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NONLOCAL-INTERACTION EQUATIONS ON UNIFORMLY PROX-REGULAR SETS

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ABSTRACT. We study the well-posedness of a class of nonlocal-interaction equations on general domains $\Omega \subset \mathbb{R}^d$, including nonconvex ones. We show that under mild assumptions on the regularity of domains (uniform prox-regularity), for λ-geodesically convex interaction and external potentials, the nonlocal-interaction equations have unique weak measure solutions. Moreover, we show quantitative estimates on the stability of solutions which quantify the interplay of the geometry of the domain and the convexity of the energy. We use these results to investigate on which domains and for which potentials the solutions aggregate to a single point as time goes to infinity. Our approach is based on the theory of gradient flows in spaces of probability measures.

1. INTRODUCTION

1.1. **Description of the problem.** We study a continuum model of agents interacting via a potential W and subject to an external potential V confined to a closed subset $\Omega \subset \mathbb{R}^d$. Such systems arise in modeling macroscopic behavior of agents interacting in geometrically confined domains. The domain boundary may be an environmental obstacle, like a river, or the ground itself, as in the models of locust patterns discussed in [3, 17, 18]. We consider systems in which the environmental boundaries limit the movement but not the interaction between agents. To illustrate, the the agents can still see each others over a river even if they are not able to traverse it.

We describe configurations of agents as measures supported on the given domain. This allows to study both the discrete case, when an individual agent carries a positive mass, and the continuum limit in which a system with many agents is described by a function giving the density of agents. The measure describing the agents interacting over time satisfies a nonlocal-interaction equation in the sense of weak measure solutions. The theory of weak measure solutions to nonlocal-interaction equations was developed in [2, 5]. In [22] systems of interacting agents on domains with boundary were considered in a setting which allowed for heterogeneous environments, but required the sets to be convex with C^1 boundary. Here we consider general domains which are not required to be convex and whose boundary may not be differentiable. The geometrical confinement introduces a constraint on the possible velocity fields of the agents at the boundary. We consider the situation in which there is no additional friction at the boundary. More precisely, for smooth domains, the velocity of the agents at the boundary is the projection of what the velocity would be, for the given configuration if there was no boundary, to the half plane of vectors pointing inside the domain. That is, inward pointing velocities at the boundary are unchanged, while the outward pointing velocities are projected on the tangent plane to the boundary. The measure $\mu(\cdot)$ describing the

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agent configuration becomes a distributional solution of the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\mu(t,x) - \operatorname{div}\left(\mu(t,x)P_x\left(-\displaystyle\int_\Omega \nabla W(x-y)d\mu(t,y) - \nabla V(x)\right)\right) = 0, \\ \mu(0) = \mu_0, \end{cases} \tag{1.1}$$

where P_x is the projection of the velocities to inward pointing ones.

When considering domains which are not C^1 the question is what should the velocity of agents be at a boundary point where the domain is not differentiable. Similar questions have been encountered in studies of differential inclusions on moving domains (general sweeping processes), see [8, 9, 19] and references therein. We rely on notions developed there to properly define the cone of admissible directions at a boundary points and the proper way to project the velocity to the allowable cone. In particular we consider the equation (1.1) with projection P_x defined in (1.6) ($P_x = P_{T(\Omega,x)}$).

While one would like to consider very general domains there are limits to possible domains on which a well-posedness of measure solutions can be developed. Namely, if the domains have an inside corner, then it is not possible for the measure solutions of (1.1) to be stable, as we discuss in Remark 1.11. It turned out that a class of domains which is rather general and allows for a well-posedness theory are the (uniformly) prox-regular domains (see Definition 1.3). Prox-regular domains are the sets which have an outside neighborhood such that for each of its points there exists a unique closest point on the boundary. In particular prox-regular domains can have outside corners and outside cusps, but not inside corners.

