A Political Theory of Progressive Income Taxation

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ABSTRACT

Progressivity in both marginal and average tax rates seems to be a universal phenomenon. Yet the optimal taxation literature does not, by and large, imply progressivity and most of the investigations of politically determined tax schedules deal with linear taxation.

This paper considers the determination of the tax structure by majority rule when tax schedules are characterized by three parameters and individuals differ in ability. It is shown that if the rankings of income and of ability always coincide the set of schedules preferred by the individual with median ability is contained in the set of local majority winning schedules. Under some additional restrictions this result is extended to any (global) movement along the tax possibility frontier.

The paper uses this result to demonstrate that the existence of both marginal and average progressivity of tax schedules can be explained as the outcome of a majority voting process in which the median individual is pivotal with respect to the tax schedule chosen by the political process. Marginal progressivity is more likely the stronger the right hand skewness and the larger the variance in the distribution of abilities. It is also more likely if the labor supply response to increased tax burdens of high ability individuals is smaller in absolute value than that of low ability individuals.

The paper also discusses conditions for uniqueness of the tax schedule.
A Political Theory of Progressive Income Taxation

By Alex Cukierman and Allan H. Meltzer*
Revised December 1988

I. Introduction

Progressive income taxation is found in all developed and in many developing countries. In most of these countries both average and marginal income tax rates increase with the level of income. Yet, this ubiquitous phenomenon has proved troublesome for economists and social scientists. Despite numerous attempts to make the case for or against progressivity, and many strong statements on both sides, the rational case for progressivity has proved elusive.

A generation ago, Blum and Kalven (1953) pointed out that most arguments for progressivity have a weak foundation. Many rely on comparisons of marginal utility of income, across individuals, an assumption that economists reject along with other interpersonal comparisons of utility. After considering the arguments, Blum and Kalven (1953, p. 71) concluded that the strongest case for progressive taxes depends on the argument that progressivity provides revenues for redistribution through the government budget. There is now broad agreement that the case for redistribution and tax progressivity cannot be made on

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*Tel-Aviv University and Carnegie Mellon University respectively. This is a substantially revised version of the paper "A Positive Theory of Progressive Income Taxation." We benefited from useful discussion with Tom Romer and Howard Rosenthal and from the criticisms of several referees on the previous version. The usual disclaimer applies. Parts of the previous version were written while Meltzer was a visiting scholar at the Bank of Japan. Their support is gratefully acknowledged. The previous version was presented at the 5th World Congress of the Econometric Society in Cambridge Massachusetts.
strictly economic, non-political grounds.\textsuperscript{1}

Despite this broad agreement, much recent research on tax progressivity focuses on the conditions that would lead to the choice of progressive taxes as an optimal form of taxation. This work is normative, not positive, and much of it is based on the utilitarian principle of maximizing the sum of individual utilities. Pigou (1947) showed this principle leads to extreme progressivity; his optimal tax policy is full equilization of after tax incomes. Mirrlees (1971) showed that Pigou's conclusion changes considerably when there are incentive effects of taxation on the choice between labor and leisure. Tax rates are much lower, and tax schedules are either linear or rates fall as income rises. Mirrlees (1971), Phelps (1973), Sadka (1976).\textsuperscript{2} Atkinson (1973) imposed a social utility function; the government chooses to alter the distribution of income by lowering inequality, as Simons (1938, pp. 18-19) had urged. Progressivity can be obtained in this case if the government is willing to move the economy to a Pareto inferior position.

The optimal tax literature typically imposes a utility function and derives the tax function. An alternative approach taken by Romer (1975) and Roberts (1977) is to specify a tax function and allow the voters to choose the parameters of this function through majority rule. Romer (1975) shows that, if individuals differ in ability, the decisive voter is the individual with median ability. The decisive voter chooses increasing average progressivity, in Romer (1975), if the utility function is Cobb-Douglas. Meltzer and Richard (1981) extend this result for net redistribution to a large class of utility functions and, in Meltzer and Richard (1983), provide evidence that the model is broadly consistent with U.S. data.

Tax schedules and so-called effective tax rates typically rise with the

\textsuperscript{1}See the comments by James Tobin, Allen Wallis, Oswald Brownlee, Norman Ture and Richard Musgrave in Campbell (1977).

\textsuperscript{2}An early survey of this literature is in Atkinson (1973). Mirrlees (1971, p. 186) own summary is: "The optimum tax schedule depends upon the distribution of skills in such a complicated way that it is not possible to say in general whether marginal rates should be higher for high income, low income or intermediate income groups."

Linearity of the optimal schedule extends to the case in which individuals respond to higher tax rates by working in an untaxed sector of the economy (Kramer and Snyder (1983, 1984)).
### Table 1

Average Tax Rates of a Family with Two Children
Filing a Joint Return at Various Income Levels in 1974*

<table>
<thead>
<tr>
<th>Normalized gross income level</th>
<th>100</th>
<th>200</th>
<th>400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Australia</td>
<td>7.4</td>
<td>20.1</td>
<td>35.7</td>
</tr>
<tr>
<td>Austria</td>
<td>13.2</td>
<td>18.6</td>
<td>24.1</td>
</tr>
<tr>
<td>Belgium</td>
<td>15.2</td>
<td>22.5</td>
<td>30.4</td>
</tr>
<tr>
<td>Canada</td>
<td>9.8</td>
<td>19.5</td>
<td>28.4</td>
</tr>
<tr>
<td>Denmark</td>
<td>31.0</td>
<td>43.3</td>
<td>52.7</td>
</tr>
<tr>
<td>Finland</td>
<td>22.9</td>
<td>32.5</td>
<td>43.5</td>
</tr>
<tr>
<td>France</td>
<td>8.4</td>
<td>12.8</td>
<td>16.9</td>
</tr>
<tr>
<td>Germany</td>
<td>22.9</td>
<td>29.5</td>
<td>35.4</td>
</tr>
<tr>
<td>Ireland</td>
<td>15.7</td>
<td>24.1</td>
<td>36.5</td>
</tr>
<tr>
<td>Italy</td>
<td>7.7</td>
<td>12.4</td>
<td>23.7</td>
</tr>
<tr>
<td>Japan</td>
<td>8.4</td>
<td>12.1</td>
<td>18.2</td>
</tr>
<tr>
<td>New-Zealand</td>
<td>15.6</td>
<td>24.7</td>
<td>36.0</td>
</tr>
<tr>
<td>Norway</td>
<td>21.2</td>
<td>31.3</td>
<td>44.0</td>
</tr>
<tr>
<td>Spain</td>
<td>7.0</td>
<td>8.4</td>
<td>15.0</td>
</tr>
<tr>
<td>Sweden</td>
<td>24.4</td>
<td>36.2</td>
<td>50.5</td>
</tr>
<tr>
<td>Switzerland</td>
<td>17.3</td>
<td>25.9</td>
<td>34.6</td>
</tr>
<tr>
<td>United Kingdom</td>
<td>13.7</td>
<td>26.0</td>
<td>30.9</td>
</tr>
</tbody>
</table>

Source: Table 16(b) of The Tax/Benefit Position of Selected Income Groups in OECD Member Countries 1972-1976. A report by the Committee on Fiscal Affairs OECD, Paris 1978, p. 110.

*Gross incomes are expressed as percentages of an average production worker's earnings within each country. The tax rates in the table are for a family income that is contributed in equal shares by both spouses and they include both personal income taxes and social security contributions. Data are available in the source for other earning profiles.
level of income. Marginal tax rates often rise also. Table 1 compares the effective tax rate paid by a family with two children that earns the mean level of income to the tax rates paid by a comparable family that earns two or four times mean income. In most countries, the average effective tax rate increases as income rises.3

Table 1 also shows that countries have very different tax functions. Denmark, Germany and Sweden have similar per capita income and all three have relatively high average tax rates at mean income. Progressivity differs, however; it is greater in Sweden and Denmark than in Germany. Australia, Japan and Canada have relatively low tax rates and, again, marked differences in progressivity. The Australian increase in tax rates with income is similar to Sweden's, but the average rates are much lower.

Differences in tax rates are often associated with differences in spending for redistribution. Meltzer and Richard (1981), using a linear tax function, show that the decisive voter's choice of per capita transfer payments determines the tax rate. Net tax payments are negative at low incomes and for non-workers, and rates rise with the level of income in their analysis.

This paper provides a positive theory of progressivity for marginal and average tax rates within a majority rule framework by using a tax function that permits marginal progressivity, linearity or regressivity. The analysis suggests some reasons for observed differences in marginal and average tax rates.

The paper provides a set of conditions under which majority voting implies marginal progressivity. We assume that each person has the same utility function, but people differ in ability and therefore in productivity. Under majority rule, the choice of progressivity depends, partially, on the response of labor supply to tax rates. If people with high ability show small response of labor supply to tax rates, majority rule is likely to produce marginal progressivity. Marginal progressivity is not restricted to this case, however. Even when the response of work effort to an increase in tax burden is not systematically related to income levels, the decisive voter can

3We believe this is true in the U.S. also, although the table is restricted to countries for which OECD attempts to provide comparable data.
choose marginal progressivity. He is more likely to do so the higher the variance of gross incomes and the more skewed to the right is the distribution of gross incomes. Studies of the distribution of income show that the distribution is indeed skewed to the right. The variance and the degree of positive skewness in the distribution of gross incomes are in turn larger the larger the variance and the degree of positive skewness in the distribution of abilities.

In order to allow the majority rule process to pick a (possibly) progressive tax schedule it is necessary to extend the family of linear tax schedules used in previous literature to a three parameter family of tax schedules. Even after taking into consideration that the government's budget constraint determines one parameter as a function of the other two, majority rule still needs to pick two parameters out of the set of feasible pairs of such parameters. It is well known that majority voting over a multi-dimensional issue space of this kind may induce collective intransitivities that preclude the existence of a majority winner. This is an aspect of Arrow's (1951) impossibility theorem. Hence it is necessary to determine if a majority winning tax schedule exists. This task logically precedes that of finding conditions for progressivity since, in the absence of a majority winner, majority rule does not produce a well defined choice of tax schedule.

The paper shows that if the ranking of incomes is independent of tax schedules, as in Roberts (1977), the set of local majority winning schedules is non empty. Moreover this set contains the set of feasible schedules most preferred by the individual who is at the median of the distribution of abilities. Hence, the median ability individual is locally decisive. Provided the proportion of individuals with intermediate levels of ability is sufficiently large this individual is also globally decisive for a wide range of utility functions.

