and hence there exist a subsequence of $\{v_k\}$ (not relabeled) and some $u_n \in H^2(\Omega)$ such that

$$\begin{aligned} v_k &\to u_n \quad \text{in } L^2(\Omega), \\ v_k &\to u_n \quad \text{pointwise a.e. in } \Omega, \\ \nabla v_k &\to \nabla u_n \quad \text{in } L^2(\Omega), \\ \nabla^2 v_k &\rightharpoonup \nabla^2 u_n \quad \text{in } L^2(\Omega). \end{aligned}$$

We claim that the above convergences imply that $J_{\varepsilon,n}(u_n;\Omega) = m_n$. Indeed, by Fatou's Lemma and lower semicontinuity of L^2 norm with respect to weak convergence, we have

$$m_n = \liminf_{k\to\infty} J_{\varepsilon,n}(v_k;\Omega) \geq J_n(u_n) \geq m_n.$$

It follows that for all $w \in H^2(\Omega)$ and all $t \in \mathbb{R}$,

$$J_{\varepsilon,n}(u_n;\Omega) \le J_{\varepsilon,n}(u_n + tw;\Omega),$$

and hence the real valued function $\omega(t) := J_{\varepsilon,n}(u_n + tw;\Omega)$ has a minimum at $t = 0$, so that $\omega'(0) = 0$. Standard arguments show that for every $w \in H^2(\Omega)$,

$$\begin{aligned} 0 = &\int_\Omega \left(\frac{1}{\varepsilon}W'(u_n)w - 2\varepsilon q\nabla u_n \cdot \nabla w + 2\varepsilon^3\nabla^2 u_n \cdot \nabla^2 w\right) \\ &+ \frac{1}{\tau}\int_\Omega (u_n - u_{n-1})w, \end{aligned} \tag{4.3.5}$$

where $W'(u_n)w$ is well–defined by the embedding of $H^2(\Omega)$ into $L^\infty(\Omega)$ for $d \le 3$, and $\nabla^2 u_n \cdot \nabla^2 w = \sum_{i,j} \frac{\partial^2 u_n}{\partial x_i \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_j}$ is the Frobenius inner product. In particular, this shows that u_n is a weak solution of the equation

$$-\frac{1}{\varepsilon}W'(u_n) - 2\varepsilon q\Delta u_n - 2\varepsilon^3\Delta^2 u_n = \frac{1}{\tau}(u_n - u_{n-1}) \quad \text{in } \Omega.$$

Since Ω has finite measure, choosing $w = 1$ in (4.3.5) gives

$$0 = \int_\Omega \frac{1}{\varepsilon}W'(u_n)dx + \frac{1}{\tau}\int_\Omega (u_n - u_{n-1})dx.$$

Step 2: Apriori bounds. For $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$, $n = 1, \dots, \ell$, we define

$$u^\tau(x,t) := u_n(x) + (t - \tau_n)\frac{u_n(x) - u_{n-1}(x)}{\tau}. \tag{4.3.6}$$

The goal of this step is to find apriori bounds on u^τ.

Since $J_{\varepsilon,n}(u_n;\Omega) = m_n$, it follows that $J_{\varepsilon,n}(u_n;\Omega) \le J_{\varepsilon,n}(u_{n-1};\Omega)$, which implies

$$\begin{aligned} \frac{1}{2\tau}\int_\Omega (u_n - u_{n-1})^2 dx \le &\int_\Omega \left(\frac{1}{\varepsilon}(W(u_{n-1}) - W(u_n)) - \varepsilon q(|\nabla u_{n-1}|^2 - |\nabla u_n|^2)\right) dx \\ &+ \int_\Omega \varepsilon^3(|\nabla^2 u_{n-1}|^2 - |\nabla^2 u_n|^2)dx. \end{aligned}$$

Summing over $n = 1, \dots, \ell$, we get

$$\begin{aligned} \frac{1}{2\tau}\sum_{n=1}^{\ell}\int_\Omega (u_n - u_{n-1})^2 dx \le &\int_\Omega \left(\frac{1}{\varepsilon}(W(u_0) - W(u_\ell)) - \varepsilon q(|\nabla u_0|^2 - |\nabla u_\ell|^2)\right) dx \\ &+ \int_\Omega \varepsilon^3(|\nabla^2 u_0|^2 - |\nabla^2 u_\ell|^2)dx. \end{aligned} \tag{4.3.7}$$