Our main result is the well-posedness of weak measure solutions, described in Definition 1.1, of the nonlocal-interaction equation (1.1) on uniformly prox-regular domains. To show it we rely on further structure the equation possesses. Namely to the interaction W, we associate interaction energy

$$\mathscr{W}(\mu) = \frac{1}{2}\int_\Omega\int_\Omega W(x-y)d\mu(x)d\mu(y),$$

the and to potential V, the potential energy

$$\mathscr{V}(\mu) = \int_\Omega V(x)d\mu(x).$$

We define the energy $\mathscr{E}$ by

$$\mathscr{E}(\mu) = \mathscr{W}(\mu) + \mathscr{V}(\mu). \tag{1.2}$$

The energy $\mathscr{E}$ is a dissipated quantity of the evolution (1.1), and furthermore the equation can be interpreted as the gradient flow of the energy with respect to the Wasserstein metric. Our strategy is to first show the existence of the gradient flow solutions to (1.1) in the space of probability measures endowed with the Wasserstein metric. The gradient flow in the space of probability measures endowed with Wasserstein metric was first used in [11] for Fokker-Planck equations. The gradient-flow approach to well-posedness of nonlocal-interaction equations was developed in [2, 5] and extended to C^1 domains with boundary in [22]. Furthermore the gradient flow approach was used to study systems in which there are state constraints that determine the set of possible velocities, in particular in crowd motion models [1, 12, 13] where the constraint on the L^∞-norm of the density of agents, which leads to an L^2-projection of velocity field.

After establishing the well-posedness of gradient-flow solutions we show the well-posedness of weak measure solutions.

To show the existence of gradient flow solutions, we use particle approximations, that is we use a sequence of delta masses $\mu_0^n = \sum_{j=1}^{k(n)} m_j \delta_{x_j^n}$ to approximate the initial data μ_0 and solve (1.1) with initial data μ_0^n. Here the notion of gradient flow solutions (and weak measure solutions) provides the advantage that we can work with delta measures, which makes the particle approximation meaningful. Since (1.1) becomes a system of ordinary differential equations, we solve the ODE system and prove that the solutions $\mu^n(\cdot)$ converges to some $\mu(\cdot)$ by establishing the stability property of solutions to (1.1) with different initial data. We then show that the limit curve $\mu(\cdot)$ is a gradient flow solution to (1.1) with initial data μ_0 by proving that $\mu(\cdot)$ achieves the maximal dissipation of the associated energy, and is thus the steepest descent of the energy.

The novelty here is that even though the domain Ω is only prox-regular (not necessarily convex or C^1) and the velocity field is discontinuous (due to the projection P), the ODE systems are still well-posed (refer to Theorem 2.6) and the stability of solutions $\mu^n(\cdot)$ in Wasserstein metric d_W is valid with explicit dependence on the prox-regularity constant (refer to Proposition 3.1). Under semi-convexity assumptions on the potential functions W and V, this enables us to show the well-posedness, that is existence and stability of weak measure solutions to (1.1) in three different cases: Ω bounded and prox-regular (Theorem 1.5 and Thorem 1.6), Ω unbounded and convex (Theorem 1.9), and Ω unbounded and prox-regular with compactly supported initial data μ_0 (Theorem 1.10). We can also generalize the well-posedness results to time-dependent interaction and external potentials $W = W(t,x), V = V(t,x)$ (Remark 5.3). We also give sufficient conditions on the shape of Ω to ensure the existence of an interaction potentials W such that solutions $\mu(\cdot)$ to (1.1) aggregate to a single delta mass of as time goes to infinity (Theorem 1.13 and Remark 6.1) .

