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Umut A. Acar  
*Toyota Technological Institute at Chicago*

Guy E. Blelloch  
*Carnegie Mellon University*

Matthias Blume  
*Toyota Technological Institute at Chicago*

Robert Harper  
*Carnegie Mellon University*

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Umut A. Acar∗ Guy E. Blelloch† Matthias Blume‡ Robert Harper§

Abstract

This papers proposes techniques for writing self-adjusting programs that can adjust to any change to their data (e.g., inputs, decisions made during the computation) etc. We show that the techniques are efficient by considering a number of applications including several sorting algorithms, and the Graham Scan, and the quick hull algorithm for computing convex hulls. We show that the techniques are practical by showing that self-adjusting programs can be trivially transformed into a kinetic programs that maintain their property as their input move continuously. We show that the techniques are practical by implementing a Standard ML library for kinetic data structures and applying the library to kinetic convex hulls. We show that the kinetic programs written with the library are more than an order of magnitude faster than recomputing from scratch. These results rely on a combination of memoization and dynamic dependence graphs. We show that the combination is sound by presenting a semantics based on abstraction of memoization via an oracle.

1 Introduction

Traditionally, a program is thought as a function that maps static inputs to an output. The inputs are static in the sense that they remain the same over time. In many real-world applications, however, computations must interact with a data that changes over time. For example, the input to a compiler evolves over time as the programmer modifies the code. Similarly, the model of the world that a search-and-rescue robot maintains in order to maps its route changes when an obstacle appears on the planned route.

One way to deal with this type of problems is the ad hoc approach, where the programmer devises their programs by considering all the possible ways that the data can change. This is how these problems are addressed in the algorithms community. There researchers devise dynamic algorithms that can adjust to discrete changes to their data such as insertions/deletions. For example, a dynamic convex hull algorithm can maintain the convex hull of a set of points as the user inserts/deletes points into/from the input. To deal with motion, researchers devise kinetic algorithms (or data structures) that maintain properties of moving objects. For example, a kinetic convex-hull algorithm maintains the convex hull of a set of moving points.

The advantage to the ad hoc approach is that the resulting algorithms can be very fast, even optimal. The disadvantage is that these algorithms are not composable (imagine composing two hash tables)! Furthermore these problems can be very difficult to solve, even for problems that are simple when in the static case (when inputs don’t change or move). As an example, consider the dynamic and kinetic convex hulls. These problems has been studied for more than two decades now [34, 33, 12, 14].

Another way to address these problems is to come up with general-purpose techniques and programming abstractions based on these techniques. In the programming-languages community, researchers developed techniques for these problems in the context of incremental computation. Previously proposed techniques were based on static dependence graphs [17, 39, 26, 45], memoization [36, 37, 31, 30], and partial evaluation [20, 19, 42]. Recent work on dynamic dependence graphs generalized dependence-graphs based approaches so that they can be applied to any purely functional program [4, 6].

Unfortunately, none of these techniques are capable of adjusting computations to general changes efficiently. For example, for incremental list sorting with memoization, the best known bound is linear [30]. Dynamic dependence graphs can achieve efficient updates for certain kinds of computations such as certain parallel computations [6, 7] and for certain kinds of changes. For example, with the quick sort algorithm, dynamic dependence graphs can achieve logarithmic time updates for insertions/deletions at the end of the input list [4], but for most changes, they are no better than rerunning from scratch and require Θ(n log n) time. Especially when a program involves heavy use of composition, these techniques quickly degenerate into from-scratch execution.

This paper addresses this long-standing problem by devising techniques for writing programs that self-adjust to changes. The techniques enable transforming an ordinary program that assumes that its data is static to a program that can self-adjust to any change to its input. The key properties of the techniques are that they are general purpose and efficient. They apply to any purely functional program, and enable programs to adjust to any change to their data whatsoever. The techniques yield program that are efficient even when compared to dynamic and kinetic algorithms developed in the algorithms community both in theory and in practice.

The contributions of this paper are the following:

1. A technique for combining dynamic dependence graphs and memoization.
2. Soundness proof of the combination.
3. Applications to several dynamic problems including sorting, and convex hull under insertions/deletions.
4. Applications to kinetic data structures and experimental results.

Key to the effectiveness of the technique is a combination of dynamic dependence graphs (DDGs) [4] and memo-
This combination of DDGs and memoization is highly nontrivial. This is because of the interaction between memoization, which requires purely functional code, and the implementation of DDGs, which critically rely on side effects. We prove that the combination is correct by presenting a semantics for evaluation and change propagation based on an abstract oracle that models memoization. In this approach, an evaluation returns a trace that represents the relationships between the computation data and evaluated expressions. During the evaluation, every function application consults an oracle which can either return a computation as a result or can return no results. If the oracle returns a computation, then it is also required to return a set of changes that characterize the differences between the evaluation store of the returned computation and the current store. If at a function application, the oracle returns no results, then the function application is evaluated as usual. If the oracle returns a computation, then change propagation is applied to the returned computation to adjust it to the current evaluation store and the resulting computation is re-used in place of the function application. Based on this semantics, we show that our techniques are sound by showing that they yield the same result and store and as a from-scratch evaluation where no result re-use and no change-propagation takes place.

To demonstrate the effectiveness of the technique, we consider a number of applications including combining values in a list (a.k.a., “reduce” operation), quick sort and merge sort algorithms, the Graham’s Scan [22] and quick hull algorithms [11] for computing convex hulls, the tree-contraction algorithm of Miller and Reif [32]. For all these algorithms, we state bounds that are within an expected constant factor of the best bounds obtained in the algorithms community. To appreciate the complexity of these problems, it should suffice to look at any one of these papers on dynamic convex hulls [34, 33, 14], or Jacob’s thesis [27] which solely addresses this problem, and the papers on dynamic trees [40, 41, 15, 38, 25, 44, 9, 21, 10, 43].

To demonstrate the practical effectiveness of the techniques, and their flexibility, we consider kinetic data structures. We describe a library that makes it nearly trivial to transform self-adjusting programs into kinetic programs by making small changes to the code. We apply the library to kinetic convex hull problems and show experimental results.

The experiments show that the kinetic programs obtained using our techniques adjust to continuous changes by nearly a factor of 50 faster than computing from-scratch. For discrete changes (insertions/deletions), we show elsewhere that the techniques are two orders of magnitude faster than recomputing from scratch. The techniques are also compared to special-purpose dynamic algorithms elsewhere [7]. There, we show that the approach performs very well even for sophisticated problems.

2 Overview of the paper

The paper consists of two parts. The first part studies the proposed technique in the context of a Standard ML library. Section 3.1 briefly describes the library, Section ?? shows how to transform the an implementation of the ordinary quick-hull algorithm [11] to a self-adjusting algorithm. Section 3.3 describes the combination of DDGs and memoization in an informal (algorithmic) setting, and Section 3.4 describes the change propagation algorithm. Section ?? describes the library for kinetic data structures and applications to kinetic convex hulls.

The second part of the paper is devoted to proving the soundness of the combination of memoization and DDGs.

3 A Framework for Self-Adjusting Programming

This section presents an informal (algorithmic) overview our approach in the context of an ML library, considers several applications and shows asymptotic bounds for these applications. As a practical example, we consider the quick-hull algorithm for computing convex hulls and show how to make the algorithm self-adjusting under both dynamic changes (insertions/deletions of points), and kinetic changes (continuous motion).

3.1 The Library

Figure 1 shows the interface to the library. To support constant-time equality tests, we rely on tag equality based on boxes [5]. The BOXED_VALUE module supplies functions for operating on boxed values. Since boxing is very well understood, we do not discuss it in detail here. The signature supplies the new function for creating boxed values, the indexOf function for accessing the index (a.k.a., tag or hash) of a boxed value, the eq function for comparing two boxed values based on their indices. The COMBINATORS module supplies operations for modifiable references (modifiables for short) and changeable computations. Every execution of a changeable computation of type α cc starts with the creation of a fresh modifiable of type α modref. The modifiable is written at the end of the computation. For the duration of the execution, the reference never becomes explicit. Instead, it is carried “behind the scenes” in a way.

```
signature BOXED_VALUE = sig type t = ... end
structure Box:BOXED_VALUE = struct type t = ... end

signature COMBINATORS = sig
  type α modref
  type α cc

  val modref : α cc → α modref
val write : (α * α → bool) → α → α cc
val read : β modref → (β → α cc) → α cc

val mkLift : (α*α → bool) →
  (int list * α) →
  (α modref → β) → β
val mkLiftCC : ((α*α → bool) * (β*β → bool)) →
  (int list * α) →
  (α modref → β cc) → β cc

val change : 'a modref * 'a -> unit
val propagate: unit -> unit
end
```

Figure 1: The interface to the ML library.
that is strongly reminiscent of a monadic computation.