By the interpolation inequality in Lemma 2.5.1,

$$\int_\Omega \left(\frac{1}{\varepsilon}W(u_\ell) - \varepsilon q|\nabla u_\ell|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx \geq \sigma \int_\Omega \left(\frac{1}{\varepsilon}W(u_\ell) + \varepsilon|\nabla u_\ell|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx,$$

where $\sigma \in (0,1)$ was chosen above. Thus, the previous inequalities imply

$$\begin{aligned}\frac{1}{2\tau}\sum_{n=1}^{\ell}\int_\Omega (u_n - u_{n-1})^2\,dx + \sigma \int_\Omega \left(\frac{1}{\varepsilon}W(u_\ell) + \varepsilon|\nabla u_\ell|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx \\ \leq \int_\Omega \left(\frac{1}{\varepsilon}W(u_0) + \varepsilon|\nabla u_0|^2 + \varepsilon^3|\nabla^2 u_0|^2\right) dx = \frac{M_\varepsilon}{2},\end{aligned} \tag{4.3.8}$$

see (4.3.2). By (4.3.6), for every $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$,

$$\begin{aligned} u_t^\tau(x,t) &= \frac{u_n(x) - u_{n-1}(x)}{\tau}, \\ \nabla u^\tau(x,t) &= \nabla u_n(x) + (t-\tau_n)\frac{\nabla u_n(x) - \nabla u_{n-1}(x)}{\tau}, \\ \nabla^2 u^\tau(x,t) &= \nabla^2 u_n(x) + (t-\tau_n)\frac{\nabla^2 u_n(x) - \nabla^2 u_{n-1}(x)}{\tau}, \end{aligned} \tag{4.3.9}$$

so that by (4.3.8) we have

$$\frac{1}{2}\int_{\Omega_T} (u_t^\tau(x,t))^2\,dxdt + \sigma \int_\Omega \left(\frac{1}{\varepsilon}W(u_\ell) + \varepsilon|\nabla u_\ell|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx \leq \frac{M_\varepsilon}{2} \tag{4.3.10}$$

which implies

$$\int_{\Omega_T} (u_t^\tau(x,t))^2\,dxdt \leq M_\varepsilon, \tag{4.3.11}$$

for every $\tau > 0$. Since u^τ is absolutely continuous, for every $0 \leq t_1 < t_2 \leq T$,

$$\begin{aligned}\int_\Omega (u^\tau(x,t_2) - u^\tau(x,t_1))^2\,dx &= \int_\Omega \left(\int_{t_1}^{t_2} u_t^\tau(x,t)dt\right)^2 dx \\ &\leq (t_2 - t_1)\int_{\Omega_T} (u_t^\tau(x,t))^2\,dxdt \\ &\leq M_\varepsilon(t_2 - t_1).\end{aligned} \tag{4.3.12}$$

Taking $t_1 = 0$ and noticing that $u^\tau(x,0) = u_0(x)$, we get

$$\int_\Omega (u^\tau(x,t) - u_0)^2\,dx \leq M_\varepsilon t \tag{4.3.13}$$

for every $\tau > 0$ and all $t \in (0,T)$. In turn, by convexity of the function $z \mapsto z^2$,

$$\int_\Omega (u^\tau(x,t))^2\,dx \leq 2M_\varepsilon t + 2\int_\Omega u_0^2(x)dx \tag{4.3.14}$$

for every $\tau > 0$ and all $t \in (0, T)$.

Moreover, by (4.3.9), for $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n]$,

$$|\nabla u^\tau(x,t)| \leq 2|\nabla u_n(x)| + |\nabla u_{n-1}(x)|,$$
$$|\nabla^2 u^\tau(x,t)| \leq 2|\nabla^2 u_n(x)| + |\nabla^2 u_{n-1}(x)|,$$

and by (4.3.8) and arbitrariness of ℓ we get

$$\int_\Omega |\nabla u^\tau(x,t)|^2 dx \leq \frac{3M_\varepsilon}{\sigma}, \quad \int_\Omega |\nabla^2 u^\tau(x,t)|^2 dx \leq \frac{3M_\varepsilon}{\sigma}. \tag{4.3.15}$$

Step 3: Convergence as $\tau \to 0^+$. In the previous step we have shown that $\{u^\tau\}$ is bounded in $L^2((0,T); H^2(\Omega))$ and $\{u_t^\tau\}$ is bounded in $L^2((0,T); L^2(\Omega))$. Since these spaces are reflexive, there exist a subsequence of $\{u^\tau\}$ (not relabeled) and u such that $u^\tau \rightharpoonup u$ in $L^2((0,T); H^2(\Omega))$ and in $H^1((0,T); L^2(\Omega))$. Using the fact that the embeddings $H^2(\Omega) \hookrightarrow H^1(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact, it follows by the compactness theorem of Aubin and Lions (see e.g. [6]) and a diagonal argument, that, up to a further subsequence, $u^\tau \to u$ in $L^2((0,T); L^2(\Omega))$. In turn, for $\mathcal{L}^1$ a.e. $t \in (0,T)$ we have that $u^\tau(\cdot, t) \to u(\cdot, t)$ in $L^2(\Omega)$. We are now ready to let $\ell \to \infty$, or equivalently, $\tau \to 0^+$ in (4.3.11), (4.3.13), (4.3.15), and deduce the corresponding apriori bounds.