The labor-leisure choice of individuals with different abilities, who face a tax schedule with a given degree of progressivity or regressivity, is analyzed in Section II. Section III introduces the government budget constraint and shows that, when the ranking of gross incomes is independent of the tax schedule the median ability individual is locally decisive for the choice of tax schedule. Conditions under which this individual is also globally decisive over all feasible tax schedules are discussed in Section IV. Section V characterizes the choice of tax schedule by the median ability
individual for the case in which he works and provides conditions under which it is likely that he will pick a progressive tax schedule. Section VI briefly considers the same issues for the case in which the decisive voter does not work. Section VII discusses conditions for uniqueness of the political equilibrium and some limited comparative statics. This is followed by concluding remarks.

II. The Private Economy

The economy consists of a large number of individuals who differ in ability and, therefore, in their real wage rate. Each individual takes his wage rate and the tax schedule as given and chooses the amount of leisure, work and consumption to maximize utility. Utility of the representative individual is given by a strictly concave function $u(c, l)$ of consumption, $c$ and leisure $l$. Consumption is a normal good and the marginal utility of consumption or leisure is infinite when the level of consumption or leisure is zero respectively.

Individual incomes reflect differences in individual productivity and the use of a common, constant return to scale technology to produce the consumption good. An individual with productivity $x$ earns pretax income, $y$;

$$y(x) = xn(x),$$

where $n(x)$ is the amount of work he supplies. Each individual is endowed with one unit of time that he can allocate to either leisure, $l(x)$, or to the production of the consumption good, so $l(x) = 1-n(x)$.

Tax revenues finance a fixed level of government expenditures $G$. The total tax paid by an individual with gross income $y$ is

$$T(y) = -r + xy + ay^2$$

where $r$, $\tau$ and $a$ are parameters of the tax schedule that are determined by the political process. The corresponding marginal tax schedule is

$$T'(y) = \tau + 2ay.$$
The parameter \( a \) measures the degree of marginal progressivity of the tax schedule. The marginal tax rate increases with income when \( a \) is positive and decreases with income when \( a \) is negative.\(^4\) For \( a = 0 \), equation (2) reduces to the widely used linear income tax schedule (Sheshinski (1972), Romer (1975), Roberts (1977)), and the marginal tax rate is constant at \( \tau \). A positive \( r \) corresponds to the case in which low income people get a subsidy by means of a negative income tax or cash transfer. This is the case discussed by Meltzer and Richard (1981). When \( a = 0 \) and \( r > 0 \), our model reduces to theirs.

We restrict the tax schedule in three ways. First negative marginal tax rates are excluded. Second, marginal tax rates cannot exceed 100%. Third, no individual pays more than 100% in taxes, so \( r \geq 0 \). Equation (4) summarizes the first two restrictions. The restrictions

\[
0 \leq T'(y) = \tau + 2ay \leq 1 \tag{4}
\]

implicitly impose upper and lower bounds on the degree of marginal progressivity (\( a > 0 \)) and marginal regressivity (\( a < 0 \)) respectively.\(^5\)

There is no saving; consumption equals disposable income as shown in (5).

\[
c(x) = r + (1-\tau)x - a(x^2). \tag{5}
\]

Given the wage rate, \( x \), and the tax parameters \( r, \tau \) and \( a \), individuals choose the allocation of their time and their consumption by solving

\[
\max_n \{ r + (1-\tau)x - a(x^2), 1-n \}. \tag{6}
\]

\(^4\)A sufficient condition for both marginal and average progressivity is \( a > 0 \) and \( r \geq 0 \).

\(^5\)Let \( x_u = \max x \) be the highest ability level in the population. Since each individual supplies at most one unit of labor, the maximum income of any individual is \( x_u \). A sufficient condition for the inequalities in equation 4 is:

\[
0 \leq \tau + 2ax_u \leq 1 \quad \text{for all } y \leq x_u.
\]

This expression imposes the following bounds on \( a \)

\[
-\tau/2x_u \leq a \leq (1-\tau)/2x_u.
\]
When \( r = 0 \), there is no redistribution. Everyone works because we have assumed that the marginal utility of consumption is infinite when consumption is zero. Since the marginal utility of leisure is also infinite at zero leisure, the solution to the problem in (6) is an internal one for all \( x \). The first order condition is

\[
x(1-T'(y))u_c(c(x), 1-n(x)) - u_x(c(x), 1-n(x)) = 0,
\]

where

\[
1-T'(y) = 1-\frac{2axn}{(1-\tau)}
\]

and \( c(x) \) is given by (5). Equation (7) determines \( n \) as a function of individual productivity, \( x \), and the tax parameters \( r, \tau \) and \( a \).

Individuals with productivity below a minimum level, denoted \( x_0 \), do not work. Their earned income is zero, and their consumption is \( r \). The value of \( x_0 \), which divides the population into workers and non-workers, is found from (7) to be

\[
x_0 = \frac{u_x(r, 1)}{(1-\tau) u_c(r, 1)}.
\]

The value of \( x_0 \) depends on \( r \) and \( \tau \) but not on \( a \) since, at \( x_0 \), the person chooses full-time leisure and \( xn = 0 \). The effects of \( r \) and \( \tau \) on \( x_0 \) are given by (10) and (11).

\[
\frac{\partial x_0}{\partial r} = \frac{u_x(r, 1) - (1-\tau)x_0u_{cc}(r, 1)}{(1-\tau)u_c(r, 1)},
\]

\[
\frac{\partial x_0}{\partial \tau} = \frac{x_0}{1-\tau}.
\]

The second order condition for a maximum is

\[
D = x^2b^2u_{cc} - 2xbu_{c\ell} + u_{\ell\ell} - 2ax^2u_c < 0.
\]
When the redistribution parameter is zero, everyone works; the "last" worker is the individual with lowest ability in the population, \( x^* \).

To characterize the political equilibrium of the next section it is necessary to find conditions under which individuals with higher ability also have higher gross incomes or, to be more precise, the conditions under which the ordering of productivity corresponds to the ordering of earned income. The relation of the two orderings depends on the properties of the utility function. Differentiating (1) and (7) with respect to \( x \), combining the resulting expressions and rearranging gives

\[
\frac{\partial y}{\partial x} = - \frac{1}{D} \left[ n(xbu_{x^c} - u_{x^c}) + x b u_c \right].
\] (12)

Since \( D < 0 \) (see footnote 6), a sufficient condition for gross income to be increasing in productivity is that the bracketed term on the right hand side of (12) be positive. An alternative sufficient condition is

\[
\eta_{lx} < \frac{n}{x}
\] (12a)

where \( \eta_{lx} \) is the elasticity of leisure with respect to productivity.\(^7\) This condition does not allow the income effect on leisure to be too large.

III. The Political Process and the Government Budget Constraint

Choice of the parameters of the tax function is a political economy decision. In our analysis, the parameters of the tax schedule are determined by majority voting. Therefore only tax schedules that cannot be defeated through majority rule by any other schedule can emerge in political equilibrium. Voters are informed about their tastes and about the effects of taxes and redistribution on earned income implied by the model in the previous section. Taxes are levied to finance a fixed level of per capita government expenditure, \( G \), and the amount of lump sum per capita redistribution, in the

\[\text{Footnote 6:} ay/ax = n(1+\eta_{nx}) \text{ where } \eta_{nx} \text{ is the elasticity of labor supply with respect to productivity. A sufficient condition for } ay/ax > 0 \text{ is } \eta_{nx} > -1 \text{ which is equivalent to (12a) since } \eta_{nx} = - (z/n)\eta_{lx}.\]
form of demogranants, that results from the political process.

Let \( F(x) \) denote the distribution function of individual productivity so \( F(x) \) is the fraction of the population with productivity not greater than \( x \). Budgetary deficits and surpluses are not possible so the budget is balanced by taxes on the incomes of those who work

\[
\int_0^x [\tau n(x) + a(n(x))]^2 dF(x) = G + r. \tag{13}
\]

Since \( G \) is taken to be exogenous it is set to zero for simplicity. Equation (13) implicitly determines \( r \) as a function of \( \tau \) and \( a \). This function is denoted

\[
r = r(\tau, a). \tag{14}
\]

Provided the utility function and the distribution function of individual ability are continuous so is the function \( r(\cdot) \). Following Romer (1975) we refer to (14) as the tax possibility frontier (TPF). Ignoring effects on incentives and income an increase in either the flat component of the tax structure, \( \tau \), or in the degree of progressivity, \( a \), increase the amount of per capita redistribution, \( r \). When incentive effects are taken into account, an increase in either \( \tau \) or \( a \) may, particularly if the tax burden is already high, decrease the tax base so much that redistribution must decrease to balance the budget. The derivatives of \( r \) with respect to \( \tau \) and \( a \) (denoted \( r_\tau \) and \( r_a \)) may, therefore, be either positive or negative. The range along the TPF where either or both of \( r_\tau \) and \( r_a \) are nonpositive is inefficient since from any point in this range, it is possible to increase redistribution without increasing tax burdens. Since all individuals like such changes majority rule will never induce a political equilibrium along the inefficient range of the TPF. This is summarized in the following proposition

**Proposition 1:** Under majority rule the voting equilibrium is never in the range of the TPF along which \( r_\tau \leq 0 \) or \( r_a \leq 0 \) or both.

---

8The case in which there is, an endogenously determined, public good is briefly discussed in Section VIII.
From equation (13) the derivatives of the TPF with respect to \( \tau \) and \( \alpha \) are:

\[ r_\tau = \frac{1}{H} \int_0^1 y(x) + xT'[y(x)] \frac{\partial n}{\partial \tau}(x) dF(x) \]  

(a)

\[ r_\alpha = \frac{1}{H} \int_0^1 (y(x))^2 + xT'[y(x)] \frac{\partial n}{\partial \alpha}(x) dF(x) \]  

(b) (15)

\[ H = 1 - \int_0^1 xT'[y(x)] \frac{\partial n}{\partial \tau}(x) dF(x) \]  

(c)

where

\[ \frac{\partial n}{\partial z}(x), \quad z = \tau, \alpha, r \]

are the responses of the labor supply of an individual with ability \( x \) to ceteris paribus changes in the tax parameters \( \tau \), \( \alpha \) and \( r \). Assuming that a ceteris paribus increase in redistribution decreases individual labor supply (or does not increase it) \( \frac{\partial n}{\partial r} \leq 0 \). Since \( T'[\cdot] \geq 0 \) this implies that \( H \) is bounded away from zero and positive. Inspection of equations (15a) and (15b) reveals that since \( y(x), x, \frac{\partial n}{\partial \tau}, \frac{\partial n}{\partial \alpha} \) and \( T'[\cdot] \) are all bounded from above so are the numerators of those equations. Since \( H \) is bounded away from zero this implies that \( r_\tau \) and \( r_\alpha \) are finite. Hence there is only one value of \( r \) that corresponds to each \( (\tau,\alpha) \) pair-so the function \( r(\tau,\alpha) \) is single valued.