*Changeable computations* of type $\alpha \rightarrow \text{cc}$ are constructed using *write*, *read*, and *mkLiftCC* functions. The *modref* function executes a given computation on a freshly generated modifiable and returns that modifiable. The *write* function creates a trivial computation that writes the given value into the underlying modifiable. To avoid unnecessary propagation of changes, old and new values are compared for equality at the time of *write* by using the supplied equality predicate. The *read* combinator takes an existing modifiable reference and a receiver for the value read. The result of the *read* combinator is a changeable computation that reads from the modifiable and applies the receiver on the value read.

Functions *mkLift* and *mkLiftCC* support memoization. They take one and two comparison operations respectively and return lift functions that are used for memoizing expressions. A *lift function* memoizes an function based on its strict and one non-strict argument. It performs a memo look up only on the strict arguments and writes non-strict arguments into modifiable references and passes these modifiables to the memoized function. In case of a memo hit, a lift function adjusts the re-used computation to the values of the non-strict arguments via a change propagation on the DDG of re-used computation. In its simplest form, a lift function takes a list of types of strict arguments, and a non-strict argument of type $\alpha$, and a function of type $\alpha \rightarrow \beta$ (or $\alpha \rightarrow \beta \text{ cc}$) and returns value of type $\beta$ (or $\beta \text{ cc}$). A *mkLift* function takes an equality test of type $\alpha \times \alpha \rightarrow \text{bool}$ for the non-strict argument and yields a lift function. Similarly a *mkLiftCC* function takes an equality test of type $\alpha \times \alpha \rightarrow \text{bool}$ for the non-strict argument, and an equality function of type $\beta \times \beta \rightarrow \text{bool}$ for the underlying value of the changeable computation, and yields a lift function. In general, a lift function can have many non-strict arguments. Due to typing restrictions, these will have to be supported separately. The library therefore contains *mkLift2*, *mkLiftCC2*, *mkLift3*, *mkLiftCC3*, etc. to support more than one non-strict argument; these are now shown here for brevity.

For changing the values of modifiables and propagating the changes, the library also support two meta-operations: *change* and *propagate*. It is only safe to use these operations at the top level, not within a program.

### 3.2 Making an application self-adjusting

Using the ML library, the programmer can transform an ordinary (non-self-adjusting) program into a self-adjusting program in two steps. These step involve applying modifiable references and memoization respectively. The first step involves determining and placing changeable parts of the input data into modifiables and applying the *modref*, *read*, *write* operations to the rest of the code. This transformation is guided by the types: values stored in modifiables can only be accessed by a *read* operation, which must end with a *write*, and which must take place within the context of a *modref*. This transformation is described in more detail in previous work [4].

The second step of the transformation involves applying memoization. To memoize an expression, the programmer determines its strict and non-strict arguments, creates a lift function for the expression, and applies the lift function to the expression. Although the programmer can choose any subset of the free variables to be strict or non-strict, we found that the following *strictness principle* leads to programs whose behavior can be analyzed using high-level analysis techniques based on stability analysis. Consider an expression $e$, if an argument $x$ appears in $x$ at an argument position, i.e., is only passed as an argument to some other function, then $x$ is non-strict; otherwise $x$ is strict. Intuitively, the arguments that contribute directly to the result of the expressions are considered strict, whereas arguments that contribute to the result through some other function are considered non-strict.

Figure ?? shows the code for ordinary quick hull algorithm (left) and its self-adjusting version (right). The quick hull algorithm find the convex hull of a set of points, i.e., the smallest polygon enclosing the points. Both versions are is based on a Geo structure that supplies geometry primitives. The ordinary version is a standard implementation of the quick-hull algorithm [7]. It relies on a List structure that supports the filter and min functions for filtering a list and finding the minimum element of a list based on the supplied functions. To make the code for quick hull self-adjusting, we follow the two step transformation process. We first change the input of *qhull* to a modifiable list (*MOD_LIST*) where each tail is placed into a modifiable reference. We then insert *mod*, *read*, and *write* combinators into appropriate places in the code. Next, we apply memoization by considering each function separately. Note that the function *qhull* performs trivial work, so it need not be memoized. For *split*, we note that the NIL branch performs trivial work, and therefore need not be memoized. To memoize the CONS branch, we determine the free variables of the branch—they are $1, p, q, \text{ hull}$. By applying the strictness principle, we find that 1 and hull are non-strict, and $p$ and $q$ are strict arguments. Using *mkLiftCC2*, we construct a lift function for lifting these variables, and apply the function to the branch. The self-adjusting version adjusts to “discrete changes”, i.e., insertions and deletions to its input automatically. After running *qhull* with some modifiable list 1, the list can be changed arbitrarily using the Comb.propagate function and the convex hull can be updated automatically by running Comb.propagate. We describe how transform this code into a kinetic program in Section 3.7.

### 3.3 Combining Memoization and DDGs

As a self-adjusting program executes, it creates a record of the operations on modifiables in the form of a dynamic dependence graph (DDG) and remembers memoized computations. Dynamic dependence graphs have been described in previous work [4]. Here, we give an overview of DDGs and focus on the how DDGs may be combined with memoization.

A dynamic dependence graph consists of nodes and edges. The execution of *modref* adds a node, and an evaluation of *read* adds an edge to the DDG. In a *read*, the node being read becomes the source, and the target of the read (the modifiable that the reader finished by writing to) becomes the target. Each edge is tagged with the reader, which is a closure, and a containment hierarchy is maintained on the edges. A read $e$ is contained within another read $e'$ if $e$ was created during the execution of $e'$.

Containment hierarchy is represented using time-stamps. Each edge in is tagged with a *start time stamp* and a *stop time stamp* corresponding to the execution (time) interval of that edge. Time intervals are then used to identify the containment relationship of reads: a read $R_e$ is
fun split (p,q,pts,hull) = 
  let
    val l = List.filter (Geo.above (p,q)) pts
    in
      case l of
        nil => cons(p,hull)
      | cons _ =>
        let
          val max = List.min (Geo.closer (p,q)) l
          val r = split (max,q,l,hull)
          in
            splitM (p,max,l,r)
          end
    end
  in
    splitM (p,q,pts,hull)
  end

fun qhull l = 
  case l of
    nil => nil
  | cons _ =>
    let
      val m = List.min Geo.toLeft l
      val x = List.min Geo.toRight l
      val hull = cons(x,nil)
      in
        split (m,x,l,hull)
      end
  end

signature MOD_LIST = sig
  datatype α cell =
    NIL
  | CONS of (α * α cell modref)
  type α t = α cell modref
  val eq: α Box.t cell * α Box.t cell -> bool
end
structure ML:MOD_LIST = struct...

fun split (p,q,pts,hull) = 
  let
    val lift = mkLift (ML.eq,ML.eq,ML.eq)
    in
      splitM (p,q,pts,hull)
    end

fun qhull l = modref (read l (fn cl =>
  case cl of
    ML.NIL => write ML.eq ML.NIL
  | ML.CONS => read hull (fn ch =>
    lift ([Box.i p,Box.i q],cl,ch) (fn (l,hull) =>
      let
        val max = ML.min (Geo.closer (p,q)) l
        val r = modref (read max (fn max =>
          splitM (max,q,l,hull)))
        in
          read max (fn max=>splitM (p,max,l,r))
        end))
    end)))

Figure 2: The code for ordinary quick hull (left), and its self-adjusting version.
** (V, E) = G is an DDG
** \( \phi \) is the memo finger
** currentTime is the current time
***)

```plaintext
1 lift (i,a) f =
2   if (lookup(f,i,currentTime,\phi) then
3       (b, (t1, t2), m) ← find(f,i,currentTime,\phi)
4       \( E' \leftarrow \{ e \in E \mid \text{currentTime} \leq t_e(e') \leq \text{currentTime} \} \)
5       \( V \leftarrow V \cup \{ v \mid \exists v'. (v, v') \in E' \lor (v', v) \in E' \} \)
6       change (m, a)
7       propagateUntil (t2)
8       currentTime ← t2
9       return b
10  else
11     m ← newModifyable ()
12     change (m, a)
13     t1 ← currentTime
14     b ← f (m)
15     t2 ← currentTime
16     remember f with i yields (b, (t1, t2), m)
```

Figure 3: The lift operation.

- **Memo match:** Let \( (t_1, t_2) \) be the time interval of the memo entry found. To re-use the computation, the system deletes all the edges whose time stamps are contained within the interval \( (\text{currentTime}, t_1) \) and the vertices adjacent to these edges. Then, the system copies the values of the non-strict arguments to the modifiables stored in the memo entry matched and run change propagation on the re-used computation by calling \texttt{propagateUntil} \( (t_2) \).