Step 4: u is a weak solution of the Swift–Hohenberg equation.
We let $x \in \Omega$ and $t \in (\tau_{n-1}, \tau_n)$, $n = 1, \ldots, \ell$, and define

$$\tilde{u}^\tau(x,t) := u_n(x). \tag{4.3.16}$$

We claim that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T); H^2(\Omega))$ as $\tau \to 0^+$.

Given $t \in (0, T]$, we find n such that $t \in (\tau_{n-1}, \tau_n]$ and we notice that

$$\tilde{u}^\tau(x,t) - u^\tau(x,t) = u_n(x) - u^\tau(x,t) = u^\tau(x, \tau_n) - u^\tau(x,t).$$

By (4.3.12),

$$\begin{aligned}\int_\Omega |\tilde{u}^\tau(x,t) - u^\tau(x,t)|^2 dx &= \int_\Omega |u^\tau(x, \tau_{n-1}) - u^\tau(x,t)|^2 dx \\ &\leq M_\varepsilon(t - \tau_{n-1}) \leq M_\varepsilon \tau \to 0,\end{aligned} \tag{4.3.17}$$

as $\tau \to 0^+$. This shows that $\tilde{u}^\tau(\cdot, t) - u^\tau(\cdot, t) \to 0$ in $L^2(\Omega)$ as $\tau \to 0^+$. Moreover, given $\phi \in L^2(\Omega \times (0,T))$, we have

$$\begin{aligned}\int_{\Omega_T} \tilde{u}^\tau(x,t)\phi(x,t) dx dt &= \int_{\Omega_T} (\tilde{u}^\tau(x,t) - u^\tau(x,t))\phi(x,t) dx dt \\ &\quad + \int_{\Omega_T} u^\tau(x,t)\phi(x,t) dx dt.\end{aligned} \tag{4.3.18}$$

By Hölder's inequality and (4.3.17), the first integral on the right-hand side of (4.3.18) converges to zero. Using the fact that $u^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$ in the second integral, we deduce that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T);L^2(\Omega))$.
Moreover, by (4.3.15) and the fact that $\tilde{u}^\tau(x,t) = u^\tau(x,\tau_n)$ for $t \in (\tau_{n-1},\tau_n]$,

$$\int_\Omega |\nabla \tilde{u}^\tau(x,t)|^2 dx \leq \frac{3M_\varepsilon}{\sigma}, \quad \int_\Omega |\nabla^2 \tilde{u}^\tau(x,t)|^2 dx \leq \frac{3M_\varepsilon}{\sigma}, \tag{4.3.19}$$

for all $\tau > 0$ and all $t \in (0,T)$. Hence, up to a subsequence, $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T);H^2(\Omega))$. Furthermore, by (4.3.5), for every $w \in L^2((0,T);H^2(\Omega))$,

$$\begin{aligned} 0 = &\int_\Omega \left(\frac{1}{\varepsilon}W'(\tilde{u}^\tau(x,t))w - 2\varepsilon q \nabla \tilde{u}^\tau(x,t)\cdot\nabla w + 2\varepsilon^3 \nabla^2 \tilde{u}^\tau(x,t)\cdot\nabla^2 w\right) dx \\ &+ \int_\Omega u_t^\tau(x,t) w \, dx. \end{aligned}$$

Integrating in time over (t_1,t_2) gives

$$\begin{aligned} 0 = &\int_{t_1}^{t_2}\int_\Omega \left(\frac{1}{\varepsilon}W'(\tilde{u}^\tau(x,t))w - 2\varepsilon q \nabla \tilde{u}^\tau(x,t)\cdot\nabla w + 2\varepsilon^3 \nabla^2 \tilde{u}^\tau(x,t)\cdot\nabla^2 w\right) dxdt \\ &+ \int_{t_1}^{t_2}\int_\Omega u_t^\tau w \, dxdt. \end{aligned}$$

We note that from (4.3.8) we have

$$\begin{aligned} \int_\Omega (u_n - u_0)^2 \, dx &= \int_\Omega (u_n - u_{n-1} + u_{n-1} - \ldots + u_1 - u_0)^2 \, dx \\ &\leq \ell \sum_{k=1}^{\ell} \int_\Omega (u_k - u_{k-1})^2 \, dx \leq \ell\tau M_\varepsilon = TM_\varepsilon \end{aligned}$$

where we have used the convexity of the function $z \mapsto z^2$ and the fact that $\tau = T/\ell$, and this implies

$$\int_\Omega |u_n|^2 \, dx \leq C \tag{4.3.20}$$

for some constant $C > 0$. Moreover, arguing as in (4.3.7), it follows that

$$\begin{aligned} \int_\Omega \left(\frac{1}{\varepsilon}W(u_n) - \varepsilon q|\nabla u_n|^2 + \varepsilon^3|\nabla^2 u_\ell|^2\right) dx &\leq \int_\Omega \left(\frac{1}{\varepsilon}W(u_0) - \varepsilon q|\nabla u_0|^2 + \varepsilon^3|\nabla^2 u_0|^2\right) dx \\ &\leq \frac{M_\varepsilon}{2} \end{aligned}$$