It is well known that majority rule induces collective intransitivities that can lead to cycles in the composition of the majority, when voters make multidimensional choices. This is an aspect of Arrow's (1951) impossibility theorem. In the context of the present model voters' preferences are affected by the three parameters \( r, \tau \) and \( \alpha \) that define a tax schedule. The TPF in (14) reduces the dimensionality of the problem to the two parameters \( \tau \) and \( \alpha \). But as of itself this does not assure that there exists a tax schedule such that there is no other schedule along the TPF that is preferred to it by a

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9 The derivations are in part 1 of the appendix.
majority. There may in general be no Condorcet winner or, in the terminology of game theory, the core of the majority rule game over tax schedules may be empty.

Proposition 1 implies that if there exists a majority winning tax schedule it must be in the range of the TPF in which \( r_\tau > 0 \) and \( r_\alpha > 0 \). The following discussion establishes conditions for the existence of a majority political equilibrium and characterizes it. Let \( s \equiv (r, \tau, \alpha) \) be a tax schedule. Let \( S_m \) be the set of tax schedules along the TPF most preferred by the individual with median ability. This individual will be referred to as "the median individual" or simply "the median." Let \( s_m \) be a tax schedule in \( S_m \). The set \( S_m \) may, but does not have to, include several tax schedules. Obviously when it does the median is indifferent between those different schedules. The following two definitions help to organize the discussion that follows.

**Definition 1:** A tax schedule \( s \) is a local majority winner if it is on the TPF and if there is no other schedule in the neighborhood of this schedule on the TPF that is strictly preferred by a majority.

**Definition 2:** A tax schedule is a global majority winner if it is on the TPF and if there is no other schedule on the TPF that is strictly preferred by a majority.

The local majority winner concept is useful as a device for the introduction of the apparatus required to characterize the conditions under which a global majority winner exists. In addition it is of intrinsic interest when the political system contemplates only small changes in the status quo as in Kramer and Klevorick (1974) and Romer (1977).

The following discussion, that culminates in theorem 1, demonstrates that all the schedules in the set \( S_m \) are local majority winners. Let

\[ S_m \equiv (r_m, \tau_m, \alpha_m) \]  

be any schedule in the set \( S_m \). Let

\[ dr \equiv r - r_m; \quad d\tau \equiv \tau - \tau_m; \quad d\alpha \equiv \alpha - \alpha_m \]
be a local deviation of $s$ from $s_m$ along the TPF. Let

$$I(r,\tau,\alpha;x) = \max \{ru + (l-\tau)x - \alpha(xn)^2, 1-n\}$$

(18)

be the indirect utility function of an individual with ability $x$. Since $u(\cdot)$ and the individual's budget constraint are continuous and the range of $u(\cdot)$ is compact $I(\cdot)$ is a continuous function of the parameters $r, \tau$, and $\alpha$. In addition the implicit function theorem guarantees the differentiability of $I(\cdot)$ with respect to those parameters.\textsuperscript{10} Differentiating (18) totally with respect to a combined local change in the parameters of the tax schedule and using the implicit function theorem

$$dI(y) = u_c[dr - yd\tau - y^2d\alpha].$$

(19)

Lemma 1: The schedule $s_m$ is a local majority winner whenever either of the following holds

(i) $d\tau > 0$ and $d\alpha > 0$
(ii) $d\tau > 0$ and $d\alpha = 0$
(iii) $d\tau = 0$ and $d\alpha > 0$
(iv) $d\tau < 0$ and $d\alpha < 0$

Proof: Since $s_m \succ s_m$, the median either dislikes, or is indifferent to, the change. Hence, from (19)

$$dI(y_m) = u^m_c[dr - y_md\tau - y^2_md\alpha] \leq 0$$

(20)

where the subscript or superscript designates that the appropriate quantity refers to the median. In case (i) since $d\tau > 0$ and $d\alpha > 0$

$$dI(y) < dI(y_m) \leq 0 \text{ for all } y > y_m.$$  

(21)

Since gross income is increasing in productivity all the individuals with ability above that of the median dislike the change. If the median dislikes

\textsuperscript{10}Varian (1978) p. 267.
the change too there is a majority against it. If he is indifferent there is a tie between those who like and those who dislike the change. In either case there is no schedule that is preferred to $s_m$ by a majority. In cases (ii) and (iii) the inequality in (21) is still satisfied for all $y>y_m$. Hence the same considerations apply and there is no majority for the change.

In case (iv)

$$dI(y) < dI(y_m) \leq 0 \text{ for all } y<y_m.$$ (22)

Hence if the median dislikes the change there is a majority against it and if he is indifferent there is no majority for it. In either case $s_m$ is a local majority winner for the set of changes specified in the lemma. Q.E.D.

**Lemma 2:** The schedule $s_m$ is a local majority winner against any change in schedule such that $d_r<0, \ da>0$.

Proof; The change in welfare experienced by an individual with income $y$ as a result of the shift from $s_m$ to $s$ is given by equation (19). Since $u_c>0$ for all $y$ the change in welfare is positive, negative or zero depending on whether the following second degree polynomial in $y$ is positive, negative or zero

$$P(y) = -(da)y^2 - (d_r)y + dr.$$ (23)

The roots of $P(y)$ are

$$y_{c1} = \frac{1}{2}[- \frac{d_r}{da} - \sqrt{\left(\frac{d_r}{da}\right)^2 + 4 \frac{dr}{da}}] \quad (a)$$

$$y_{c2} = \frac{1}{2}[- \frac{d_r}{da} + \sqrt{\left(\frac{d_r}{da}\right)^2 + 4 \frac{dr}{da}}]. \quad (b)$$

Since $-da<0$ the polynomial $P(y)$ has a maximum and looks like an inverted U. If both roots are imaginary $P(y)$ is either positive or negative for all $y$-s. When $P(y_m) < 0$ this implies that $P(y)<0$ for all $y$ and all voters dislike the
change. If both roots are real and distinct (see panel a of figure 1)

\[ P(y) < 0 \quad \text{for all} \quad y > y_{c2} \quad \text{and} \quad y < y_{c1} \]  
(a)

\[ P(y) > 0 \quad \text{for all} \quad y_{c1} < y < y_{c2} \]  
(b) (25)

\[ P(y) = 0 \quad \text{for} \quad y = y_{c1} \quad \text{and} \quad y = y_{c2}. \]  
(c)

Since \( P(y_{m}) < 0 \), \( y_{m} \) must be in the range defined by equation (25a). If \( y_{m} > y_{c2} \) at least all individuals with incomes above \( y_{m} \) dislike the change too and if \( y_{m} < y_{c1} \) at least all individuals with incomes below \( y_{m} \) dislike the change. In either case there is a majority against the change.

When \( P(y_{m}) = 0 \), \( y_{m} \) is equal to either \( y_{c1} \) or \( y_{c2} \). If \( y_{c1} \) and \( y_{c2} \) are distinct the previous argument implies that there is always a majority against the change. If \( y_{c1} \) and \( y_{c2} \) collapse to one root \( y_{m} \) equals it's common value and \( P(y) \) touches the horizontal axis only at this point. All other points are therefore below the horizontal axis since \( P(y) \) has a maximum at \( y_{m} \). Hence everybody except for the median strictly dislikes the change. It follows that \( s_{m} \) is a local majority winner for all changes of the type \( d_{t} < 0 \) and \( d_{a} > 0 \).

Q.E.D.

Lemma 3: The schedule \( s_{m} \) is a local majority winner against any change in schedule such that

\[ dr \leq 0, \quad dt > 0, \quad da < 0. \]

Proof: When \( da < 0 \), \( P(y) \) has a minimum and looks like a U (panel b of figure 1). The value of \( P(y) \) at the minimum is

\[ \frac{1}{4} \left( \frac{d_{t}}{da} \right)^{2} + dr \]  
(26)

which is negative since \( da < 0 \) and \( dr \leq 0 \). Since for extreme values of \( y \), \( P(y) \) eventually becomes positive the two roots -- \( y_{c1} \) and \( y_{c2} \) -- are real and distinct.

Since \( dr \leq 0 \) and \( da < 0 \) (24a) implies that \( y_{c1} \leq 0 \). Since \( P(y_{m}) \leq 0 \), \( y_{c1} \leq y_{m} \leq y_{c2} \). Hence at least all the individuals with incomes below \( y_{m} \) weakly dislike the change. Since the median also weakly dislikes the change there is
a. The case $dr < 0$, $d\alpha > 0$ (Lemma 2)

b. The case $dr \leq 0$, $d\tau > 0$, $d\alpha < 0$ (Lemma 3)

c. The case $dr > 0$, $d\tau > 0$, $d\alpha < 0$ (Lemma 4)

Figure 1
no majority in favor of the change and \( s_{m} \) is a local majority winner. Q.E.D.

From the TPF in (14)

\[
dr = r_{\tau} d_{T} + r_{a} da.
\]

(27)

Dividing (27) by \( da \) and substituting it into equations (24a) and (24b)

\[
y_{c1}(a) = \frac{1}{2} [a - \sqrt{\frac{a^{2} - 4r_{\tau}a + 4r_{a}}{2}}]
\]

(a)

\[
y_{c2}(a) = \frac{1}{2} [a + \sqrt{\frac{a^{2} - 4r_{\tau}a + 4r_{a}}{2}}]
\]

(b) (28)

\[
a = -\frac{dr}{da}.
\]

(c)

The only type of change in schedules not covered by Lemmas 1 to 3 is the case \( dr>0, d_{T}>0, da<0 \). The next lemma demonstrates that if there is an internal solution along the TPF for \( s_{m} \) this case can be disregarded.