1.2. **Description of weak measure solutions.** Let $\mathscr{P}(\Omega)$ be the space of probability measures on Ω and let

$$\mathscr{P}_2(\Omega) = \left\{ \mu \in \mathscr{P}(\Omega) : \int_\Omega |x|^2 d\mu(x) < \infty \right\},$$

the space of probability measures with finite second moment. $\mathscr{P}_2(\Omega)$ is a complete metric space endowed with the 2-Wasserstein metric

$$d_W^2(\nu,\mu) = \min\left\{ \int_{\Omega\times\Omega} |x-y|^2 d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu) \right\}, \tag{1.3}$$

where $\Gamma(\mu,\nu)$ is the set of transportation plans between ν and μ, that is the set of joint probability distributions on $\Omega\times\Omega$ with first marginal μ and second marginal ν:

$$\Gamma(\mu,\nu) = \{\gamma \in \mathscr{P}(\Omega\times\Omega) : (\pi_1)_\sharp\gamma = \mu, (\pi_2)_\sharp\gamma = \nu\}.$$

We refer to books [20, 21] for theory of optimal transport. We denote the set of optimal transport plans between μ and ν (with respect to 2-Wasserstein metric) by $\Gamma_o(\mu,\nu)$, that is

$$\Gamma_o(\mu,\nu) = \left\{ \gamma \in \Gamma(\mu,\nu) : \int_{\Omega\times\Omega} |x-y|^2 d\gamma(x,y) = d_W^2(\mu,\nu) \right\}. \tag{1.4}$$

We now give the definition of weak measure solutions to the continuity equation.

Definition 1.1. A locally absolutely continuous curve $\mu(\cdot) \in \mathscr{P}_2(\Omega)$ is a *weak measure solution* to (1.1) with initial value μ_0 if

$$v(t,x) = -P_x\left(\int_\Omega \nabla W(x-y) d\mu(y) + \nabla V(x) \right) \in L^1_{loc}([0,+\infty); L^2(\mu(t)))$$

and

$$\int_0^\infty\int_\Omega \frac{\partial \phi}{\partial t}(t,x)d\mu(t,x)dt + \int_\Omega \phi(0,x)d\mu_0(x) + \int_0^\infty\int_\Omega \langle \nabla\phi(t,x), v(t,x)\rangle\, d\mu(x) = 0\,,$$

for all $\phi \in C_c^\infty([0,\infty)\times\Omega)$. The projection P_x is described below and formally defined in (1.6) (with $P_x = P_{T(\Omega,x)}$).

Note that the test function ϕ does not have to be zero on the boundary of Ω, and thus the no-flux boundary condition is imposed in a weak form.

We now define the projection P_x. When $\partial\Omega \in C^1$ is smooth and oriented, the definition of P_x is given in [22, 23], and it is given by $P_x(v) = v - \langle v, \nu(x)\rangle \nu(x)$ if $\langle v, \nu(x)\rangle > 0$ and $P_x(v) = v$ otherwise, where $\nu(x)$ is the unit outward normal vector to the boundary at $x \in \partial\Omega$. When Ω is only prox-regular, to define P_x, we need to recall some notations from non-smooth analysis, see [4, 7], in order to replace the normal vector field and the inward and outward directions.

Definition 1.2. Let S be a closed subset of $\mathbb{R}^d$. We define the *proximal normal cone* to S at x by,

$$N^P(S,x) = \left\{ v \in \mathbb{R}^d \,:\, \exists \alpha > 0,\ x \in P_S(x+\alpha v)\right\},$$

where

$$P_S(y) = \left\{ z \in S \,:\, \inf_{w\in S}|w-y| = |z-y| \right\}$$

is the projection of y onto S.

Note that for $x \in S\setminus \partial S, N^P(S,x) = \{0\}$ and by convention for $x \notin S, N^P(S,x) = \emptyset$. The notion of normal cone extends the concept of outer normal of a smooth set in the sense that if S is a closed subset of $\mathbb{R}^d$ with boundary ∂S an oriented C^2 hypersurface, then for each $x \in \partial S$, $N^P(S,x) = \mathbb{R}^+\nu(x)$ where $\nu(x)$ is the unit outward normal to S at x. We now recall the notion of uniform prox-regular sets.

Definition 1.3. Let S be a closed subset of $\mathbb{R}^d$. S is said to be η*-prox-regular* if for all $x \in \partial S$ and $v \in N^P(S,x), |v| = 1$ we have

$$B_\eta(x+\eta v)\cap S = \emptyset,$$

where $B_\eta(y)$ denotes the open ball centered at y with radius $\eta > 0$.