for all $n \in \{0,\ldots,\ell\}$, and in turn, by the interpolation inequality in Lemma 2.5.1,

$$\int_\Omega |\nabla u_n|^2 dx \leq C \quad \text{and} \quad \int_\Omega |\nabla^2 u_n|^2 dx \leq C \tag{4.3.21}$$

for some constant $C > 0$ and for all $n \in \{0, \dots, \ell\}$. Using (4.3.20), (4.3.21) and the Sobolev embedding theorem, we have

$$||u_n||_{L^\infty(\Omega)} \leq C||u_n||_{H^2(\Omega)} \leq C, \tag{4.3.22}$$

where $C > 0$ changes from side to side. By the Mean Value Theorem, (4.3.22), and the fact that W is C^2, we deduce

$$\begin{aligned}\int_\Omega (W'(\tilde{u}^\tau(x,t)) - W'(u(x,t)))w dx &\leq \max_{-C^\star \leq \xi \leq C^\star} |W''(\xi)| \int_\Omega |\tilde{u}(x,t) - u(x,t)||w| dx \\ &\leq C \int_\Omega |\tilde{u}(x,t) - u(x,t)||w| dx.\end{aligned} \tag{4.3.23}$$

Letting $\tau \to 0^+$ and using the facts that $\tilde{u}^\tau \rightharpoonup u$ in $L^2((0,T); H^2(\Omega))$, $u^\tau \rightharpoonup u$ in $H^1((0,T); L^2(\Omega))$ we get

$$\begin{aligned}0 = &\int_{t_1}^{t_2} \int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w - 2\varepsilon q \nabla u(x,t) \cdot \nabla w + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w \right) dxdt \\ &+ \int_{t_1}^{t_2} \int_\Omega u_t(x,t) w \, dxdt.\end{aligned} \tag{4.3.24}$$

In particular, let $\{w_k\} \subset H^2(\Omega)$ be dense. Using the fact that $u(\cdot, t) \in H^2(\Omega)$ and $u_t \in L^2(\Omega)$ for $\mathcal{L}^1$ a.e. $t \in (0,T)$, by the arbitrariness of t_1 and t_2, we find that

$$\begin{aligned}0 = &\int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w_k - 2\varepsilon q \nabla u(x,t) \cdot \nabla w_k + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w_k \right) dx \\ &+ \int_\Omega u_t(x,t) w_k \, dx\end{aligned}$$

for $\mathcal{L}^1$ a.e. $t \in (0,T)$, where the measure–zero set depends on k. Since $\{w_k\}$ is countable, we can find a set $E \subset (0,T)$ with $\mathcal{L}^1(E) = 0$ such that the previous equality holds for all $t \in (0,T) \setminus E$ and all k.

Since $u(\cdot,t) \in H^2(\Omega)$, then $u(\cdot,t) \in L^\infty(\Omega)$ and, again by Mean Value Theorem and the fact that W is C^2, it follows that $W'(u(\cdot,t)) \in L^2(\Omega)$. This, together with the density of $\{w_k\}$ in $H^2(\Omega)$, and the fact that $u_t \in L^2(\Omega)$ for $t \in (0,T) \setminus E$, implies that

$$\begin{aligned}0 = &\int_\Omega \left(\frac{1}{\varepsilon} W'(u(x,t))w - 2\varepsilon q \nabla u(x,t) \cdot \nabla w + 2\varepsilon^3 \nabla^2 u(x,t) \cdot \nabla^2 w \right) dx \\ &+ \int_\Omega u_t(x,t) w \, dx\end{aligned} \tag{4.3.25}$$

for all $t \in (0,T) \setminus E$ and all $w \in H^2(\Omega)$. Hence u is a weak solution of equation (4.3.1) and since Ω has finite measure, taking $w = 1$ leads to

$$0 = \int_\Omega \frac{1}{\varepsilon} W'(u(x,t)) dx + \int_\Omega u_t(x,t) \, dx, \tag{4.3.26}$$

which implies

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx + \int_0^t \int_\Omega \frac{1}{\varepsilon} W'(u(x,s))dxds.$$

$\square$

Chapter 5

Ongoing and future research

We present here two ongoing research projects ensuing from the main topic of this thesis. In the first section we describe a model that could be employed in the context of phase transitions. It is connected to a higher order energy like (2.5.1), and contains a mixed term of the kind $u'u$.

The second section is devoted to the introduction of the fractional counterpart of the isoperimetric function, used in the first part of this thesis.