Lemma 4: If the schedule \( s_{m} \) preferred by the median individual does not occur along the boundary\(^{11} \) of the TPF, changes of the type \( dr>0, d_{T}>0, da<0 \) cannot occur.

Proof: Since \( da<0 \) and \( dr>0 \), \( \frac{dr}{da} < 0 \). It follows from equation (24a) that \( y_{c1}(a) > 0 \) for all \( a \)-s in the range defined by the conditions of the lemma. Since \( da<0 \), \( P(y) \) has a minimum and can therefore be drawn as in panel c of figure 1. Equation (27) implies that

\[
\text{dr>0 is equivalent to } a > r_{a}/r_{\tau}.
\]

\(^{11}\)The boundary of the TPF is defined by the restrictions \( r>0 \) and \(-\tau/2x_{u} \leq a \leq (1-\tau)/2x_{u} \). Hence the boundary is attained when \( r = 0 \) or \( a = -\tau/2x_{u} \) or \( a = (1-\tau)/2x_{u} \).
Since $P(y_m) \leq 0$, $y_m < y_{c2}(a)$. Rearranging, this is equivalent to

$$a \leq \frac{r - y_m^2}{r - y_m}.$$  

Hence we need to examine only cases for which $a$ is in the range

$$\frac{r_a}{r} < a \leq \frac{r_a}{r} - y_m$$  \hspace{1cm} (29)

since in all other cases either $dr$ is not positive or the median likes the change contradicting the fact that $s_m$ is weakly preferred by the median to all other schedules.

Since $s_m$ occurs at an internal point along the TPF it must satisfy (applying the envelope theorem to (18)) the following two first order conditions

$$\frac{dI_m}{d\alpha} = -u_m^C(\cdot)y_m[\frac{d\tau}{d\alpha} + y_m] = 0$$  \hspace{1cm} (a) \hspace{1cm} (30)

$$\frac{dI_m}{d\tau} = u_m^C(\cdot)[1-y_m\frac{d\tau}{d\tau}] = 0$$  \hspace{1cm} (b)

where $d\tau/d\alpha$ is the change in $\tau$ as a result of a change in $\alpha$ along the TPF for a given $r$ and $d\tau/dr$ is the change in $\tau$ as a result of a change in $r$ along the TPF holding $\alpha$ constant. The index $m$ designates that the appropriate quantity refers to the median ability individual. For changing $\tau$ and $\alpha$ and a constant $r$ equation (27) implies

$$\frac{d\tau}{d\alpha} = -\frac{r_a}{r}.$$  \hspace{1cm} (31)

If the median works $u_m^C(\cdot)y_m > 0$ and equations (30a) and (31) imply
Substituting (32) into the upper limit of the interval in (29)

\[ r_a - \frac{y_m}{r_{\tau}} = \frac{r_a - (\frac{r_{\tau}}{r_{\tau}})^2}{r_{\tau} - \frac{r_a}{r_{\tau}}} = \frac{r_a^2 - r_{\tau}^2}{r_{\tau}^3 - r_a r_{\tau}} = \frac{r_a}{r_{\tau}}. \]

Hence the interval in (29) reduces to

\[ \frac{r_a}{r_{\tau}} < a \leq \frac{r_a}{r_{\tau}} \]

implying that there is no value of a that simultaneously satisfies \( dr > 0 \) and \( P(y_m) \leq 0 \). Hence changes of the type \( dr > 0 \), \( dv > 0 \), \( da < 0 \) can be disregarded.

If the median does not work \( s_m \) is characterized by the two first order conditions \( r_a = r_{\tau} = 0 \). It follows from equation (27) that \( dr = 0 \) for all \( da \) and \( dv \) when the median does not work. Hence changes of the type \( dr > 0 \) are not possible. Q.E.D.

**Theorem 1:** Let \( S_m \) be the set of tax schedules along the TPF such that there is no other tax schedule along the TPF that is strictly preferred by the individual with median ability. Let \( s_m \) be an element of \( S_m \). If \( s_m \) does not occur on the boundary of the TPF then \( s_m \) is a local majority winner.\(^{12}\)

---

\(^{12}\)The set \( S_m \) does not always contain all the local majority winning tax schedules. For example there may exist a tax schedule, \( s_\phi \), along the TPF and not in \( S_m \) that is strictly preferred by the median individual to any schedule in the vicinity of \( s_\phi \). This occurs when none of the schedules in \( S_m \) is in the vicinity of \( s_\phi \). Since \( s_\phi \) does not belong to \( S_m \) the median individual strictly prefers any of the schedules in \( S_m \) to \( s_\phi \). Nonetheless \( s_\phi \) is a local majority winner since the arguments leading to theorem 1 imply that if the median individual weakly prefers \( s_\phi \) to any schedule in the neighborhood of \( s_\phi \) there is a weak majority in favor of \( s_\phi \) whenever \( s_\phi \) is confronted with any of the schedules in the vicinity of \( s_\phi \). Hence the set of local majority winners may be larger than \( S_m \). But (subject to the conditions of Section IV below) the set of global majority winners is identical to \( S_m \) since any of the (more distant) schedules in \( S_m \) is weakly preferred by a majority to \( s_\phi \).
Proof: The proof is a direct consequence of lemmas 1 through 4.

Suppose that a given schedule, $s_1$, along the TPF is the status quo. Now an alternative schedule, $s_2$, is proposed. If $s_2$ is strictly preferred by a majority it becomes the status quo schedule. In all other cases, including ties, the original status quo $s_1$ remains in place unless another schedule that is strictly preferred by a majority is found. The process ends when no schedule that is strictly preferred by a majority to the status quo can be found. With this kind of mechanism for bringing schedules up for voting theorem 1 implies that, once a schedule $s_m \not\in S_m$ becomes the status quo, the process stops and $s_m$ is adopted provided only schedules in the neighborhood of the status quo are considered as alternatives at each stage. If $S_m$ is not a singleton this does not pin down which of the schedules in $S_m$ will be adopted. But it limits the search for equilibrium schedules to the set that is most preferred by the individual with median ability. This characteristic of the local majority winning set of schedules proves useful for the characterization of the equilibrium degree of progressivity in Section V.

IV. Conditions for the Existence of a Globally Winning Tax Schedule

The result of theorem 1 can be extended to any (global) change in schedule along the TPF provided some additional conditions are satisfied. The following discussion leads to a statement of the precise conditions. Let

\[ n(s,x), y(s,x) \] (13a)

be the labor input and the gross labor income of an individual with productivity $x$ when the tax schedule is $s$. Let

\[ \Delta \tau \equiv \tau - \tau_m, \quad \Delta \tau \equiv \tau - \tau_m, \quad \Delta \alpha \equiv \alpha - \alpha_m \] (34)

be any global movement away from $s_m$ along the TPF. The following lemma establishes the intuitive fact that if an individual dislikes this change his net income or consumption, holding his employment fixed at $n(s_m, x)$, must be lower after the change.
Lemma 5; If an individual with productivity \( x \) dislikes (or is indifferent to) the shift from \( s_m = \{r_m, \tau_m, a_m\} \) to \( s = \{r, \tau, a\} \) then, holding his labor input constant at \( n(s_m, x) \), his net income or consumption is lower at \( s \) than at \( s_m \).

Proof; Given that the individual's employment remains at \( n(s_m, x) \) the change in his consumption is, from equation (5)

\[
\Delta a\left[y(s_m, x)\right]^2 - \Delta \tau y(s_m, x) + \Delta r.
\]  

Suppose the expression in (35) is non negative contrary to the assertion of the lemma. Since leisure remains the same the welfare of the individual obviously increases if (35) is positive. When the individual is allowed to adjust his labor input his welfare increases even further since he is allowed to take advantage of additional substitution possibilities between labor and leisure that were not available when labor was frozen as \( n(s_m, x) \). If the expression in (35) is zero he has the same amount of leisure and of consumption than prior to the change so his welfare does not change. But once he is allowed to adjust his labor his welfare increases. Hence if the expression in (35) is non negative the individual's welfare increases as a result of the change contradicting the assumption that the individual dislikes the change. It follows that given that labor is frozen at \( n(s_m, x) \) consumption is lower at \( s \) than at \( s_m \) if the individual dislikes the change. Q.E.D.

An important consequence of lemma 5 is that if the median dislikes the switch from \( s_m \) to \( s \) his net income at the pre-change level of work must be reduced by the change.

Theorem 2 below presents sufficient conditions for the existence of a global majority winner \( (s_m) \) and characterizes it. The statement of the theorem requires some preliminary discussion to which we turn next. Let \( p^q(y) \) be the polynomial in (23) with the local changes \( (dr, d\tau, da) \) replaced everywhere by global changes \( -(\Delta r, \Delta \tau, \Delta a) \). Let

\[
\bar{r}_\tau = \frac{r(\tau, a_m) - r(\tau_m, a_m)}{\tau - \tau_m}
\]  

\[
\bar{r}_a = \frac{r(\tau, a) - r(\tau, a_m)}{a - a_m}
\]  

(36)
Then by definition

$$\Delta r = \tilde{\tau} \Delta \tau + \tilde{\alpha} \Delta \alpha.$$  \hspace{1cm} (37)

Let \( y_{c1}^g \) and \( y_{c2}^g \) be the roots of the polynomial \( P^g(y) \). The explicit form of those roots is the same as in equations (24) except for the fact that the small changes denoted by "d" are now replaced by global changes denoted by "A". In the case \( \Delta \alpha < 0 \) the polynomial has a minimum and is described qualitatively by either panel b or c of figure 1. Dividing (37) by \( \Delta \alpha \) and substituting the result into the expressions for \( y_{c1}^g \) and \( y_{c2}^g \) we obtain

\[
y_{c1}^g(b) = \frac{1}{2} \left[ b - \sqrt{b^2 - 4 \tilde{\tau} b + 4 \tilde{\alpha}} \right] \hspace{1cm} (a)
\]

\[
y_{c2}^g(b) = \frac{1}{2} \left[ b + \sqrt{b^2 - 4 \tilde{\tau} b + 4 \tilde{\alpha}} \right] \hspace{1cm} (b) \hspace{1cm} (38)
\]

\[
b = - \frac{\Delta \tau}{\Delta \alpha} . \hspace{1cm} (c)
\]

At pre-change labor inputs all individuals with gross labor incomes in the range

\[
\text{Max}[0, y_{c1}] < y < y_{c2}
\]

experience a drop in net income as a result of the change. But those who are just a little above \( \text{Max}[0, y_{c1}] \) or just a little below \( y_{c2} \) may actually experience a small increase in welfare (in spite of the income drop) once they adjust their labor inputs optimally. Hence the set of pre-change gross incomes at which welfare decreases as a result of the change is strictly contained in the set defined by (39). Condition (i) of theorem 2 below and the fact that \( P^g(y) \) decreases as one moves away from \( y_{c1}^g \) and \( y_{c2}^g \) towards \( b/2 \) imply that the smaller set is convex. Hence there exist small but positive numbers \( \epsilon_1(a) \) and \( \epsilon_2(a) \) such that welfare decreases for all individuals with pre-change gross income in the range.
Max\{0, y^g_{C1}(b)+\varepsilon_1(b)\} < y < y^g_{C2}(b)-\varepsilon_2(b). \hspace{1cm} (40)

Given a tax schedule an individual's level of labor input and therefore his gross income is determined by his productivity, x. Hence gross income of an individual with productivity x when the tax schedule is $s_m$ can be written

$$y = y(s_m, x).$$ \hspace{1cm} (41)

Let

$$x = \hat{x}(s_m, y) \equiv x[y]$$ \hspace{1cm} (42)

be the inverse function to (41). Given $s_m$, it expresses x as a function of y. Since, given the tax schedule, the ranking of individuals by productivity coincides with their ranking by gross income $x(y)$ is an increasing function of y.