Memo lookups are different than ordinary memo lookups. They take place with respect to a time stamp, called the \( \text{(memo) finger} \), denoted \( \phi \), that the system maintains. The finger is initialized to the first time stamp ever created, and is changed only by the change propagation algorithm. In a memo lookup, the systems seeks an entry that has the same strict arguments as the expression being evaluated (this is checked by comparing the index lists) and that has a time interval \( (t_1, t_2) \) that lies within the interval \( \text{currentTime}, \phi \) defined by the current time and the finger. If such an entry

Since the values stored in modifiables can be side effected by the \texttt{write} operation and by the \texttt{change} meta-operation, and since the system uses shallow equality (by comparing indices of boxed values) for memo lookups, \texttt{propagateUntil} must be run on the re-used computation. This ensure that the re-used computation is adjusted to the changes.

### 3.4 Change Propagation

Given a dynamic dependence graph and a set of changed modifiables, the \textit{change-propagation algorithm} updates a self-adjusting computation by propagating changes through the DDG. The change-propagation algorithm in the context of DDGs have been described in previous work [4]. In this section, we extend the algorithm to support memoization.

Figure 4 shows the pseudo-code for \texttt{propagate} and \texttt{propagateUntil} functions. The \texttt{propagateUntil} function is only for internal use. The function \texttt{propagate} is implemented by calling \texttt{propagateUntil} with the largest time stamp, denoted \( t_\infty \). The algorithm maintains a priority queue of edges that are affected, i.e., whose values have changed. The queue is prioritized on the time stamp of each edge, and is initialized with the out-edges of the changed modifiables. During change propagation, the queue is populated by \texttt{write} operations; if the contents of a modifiable is changed as a result of a \texttt{write} all reads of that modifiable are inserted into the priority queue. Each iteration of the while loop processes one affected edge by re-evaluating the reader associated with the edge. Before re-evaluating the reader, the algorithm remembers the current memo finger \( \phi \) and sets it to \( t_2 \) (the end time-stamp of the edge), and set \texttt{currentTime} back to \( t_1 \) (the start time of the edge). After the re-evaluation, the algorithm deletes all the edges whose time intervals are within the interval defined by \texttt{currentTime} and \( t_2 \), and restores the memo finger.

Setting the finger \( \phi \) to \( t_2 \) makes the results from the previous evaluation of the read available for re-use. As described in the Section 3.3, a computation is re-used only if its interval is between the \texttt{currentTime} and the finger. Note that since \( \phi \) is initialized to the first time stamp and is only changed by change-propagation, result re-use takes place only during change propagation.

For correctness, it is critical that all the edges and vertices that are not re-used during change-propagation be
The combination and the change-propagation algorithm described in Sections 3.3 and 3.4 respectively is that they are efficient and their performance can be determined using algorithmic analysis. These restrictions are between the re-executed edge and the re-used computation. Therefore, when re-execution of an edge completes, all unused edges are between the currentTime and the end time stamp of the re-executed edge $t_2$.

3.5 Asymptotic Complexity of Change Propagation

The key property of the combination and the change propagation algorithm described in Sections 3.3 and 3.4 respectively is that they are efficient and their performance can be determined using algorithmic analysis.

Chief to the efficiency of the techniques is two restrictions stated. These bounds are experimentally verified in the thesis and elsewhere [2, 3].

as well verified experimentally in the thesis and elsewhere [2, 3].

Based on the change propagation algorithm, we analyzed a number of applications including combining values in a list, the quick sort and merge sort algorithms, the Graham scan algorithm for convex hulls [22], and the tree-contraction algorithm [32] and showed that these algorithms adjust to changes efficiently. These bounds are within a constant factor of the best known bounds. We state the theorems here and refer the interested reader to Acar’s thesis for the proofs [2]. All the algorithms are randomized and all expectations are taken over internal randomization unless otherwise stated. These bounds are experimentally verified in other work [2, 7, 3].

Theorem 1 (Self-Adjusting List Combine)
The list combine algorithm self-adjusts to insertions/deletions in expected $O(n \log n)$ time.

Theorem 2 (Self-Adjusting Quick Sort)
Quick sort self-adjusts to an insertion/deletion at a uniformly random position in expected $O(n \log n)$ time. The expectation is taken over all permutations of the input.

Theorem 3 (Self-Adjusting Merge Sort)
Merge sort algorithm self-adjusts to insertions/deletions anywhere in the input in expected $O(n \log n)$ time.

Theorem 4 (Self-Adjusting Graham’s Scan)
The Graham’s Scan algorithm self-adjusts to an insertion/deletion at a uniformly randomly chosen location in expected $O(n \log n)$ time.

Theorem 5 (Dynamic Trees)
The tree contraction algorithm adjusts to a single edge insertion/deletion in expected $O(n \log n)$ time.

As with the ordinary quick hull algorithm, it is not possible to give tight bounds for the self-adjusting version of quick hull—the algorithm has poor worst case stability. As with the ordinary quick hull algorithm, however, the self-adjusting version performs well in practice.

3.6 Implementation

The full code for the SML library (Section 3.1) is presented elsewhere [3, 2]. Our experiments with this implementation show that its has a constant factor overhead of eight over from-scratch execution, and that change-propagation can yield speed up of up to two orders of magnitude over from-scratch execution [3, 2].

3.7 Kinetic Data Structures

Kinetic algorithms or data structures maintain a combinatorial property of continuously moving objects. For example a kinetic convex hull algorithm maintains the convex hull of a set of points as the points move continuously in the plane. Since their invention by Basch, Guibas, and Hershberger [12], kinetic algorithms have been studied extensively [13, 8, 23, 12, 24, 29, 16, 28, 1] maintain properties of moving objects. In this section, we show how a self-adjusting program can be transformed into a kinetic program trivially and show some preliminary experimental results.
Kinetic data structures are based on the idea of creating a certificate for each predicate computed by a static algorithm. Each certificate is tagged with a failure time at which it is known to change value. The certificates generated by a computation can be thought as a proof of the property being computed. To simulate motion, the certificates are placed in a priority queue of an event scheduler. The event scheduler simulates motion in a stepwise fashion. At each step, the scheduler removes the earliest failing certificate from the queue, advances the simulation time to the failure time of the certificate and runs an update algorithm. The update algorithm updates the computation according to the new value of the certificate.

The key component of a kinetic algorithm is the update algorithm. In all previous work, update algorithms are designed on a per problem basis [13, 23, 8, 29, 16, 28]. We use the change-propagation algorithm as a general-purpose update algorithm.

Based on this idea we implemented an ML-library for kinetic data structures. Using our library, the programmer can transform a self-adjusting program into a kinetic program in three steps. The first involves replacing each comparison with a library supplied kinetic comparison. A kinetic comparison generates a certificate for the comparison. The second step involves applying the library supplied certify function to each kinetic comparison. The certify function writes the value of the certificate into a modifiable and inserts that modifiable into the event scheduler’s queue based on its failure time. The third step involves modifying the self-adjusting program so that the values returned by certify functions, which are modifiables, are read. For example, given our Kinetic library, and a library KGeo of certifying comparisons, the main change to the code for the quick-hull algorithm is to replace Geo.above, Geo.closer ... with (Kinetic.certify o KGeo.above), (Kinetic.certify o KGeo.closer) ... respectively.

To determine the effectiveness of our approach, we performed some experiments with the kinetic quick-hull program. Figure 5 shows snapshots of a simulation with our kinetic quick hull program. All points move inside the box with randomly determined initial velocities and directions. Every time a kinetic event happens, i.e., a certificate fails, the simulation inserts a random set of points and deletes a random set of points. In the example simulation, insertions are four times more likely than deletions. This ability to support discrete and continuous changes simultaneously is unique to our approach.

Figure 7 show the average time per kinetic event with the our kinetic quick-hull program for inputs sizes of up to 5000 points. We measure this by assigning random velocity vector to points uniformly randomly selected within a bounding box and performing a simulation until there are no more kinetic events. A brute-force way to implement kinetic algorithms is to run the computation from scratch every time the value of a comparison changes. Figure 7. In our experiments, we measured the ratio of time for a from-scratch execution to the time for change propagation (this is the speed up obtained by change propagation) to be 40. These timings also include the time spent by the SML/NJ garbage collector which can be as high as 60% of the total time. This is due to known problems in the interaction of SML/NJ garbage collector and impure program (programs with side effects). Note that our techniques heavily rely on side effects—ordinary references are used to implement modifiable references.