5.1 A new model for phase transitions

The fifth–order Korteveg–de Vries equation

$$u_t + \alpha u''' + \beta u^{(v)} = h'(u, u', u'')$$

where $'$ denotes derivatives with respect to the spatial variable, and $z \mapsto h(z)$ is a smooth map, $\alpha, \beta \in \mathbb{R}$ with $\beta \neq 0$, models the behavior of waves in various frameworks, such as in plasma physics, see [10], [18], [69]. This equation belongs to the family

$$u_t + \gamma u^{(v)} + \zeta u''' - \xi\{2uu'' + |u'|^2\}' + 2\kappa uu' + 3ru^2u' = 0$$

where $\zeta, \xi, \kappa, r \in \mathbb{R}$ and $\gamma \in \mathbb{R}^+$ are constants, and such family arises in the context of water waves, used as a description of long waves in shallow water under gravity, see e.g. [10], [18], [25], [31]. Looking for traveling waves $u(x,t) = w(y)$, where $y = x - ct$, one obtains, after appropriate scaling, the equation

$$u^{(iv)} + qu'' - \mu\{2uu'' + |u'|^2\} + f(u) = 0,$$

in which f is a third order polynomial, $q, \mu \in \mathbb{R}$, see [68], page 8 for more details. This last equation can be seen, in the one–dimensional setting, as the Euler–Lagrange equation of the functional

$$\int_I \left(\frac{1}{2}|u''|^2 - \frac{q}{2}|u'|^2 + \mu u|u'|^2 + W(u) \right) dx \tag{5.1.1}$$

where $I \subset \mathbb{R}$ is an open, bounded interval and W is the primitive of f. As a first simplification, considering all the values of $\mu, q > 0$ such that $\mu = q/2$, we have

$$\int_I \left(\frac{1}{2}|u''|^2 - \frac{q}{2}|u'|^2 + \frac{q}{2}u|u'|^2 + W(u) \right) dx = \int_I \left(\frac{1}{2}|u''|^2 + \frac{q}{2}|u'|^2(u-1) + W(u) \right) dx.$$

An interesting problem is to understand, after appropriate rescaling, the asymptotic behavior of energies of this type. More specifically, the Γ–convergence analysis of such functionals, where terms with a sign are present, heavily relies on nonlinear interpolation results as in the case of the analogous model introduced earlier, see (2.5.1), (2.5.1), and (2.5.2). The aforementioned nonlinear interpolation inequality result would read as follows in our case. With I as above, we would like to show that there exists a constant $c > 0$ such that

$$\int_I \frac{1}{2}|u''|^2 + W(u)\,dx \geq c \int_I |u'|^2(1-u)\,dx \tag{5.1.2}$$

for all $u \in H^2(I)$.

As a first step in this direction, one can tackle the study of the functional

$$\int_I |u''|^2 + k|u'|^s|u-1|^t + W^\star(u)\,dx$$

where $k > 0$ is a parameter, $s, t > 0$ are values to be investigated and $z \mapsto W^\star(z)$ is a double–well potential. We expect the the constraints on the exponents s, t to follow from applications of Young's inequality and classical interpolation inequalities, see [64]. This functional could prove useful in the theory of phase separation in composite materials, being close to models recently taken into consideration, see [21], [23].

We expect to be able to prove the following.

Theorem 5.1.1. *Let $I \subset \mathbb{R}$ be an open, bounded interval. Then there exist $s, t > 0$ and a constant $k_2 > 0$, such that*

$$k_2 \int_I |u'|^s|u|^t\,dx \leq |I|^{4-s} \int_I |u''|^2\,dx + \frac{1}{|I|^s} \int_I W^\star(u)\,dx$$

for all $u \in H^2(I)$, where $z \mapsto W^\star(z)$ is the double–well potential with wells at ± 1 previously mentioned.

A compactness result for the rescaled functional would follow by using the previous theorem.

Theorem 5.1.2 (Compactness)**.** *Let $I \subset \mathbb{R}$ be an open, bounded interval and let $-\infty < k < k_0$. Let $\varepsilon_n \to 0^+$ and consider $\{u_n\} \subset H^2(I)$ such that*

$$\sup_n \int_I \left(\frac{1}{\varepsilon} W^\star(u_n) - k|u_n'|^{s-1}|1-u_n|^t + \varepsilon^3 |u_n''|^2 \right) dx < \infty.$$

Then there exist a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u \in BV(I; \{-1, +1\})$ such that

$$u_{n_k} \to u \text{ in } L^1(I).$$

The study of the Γ–limit is the next natural step. Moreover, in order to analyze the original functional (5.1.1), it is necessary to understand the role of the constraint on the exponents s, t.

5.2 The fractional isoperimetric function

The fractional version of the classical isoperimetric function (1.2.2) can be defined as

$$\mathcal{I}(r) := \inf \{P_s(E;\Omega) : E \subset \Omega, \mathcal{L}^n(E) = r\}, \quad s \in (0,1), \quad r \in [0, \mathcal{L}^n(\Omega)] \tag{5.2.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth, bounded, open set, E is a measurable set and $P_s(E;\Omega)$ denotes the fractional perimeter of E relative to Ω. The fractional perimeter was introduced for the first time in [15], defined by the fractional s–Sobolev seminorm of the characteristic function χ_E of E, and in our case it reads

$$P_s(E;\Omega) := \int_E \int_{E^c\cap\Omega} \frac{1}{|x-y|^{n+s}} dx\, dy, \quad s \in (0,1). \tag{5.2.2}$$

One way to study the regularity of the map $r \mapsto \mathcal{I}(r)$ for a fixed value of s, is to analyze the first and second variations of (5.2.2) and then, using a similar approach to the one adopted in [75], study the second derivative of $P_s(E;\Omega)$ with respect to the volume variable.