**Theorem 2:** Let $\Delta r \equiv r - r_m$, $\Delta \tau \equiv \tau - \tau_m$, $\Delta \alpha \equiv \alpha - \alpha_m$ be any global change in the tax schedule along the TPF starting from a status quo at $s_m = (r_m, \tau_m, \alpha_m)$. Let the conditions of theorem 1 be satisfied. Then $s_m$ is a global majority winner provided the following additional conditions are satisfied;

(i) If individual i dislikes the change all individuals whose net income decreases by more than that of individual i, when the levels of employment are held fixed at their pre-change sizes, dislike the change too.

(ii) \hspace{1cm} $F[x(y^g_{C2}(b)-\varepsilon_2(b))] - F[x(\text{Max}\{0, y^g_{C1}(b)+\varepsilon_1(b)\})] \geq \frac{1}{2}$ \hspace{1cm} (43)

for all b-s that correspond to changes of the type $\Delta r \leq 0$, $\Delta \tau > 0$, $\Delta \alpha < 0$.

(iii) \hspace{1cm} $F[x(y^g_{C2}(b)-\varepsilon_2(b))] - F[x(y^g_{C1}(b)+\varepsilon_1(b))] \geq \frac{1}{2}$ \hspace{1cm} (44)

for all b's that satisfy

$$\frac{\bar{a}}{r_\alpha} < b < \frac{r^2_\tau - (r_\alpha/\bar{a})r_\alpha \bar{a}}{r^2_\tau - (r_\alpha/\bar{a})r_\alpha \bar{a}} \hspace{1cm} (45)$$
and that correspond to changes of the type

\[ \Delta \tau > 0, \Delta \tau > 0, \Delta \alpha < 0. \]

Proof; In part 2 of the appendix.

Theorem 2 implies that every \( s_m \not\in S_m \) is a global majority winner. In addition since any tax schedule on the TPF but not in \( S_m \) is not strictly preferred by a majority the set \( S_m \) contains all the majority winning schedules.\(^{13}\)

Some discussion of the conditions in theorem 2 is in order. Condition (i) is basically an implicit restriction on the possible set of utility functions. Condition (ii) is always satisfied when \( y_{\mathcal{E}_1}(b) + \varepsilon_1(b) \leq 0 \) since \( 0 \leq y_m \leq y_{\mathcal{E}_2}(b) - \varepsilon_2(b) \). The reason is that in all those cases at least half of the voters do not strictly prefer the change. This is more likely to be the case when \( \varepsilon_1(b) \) is small. In turn, this will be the case when the utility difference between a compensating change in income that maintains the same level of net income and a compensating change that maintains the same level of utility is small. Finally condition (iii), that needs to be satisfied only in a rather limited range, requires that the density of individuals in the range of abilities around the median be sufficiently large. In more intuitive but less exact terms it requires that the middle class be a sufficiently large fraction of society. Note that when \( r_\alpha / \bar{r}_\alpha = r_\tau / \bar{r}_\tau \) the range of \( b \) over which this condition is required to hold is empty so that the condition is not binding.\(^{14}\) When \( r_\alpha / \bar{r}_\alpha \neq r_\tau / \bar{r}_\tau \) the range in (45) is narrower (and the condition in (44) therefore less restrictive) the nearer are the ratios \( r_\alpha / \bar{r}_\alpha \) and \( r_\tau / \bar{r}_\tau \) to each other. Those ratios tend to be similar when the differences between the global and the local curvatures of the TPF along the \( \tau \) and the \( \alpha \) directions are of similar magnitudes.

\(^{13}\)In game theoretic terms this means that \( S_m \) and the core of the majority rule game over tax schedules coincide.

\(^{14}\)In particular this condition is satisfied when \( r_\alpha = \bar{r}_\alpha \) and \( r_\tau = \bar{r}_\tau \) so that the TPF is a plane in the range between \((\tau_m, \alpha_m)\) and \((\tau, \alpha)\).
V. Majority Rule and Income Tax Progressivity When the Decisive Voter Works

The previous two sections have established the decisiveness of the median ability individual for the choice of tax schedule. Hence the choice of schedule under majority rule depends on the position of the median or decisive voter in the income distribution, and therefore on his level of ability or productivity. This section establishes conditions under which a decisive voter who works chooses to tax incomes at marginally progressive rates. The following section analyzes the tax schedule chosen by a voter who subsists on transfer payments and does not work.

The decisive voter's problem is to choose the parameters of the TPF by maximizing his indirect utility (18)

\[ I[r, \tau, \alpha ; x_m] \]  \hspace{1cm} (46)

subject to the TPF in (14). Here \( x_m \) is the productivity of the person at the median of the ability distribution. Substituting (14) into (46), differentiating totally with respect to \( \alpha \) and \( r \) and using the envelope theorem, we obtain

\[ \frac{dI_m}{d\alpha} = u_c[r_a - y_m^2] \]  \hspace{1cm} (a)

\[ \frac{dI_m}{d\tau} = u_c[r_\tau - y_m] \]  \hspace{1cm} (b)

The subscript \( m \) denotes the voter with median ability and income, so \( y_m \) is the pretax income of the decisive voter and \( I_m \) is his (indirect) utility. We focus on the case in which the tax schedule that is preferred by the median is not on the boundary of the TPF. In this case the parameters of \( s_m \) are obtained by equating equations (47a) and (47b) to zero. The resulting equations together with the TPF constitute three equations that, in principle,
can be used to determine the three parameters $r_m$, $\tau_m$ and $\alpha_m$.\footnote{Similarly corner solutions can be obtained by incorporating the constraints $r \geq 0$ and $-\tau/2x_u \leq \alpha \leq (1-\tau)/2x_u$.}

We are less interested in explicit solutions for these three parameters than in the conditions under which majority rule results in rising marginal tax rates. Earlier studies that used a political economy framework, by Romer (1975), Roberts (1977) and Meltzer and Richard (1981, 1983) ruled out marginal progressivity by imposing a linear tax function. The tax function used here can accommodate either marginal progressivity ($\alpha > 0$), marginal regressivity ($\alpha < 0$) or a constant marginal tax rate ($\alpha = 0$). We can derive a sufficient condition for marginal progressivity by requiring that the expression in (47a) be positive for all $\alpha < 0$. Using (15b) in (47a) and noting that $u_c > 0$ this is equivalent to the condition

\[
\frac{x_0 \int [(y(x,r,a))^2 + xT'[y(x,r,a)]} \frac{\partial n}{\partial a}(x,r,a)]dF(x)}{1 - x_0 \int xT'[y(x,r,a)] \frac{\partial n}{\partial r}(x,r,a)dF(x)} > (y(x_m,r,a))^2 \text{ for all } \alpha \leq 0 \text{ and } \tau. \tag{48}
\]

The dependence of the various terms in (48) on the productivity class, $x$, and on the parameters $r$ and $\alpha$ is recognized explicitly in the notation. By contrast the parameter $\tau$ which is held constant in (47), and therefore in (48), is subsumed into the functional forms.

Equation (48) assures that all (of the possibly many) equilibria are in the progressive range ($\alpha > 0$). The reason is that increases in $\alpha$ increase utility for all $\alpha < 0$, so the decisive voter will never stop at a non-positive value of $\alpha$. Using basic formulas for the mean and the variance and rearranging, the condition in (48) can be restated as;

\[
C(r,a) = [(1-F(x_0))(V(r,a)+\bar{y}(r,a))^2 - (y(x_m,r,a))^2 \\
+ x_0 \int xT'[y(x,r,a)] \frac{\partial n}{\partial a}(x,r,a) + (y(x_m,r,a))^2 \frac{\partial n}{\partial r}(x,r,a)]dF(x) > 0 \text{ for all } \alpha \leq 0 \text{ and } \tau. \tag{49}
\]
Here \( \bar{y}(\cdot) \) and \( V(\cdot) \) are the mean and the variance of income of the working population. At \( \alpha=0 \) (a linear tax schedule) condition (49) reduces to

\[
C(r,0) = [1-F(x_0)][V(r,0)+\{\bar{y}(r,0)\}^2]-\{y(x_m,r,0)\}^2
\]

\[
+ \tau^x \int x_0^x \frac{an(x,r,0)+\{y(x_m,r,0)\}^2}{\partial x} \frac{an(x,r,0)}{\partial x} dF(x) > 0 \text{ for all } \tau. \quad (49a)
\]

Other things equal conditions (49) and (49a) are more likely to be satisfied the larger the variance of income \( V(\cdot) \) and the larger mean income \( \bar{y}(\cdot) \) in relation to median income \( y(x_m,r,0) \). Given \( \alpha \) and \( r \), \( V(\cdot) \) is normally larger the larger the variance of abilities. The mean-median income spread is normally larger the larger is the degree of positive skewness in the distribution of abilities. This leads to the following proposition.

**Proposition 2:** Majority rule is more likely to produce a progressive tax structure the more spread out is the distribution of abilities and the larger the degree of positive skewness of this distribution.