4 The Language

4.1 Abstract syntax

| Types | τ ::= int | unit | τ1 × τ2 | τ1 + τ2 |
|       | τ mod | τ1 ↠ τ2 | τ1 ↛ τ2 |
| Values | v ::= O | n | x | f | (τ1, v2) | inl_τ1 v1 + v2 | inr_τ1 v1 + v2 |
|        | functor f(x : τ1) : τ2 is e1 end |
|        | func f(x : τ1) : τ2 is e1 end |
| Op.   | o ::= not | + | - | = | < | ...
| Exp.  | e ::= e1 | e2 |
| St. Exp. | es ::= v | o(v1, ..., vn) | mod e1 | apply_s(v1, v2) |
|        | let x be e1 in e3 end |
|        | let x1 × x2 be v in e3 end |
|        | case v of inl(x1 : τ1) ⇒ e4 |
|        | inr(x2 : τ2) ⇒ e′ |
| Ch. Exp. | ec ::= write(v) | apply_c(v1, v2) |
|        | let x be e1 in e3 end |
|        | let x1 × x2 be v in e3 end |
|        | read v as x in e3 end |
|        | case v of inl(x1 : τ1) ⇒ e5 |
|        | inr(x2 : τ2) ⇒ e′ |

Program p ::= ec

Figure 7: The abstract syntax of SLf.
we have described earlier [4]. We use the meta-variables $x$, $y$, and $z$ (and variants) to range over an unspecified set of variables, and the meta-variable $l$ (and variants) to range over a separate, unspecified set of locations—the locations are modifiable references. As before, the syntax of the presentation of the dynamic semantics.

The types language includes the base types \texttt{int} and \texttt{unit}, sums and products, the stable function type $\tau_1 \rightarrow\!\!\!\rightarrow \tau_2$, the changeable function type $\tau_1 \odot\!\!\!\odot \tau_2$, and the type $\tau \mod$ of modifiable references of type $\tau$. Extending SLf with recursive types presents no fundamental difficulties, but they are omitted here for the sake of brevity.

Expressions are classified into three categories: values, stable expressions, and changeable expressions. Values are constants, variables, locations, and the introduction forms for sums, products, and functions. The value of a stable expression is not sensitive to modifications to the inputs, whereas the value of a changeable expression may, directly or indirectly, be affected by them.

The familiar mechanisms of functional programming are embedded in SLf as stable expressions. We have a sequential \texttt{let} construct for ordering evaluation, elimination forms for products and sums, applications of stable functions, and the creation (including initialization) of new modifiables. The body of a stable function must itself be a stable expression. The expression \texttt{mod $\chi$, $e$} allocates a new modifiable reference and initializes it by executing the changeable expression $e$. Note that the modifiable itself is stable, even though its contents is subject to change.

Changeable expressions always execute in the context of an enclosing \texttt{mod-expression} which provides the implicit target location that every such execution ultimately writes to. The changeable expression \texttt{write $\chi$, ($v$)} writes the value $v$ of type $\tau$ into the target. The expression \texttt{read $\tau$ as $x$ in $c$ end} binds the contents of the modifiable $v$ to the variable $x$, then continues evaluation of $c$. A \texttt{read} is considered changeable because the contents of the modifiable on which it depends is subject to change. A changeable function is stable as a value, but its body is changeable; correspondingly, the application of a changeable function is a changeable expression. Like in the stable case there are also changeable versions of the sequential \texttt{let} and the elimination forms for sums and products.

The static semantics of SLf is shown in Appendix A.

4.2 Dynamic semantics

The evaluation judgements of SLf have one of two forms. The judgement $\sigma, \chi, e :: \downarrow^\beta \upsilon, \sigma', \chi', \tau_s$ states that evaluation of the stable expression $e$, relative to the input store $\sigma$, yields the value $\upsilon$, the trace $\tau_s$, and the updated store $\sigma'$. Moreover, it relates change sets $\chi$ and $\chi'$, which are sets of locations: if $\chi$ contains those locations of $\sigma$ relevant to $e$ that potentially have been modified (relative to a hypothetical previous run), then $\chi'$ accounts for (at least) all the locations of $\sigma'$ that have changed relative to the same previous run.

Similarly, the judgement $\sigma, \chi, l :: \downarrow^\beta \upsilon, \sigma', \chi', \tau_c$ states that evaluation of the changeable expression $e$, relative to the input store $\sigma$, writes its value to the target $l$, and yields the trace $\tau_c$, and the updated store $\sigma'$. The meaning of the change sets is analogous to the stable case.

In the dynamic semantics, a store, $\sigma$, is a finite function mapping each location in its domain, $\text{dom}(\sigma)$, to either a value $\upsilon$ or a “hole” $\Box$. The defined domain, $\text{gen}(\sigma)$, of $\sigma$ consists of those locations in $\text{dom}(\sigma)$ not mapped to $\Box$ by $\sigma$. When a location is created, it is assigned the value $\Box$ to reserve that location while its value is being determined. With a store $\sigma$, we associate a location typing $\Lambda$ and write $\sigma : \Lambda$, if the store satisfies the typing $\Lambda$.

A trace is a finite data structure recording the adaptive aspects of evaluation. Like the expressions whose evaluations they describe, traces come in stable and changeable varieties:

The abstract syntax of traces is given by the following grammar:

\begin{align*}
\text{Stable} & \quad T_s ::= \epsilon | \text{mod}, l :: T_s | \odot T_s | \text{let} T_s, T_s \nonumber \\
\text{Changeable} & \quad T_c ::= \text{write} \tau | \odot T_c | \text{let} T_c, T_c | \text{read}\tau_{\Rightarrow x}, T_c \nonumber
\end{align*}

Notice that every evaluation step extends the trace, although sometimes just with a dummy item. We use this technical trick to be able to perform proofs by induction on the length of the trace.

A stable trace records the sequence of allocations of modifiables that arise during the evaluation of a stable expression. The trace $\text{mod}, l :: T_s$ records the allocation of the modifiable, $l$, its type, $\tau$, and the trace of the initialization code for $l$. The trace $\text{let} T_s, T_s$ results from evaluation of a $\text{let}$ expression in stable mode, the first trace resulting from the bound expression, the second from its body.

A changeable trace has one of three forms. A write, $\text{write} \tau$, records the storage of a value of type $\tau$ in the target. A sequence $\text{let} T_s, T_c$ records the evaluation of a $\text{let}$ expression in changeable mode, with $T_s$ corresponding to the bound stable expression, and $T_c$ corresponding to its body. A read $\text{read}\tau_{\Rightarrow x}, T_c$ trace specifies the location read ($l$), the context of use of its value ($x.e$) and the trace, $T_c$, of the remainder of evaluation with the scope of that read. This records the dependency of the target on the value of the location read.

The dynamic dependency graphs described in Section ?? may be seen as an efficient representation of traces. Time stamps may be assigned to each read and write operation in the trace in left-to-right order. These correspond to the time stamps in the DDG representation. The containment hierarchy is directly represented by the structure of the trace. Specifically, the trace $T_s$ (and any read in $T_s$) is contained within the read trace $\text{read}\tau_{\Rightarrow x}, T_c$.

Stable Evaluation. The evaluation rules for stable expressions are given in Figure 8. Most of the rules are standard for a store-passing semantics. For example, the $\text{let}$ rule sequences evaluation of its two expressions, and performs binding by substitution. Less conventionally, it yields a trace consisting of the sequential composition of the traces of its sub-expressions and maintains a change set.

The most interesting rule is the evaluation of $\text{mod}, e$. Given a store $\sigma$, a fresh location $l$ is allocated and initialized to $\Box$ prior to evaluation of $e$. The sub-expression $e$ is evaluated in changeable mode, with $l$ as the target. Pre-allocating $l$ ensures that the target of $e$ is not accidentally re-used during evaluation; the static semantics ensures that $l$ cannot be read before its contents is set to some value $\upsilon$.

Each location allocated during the evaluation of a stable expression is recorded in the trace and is written to: If $\sigma, \chi, e :: \downarrow^\beta \upsilon, \sigma', \chi', T_s$, then $\text{dom}(\sigma') = \text{dom}(\sigma) \cup \text{gen}(T_s)$, and $\text{gen}(\sigma') = \text{gen}(\sigma) \cup \text{gen}(T_s)$. Furthermore, all locations
\[
\sigma, \gamma, v \vdash^\beta v, \sigma, \gamma \quad \text{(value)}
\]
\[
\sigma, \gamma, o(v_1, \ldots, v_n)) \vdash^\beta v, \sigma, \gamma \quad \text{(primitives)}
\]
\[
\sigma[l \mapsto \Box, \sigma, \gamma, l \vdash e \vdash^C \sigma', \gamma, T] \quad \text{(mod)}
\]
\[
(v_1 = \text{func } f(x : \tau_1) : \tau_2 is e \text{ end})
\]
\[
\sigma, [v_1/f, v_2/x] \vdash^\beta v, \sigma, \gamma, \tau' \quad \text{(apply/miss)}
\]
\[
\sigma, \gamma, \text{apply}_g(v_1, v_2) \vdash^\beta v, \sigma', \gamma', \circ T_s \quad \text{(apply/apply/miss)}
\]
\[
\sigma, \text{let } x \to e_1 in e_2 = e_2 \quad \text{(let)}
\]
\[
\sigma, \text{let } x \to \text{case } e_1 \text{ of } e_2 \quad \text{(case/let)}
\]
\[
\sigma, \text{case inl}_{x + \tau_1} of \inl(x_1: \tau_1) \Rightarrow e_1 \quad \text{(case/inl)}
\]
\[
\sigma, \text{case inr}_{x + \tau_1} of \\inr(x_1: \tau_1) \Rightarrow e_1 \quad \text{(case/inr)}
\]

**Figure 8:** Evaluation of stable and changeable expressions.

Read during evaluation are defined in the store, \(\text{dom}(T_s) \subseteq \text{gen}(\sigma')\).