In the first part of this thesis we have used the standard isoperimetric function and its local counterpart in connection to the speed of evolution of solutions of PDEs arising as gradient flows of the Cahn-Hillard energy, see [55], [60]. An interesting problem is the search for relations, if any, between (5.2.1) and the fractional version of the Allen–Cahn equation

$$u_t = \frac{1}{\varepsilon}\left(\mathcal{S}u - \frac{1}{\varepsilon^{2s}} W'(u) + \sigma(t,x)\right), \tag{5.2.3}$$

whose solutions represent the atom dislocation in a crystal for a general type of Nabarro–Peierls model, and where $\varepsilon > 0$ is a small parameter, $\mathcal{S}$ is an integro–differential operator as in (5.2.2), $z \mapsto W(z)$ is a periodic potential, and σ is an external stress. See [67] and references therein for more details.

Bibliography

[1] E. ACERBI, N. FUSCO, AND M. MORINI, *Minimality via second variation for a nonlocal isoperimetric problem*, Comm. Math. Phys., 322 (2013), pp. 515–557.

[2] N. D. ALIKAKOS, P. W. BATES, AND X. CHEN, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Rational Mech. Anal., 128 (1994), pp. 165–205.

[3] N. D. ALIKAKOS, L. BRONSARD, AND G. FUSCO, *Slow motion in the gradient theory of phase transitions via energy and spectrum*, Calc. Var. Partial Differential Equations, 6 (1998), pp. 39–66.

[4] N. D. ALIKAKOS AND G. FUSCO, *Slow dynamics for the Cahn-Hilliard equation in higher space dimensions. I. Spectral estimates*, Comm. Partial Differential Equations, 19 (1994), pp. 1397–1447.

[5] ——, *Slow dynamics for the Cahn-Hilliard equation in higher space dimensions: the motion of bubbles*, Arch. Rational Mech. Anal., 141 (1998), pp. 1–61.

[6] J.-P. AUBIN, *Un théorème de compacité*, C. R. Acad. Sci. Paris, 256 (1963), pp. 5042–5044.

[7] C. BAVARD AND P. PANSU, *Sur le volume minimal de* $\mathbf{R}^2$, Ann. Sci. École Norm. Sup. (4), 19 (1986), pp. 479–490.

[8] G. BELLETTINI, A. HASSEM NAYAM, AND M. NOVAGA, Γ*-type estimates for the one-dimensional Allen-Cahn's action*, Asymptotic Analysis, 94 (2015), pp. 161–185.

[9] G. BELLETTINI, A.-H. NAYAM, AND M. NOVAGA, Γ*-type estimates for the one-dimensional Allen-Cahn's action*, Asymptot. Anal., 94 (2015), pp. 161–185.

[10] T. B. BENJAMIN, J. L. BONA, AND D. K. BOSE, *Solitary-wave solutions of nonlinear problems*, Philos. Trans. Roy. Soc. London Ser. A, 331 (1990), pp. 195–244.

[11] H. BRÉZIS, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973. North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

[12] L. BRONSARD AND R. V. KOHN, *On the slowness of phase boundary motion in one space dimension*, Comm. Pure Appl. Math., 43 (1990), pp. 983–997.

[13] ——, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations, 90 (1991), pp. 211–237.

[14] L. BRONSARD AND B. STOTH, *Volume-preserving mean curvature flow as a limit of a nonlocal Ginzburg-Landau equation*, SIAM J. Math. Anal., 28 (1997), pp. 769–807.

[15] L. CAFFARELLI, J.-M. ROQUEJOFFRE, AND O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math., 63 (2010), pp. 1111–1144.

[16] J. CARR AND R. L. PEGO, *Metastable patterns in solutions of $u_t = \epsilon^2 u_{xx} - f(u)$*, Comm. Pure Appl. Math., 42 (1989), pp. 523–576.

[17] ——, *Very slow phase separation in one dimension*, in PDEs and continuum models of phase transitions (Nice, 1988), vol. 344 of Lecture Notes in Phys., Springer, Berlin, 1989, pp. 216–226.

[18] A. R. CHAMPNEYS AND M. D. GROVES, *A global investigation of solitary-wave solutions to a two-parameter model for water waves*, J. Fluid Mech., 342 (1997), pp. 199–229.

[19] X. CHEN, *Generation and propagation of interfaces for reaction-diffusion equations*, J. Differential Equations, 96 (1992), pp. 116–141.