Proposition 2 implies that, other things equal, voters are more likely to choose a progressive tax structure the larger the spread in the distribution of abilities and the more the distribution is skewed toward high productivity. The intuition underlying these results is familiar. Individuals with higher ability have a larger share of income than of votes. They earn higher incomes because their productivity is higher and, often, because they work more hours. Even if they work fewer hours, income and productivity are positively related in our model. Since the number of people with high productivity is relatively small, the median voter can readily form a majority that agrees to lower the flat tax rate \( \tau \) and raise the degree of progressivity (or lower regressivity). The median voter chooses to do this if he can reduce his own tax burden without reducing the transfers he, and all others, receive from the budget.

A larger spread in the distribution of income increases the likelihood of marginal progressivity. The reason is that a wide spread of the distribution increases the tax base at relatively high incomes. The larger tax base increases the likelihood that a compensated increase in \( \alpha \) will generate
sufficient additional tax revenues to reduce the average tax paid by the
decisive voter.

Effects on incentives modify the results just discussed. Income taxes alter labor-leisure choices and change the tax base. Some notion about the effects of incentives can be obtained by examining the conditions for \( a > 0 \) when the first order condition in (47b) for an optimal choice of \( \tau \) is satisfied. Since \( u_c > 0 \) this condition implies \( y_m = r_T \). Hence a sufficient condition for \( a > 0 \) is, from (47a),

\[
r_a - r_T^2 > 0 \quad \text{for all } a \leq 0. \tag{50}
\]

Using equations (15) in (50), noting that \( H > 0 \) and rearranging this is equivalent to

\[
x_0 \int [y^2 + xT' \frac{an}{\alpha_a}] dF (1 - \frac{x_0 \int xT' \frac{an}{\alpha_T} dF}{x_0 \int (y + xT' \frac{an}{\alpha_T}) dF}) > 0 \quad \text{for all } a \leq 0 \tag{51}
\]

where the dependence on \( x \) and \( s \) has been suppressed for notational simplicity. Since \( an/\alpha_r < 0 \)

\[
H = 1 - \frac{x_0 \int xT' \frac{an}{\alpha_T} dF}{x_0 \int (y + xT' \frac{an}{\alpha_T}) dF} > 0.
\]

Since \( r_a \) and \( r_T \) are positive this implies that

\[
x_0 \int [y^2 + xT' \frac{an}{\alpha_a}] dF; \quad x_0 \int [y + xT' \frac{an}{\alpha_T}] dF
\]

are both positive. Since \( an/\alpha_a \) and \( an/\alpha_T \) are both negative this implies that a positive \( a \) is more likely the lower \( |an/\alpha_a| \) and the higher \( |an/\alpha_T| \) and \( |an/\alpha_r| \) in the range \( a \leq 0 \). This is summarized in the following proposition;

**Proposition 3:** Majority rule is more likely to produce a progressive tax
structure the lower the disincentive effects of an increase in α and the higher the disincentive effects of an increase in either τ or r on employment.

The intuition underlying proposition 3 is the following; An increase in α can be used to either increase redistribution, r, or to decrease the flat component of the tax structure, τ, or to do a bit of both. To understand the intuition it is convenient to consider pure cases in which the increase in α is used either to reduce τ or to increase r. In the first case the tax burden on low ability individuals is alleviated and the tax burden on high ability individuals is increased. This is a good strategy for the median when the tax base is not decreased by much. This is the case if employment is relatively insensitive to the increase in α (|an/αa| small) and relatively sensitive to the increase in τ (|an/ατ| large). This will be the case, in turn, if the labor supply of high income individuals is less sensitive to a change in the marginal rate of taxation than the labor supply of low income individuals.

Consider now an increase in α that is used solely to increase redistribution r. As can be seen from equations (15a) and (15b) large values of |an/ar| reduce both r_a and r_τ because of the larger negative incentive effects of an increase in redistribution on work. But, as can be seen from (50), the median finds it worthwhile to increase α whenever r_a is larger than r_τ^2. Since H>1 larger |an/ar| reduce r_τ^2 by more than they reduce r_a. The incentive to increase α is therefore larger when |an/ar| is larger.

Condition (49) is sufficient for global progressivity whereas condition (49a) assures progressivity only locally. Since condition (49a) is simpler it is useful to know under what circumstances this condition alone is sufficient for global progressivity. If C(r,α) is a decreasing function of α condition (49a) alone is sufficient since for negative α-s C(·) is a fortiori positive. We turn therefore to a discussion of the channels through which an increase in α affects C(·).

An increase in α corresponds to a movement towards higher values of α and r along the TPF keeping τ constant. The increase in α produces (at the original levels of income) an increase in the tax burdens of all working individuals so they reduce their levels of work. The increase in r causes a further reduction in the labor input of all individuals. If an/αα and an/ar do not differ much across individuals the work levels of different individuals decrease by roughly similar amounts but the incomes of abler individuals
decrease by more. As a consequence a movement towards higher values of $\alpha$ and $r$ along the TPF produces a decrease in both the mean and the variance of income. The increase in $\alpha$ increases the tax burden of individuals with higher incomes by more. If, as a result, $|\alpha n/a_\alpha|$ is larger for individuals with larger incomes the downward effect on $V$ and $\bar{y}$ is even stronger. The increase in $r$ also raises the threshold productivity level below which individuals choose to remain idle. This raises $x_0$ and lowers $1-F(x_0)$.

On the other hand the increase in $\alpha$ and $r$ by decreasing $y(x_m,r,\alpha)$ and by raising the lower limit of the integral in (49) tends to increase $C(\cdot)$ when $\alpha$ and $r$ increase along the TPF. If the elasticity of individual labor supply with respect to the combined change in $\alpha$ and $r$ is smaller than 1 in absolute value $T'[\cdot]$ increases\(^{16}\) counteracting some of those effects by increasing the weights on the negative $\alpha n/a_\alpha$ terms. The upshot is that when the negative effects through $V$, $\bar{y}$, $1-F(x_0)$ and possibly $T'[\cdot]$ dominate the other positive effects for negative $\alpha$-s

$$\frac{\partial C(r,\alpha)}{\partial \alpha} < 0, \text{ for } \alpha < 0.$$  \hspace{1cm} (50)

When the condition in (50) is satisfied (49a) is sufficient to assure $\alpha > 0$.

VI. Majority Rule and Tax Progressivity When the Decisive Voter Does Not Work

In most democracies, the decisive voter works. He may receive transfers, but he also pays taxes. His decision to impose marginal progressivity depends on the balancing of the gains and losses he experiences and, thus, on parameters of the distribution of income and the labor supply response of those who pay the highest marginal rates. A non-worker faces a simpler problem. He pays no taxes, so his interest in marginal progressivity is greater. His own welfare depends on the transfers he receives, but these

\[^{16}\text{The total change in } T'[\cdot] \text{ at income } y \text{ is } 2y(1+\eta_{n\alpha}) \text{ where } \eta_{n\alpha} \text{ is the (negative) elasticity of labor supply with respect to the combined increase in } \alpha \text{ and } r \text{ along the TPF. Hence if } |\eta_{n\alpha}|<1, T'[\cdot] \text{ increases as a result of the change.}\]
transfer do not increase with marginal progressivity if disincentive effects on income earners are strong. Consequently, a rational decisive voter who does not work never chooses a value of \( a \) that lowers redistribution.

The utility of a decisive voter who does not work increases monotonically with \( r \). To maximize utility, he chooses the point on the TPF at which \( r \) is maximized. Formally, we can state his problem as

\[
\text{Max } r(\tau, a). \tag{51}
\]

The solution to this problem yields the following two first order conditions

\[
\begin{align*}
\frac{r}{a}(\tau, a) &= 0 \quad \text{(a)} \\
\frac{r}{\tau}(\tau, a) &= 0 \quad \text{(b)}
\end{align*}
\]

from which, together with (14), it is possible to solve in principle for the decisive voter's preferred tax schedule \( s_m \). Using (15b) and the fact that \( H \) is positive a sufficient condition for progressivity is therefore

\[
\int_{x_0}^{x_T} (y + x T') \frac{\partial n}{\partial a} dF > 0 \quad \text{for all } a < 0 \text{ and } \tau \tag{53}
\]

which is equivalent to

\[
K(r, a) = (1 - F(x_0))(V + y^2) - \int_{x_0}^{x_T} x T' \frac{\partial n}{\partial a} dF > 0 \quad \text{for all } a < 0 \text{ and } \tau. \tag{54}
\]

At \( a = 0 \) this condition reduces to

\[
(1 - F(x_0))(V + y^2) - \int_{x_0}^{x_T} x \frac{\partial n}{\partial a} dF > 0 \quad \text{for all } \tau. \tag{54a}
\]

We saw in the previous section that when, given \( \tau \), we move in the direction of
a higher $\alpha$ and a higher $r$ along the TPF $(1-F(x_0))(V+y^2)$ goes down. Hence if the last term in (54) does not go down $K(r(\alpha,\tau),\alpha)$ is (given $\tau$) a decreasing function of $\alpha$. This leads to the following proposition.

**Proposition 4:** If the median does not work and $A(r,\alpha) = \frac{x}{x_0} \int |\eta_{n\alpha}| \, dF$ is a non decreasing function of $\alpha$ for all $0 \leq \tau \leq 1$ condition (54a) is sufficient to assure marginal progressivity.

$A(r,\alpha)$ will be non decreasing in $\alpha$ if $|\eta_{n\alpha}| < 1$ for all $x$ and if the reduction in $A(\cdot)$ due to the increase in $x_0$ does not dominate the increase in it because of the increase in $T'$. Those statements implicitly assume that $|\eta_{n\alpha}|$ does not depend on $\alpha$. If $|\eta_{n\alpha}|$ increases with $\alpha$ (implying that the marginal disincentive effect of $\alpha$ on work grows as the tax burden increases) $A(\cdot)$ is even more likely to increase with $\alpha$.

Proposition 4 confirms the intuition that some of the same factors that are conducive to marginal progressivity when the median works are also conducive to it when he does not work. In particular the larger the variance of abilities, the more skewed to the right is their distribution, and the smaller the disincentive effects of an increase in $\alpha$ the more likely it is that majority rule will produce a marginally progressive tax schedule.

Finally, comparison of conditions (49) and (49a) with conditions (54) and (54a) respectively suggests that if a progressive tax schedule arises when the median works it must arise a fortiori when the median does not work.