**Changeable Evaluation.** The evaluation rules for changeable expressions are also given in Figure 8. The \texttt{let} rule is similar to the corresponding rule in stable mode, except that the bound expression, \(e\), is evaluated in stable mode, whereas the body, \(e'\), is evaluated in changeable mode. The \texttt{read} expression substitutes the binding of location \(l\) in the store for the variable \(x\) in \(e\), and continues evaluation in changeable mode. The read is recorded in the trace, along with the expression that employs the value read. The \texttt{write} rule simply assigns its argument to the target. Finally, application of a changeable function passes the target of the caller to the callee, avoiding the need to allocate a fresh target for the callee and a corresponding read to return its value to the caller.

Each location allocated during the evaluation of a changeable expression is recorded in the trace and is written; the destination is also written: If \(\sigma, \text{let } l \to e \vdash^C \sigma', \gamma, T_c\), then \(\text{dom}(\sigma') = \text{dom}(\sigma) \cup \text{gen}(T_s)\), and \(\text{gen}(\sigma') = \text{gen}(\sigma) \cup \text{gen}(T_c) \cup \{l\}\). Furthermore, all locations read during evaluation are defined in the store, \(\text{dom}(T_s) \subseteq \text{gen}(\sigma')\).

### 4.3 The Oracle

Evaluation of function application (both stable and changeable) refers to an \textit{oracle}. The oracle is used to model memoization. Corresponding to a memo-table miss, the oracle can decide to give a negative answer. We write this as \(\sigma, e_s \vdash^s T\) or \(\sigma, l \to e_c \vdash^C T\), respectively. Alternatively, it can report value and trace of an earlier evaluation of a given expression, represented by judgments \(\sigma, e_s \vdash^s v, T_s, \gamma\) and \(\sigma, l \to e_c \vdash^C t_s, \gamma\). The change set \(\gamma\) is part of a positive answer contains the locations that differ between \(\sigma\) (the store in which we want to evaluate \(e_s\) now) and the
original store that was in effect when the earlier evaluation was memoized. A positive answer from the oracle still needs to be adjusted to these changes. Change propagation is explained in Section 5.

Notice that the oracle makes the dynamic semantics formally non-deterministic. Lemmas 7 and 8 in Section 6 express the important property that this non-determinism does not matter: change propagation will correctly update the positive answers of the oracle and make everything look as if the oracle always answered negatively. We write \( \sigma, \chi, e \vdash^S v, \sigma', \chi', T_e \) and \( \sigma, \chi, l \vdash e \uparrow^C \sigma', \chi', T_e \) if \( \sigma, \chi, e \vdash^S v, \sigma', \chi', T_e \) and \( \sigma, \chi, l \vdash e \vdash^C \sigma', \chi', T_e \), respectively, can be derived without ever using a hit rule. We call an evaluation that does not use hit a clean evaluation.

5 Change propagation

The change propagation rules are relatively simple: Most rules simply skip ahead. Only when a read finds that the location read has changed, it re-runs the changeable computation that is in its, thereby replacing the corresponding part of the trace. As explained earlier, the actual implementation is more efficient than what the rules suggest: By keeping the reads that need attention on a priority queue, change propagation can skip ahead arbitrarily far in a single step.

6 Correctness

As we have already pointed out, the operational semantics of SLf is non-deterministic because the evaluation of function calls has the option to execute the call normally or to use and adjust the result of an earlier such call. Of course, it is undesirable for two different evaluation paths to lead to different results. We call memoization and change propagation incorrect if this were possible.

Naively we might expect that correctness can be characterized by saying that an arbitrary evaluation should yield results (value, store, and trace) that are identical to those obtained from a clean (hit-free) evaluation:

\[
\sigma, \emptyset, e \downarrow^S v, \sigma' \downarrow^L \Rightarrow \sigma, \emptyset, e \downarrow^S v, \sigma' \downarrow^L
\]

6.1 Structural Equality and Lifting

Unfortunately, we cannot prove this because it is not even generally true. This problem is due to differences in the stores that are involved: re-executing mod-expressions will allocate fresh locations, and there is no control over their identity. However, we are not really interested in the identity of locations. We consider a value \( v_1 \) in a store \( \sigma_1 \) equal to a value \( v_2 \) in a store \( \sigma_2 \) if they are structurally equal in the sense of, e.g., Lisp’s \textbf{EQUAL} function. This works because SLf programs cannot create cyclic memory graphs. Moreover, like memory addresses of cons-cells in Standard ML, SLf programs cannot compare locations for equality and, therefore, cannot observe sharing within data structures.

We formalize the notion of structural equality using the lift operation \( e \uparrow \sigma \). Intuitively, starting with the locations in \( e \), this recursively replaces all locations \( l \) with their respective values \( \sigma(l) \). The definition of \( e \uparrow \sigma \) is shown in Figure 11. A similar definition for lifting of traces is omitted.

6.2 The Correctness Theorem

Using lift, we can now formulate the main theorem which states correctness of memoization and change-propagation. Memoization and change propagation are correct if their use is undetectable: any evaluation should produce results that are structurally equal to those obtained from a clean run:

**Theorem 6 (Correctness)**

\[
\sigma, \emptyset, e \downarrow^S v, \sigma', \omega \uparrow^L \Rightarrow v' \uparrow \sigma' = v'' \uparrow \sigma''
\]

To be able to prove this, we need to strengthen the statement of the theorem. The problem with the theorem as stated is that it considers two evaluations that start with identical expressions and identical stores. However, after taking a few steps on non-identical evaluation paths, expressions and stores can become different, and the theorem can no longer be used as the induction hypothesis. To formulate the strengthening of the theorem that will let us use a proof by induction, we need to introduce another technical device: pegging.

6.3 Pegging a Store

We need to be able to talk about expressions (and values, and traces) that are “equal up to changes given by a change set \( \chi \).” We do that by lifting them using a pegged store. A pegged store \( \sigma \uparrow \chi \) is like the original store \( \sigma \) except for locations that appear in the change set \( \chi \); locations in \( \chi \) are mapped to a dummy value \( e_\chi \):

\[
(\sigma \cdot \chi)(l) = \begin{cases} 
\bullet & l \in \chi \\
\sigma(l) & l \notin \chi
\end{cases}
\]

6.4 Change Propagation Lemmas

The strengthening of the correctness theorem (Theorem 6) consists of two lemmas, one for the stable case, and one for the changeable case. Each lemma has seven conditions (1)-(7) that must hold before it can be applied. To give some intuition, let us describe the scenario that Lemma 7 characterizes:

At some earlier time, an evaluation of \( e_1 \) in store \( \sigma_1 \) (1) has been memoized as is now reported by the oracle. The current evaluation of the same expression \( e_1 \) is performed in a store \( \sigma_2 \) which contains all the locations of \( \sigma_1 \) and also all locations—with their corresponding values—allocated during the memoized evaluation (2). This means

\footnote{For example, the dummy value could be just \( 0 \) or 0.}
that considering $T_1$ in $\sigma_3$ does not suffer from dangling references, and parts of $T_2$ that allocate new modifiables but which get skipped by change propagation are still valid in $\sigma_3$. The reported change set $\chi$ is as subset of the domain of $\sigma_1$ (3) and correctly characterizes the differences between the original store $\sigma_1$ and the current store $\sigma_3$ as far as they concern $e_1$ (4). Change propagation is performed on the trace of the original evaluation (5). Expression $e_1$ in the current store $\sigma_3$ is equal to some other expression $e_2$ in its store $\sigma_2$ (6). For comparison purposes, we consider a clean evaluation (without $\text{hit}$) of that other expression in its store (7).

Under these conditions, change propagation will correctly fix everything up: in the resulting store $\sigma'_3$ the change-propagated trace $T'_2$ looks like the clean trace $T_2$ in its store, the value $v_1$ looks like $v_2$, and the resulting change set $\chi'$ correctly characterizes the differences between $\sigma'_3$ and the original $\sigma'_1$.

We state the lemma for the stable case here. The changeable case together with proofs for both lemmas is given in Appendix B.

**Lemma 7 (Change Propagation/Stable)**

**If**

1. $\sigma_1, \chi_1, e_1 \models v_1, \sigma'_1, \chi'_1, T_1$,
2. $\sigma_3 \vDash \sigma'_1[\text{gen}(T_1)], \text{dom}(\sigma_3) \supseteq \text{dom}(\sigma_1)$,
3. $\chi \subseteq \text{dom}(\sigma_1)$,
4. $e_1 \uparrow \sigma_1 \cdot \chi = e_1 \uparrow \sigma_3 \cdot \chi$,
5. $\sigma_3, \chi, T_1 \vDash \sigma'_3, \chi', T'_1$,
6. $e_1 \uparrow \sigma_3 = e_2 \uparrow \sigma_2$,
7. $\sigma_2, \chi_2, e_2 \models v_2, \sigma'_2, \chi'_2, T_2$.