Note that an equivalent characterization, see [7, 15], is given by: S is η-prox-regular if for any $y \in S, x \in \partial S$ and $v \in N^P(S,x)$,

$$\langle v, y-x\rangle \le \frac{|v|}{2\eta}|y-x|^2. \tag{1.5}$$

Observe that if S is closed and convex, then S is ∞-prox-regular. We now turn to the tangent cones.

Definition 1.4. Let S be a closed subset of $\mathbb{R}^d$ and $x \in S$, define the *Clarke tangent cone* by

$$T^C(S,x) = \left\{ v \in \mathbb{R}^d \,:\, \forall t_n \searrow 0, \forall x_n \in S,\ \text{s.t. } x_n \to x,\ \exists v_n \to v \text{ s.t. } (\forall n)\, x_n + t_n v_n \in S \right\},$$

and denote the *Clarke normal cone* by

$$N^C(S,x) = \left\{ \xi \in \mathbb{R}^n : \langle \xi, v\rangle \le 0 \forall v \in T^C(S,x)\right\}.$$

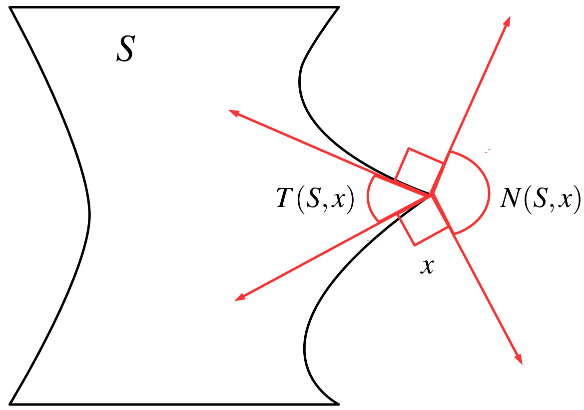

FIGURE 1. The set S is prox-regular but not convex. At the corner point $x \in \partial S$, the tangent and normal cones are denoted by $T(S,x)$ and $N(S,x)$.

Note that $T^C(S,x), N^C(S,x)$ are closed convex cones, also by convention $N^C(S,x) = \emptyset$ for all $x \notin S$. In general, we only have $N^P(S,x) \subset N^C(S,x)$ and the inclusion can be strict. However, for η-prox-regular set S, we have $N^P(S,x) = N^C(S,x)$, see [7, 15]. In that case, we put the normal cone and tangent cone as $N(S,x)$ and $T(S,x)$ respectively, and for any vector $w \in \mathbb{R}^d$, we define the projection onto the tangent cone by $P_{T(S,x)}(w)$, i.e.,

$$P_{T(S,x)}(w) = \left\{ v \in T(S,x) : |v-w| = \inf_{\xi \in T(S,x)} |\xi - w| \right\}. \tag{1.6}$$

Since $T(S,x)$ is a closed convex cone, the infimum is always attained, and $P_{T(S,x)}$ is well-defined. For notation simplicity, since the set we are considering Ω is not changing, we write P_x instead of $P_{T(\Omega,x)}$ and when the context is clear, we put P for P_x. With these preliminaries, we can now state the main results of this work.

1.3. **Main results.** For any set $A \subset \mathbb{R}^d$, we denote by $A - A = \{x - y : x, y \in A\}$, and the convex hull of A by $\mathrm{Conv}(A) = \{\theta x + (1-\theta)y : x, y \in A, 0 \le \theta \le 1\}$. For a function $f \in C^1(\mathbb{R}^d)$, we say that f is λ-geodesically convex on a convex set S if for any $x, y \in S$ we have

$$f(y) \ge f(x) + \langle \nabla f(x), y - x \rangle + \frac{\lambda}{2}|y-x|^2.$$

We call f *locally λ-geodesically convex* if there exist a sequence of compact convex sets $K_n \subset \mathbb{R}^d$ and a sequence of constants λ_n such that $K_n \subset K_{n+1}$, $\bigcup_n K_n = \mathbb{R}^d$ and f is λ_n-geodesically convex on K_n. Note that f is λ-geodesically convex on a convex set S implies for any $x, y \in S$