**VII A Remark on Uniqueness of the Political Equilibrium and on Comparative Statics**

Whether the set of schedules most preferred by the median contains one element or more depends on the characteristics of the indirect utility function and of the TPF. Inserting (14) into (18) and specializing $x$ to $x_m$

$$I[(r(\tau,\alpha),\tau,\alpha;x_m)] = \max_n u[r(\tau,\alpha)+(1-\tau)x_m n-a(x_m)^n, 1-n]. \quad (18a)$$

Subsuming $r(\tau,\alpha)$ into the functional form we can write

$$J[\tau,\alpha;x_m] = I[(r(\tau,\alpha),\tau,\alpha;x_m)]. \quad (18b)$$

32
A sufficient condition for the uniqueness of $s_m$ is that $J[\tau, \alpha; x_m]$ be a globally concave function of $\tau$ and $\alpha$. This will be the case if there is sufficient concavity in the utility function and in the TPF—$r(\tau, \alpha)$. The TPF in turn is more likely to be concave if the disincentive effects of higher tax rates become stronger as the overall burden of taxation (as measured by the $(\tau, \alpha)$ combination) rises.

In this section we assume that there is enough concavity in either $u(\cdot)$ or $r(\cdot)$ or in both to assure a unique solution for $s_m$ and focus on some comparative static implications of the analysis in the previous two sections. Given uniqueness it is possible to relate more clearly the equilibrium tax schedule to the characteristics of the distribution of abilities. In particular the analysis of the previous two sections seems to indicate that the degree of progressivity chosen by the decisive voter is larger the larger the variance of abilities and the larger the degree of positive skewness in this distribution. This intuition is not quite right. It is, however, under some qualifications. To see the necessary qualifications consider the case of a working median for a given initial distribution of abilities $F_0(x)$. For this case equation (49) implies that $s_m$ is characterized by the condition

$$C(r_m, \alpha_m) = 0.$$ 

Consider now another distribution $F_1(x)$ which has a larger variance but in which the median still has productivity $s_m$, and in which the fraction of individuals up to $x_0$ remains the same. We shall refer to such a change as an upward biased median preserving increase in spread (UBMPIS). Since the productivity of the median does not change his work level and therefore his income is the same as the one he had at the original tax schedule. So are the work levels and the incomes of all other types including in particular individuals with productivity $x_0$. Hence it is still the case that all types above $x_0$ work and all types below it do not work. Hence $F(x_0)$ remains the same after the change. Since the variance of abilities above $x_0$ is higher so is the variance of incomes above $x_0$ and $V(r_m, \alpha_m)$ is higher for the new distribution. The effect of the UBMPIS on mean income and on the last integral in (49) is ambiguous. However, it is likely to be small and whatever
its sign to be dominated by the increase in $V(\cdot)$. Assuming this is the case implies that with the distribution $F_1(x)$, $C(r_m, \alpha_m)$ is now positive and therefore that

$$\frac{dJ}{d\alpha} [r_m, \alpha_m; x_m] > 0.$$  

Since $J[\cdot]$ is concave in $\alpha$ this implies that equilibrium $\alpha$ is larger under the new distribution.

We could consider similarly an increase in the mass of individuals with high productivity keeping the median at $x_m$. Such a change will also raise $V(r_m, \alpha_m)$ and change the integral in (49). But its primary effect is likely to be an increase in $\tilde{y}(r_m, \alpha_m)$. Since now both $\tilde{y}(\cdot)$ and $V(\cdot)$ increase $dJ/d\alpha$ is even more likely to become positive at $s_m$ than in the case in which only the variance of abilities changed.

We conclude that for many types of changes in the distribution of abilities a higher variance and a higher degree of positive skewness induce a higher degree of marginal progressivity. However, this is not necessarily true for all changes in distribution that induce a higher variance and a higher positive skewness in the distribution of abilities.

VIII. A Remark on Progressivity in the Presence of a Public Good

Since individual utility from expenditures on public goods has not been modeled explicitly one may get the erroneous impression that progressivity of the tax schedule arises only when the budget is used for redistributational purposes. As a matter of fact the same elements that are conducive to progressivity when the budget is used to redistribute income are likely to lead to progressivity when it is used to provide a public good. A general demonstration of this claim is beyond the scope of this paper. Instead we illustrate it for a particular utility function in the case in which the entire budget is used to finance a public good.

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17 As a matter of fact it is possible to formalize this notion by structuring the increase in variance so as to maintain all the other terms except $V$ at their prechange value at $s_m$.  

34
Let $g$ be the amount of a public good enjoyed by a representative individual and let

$$v(c+g, z)$$

be the utility function of a typical individual. This specification which implies that $c$ and $g$ are perfect substitutes is adopted for simplicity. The TPF in (13) is replaced now by

$$g = \delta N \int_{x_0}^{x_1} (z x_n(x) + (x_n(x))^2) dF(x) = \delta Nr$$

where $N$ is the number of individuals in the economy and $\delta$ is a parameter between $1/N$ and 1. When $\delta=1$, $g$ is a pure public good. When $\delta=1/N$, $g$ is a publicly provided private good. In the more likely intermediate cases, $1/N < \delta < 1$, $g$ is a public good but not a pure public good. Private consumption is now given by

$$c(x) = (1-\tau)x_n - a(x_n)^2.$$

For $\delta=1/N$ equation (13a) implies that $g=r$ and the model of this section becomes formally identical to the model of the previous sections. Hence the same factors that were conducive to progressivity before are conducive to progressivity now as well.

For $\delta>1/N$, $g$ is larger than $r$ by some fixed factor. As a result all individuals prefer a larger budget than in the case $\delta=1/N$. The reason is that the marginal utility of the public good is now higher. However, the same conflicts of interest regarding the financing of $g$ that existed when $\delta=1/N$ are also present when $\delta>1/N$. In particular, the analysis of sections III through VI can be replicated with $r$ replaced by $\delta Nr$. The formal conditions for the existence of a majority winning schedule and for progressivity have to be adjusted to reflect this change. However, the qualitative results of propositions 2 and 3 are likely to carry over to this case too. The intuitive reasoning underlying this statement relies on the observation that an increase in the budget increases the consumption of the public good by the same amount for everybody as was the case for $\delta=1/N$. But the necessary increase in financing triggers redistributional conflicts that are essentially identical.
to those that are present when \( \delta = 1/N \) since, except for \( \delta \), the model is the same.

This example suggests that, at least for some classes of utility functions, the factors that are conducive to progressivity when the budget is used to redistribute income are also conducive to progressivity when the budget is used solely to provide a public good.

**IX. Concluding Comments**

Our intent in this paper is to develop a positive theory of income taxation that generates tax schedules exhibiting marginal and average tax rates that rise with income. This feature is commonly found in many democratic countries, and in some states of the U.S.

Economists and others have long speculated on the desirability of progressive taxes. Efforts to use theories of optimal taxation to explain the existence of progressivity have not been completely successful. Maximization of a Benthamite criterion very seldom leads to progressive tax structures. An alternative view, taken in this paper, is that tax schedules are the outcome of a political equilibrium in which the majority imposes its will on the minority. Being motivated by self interest all individuals would like to pay no taxes and to obtain positive redistribution from the government. Obviously this is not feasible for everybody. But in a democratic society in which tax schedules are determined by majority rule the low and middle income majority can impose a certain level of redistribution on the more affluent minority. This paper provides conditions under which this type of democracy leads to progressivity in the taxation of income.

A fundamental problem which arises once tax schedules are specified in a way that is sufficiently flexible to allow progressivity is that a majority winning tax schedule need not exist. The paper derives conditions for the existence of a winning schedule within the set of quadratic schedules. An important condition for existence is that the ranking of gross incomes and of abilities be the same for all tax schedules. Under this condition and some additional restrictions on the utility function and on the distribution of abilities the individual with median ability is shown to be decisive. This result paves the way for finding conditions for progressivity since it reduces this task to that of finding conditions under which the median ability
individual prefers progressivity. In contrast to the optimal taxation literature it is found that the set of circumstances under which the decisive voter, who maximizes utility, imposes progressivity is non negligible. Marginal progressivity is more likely (1) the larger the spread of the distribution of abilities in the population, (2) the smaller the labor supply response of the relatively more productive to an increase in marginal tax rates and (3) the larger the difference between the ability of the decisive (median) voter and the mean ability of the community. (4) When the median works, the larger the disincentive effects of redistribution on labor force participation. These conditions do not have to be satisfied separately; their combined effect is sufficient.

The tax schedules that result from our analysis reflect, mainly, skewness of the distribution of income that puts average income above median income, the variance of the distribution, and the effects of tax rates and redistribution on incentives to work. If these factors were identical across countries, and the franchise approximately the same, we would predict common tax schedules in democratic countries. Differences in tax schedules principally reflect possible differences in the voting rule that determines the franchise, differences in the distribution of income and ability and difference in the effect of incentives, as measured in our analysis by the marginal effect of tax rates on labor supply.

Although the focus of the paper is on deriving conditions for progressivity when the budget is used to redistribute income it is likely that similar conditions are conducive to progressivity when the budget is used to provide a public good. As illustrated in Section VII even the provision of a pure public good is not independent, because of the need to decide on ways to finance it, from redistributional considerations.

Some limitations of the analysis should be noted. Our model is static; income must be interpreted as lifetime income. The budget is always balanced. We neglect migration and other open economy considerations that can limit progressivity over the time frame to which our model is most

\footnote{A recent application of the majority rule paradigm to intertemporal redistribution and the determination of the public debt and deficits appears in Cukierman and Meltzer (1988).}
applicable. There is no capital, and therefore there are no taxes on capital.

Despite these limitations, the political economy model appears to be useful for understanding the determination of tax schedules and for showing that majority rule implies marginal progressivity to finance income redistribution and other expenditures under a relatively wide set of circumstances. The median or decisive voter chooses rising marginal tax rates if he can thereby reduce his own taxes without lowering the transfers he receives, or if he can increase the transfer he receives without increasing his own tax rates. Casual observations for many economies suggest that the voting process produces an outcome of this kind.