**then**

$I. \quad T_1 \uparrow \sigma'_1 = T_2 \uparrow \sigma'_2$,
$II. \quad v_1 \uparrow \sigma'_1 = v_2 \uparrow \sigma'_2$,
$III. \quad v_1 \uparrow \sigma'_1 \cdot \chi' = v_1 \uparrow \sigma'_2 \cdot \chi'$.

**6.5 Proof of correctness**

Theorem 6 is an instance of Lemma 7 by considering a change set that is empty:

**Proof:** We set $e_1 = e_2 = e$, $v_1 = v'$, $v_2 = v''$, $\sigma_1 = \sigma_2 = \sigma$, $\sigma'_1 = \sigma_3 = \sigma'$, $\sigma'_2 = \sigma''$, $\chi_1 = \chi_2 = \emptyset$, $T_2 = T'$. With $\chi = \emptyset$ we have $T_1 = T_1' = T$ and $\sigma_3 = \sigma'_3 = \sigma'$. It is easy to show that conditions (1)-(7) of Lemma 7 are satisfied and that its conclusions imply those of the theorem.

**References**


\[
\begin{array}{rcl}
\tau \cdot \sigma &=& \tau \\
\sigma(f(x:\tau_1)) &=& \sigma(f(x:\tau_2)) \\
\sigma(f(x:\tau_1)) &=& \sigma(f(x:\tau_2)) \\
\end{array}
\]

| o(v_1, \ldots, v_n) | \sigma &=& o(v_1 \mid \sigma, \ldots, v_n \mid \sigma) \\
| mod, e, \sigma &|& mod, e \mid \sigma \\
| applyG(v_1, v_2) | \sigma &=& applyG(v_1 \mid \sigma, v_2 \mid \sigma) \\
| let x = e \mid \sigma &|& let x = e \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| let x_1 \times x_2 = v \mid \sigma &|& let x_1 \times x_2 = v \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| case v \cdot (\inl(x_1;\tau_1)) \Rightarrow e_\sigma &=& case v \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| let x \mid \sigma &|& let x \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| read v \mid \sigma &|& read v \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| case v \cdot (\inl(x_1;\tau_1)) \Rightarrow e_\sigma &=& case v \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| let x_1 \times x_2 = \inl(x_1;\tau_1) \Rightarrow e_\sigma &=& case v \mid \sigma \mid \sigma \mid \sigma \mid \sigma \\
| \end{array}

Figure 11: The lift operation


A Static semantics

The SLf type system is inspired by the type theory of modal logic given by Pfenning and Davies [35]. We distinguish two modes, the stable and the changeable, corresponding to the distinction between terms and expressions, respectively, in Pfenning and Davies’ work.

The judgement $\Lambda; \Gamma \vdash e : \tau$ states that $e$ is a well-formed stable expression of type $\tau$, relative to $\Lambda$ and $\Gamma$. The judgement $\Lambda; \Gamma \vdash c : \tau$ states that $e$ is a well-formed changeable expression of type $\tau$, relative to $\Lambda$ and $\Gamma$. Here $\Lambda$ is a location typing and $\Gamma$ is a variable typing; these are finite functions assigning types to locations and variables, respectively.

The typing judgements for stable and changeable expressions are shown in Figure 12 respectively. For primitive functions, we assume a typing relation $\Gamma_o$. For stable expression, the interesting rules are the $\text{mod}$ and the changeable functions. The bodies of these expressions are changeable expressions and therefore they are typed in the changeable mode. For changeable expressions, the interesting rule is the $\text{let}$ rule. The body of $\text{let}$ is a changeable expression and thus typed in the changeable mode; the expression bound, however, is a stable expression and therefore it typed in the stable mode. The $\text{mod}$ and $\text{let}$ rules therefore provide inclusion between two modes.

B Proofs

The proof of the change-propagation lemmas relies on several properties of the lift operation, in particular the following:

Theorem 9

Let $e$ be an expression and $\sigma, \sigma'$ be two stores and $\chi, \chi'$ two changed sets. If

1. $\sigma' \supseteq \sigma$,
2. $\chi' \supseteq \chi$, and
3. $\chi' \setminus \chi \subseteq \text{dom}(\sigma') \setminus \text{dom}(\sigma)$

then $e \uparrow \sigma \cdot \chi = e \uparrow \sigma' \cdot \chi'$.

B.1 Proof of change propagation lemmas

Lemma 7 (Change Propagation/Stable)

If

(A) $\sigma_1, \chi_1, e_1 \downarrow^S v_1, \sigma'_1, \chi'_1, T_1$,
(B) $\sigma_3 \supseteq \sigma'_3 \uparrow \tau (\text{gen}(T_1), \text{dom}(\sigma_3)) \supseteq \text{dom}(\sigma_1)$,
(C) $\chi \subseteq \text{dom}(\sigma_1)$,
(D) $e_1 \uparrow \sigma_1 \cdot \chi = e_1 \uparrow \sigma_3 \cdot \chi$,
(E) $\sigma_3, \chi, T_1 \uparrow \chi, \sigma'_3, \chi', T'_1$,
(F) $e_1 \uparrow \sigma_3 = e_2 \uparrow \sigma_2$,
(G) $\sigma_2, \chi_2, e_2 \downarrow^S v_2, \sigma'_2, \chi'_2, T_2$

then

1. $T'_1 \uparrow \sigma'_2 \uparrow \sigma_2$,
2. $v_1 \uparrow \sigma'_2 = v_2 \uparrow \sigma_2$,
3. $v_1 \uparrow \sigma_1 \cdot \chi' = v_1 \uparrow \sigma'_1 \cdot \chi'$.

Lemma 8 (Change Propagation/Changeable)

If

(A) $\sigma_1, \chi_1, l_1 \leftarrow e_1 \downarrow^c \sigma'_1, \chi'_1, T_1$,
(B) $\sigma_2, \chi_2, l_2 \leftarrow e_2 \downarrow^c \sigma'_2, \chi'_2, T_2$,
(C) $\chi \subseteq \text{dom}(\sigma_1)$ s.t. $e_1 \uparrow \sigma_1 \cdot \chi \geq e_2 \uparrow \sigma_2$, 
\[ \frac{\lambda; \Gamma \vdash \text{unit} : \text{unit} \quad \lambda; \Gamma \vdash n : \text{int} : \text{int}}{\lambda; \Gamma \vdash (x = \tau)} \quad \frac{(\lambda(l) = \tau)}{\lambda; \Gamma \vdash l : \tau \text{ mod}} \]

<table>
<thead>
<tr>
<th>(variable)</th>
<th>(unit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(pair)</td>
<td>(number)</td>
</tr>
</tbody>
</table>
| \[ \lambda; \Gamma \vdash v_1 : \tau_1 \quad \lambda; \Gamma \vdash v_2 : \tau_2 \]
| \[ \lambda; \Gamma \vdash (v_1, v_2) : \tau_1 \times \tau_2 \] |
| \[ \lambda; \Gamma \vdash v : \tau \]
| \[ \lambda; \Gamma \vdash \text{inr} \_1 + v : \tau_1 + \tau_2 \] |
| \[ \lambda; \Gamma \vdash e : \tau \quad \lambda; \Gamma, x : \tau \vdash e' : \tau' \]
| \[ \lambda; \Gamma \vdash \text{let } x \text{ be } e \text{ in } e' \text{ end : } \tau' \] |
| \[ \lambda; \Gamma \vdash v : \tau \times \tau_2 \quad \lambda; \Gamma, x_1 : \tau_1, x_2 : \tau_2 \vdash c e : \tau \]
| \[ \lambda; \Gamma \vdash \text{let } x_1 \times x_2 \text{ be } v \text{ in e end : } \tau \] |
| \[ \lambda; \Gamma, f : \tau_1 \rightarrow \tau_2 \vdash f(x : \tau_1) : \tau_2 \text{ is } e \text{ end : } \tau_1 \rightarrow \tau_2 \] |
| \[ \lambda; \Gamma \vdash \text{func } f(x : \tau_1) : \tau_2 \text{ is } e \text{ end : } \tau_1 \rightarrow \tau_2 \] |
| \[ \lambda; \Gamma \vdash \text{case } v \text{ of } \text{inl } (x_1 : \tau_1) \Rightarrow e_1 : \tau \quad \text{inr } (x_2 : \tau_2) \Rightarrow e_2 \] |

Figure 12: Typing of stable and changeable expressions.
(D) \( \sigma_3 \supseteq \sigma_1^\text{gen}(T_1), \text{dom}(\sigma_3) \supseteq \text{dom}(\sigma_1) \),

(E) \( e_1 \uparrow \sigma_3 = e_2 \uparrow \sigma_2 \),

(F) \( e_1 \uparrow \sigma_1 \cdot \chi = e_1 \uparrow \sigma_3 \cdot \chi \),

(G) \( \sigma_3, T_1, \chi \not\supseteq \sigma_1', T_1', \chi' \),

then

1. \( T_1' \uparrow \sigma_1' = T_2 \uparrow \sigma_2' \),
2. \( l_1 \uparrow \sigma_1' = l_2 \uparrow \sigma_2' \),
3. \( l_1 \uparrow \sigma_1' \cdot \chi' \geq l_2 \uparrow \sigma_2' \),
4. \( l_1 \uparrow \sigma_1' \cdot \chi' = l_1 \uparrow \sigma_3' \cdot \chi \).