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \ge \lambda |x-y|^2.$$

The main assumptions depend on the domain Ω and the support of initial data. In fact, we separate our results in three cases: Ω bounded, Ω unbounded and convex, and Ω unbounded with compactly supported initial data. The assumptions are very similar in nature based on the convexity of the potentials V and W and on their growth behavior at ∞ in the unbounded cases. We assume that both potentials V and W are λ_V- and λ_W-convex respectively, possibly locally convex. Finally, in case V and W are λ-locally convex, we can assume, without loss of generality, that V and W share the same sequence of compact convex sets, K_k in the definition of locally λ-geodesic convexity, i.e., $K_k \subset K_{k+1}$, $\bigcup_{k\in\mathbb{N}} K_k = \mathbb{R}^d$ with V and W being $\lambda_{V,k}$ and $\lambda_{W,k}$-geodesically convex on K_k.

In case Ω is bounded, we assume that

(M1) $\Omega \subset \mathbb{R}^d$ is η-prox-regular with $\eta > 0$.
(A1) $W \in C^1(\mathbb{R}^d)$ is λ_W-geodesically convex on $\mathrm{Conv}(\Omega-\Omega)$ for some $\lambda_W \in \mathbb{R}$.
(A2) $V \in C^1(\mathbb{R}^d)$ is λ_V-geodesically convex on $\mathrm{Conv}(\Omega)$ for some $\lambda_V \in \mathbb{R}$.

We call a locally absolutely continuous curve $\mu(t) \in \mathscr{P}_2(\Omega)$ a gradient flow of the energy functional $\mathscr{E}$ defined in (1.2) if for a.e $t > 0$

$$v(t) \in -\partial\mathscr{E}(\mu(t)),$$

where $\partial\mathscr{E}(\mu(t))$ is the subdifferential of $\mathscr{E}$ at $\mu(t)$ (as given in Definition 3.3) and $v(t)$ is the tangent velocity of the curve $[0,\infty) \ni t \mapsto \mu(t) \in \mathscr{P}_2(\Omega)$ at $\mu(t)$, which we recall in Section 3.

For a locally absolutely continuous curve $[0,T] \ni t \mapsto \mu(t) \in \mathscr{P}_2(\Omega)$ with respect to 2-Wasserstein metric d_W, we denote its metric derivative by

$$|\mu'|(t) = \limsup_{s\to t} \frac{d_W(\mu(t),\mu(s))}{|s-t|}. \tag{1.7}$$

The main results of this paper is the well-posedness of weak measure solutions: existence and stability, with arbitrary initial data. We establish it using an approximation scheme and the theory of gradient flows in spaces of probability measures.

Theorem 1.5. *Assume Ω is bounded and satisfies (M1) and W,V satisfy (A1), (A2). Then there exists a locally absolutely continuous curve $\mu(\cdot) \in \mathscr{P}_2(\Omega)$ such that $\mu(\cdot)$ is a gradient flow with respect to $\mathscr{E}$. Moreover, $\mu(\cdot)$ is a weak measure solution to* (1.1)*.*

Furthermore for a.e. $t > 0$

$$|\mu'|^2(t) = \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(t,x), \tag{1.8}$$

and for any $0 \le s \le t < \infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(r,x) dr. \tag{1.9}$$

Theorem 1.6. *Assume Ω is bounded and satisfies (M1) and W,V satisfy (A1), (A2). Let $\mu^1(\cdot), \mu^2(\cdot)$ be two weak measure solutions to* (1.1) *with initial data μ_0^1, μ_0^2 respectively. Then*

$$d_W\left(\mu^1(t),\mu^2(t)\right) \le \exp\left(\left(-\lambda_W^- - \lambda_V + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)} + \|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right) t\right) d_W\left(\mu_0^1,\mu_0^2\right). \tag{1.10}$$