We conclude by offering a reflection on empirical testing of the theory. The fact that the distribution of abilities is by and large non observable appears as an obstacle to testing the implications that depend on the characteristics of this distribution. But since the variance and the degree of positive skewness in the distributions of income and of abilities are usually positively related the theory implies that, ceteris paribus, countries with larger variability and a larger degree of positive skewness in the distribution of abilities should often have a more progressive tax structure.\(^1\)

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\(^1\) The proviso "often" refers to the fact that comparative statics require uniqueness. It also alludes to additional restrictions discussed in Section VII.
Appendix

1. Derivation of Equations (15)

Totally differentiating (13) with respect to $\tau$, holding $a$ constant, we obtain

$$
\frac{\partial}{\partial \tau} \int_{x_0}^{x_1} \left[ y + \tau \left( \frac{3n}{3\tau} + r \frac{3n}{3r} \right) + 2ax^2n\left( \frac{3n}{3\tau} + r \frac{3n}{3r} \right) \right] dF
$$

$$
- r_{\tau} - \left[ \tau x_0 n(x_0) + a(x_0 n(x_0))^2 \right] \left[ \frac{3x_0}{3\tau} + r \frac{3x_0}{3r} \right] = 0
$$

where the dependence of the various terms on $x$ is not made explicit in the notation. Since $n(x_0) = 0$ the last term drops. Equation (15a) follows by rearranging and by using equation (4) in the text.

Totally differentiating (13) with respect to $a$, holding $\tau$ constant, we obtain

$$
\frac{\partial}{\partial a} \int_{x_0}^{x_1} \left[ y + \tau \left( \frac{3n}{3a} + r \frac{3n}{3r} \right) + 2ax^2n\left( \frac{3n}{3a} + r \frac{3n}{3r} \right) \right] dF
$$

$$
- r_{a} - \left[ \tau x_0 n(x_0) + a(x_0 n(x_0))^2 \right] \left[ \frac{3x_0}{3a} + r \frac{3x_0}{3r} \right] = 0.
$$

Equation (15b) follows by noting that $n(x_0) = 0$, using (4), and by rearranging.


It is convenient to break the proof into several lemmas.

Lemma A1: If condition (i) of theorem 2 is satisfied $s_m$ is a global majority winner against any change in tax schedule such that $a$ and $\tau$ change in the same direction or such that only one of either $a$ or $\tau$ changes.
Proof: Since \( s_m > S_m \), the median either dislikes or is indifferent to the change. Hence by lemma 5

\[
P(y_m) = - \Delta \alpha y_m^2 - \Delta \tau y_m + \Delta r < 0 \tag{A1}
\]

where

\[
y_m = y (s_m, x_m).
\]

Multiplying (A1) by \(-1/\Delta \alpha\)

\[
Q(y_m) = y_m^2 + \frac{\Delta \tau}{\Delta \alpha} y_m - \frac{\Delta r}{\Delta \alpha} . \tag{A2}
\]

(A1) and (A2) imply

\[
Q(y_m) < 0 \quad \text{if} \quad \Delta \alpha < 0 \quad \text{(a)}
\]

\[
Q(y_m) > 0 \quad \text{if} \quad \Delta \alpha > 0. \quad \text{(b)}
\]

Note that

\[
Q'(y) = 2y + \frac{\Delta \tau}{\Delta \alpha} \tag{A4}
\]

which is positive for all \( y \geq 0 \) when \( \Delta \tau \) and \( \Delta \alpha \) have the same signs. Hence

\[
Q(y) < Q(y_m) \quad \text{for all} \quad \Delta \alpha < 0 \quad \text{and all} \quad y < y_m \quad \text{(a)}
\]

\[
Q(y) > Q(y_m) \quad \text{for all} \quad \Delta \alpha > 0 \quad \text{and all} \quad y > y_m. \quad \text{(b)}
\]

(A3) and (A5) imply

\[
P(y) < P(y_m) \quad \text{for all} \quad \Delta \alpha < 0 \quad \text{and all} \quad y < y_m \quad \text{(a)}
\]

\[
P(y) < P(y_m) \quad \text{for all} \quad \Delta \alpha > 0 \quad \text{and all} \quad y > y_m. \quad \text{(b)}
\]

But \( P(y) \) is the change in the net income of an individual with pre-change income \( y \) when he is not allowed to adjust his labor income. Condition (i) of
Theorem 2 therefore implies that

\[ \text{when } \Delta \alpha > 0 \quad s_m P_i s \quad \text{by all } y_i < y_m \quad (a) \]  
\[ \text{when } \Delta \alpha < 0 \quad s_m P_i s \quad \text{by all } y_i > y_m. \quad (b) \]

The notation \( s_m P_i s \) should be read as; "\( s_m \) is preferred to \( s \) by an individual with pre-change income \( y_i \)." Since the ranking of gross incomes is the same as that of abilities (A7) implies that there is no change in schedule, such that \( \Delta \alpha \) and \( \Delta \tau \) have the same sign, that is strictly preferred by a majority.

When \( \Delta \tau = 0 \) and \( \Delta \alpha \neq 0 \) the fact that \( s_m \not\sim S_m \) in conjunction with lemma 5 imply

\[ P(y_m) = -\Delta \alpha y_m^2 + \Delta \tau < 0. \]

This implies

\[ P(y) < P(y_m) \quad \text{for all } \Delta \alpha < 0 \quad \text{and all } y < y_m \quad (a) \]  
\[ P(y) < P(y_m) \quad \text{for all } \Delta \alpha > 0 \quad \text{and all } y > y_m. \quad (b) \]

When \( \Delta \tau \neq 0 \) and \( \Delta \alpha = 0 \) the fact that \( s_m \not\sim S_m \) in conjunction with lemma 5 imply

\[ P(y_m) = -\Delta \tau y_m^2 + \Delta \tau < 0. \]

This implies

\[ P(y) < P(y_m) \quad \text{for all } \Delta \tau < 0 \quad \text{and all } y < y_m \quad (a) \]  
\[ P(y) < P(y_m) \quad \text{for all } \Delta \tau > 0 \quad \text{and all } y > y_m. \quad (b) \]

(A8), (A9) and condition (i) of theorem (ii) imply that there is no majority that strictly prefers changes of the type \( \Delta \alpha = 0 \) and \( \Delta \tau \neq 0 \) or \( \Delta \alpha \neq 0 \) and \( \Delta \tau = 0. \) Q.E.D.
Lemma A2: If condition (i) of theorem 2 is satisfied $s_m$ is a global majority winner against any change in schedule of the type

$$
\Delta a > 0, \quad \Delta \tau < 0.
$$

Proof: The fact that $s_m = S_m$ and lemma 5 imply that (A1) holds. Since $\Delta a > 0$ this is equivalent in turn to

$$
Q(y_m) > 0.
$$

(A10)

Since $\frac{\Delta \tau}{\Delta a} < 0$ equation (A4) implies

$$
Q'(y) = \begin{cases} 
> 0 & y > \frac{1}{2} |\frac{\Delta \tau}{\Delta a}| \\
= 0 & y = \frac{1}{2} |\frac{\Delta \tau}{\Delta a}| \\
< 0 & y < \frac{1}{2} |\frac{\Delta \tau}{\Delta a}|
\end{cases}
$$

(A11)

(A10) and (A11) imply

$$
Q(y) > Q(y_m) \text{ for all } y > y_m \text{ if } y_m > \frac{1}{2} |\frac{\Delta \tau}{\Delta a}| \quad (a)
$$

$$
Q(y) > Q(y_m) \text{ for all } y < y_m \text{ if } y_m < \frac{1}{2} |\frac{\Delta \tau}{\Delta a}| \quad (b) \quad (A12)
$$

$$
Q(y) > Q(y_m) \text{ for all } y \neq y_m \text{ if } y_m = \frac{1}{2} |\frac{\Delta \tau}{\Delta a}|. \quad (c)
$$

Since $Q(y)$ and $P(y)$ are inversely related for $\Delta a > 0$ this implies that in all three cases at least fifty percent of the voters suffer (at pre-change labor inputs) a decrease in net income that is larger than $P(y_m)$. Condition (i) of theorem 2 implies therefore that at least the same number of voters dislikes changes of the type $\Delta a > 0, \Delta \tau < 0$. Since the median either dislikes or is indifferent to the change there is no change of the type $\Delta a > 0, \Delta \tau < 0$ that is preferred by a majority to $s_m$. Hence $s_m$ is a global majority winner against changes of this type.

Q.E.D.
Lemma A3: \( s_m \) is a global majority winner against any change in schedule of the type

\[
\Delta r \leq 0, \quad \Delta \tau > 0, \quad \Delta \alpha < 0
\]

if conditions (i) and (ii) of theorem 2 are satisfied.

Proof: Due to condition (i) of theorem 2 all individuals with gross incomes in the open segment defined by (40) weakly dislike the change. Condition (ii) of theorem 2 assures that for any \( b \) such that \( \Delta r \leq 0, \Delta \tau > 0, \Delta \alpha < 0 \) there is a majority that weakly dislikes the change. Q.E.D.

Lemma A4: \( s_m \) is a global majority winner against any change in schedule of the type

\[
\Delta r > 0, \quad \Delta \tau > 0, \quad \Delta \alpha < 0
\]

if conditions (i) and (iii) of theorem 2 are satisfied.

Proof: Since \( \Delta r > 0 \) and since \( P(y_m) < 0 \) so that \( y_m < y_{C2} \), an argument similar to that which led to equation (29) in the text implies that

\[
\frac{P_a}{r_a} < b < \frac{P_a - y_m^2}{r_a - y_m} \quad (A13)
\]

provided the median works. Since \( s_m \) does not occur on the boundary of the TPF \( y_m = r_a/r_\tau \) (equation (32)). Hence the condition in (A13) reduces to the restriction on \( b \) in condition (iii) of theorem 2. Since \( \Delta r > 0, \quad y_{C1} > 0 \). Condition (i) of theorem 2 implies that all individuals with prechange gross incomes in the range

\[
y_{C1}(b) + \epsilon_1(b) < y < y_{C2}(b) - \epsilon_2(b) \quad (A14)
\]

dislike the change. Condition (iii) of theorem 2 implies that for all \( b \)-s in the open interval defined by (A14) there is no majority in favor of the
change. Hence \( s_m \) is a global majority winner against all changes of the type
\( \Delta r>0, \Delta \tau>0, \Delta a<0 \), when the median works.

If the median does not work the fact that he does not prefer the change
implies that redistribution, \( r \), goes down or does not change as a consequence
of the change. Hence the case \( \Delta r>0, \Delta \tau>0, \Delta a<0 \) is not possible when the
median does not work so there is no need to consider it. Q.E.D.

The proof of theorem 2 is completed by combining lemmas A1 through A4.
References