[20] X.-Y. CHEN, *Dynamics of interfaces in reaction diffusion systems*, Hiroshima Math. J., 21 (1991), pp. 47–83.

[21] M. CHERMISI, G. DAL MASO, I. FONSECA, AND G. LEONI, *Singular perturbation models in phase transitions for second-order materials*, Indiana Univ. Math. J., 60 (2011), pp. 367–409.

[22] A. CIANCHI AND V. MAZ'YA, *Neumann problems and isocapacitary inequalities*, J. Math. Pures Appl. (9), 89 (2008), pp. 71–105.

[23] M. CICALESE, E. N. SPADARO, AND C. I. ZEPPIERI, *Asymptotic analysis of a second-order singular perturbation model for phase transitions*, Calc. Var. Partial Differential Equations, 41 (2011), pp. 127–150.

[24] E. A. CODDINGTON AND N. LEVINSON, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[25] W. CRAIG AND M. D. GROVES, *Hamiltonian long-wave approximations to the water-wave problem*, Wave Motion, 19 (1994), pp. 367–389.

[26] M. Crandall and L. Tartar, *Some relations between nonexpansive and order preserving mappings*, Proc. Amer. Math. Soc., 78 (1980), pp. 385–390.

[27] M. C. Cross and P. C. C. Hohenberg, *Pattern formation outside of equilibrium*, Rev. Mod. Phys., 65 (1993), pp. 851–1112.

[28] C. M. Dafermos and M. Pokorný, eds., *Handbook of differential equations: evolutionary equations. Vol. V*, Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2009.

[29] P. de Mottoni and M. Schatzman, *Development of interfaces in* $\mathbf{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A, 116 (1990), pp. 207–220.

[30] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[31] P. G. Drazin and R. S. Johnson, *Solitons: an introduction*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1989.

[32] S.-I. Ei and E. Yanagida, *Slow dynamics of interfaces in the Allen-Cahn equation on a strip-like domain*, SIAM J. Math. Anal., 29 (1998), pp. 555–595 (electronic).

[33] C. M. Elliott and Z. Songmu, *On the Cahn-Hilliard equation*, Arch. Rational Mech. Anal., 96 (1986), pp. 339–357.

[34] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[35] P. C. Fife and L. Hsiao, *The generation and propagation of internal layers*, Nonlinear Anal., 12 (1988), pp. 19–41.

[36] I. Fonseca, N. Fusco, G. Leoni, and M. Morini, *Motion of elastic thin films by anisotropic surface diffusion with curvature regularization*, Arch. Ration. Mech. Anal., 205 (2012), pp. 425–466.

[37] I. Fonseca and L. Tartar, *The gradient theory of phase transitions for systems with two potential wells*, Proc. Roy. Soc. Edinburgh Sect. A, 111 (1989), pp. 89–102.

[38] G. Fusco, *A geometric approach to the dynamics of* $u_t = \epsilon^2 u_{xx} + f(u)$ *for small* ϵ, in Problems involving change of type (Stuttgart, 1988), vol. 359 of Lecture Notes in Phys., Springer, Berlin, 1990, pp. 53–73.

[39] G. Fusco and J. K. Hale, *Slow-motion manifolds, dormant instability, and singular perturbations*, J. Dynam. Differential Equations, 1 (1989), pp. 75–94.

[40] N. Fusco, F. Maggi, and A. Pratelli, *The sharp quantitative isoperimetric inequality*, Ann. of Math. (2), 168 (2008), pp. 941–980.

[41] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[42] A. Giorgini, *On the Swift-Hohenberg equation with slow and fast dynamics: well-posedness and long-time behavior*, Commun. Pure Appl. Anal., 15 (2016), pp. 219–241.

[43] E. Gonzalez, U. Massari, and I. Tamanini, *On the regularity of boundaries of sets minimizing perimeter with a volume constraint*, Indiana Univ. Math. J., 32 (1983), pp. 25–37.

[44] C. P. Grant, *Slow motion in one-dimensional Cahn-Morral systems*, SIAM J. Math. Anal., 26 (1995), pp. 21–34.

[45] M. Grüter, *Boundary regularity for solutions of a partitioning problem*, Arch. Rational Mech. Anal., 97 (1987), pp. 261–270.

[46] G. Hayrapetyan and M. Rinaldi, *Slow motion for the 1D Swift–Hohenberg equation*, To appear: J. Differential Equations, (2016).

[47] P. C. Hohenberg and J. B. Swift, *Effects of additive noise at the onset of Rayleigh-Bénard convection*, Phys. Rev. A, 46 (1992), pp. 4773–4785.

[48] G. W. Hunt, M. A. Peletier, A. R. Champneys, P. D. Woods, M. A. Wadee, C. J. Budd, and G. J. Lord, *Cellular buckling in long structures*, Nonlinear Dynam., 21 (2000), pp. 3–29. The theme of solitary waves and localization phenomena in elastic structures.