Key cases of the proof are shown in Appendix B.

**Proof:**

We consider finite stores only. The proofs for Lemmas 7 and 8 proceed together by simultaneous induction on \( (|\sigma_3| \times |T_1'|) \), i.e., on pairs consisting of the number of bound locations in \( \sigma_3 \) and the length of trace \( T_1' \), under lexicographic ordering. The case for values and the case for primitives, where the trace is empty constitute the bases cases.

Notice that condition (F) in each of the two lemmas implies that we only consider expressions \( e_1 \) and \( e_2 \) which have the same syntactic structure up to values and locations. In particular, this means that their outermost syntactic forms are guaranteed to be equal.

- **values:** Assume that

  (A) \( \sigma_1, \chi, v_1 \not\supseteq v_1, \sigma_1, \chi' \cdot \varepsilon \)

  (B) \( \sigma_3 \supseteq \sigma_1^\text{gen}(e), \text{dom}(\sigma_3) \supseteq \text{dom}(\sigma_1) \)

  (C) \( \chi \subset \text{dom}(\sigma_1) \)

  (D) \( v_1 \uparrow \sigma_1 \cdot \chi = v_1 \uparrow \sigma_3 \cdot \chi \)

  (E) \( \sigma_3, \varepsilon, \chi \not\supseteq \sigma_3, \varepsilon, \chi \)

  (F) \( v_1 \uparrow \sigma_3 = e_2 \uparrow \sigma_2 \)

  (G) \( \sigma_2, v_2, \chi_2 \not\supseteq v_2, \sigma_2, \chi_2 \cdot \varepsilon \)

Then the following hold.

1. \( \varepsilon \uparrow \sigma_3 = \varepsilon \uparrow \sigma_2 \); follows trivially.
2. \( v_1 \uparrow \sigma_3 = v_2 \uparrow \sigma_2 \); follows by property F.
3. \( v_1 \uparrow \sigma_1 \cdot \chi = v_1 \uparrow \sigma_3 \cdot \chi \); follows by property D.

- **primitives:** Assuming that the following hold.

  (A) \( \sigma_1, \chi_1, o(u_1, \ldots, u_n) \not\supseteq u_1, \sigma_1, \chi_1 \cdot \varepsilon \)

  (B) \( \sigma_3 \supseteq \sigma_1^\text{gen}(e) \)

  (C) \( \chi \subset \text{dom}(\sigma_1) \)

  (D) \( o(u_1, \ldots, u_n) \uparrow \sigma_1 \cdot \chi = o(u_1, \ldots, u_n) \uparrow \sigma_3 \cdot \chi \)

  (E) \( \sigma_3, T_1, \chi \not\supseteq \sigma_3, T_1, \chi \)

  (F) \( o(u_1, \ldots, u_n) \uparrow \sigma_3 = o(w_1, \ldots, w_n) \uparrow \sigma_2 \)

(G) \( \sigma_2, \chi_2, o(w_1, \ldots, w_n) \not\supseteq w_1, \sigma_2, \chi_2 \cdot \varepsilon \)

we show that the following are true.

1. \( \varepsilon \uparrow \sigma_3 = \varepsilon \uparrow \sigma_2 \) holds trivially.
2. \( v_1 \uparrow \sigma_3 = v_2 \uparrow \sigma_2 \). By property F, we know that \( o(u_1, \ldots, u_n) \uparrow \sigma_3 = o(w_1, \ldots, w_n) \uparrow \sigma_2 \). Since the arguments to primitives cannot contain locations, \( o(u_1, \ldots, u_n) = o(w_1, \ldots, w_n) \), and thus \( v_1 = v_2 \). Since \( v_1 \) and \( v_2 \) contain no locations, the property follows.
3. \( v_1 \uparrow \sigma_1 \cdot \chi = v_1 \uparrow \sigma_3 \cdot \chi \). Like the previous property this property following from the fact that \( v_1 = v_2 \) and that \( v_1 \) and \( v_2 \) contains no locations.

**mod:** Assume that the following hold:

(A) \( \sigma_1, \chi_1, \text{mod}, e_1 \not\supseteq l_1, \sigma_1, \chi_1, \text{mod}, l_1 \leftarrow T_1 \)

(B) \( \sigma_3 \supseteq \sigma_1^\text{mod}(T_1), \text{dom}(\sigma_3) \supseteq \text{dom}(\sigma_1) \)

(C) \( \chi \subset \text{dom}(\sigma_1) \)

(D) \( \text{mod}, e_1 \uparrow \sigma_1 \cdot \chi = \text{mod}, e_1 \uparrow \sigma_3 \cdot \chi \)

(E) \( \sigma_3, \chi, \text{mod}, T_1 \not\supseteq \sigma_3', \chi', \text{mod}, T_1' \)

(F) \( \text{mod}, e_1 \uparrow \sigma_3 = \text{mod}, e_2 \uparrow \sigma_2 \)

(G) \( \sigma_2, \chi, \text{mod}, e_2 \not\supseteq l_2, \sigma_2, \chi' \cdot \text{mod}, l_2 \leftarrow T_2 \)

we show that the following are true.

1. \( \text{mod}, T_1' \uparrow \sigma_3 = \text{mod}, T_2 \uparrow \sigma_2' \),
2. \( l_1 \uparrow \sigma_3' = l_2 \uparrow \sigma_2' \),
3. \( l_1 \uparrow \sigma_1' \cdot \chi' = l_1 \uparrow \sigma_3' \cdot \chi' \).

We first consider the evaluation, and change-propagation rules for \text{mod} \ expressions (shown below) and apply the induction hypothesis to the antecedents.

\[
\frac{(l_1 \not\in \text{dom}(\sigma_1))}{\sigma_1[l_1 \rightarrow \Box_1], \chi_1, l_1 \leftarrow e_1 \not\supseteq \sigma_1', \chi_1', T_1}
\]

\[
\frac{(l_2 \not\in \text{dom}(\sigma_1))}{\sigma_2[l_2 \rightarrow \Box_1], \chi_2, l_2 \leftarrow e_2 \not\supseteq \sigma_2', \chi_2', T_2}
\]

\[
\frac{(l_3 \not\in \text{dom}(\sigma_1))}{\sigma_3[\chi_1, l_1 \leftarrow T_1 \not\supseteq \sigma_3', \chi_1', T_1']}
\]

We first show that the following are true.

(AA) \( \sigma_1, \chi_1, l_1 \leftarrow e_1 \not\supseteq \sigma_1', \chi_1', T_1 \)

(BB) \( \sigma_3 \supseteq \sigma_1^\text{mod}(T_1), \text{dom}(\sigma_3) \supseteq \text{dom}(\sigma_1) \)

(CC) \( \chi \subset \text{dom}(\sigma_1) \)

(DD) \( e_1 \uparrow \sigma_1 \cdot \chi = e_1 \uparrow \sigma_3 \cdot \chi \)

(EE) \( \sigma_3, \chi, T_1 \not\supseteq \sigma_3', T_1' \cdot \chi \)
The properties AA, EE, GG follow from A,E,G and by evaluation and change-propagation rules for mod expressions shown above. The properties BB,CC,DD,FF follow trivially from B,C,D,and F respectively. Since the $T_2$ is shorter than mod, $l_2 \leftarrow T_2$, the induction hypothesis applies, and by Lemma $S$ and we know that the following are true.

11. $T_1 \uparrow \sigma_1' = T_2 \uparrow \sigma_2'$.
22. $l_1 \uparrow \sigma_1' = l_2 \uparrow \sigma_2'$.
44. $l_1 \uparrow \sigma_1' \cdot \chi' = l_1 \uparrow \sigma_3' \cdot \chi'$.

Consider the properties 1,2, and 4 that we want to show. Of these 2, and 4 follow directly from 22, and 44. The property 1 follows directly from 11 and by the definition of the lift operation for traces. More precisely, mod, $T_1 \uparrow \sigma_1' = mod_3$, $(T_1 \uparrow \sigma_1') = mod_4$, $(T_1 \uparrow \sigma_3') = mod_4$, $T_2 \uparrow \sigma_2'$.

### Rule 1: Apply

Let $\chi \subseteq \text{dom}(\sigma_1)$.