for any $t \ge 0$ where $\lambda_W^- = \min\{\lambda_W, 0\}$. Moreover, the weak measure solution is characterized by the system of Evolution Variational Inequalities:

$$\frac{1}{2}\frac{d}{dt} d_W^2(\mu(t),\nu) + \left(\frac{\lambda_W^-}{2} + \frac{\lambda_V}{2} - \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)} + \|\nabla V\|_{L^\infty(\Omega)}}{2\eta}\right) d_W^2(\mu(t),\nu) \le \mathscr{E}(\nu) - \mathscr{E}(\mu(t)), \tag{1.11}$$

for a.e. $t > 0$ and for all $\nu \in \mathscr{P}_2(\Omega)$.

Observe that in the stability estimate for solutions (1.12), we find two contributions due to the λ-convexity of the potentials and the η-prox-regular property of the domain Ω respectively.

On $\mathbb{R}^n$ when $\mu^1(0)$ and $\mu^2(0)$ have the same center of mass λ_W^- can be replaced by λ_W in (1.12). Thus when the potential W is uniformly geodesically convex, $\lambda_W < 0$ and thus there is exponential contraction of solutions. On bounded domains this is not the case since interaction with boundary can change the center of mass of a solution. Nevertheless part of the claim can be recovered. We consider the case that $V = 0$. Denote the set of singletons by $\Xi = \{\delta_x : x \in \mathbb{R}^d\}$. Note that we included the singletons which are not in the set Ω, since the center of mass for measures in non-convex sets may lie outside Ω.

Proposition 1.7. *Assume Ω is bounded and satisfies (M1) and W satisfies (A1). Let $\mu^1(\cdot)$ be a weak measure solutions to* (1.1) *with $V = 0$. Then*

$$d_W\left(\mu^1(t),\Xi\right) \le \exp\left(\left(-\lambda_W + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{\eta}\right)t\right) d_W\left(\mu_0^1,\Xi\right). \tag{1.12}$$

for any $t \ge 0$.

The proposition implies that solution can aggregate to a point (in perhaps infinite time) even on a nonconvex domain. We ask on what domains there exists a potential for which for any initial datum this aggregation property holds. We provide a sufficient condition on the shape of Ω for aggregation to hold: Let $\operatorname{diam}(\Omega) = \sup_{x,y\in\Omega}|x-y|$.

Theorem 1.8. *Assume that Ω is bounded and satisfies (M1). If $\eta > \frac{1}{2}\operatorname{diam}(\Omega)$, then for external potential $V \equiv 0$, there exists an interaction potential W satisfying (A1) for some $\lambda_W > 0$, and constant $C(\Omega) < 0$ such that*

$$d_W(\mu(t),\Xi) \le d_W(\mu_0,\Xi)\exp\left(C(\Omega)t\right), \tag{1.13}$$

for all $t \ge 0$. In particular, the solution aggregates to a singleton:

$$\lim_{t\to\infty} d_W\left(\mu(t),\Xi\right) = \lim_{t\to\infty} d_W\left(\mu(t),\delta_{\bar{x}(t)}\right) = 0, \tag{1.14}$$

where $\bar{x}(t)$ is the center of mass of $\mu(t)$.

Note that the constant in $\eta > \frac{1}{2}\operatorname{diam}(\Omega)$ cannot be improved, as the example in Remark 6.1 shows.

We generalize the two existence and stability results to the unbounded case in two different settings. In case Ω is unbounded, and for general initial data μ_0, possibly with noncompact support, we give the global assumptions: for some constants $\lambda_W, \lambda_V \in \mathbb{R}$ and $C > 0$,

(GM1) $\Omega \subset \mathbb{R}^d$ is convex, i.e., Ω is ∞-prox-regular.
(GA1) $W \in C^1(\mathbb{R}^d)$ is λ_W-geodesically convex on $\operatorname{Conv}(\Omega-\Omega) = \Omega-\Omega$.
(GA2) ∇W has linear growth, i.e., $|\nabla W(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}^d$.
(GA3) $V \in C^1(\mathbb{R}^d)$ is λ_V-geodesically convex on $\operatorname{Conv}(\Omega) = \Omega$.
(GA4) ∇V has linear growth, $|\nabla V(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}^d$.