[49] V. Julin and G. Pisante, *Minimality via second variation for microphase separation of diblock copolymer melts*, To appear: J. Reine Angew. Math, (2013).

[50] W. D. Kalies, R. C. A. M. VanderVorst, and T. Wanner, *Slow motion in higher-order systems and Γ-convergence in one space dimension*, Nonlinear Anal., 44 (2001), pp. 33–57.

[51] M. Kuwamura, *The phase dynamics method with applications to the Swift-Hohenberg equation*, J. Dynam. Differential Equations, 6 (1994), pp. 185–225.

[52] N. Q. Le, *A gamma-convergence approach to the Cahn-Hilliard equation*, Calc. Var. Partial Differential Equations, 32 (2008), pp. 499–522.

[53] J. Lega, J. V. Moloney, and A. C. Newell, *Swift–Hohenberg Equation for Lasers*, Phys. Rev. Lett., 73 (1994), pp. 2978–2981.

[54] G. LEONI, *A first course in Sobolev spaces*, vol. 105 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2009.

[55] G. LEONI AND R. MURRAY, *Second-order Γ-limit for the Cahn-Hilliard functional*, Arch. Ration. Mech. Anal., 219 (2016), pp. 1383–1451.

[56] F. MAGGI, *Sets of finite perimeter and geometric variational problems*, vol. 135 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[57] V. MAZ'YA, *Sobolev spaces with applications to elliptic partial differential equations*, vol. 342 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg, augmented ed., 2011.

[58] L. MODICA, *Gradient theory of phase transitions with boundary contact energy*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 4 (1987), pp. 487–512.

[59] L. MODICA AND S. MORTOLA, *Il limite nella Γ-convergenza di una famiglia di funzionali ellittici*, Boll. Un. Mat. Ital. A (5), 14 (1977), pp. 526–529.

[60] R. MURRAY AND M. RINALDI, *Slow motion for the nonlocal Allen–Cahn in n-dimensions*, To appear: Calc.Var.Partial Differential Equations, (2016).

[61] J. V. NEU, *Unpublished notes*, (1984).

[62] T. N. NGUYEN, *Equations d'èvolution non locales et problèmes de transition de phase*, General Mathematics. Université Paris Sud - Paris XI, (2013).

[63] B. NICOLAENKO, B. SCHEURER, AND R. TEMAM, *Some global dynamical properties of a class of pattern formation equations*, Comm. Partial Differential Equations, 14 (1989), pp. 245–297.

[64] L. NIRENBERG, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa (3), 13 (1959), pp. 115–162.

[65] I. Y. OLEKSIV AND N. PESIN, *Finiteness of Hausdorff measure of level sets of bounded subsets of Euclidean space*, Mathematical Notes, 37 (1985), pp. 237–242.

[66] F. OTTO AND M. G. REZNIKOFF, *Slow motion of gradient flows*, J. Differential Equations, 237 (2007), pp. 372–420.

[67] S. PATRIZI AND E. VALDINOCI, *Crystal dislocations with different orientations and collisions*, Arch. Ration. Mech. Anal., 217 (2015), pp. 231–261.

[68] L. A. PELETIER AND W. C. TROY, *Spatial patterns*, Progress in Nonlinear Differential Equations and their Applications, 45, Birkhäuser Boston, Inc., Boston, MA, 2001. Higher order models in physics and mechanics.

[69] Y. POMEAU AND P. MANNEVILLE, *Wavelength selection in cellular flows*, Physics Letters A, 75 (1980), pp. 296 – 298.

[70] R. T. ROCKAFELLAR AND R. J.-B. WETS, *Variational analysis*, vol. 317 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1998.

[71] J. RUBINSTEIN AND P. STERNBERG, *Nonlocal reaction-diffusion equations and nucleation*, IMA J. Appl. Math., 48 (1992), pp. 249–264.

[72] G. R. SELL, *Smooth linearization near a fixed point*, Amer. J. Math., 107 (1985), pp. 1035–1091.

[73] H. M. SONER, *Ginzburg-Landau equation and motion by mean curvature. I. Convergence*, J. Geom. Anal., 7 (1997), pp. 437–475.

[74] P. STERNBERG, *The effect of a singular perturbation on nonconvex variational problems*, Arch. Rational Mech. Anal., 101 (1988), pp. 209–260.

[75] P. STERNBERG AND K. ZUMBRUN, *On the connectivity of boundaries of sets minimizing perimeter subject to a volume constraint*, Comm. Anal. Geom., 7 (1999), pp. 199–220.

[76] J. SWIFT AND P. C. HOHENBERG, *Hydrodynamic fluctuations at the convective instability*, Phys. Rev. A, 15 (1977), pp. 319–328.

[77] P. B. UMBANHOWAR, F. MELO, AND H. L. SWINNEY, *Localized excitations in a vertically vibrated granular layer*, Nature, 382 (1996), pp. 793–796.