### Rule 2: Let

Let $x$ be $e_1$ in $e_2$ end $\uparrow \sigma_1' \cdot \chi = let x be e_1 in e_2$ end $\uparrow \sigma_3' \cdot \chi$; this implies that $e_1 \uparrow \sigma_1' = e_1 \uparrow \sigma_3' \cdot \chi$ and $e_2 \uparrow \sigma_1' = e_2 \uparrow \sigma_3' \cdot \chi$.

We need to show that the following.

1. $\lambda T_1 T_2 \uparrow \sigma_1' = let T_3 T_4 \uparrow \sigma_2''$.
2. $\lambda u \uparrow \sigma_2'' = v \uparrow \sigma_3''$.
3. $\lambda u \uparrow \sigma_3'' \cdot \chi'' = u \uparrow \sigma_3'' \cdot \chi''$.

To this end, we consider the evaluation and the change-propagation rules for let expressions (shown below) and apply the induction hypothesis to the antecedents.

\[
\begin{align*}
\sigma_1, \chi, e_1 & \sigma_1', \chi', T_1 & \sigma_1, \chi, [u_1/x]e_2 & \sigma_1', \chi', T_2 & (\lambda/1) \\
\sigma_2, \chi, e_3 & \sigma_3', \chi', T_3 & \sigma_2, \chi', [v_1/x]e_4 & \sigma_2', \chi'', T_4 & (\lambda/2) \\
\sigma_3, \chi, T_1 & \sigma_3', \chi', T_1 & \sigma_3', \chi', T_2 & \sigma_3', \chi'', T_2 & (\lambda/c/p).
\end{align*}
\]

### Step 1: We first consider the top antecedents from each rule and show that the following hold.

(AA) $\sigma_1, \chi, e_1 \uparrow \sigma_1', \chi', T_1$; this follows by evaluation rule $\lambda/1$.

(BB) $\sigma_3 \subseteq \sigma_3'|[\text{gen}(T_1)]$. By property B, we know that $\sigma_3 \subseteq \sigma_3'|[\text{gen}(T_1)]$. Since $\sigma_2'' \subseteq \sigma_3'$, $\sigma_3 \subseteq \sigma_2''[\text{gen}(T_1)]$ holds.

(CC) $\chi \subseteq \text{dom}(\sigma_1)$; follows by property C.

(DD) $e_1 \uparrow \sigma_1' \cdot \chi = e_1 \uparrow \sigma_3' \cdot \chi$; follows directly from D.

(EE) $\sigma_3, T_1, \chi \uparrow \sigma_3', T_1, \chi'$; follows by rule $\lambda/c/p$.

(FF) $e_1 \uparrow \sigma_3' = e_1 \uparrow \sigma_2'$; follows directly from F.

(GG) $\sigma_2, \chi, e_3 \uparrow \sigma_2', \chi_2', T_3$; follows by evaluation rule $\lambda/2$.

Since the trace $T_3$ is shorter than the trace $T_3 T_4$, the induction hypothesis applies and the following are true.

11. $T_1 \uparrow \sigma_1' = T_3 \uparrow \sigma_2'$.
22. $u_1 \uparrow \sigma_3' = v_1 \uparrow \sigma_2'$.
44. $u_1 \downarrow \sigma'_1 \cdot \chi' = u_1 \downarrow \sigma'_2 \cdot \chi' \]

- **Step II:** We consider the second (bottom) antecedent from each rule and show that the following hold.

AAA. $\sigma'_1, \chi'_1, [u_1/x]e_1 \Downarrow^S u, \sigma''_1, \chi''_1, T_2$; this follows by evaluation rule let/1.

BBB. $\sigma'_3 \equiv \sigma''_1 | gen(T_2)$. By property B, we know that $\sigma_3 \equiv \sigma''_1 | gen(T_2)$. Since $\sigma'_3 \equiv \sigma'_3 | gen(T_2)$,

$\text{dom}(\sigma'_3) \supseteq \text{dom}(\sigma'_3 | gen(T_2))$. By property B, we know that $\text{dom}(\sigma'_3) \supseteq \text{dom}(\sigma''_1 | gen(T_1)) \cup \text{dom}(\sigma_1)$. This simplifies that $\text{dom}(\sigma_3) \supseteq \text{dom}(\sigma'_1 | gen(T_1)) \cup \text{dom}(\sigma'_1) = \text{gen}(T_1) \cup \text{dom}(\sigma_1)$. We know that $\text{dom}(\sigma'_1) = \text{dom}(\sigma'_1) \cup \text{gen}(T_1)$. Therefore, $\text{dom}(\sigma_3) \supseteq \text{dom}(\sigma'_1)$, since $\text{dom}(\sigma'_3) \supseteq \text{dom}(\sigma'_1), \text{dom}(\sigma'_3) \supseteq \text{dom}(\sigma'_1)$ is true.

CCC. $\chi' \subseteq \text{dom}(\sigma'_1)$. We know by property C that $\chi \subseteq \text{dom}(\sigma_1)$. We know that $\chi' \subseteq \chi \cup \text{gen}(T_1)$. It follows that $\chi' \subseteq \text{dom}(\sigma'_1) \cup \text{gen}(T_1)$. Since $\text{dom}(\sigma'_1) = \text{dom}(\sigma_1) \cup \text{gen}(T_1)$, it follows that $\chi' \subseteq \text{dom}(\sigma'_1)$.

DDD. $[u_1/x]e_2 \Downarrow \sigma'_1 \cdot \chi' = [u_1/x]e_2 \Downarrow \sigma'_3 \cdot \chi'$. We know by property 44, that $u_1 \Downarrow \sigma'_1 \cdot \chi' = u_1 \Downarrow \sigma'_3 \cdot \chi'$. Thus all we need to show is that $e_2 \Downarrow \sigma'_1 \cdot \chi' = e_2 \Downarrow \sigma'_3 \cdot \chi'$. We know, by property D, that $e_2 \Downarrow \sigma'_1 \cdot \chi = e_2 \Downarrow \sigma'_3 \cdot \chi$. By This follows by properties of lifting.

EEE. $\sigma'_1, T_2, \chi' \Downarrow^S \sigma''_3, T'_2, \chi''$; this follows by evaluation rule let/c.p.

FFF. $[u_1/x]e_2 \Downarrow \sigma'_3 = [u_1/x]e_2 \Downarrow \sigma'_2$. We know, by property 22, that $u_1 \Downarrow \sigma'_3 = u_1 \Downarrow \sigma'_2$. Thus all we need to show is that $e_2 \Downarrow \sigma'_3 = e_2 \Downarrow \sigma'_2$. By property F and properties of lifting, we know that $e_2 \Downarrow \sigma'_3 = e_2 \Downarrow \sigma'_2$.

GGG. $\sigma'_2, \chi'_2, [v_1/x]e_4 \Downarrow^S v, \sigma''_2, \chi''_2, T_4$; this follows by evaluation rule let/2.

By induction hypothesis, we therefore conclude that the following hold.

111. $T'_2 \Downarrow \sigma'_2 = T_4 \Downarrow \sigma'_2$.

222. $u \Downarrow \sigma''_3 = v \Downarrow \sigma''_3$.

444. $u \Downarrow \sigma''_3 \cdot \chi'' = u \Downarrow \sigma''_3 \cdot \chi''$.

By using properties 11, 22, 33, 44 and 111, 222, 333, 444, we will now show the We need to show that the following.

1. let $T'_1 T'_2 \Downarrow \sigma''_1 = \text{let } T_3 T_4 \Downarrow \sigma''_2$.

   We know, by property 111, that $T'_2 \Downarrow \sigma''_2 = T_4 \Downarrow \sigma''_2$. Thus all we have to show is that $T'_1 \Downarrow \sigma''_1 = T_3 \Downarrow \sigma''_2$. We know, by property 111, that $T'_1 \Downarrow \sigma''_1 = T_3 \Downarrow \sigma''_2$. We know, by property 11, that $T'_1 \Downarrow \sigma''_1 = T_3 \Downarrow \sigma''_2$. Since $\sigma''_2 \supseteq \sigma''_2$ and since $\text{dom}(\sigma''_2) \setminus \text{dom}(\sigma''_2) \cap \text{dom}(\sigma''_2) = \emptyset$, we know that $T'_1 \Downarrow \sigma''_2 = T_3 \Downarrow \sigma''_2$. Similarly since the difference between $\sigma''_1$ and $\sigma''_2$ does not intersect with that the domain of $T'_1 \Downarrow \sigma''_1 = T'_1 \Downarrow \sigma''_2$. 2.

2. $u \Downarrow \sigma''_3 = v \Downarrow \sigma''_2$. This directly follows by 222.

3. $u \Downarrow \sigma''_3 \cdot \chi'' = u \Downarrow \sigma''_3 \cdot \chi''$. This directly follows by 444.

\[2\text{This must be made precise}\]