The main result in this setting reads as:

Theorem 1.9. *Assume Ω is unbounded and satisfies (GM1) and W,V satisfy (GA1)-(GA4), then for any $\mu_0 \in \mathscr{P}_2(\Omega)$, there exists a gradient flow solution $\mu(\cdot)$ with respect to $\mathscr{E}$ such that $\mu(\cdot)$ is a* *weak measure solution to* (1.1)*. Moreover, for a.e.* $t > 0$

$$|\mu'|^2(t) = \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(t,x),$$

and for any $0 \le s \le t < \infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(r,x) dr.$$

Similarly, if $\mu^1(\cdot), \mu^2(\cdot)$ *are two weak measure solutions to (1.1) with initial data* μ_0^1, μ_0^2 *respectively, then*

$$d_W\left(\mu^1(t), \mu^2(t)\right) \le \exp\left(-\left(\lambda_W^- + \lambda_V\right) t\right) d_W\left(\mu_0^1, \mu_0^2\right). \tag{1.15}$$

for any $t \ge 0$*. Also the weak measure solution is characterized by the system of Evolution Variational Inequalities:*

$$\frac{1}{2}\frac{d}{dt} d_W^2\left(\mu(t), \nu\right) + \left(\frac{\lambda_W^-}{2} + \frac{\lambda_V}{2}\right) d_W^2\left(\mu(t), \nu\right) \le \mathscr{E}(\nu) - \mathscr{E}(\mu(t)), \tag{1.16}$$

for a.e. $t > 0$ *and for all* $\nu \in \mathscr{P}_2(\Omega)$.

Since Ω is convex means Ω is ∞-prox-regular, the stability estimate (1.15) and EVI (1.16) in the convex setting are consistent with the estimates in the η-prox-regular setting by taking $\eta = \infty$ in (1.12) and (1.11).

The convexity assumption is needed since on nonconvex unbounded domains we do not know how to control the error due to lack of convexity (as measured by the prox-regularity (1.5)) in the stability of solutions. However, we can show that control assuming compactly supported initial data. Therefore, when Ω is unbounded and the initial data μ_0 has compact support, we assume there exist some constants $\eta > 0, \lambda_W, \lambda_V \in \mathbb{R}, C > 0$ such that the following local assumptions hold

- **(M1)** $\Omega \subset \mathbb{R}^d$ is η-prox-regular.
- **(LA1)** $W \in C^1(\mathbb{R}^d)$ is locally λ-geodesically convex on $\mathbb{R}^d$.
- **(LA2)** ∇W has linear growth, i.e., $|\nabla W(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}^d$.
- **(LA3)** $V \in C^1(\mathbb{R}^d)$ is locally λ-geodesically convex on $\mathbb{R}^d$.
- **(LA4)** ∇V has linear growth, $|\nabla V(x)| \le C(1+|x|)$ for all $x \in \mathbb{R}^d$.

Note that the conditions (LA1) and (LA3) are satisfied whenever V and W are C^2 functions on $\mathbb{R}^d$, which is the case in many practical applications.

We show in this setting the following theorem about existence and stability for weak measure solutions for initial data with compact support.

Theorem 1.10. *Given that* Ω *is unbounded and satisfies (M1), and* W, V *satisfy (LA1)-(LA4). If* $\operatorname{supp}(\mu_0) \subset \Omega$ *is compact, say* $\operatorname{supp}(\mu_0) \subset B(r_0) \cap \Omega$*, then there exists a weak measure solution* $\mu(\cdot)$ *to* (1.1) *such that* $\operatorname{supp}(\mu(t)) \subset B(r(t))$ *for* $r(t) = (r_0+1)\exp(Ct)$*, where* $C = C(W,V)$ *and* $\mu(\cdot)$ *satisfies for a.e.* $t > 0$

$$|\mu'|^2(t) = \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(t,x),$$

and for any $0 \le s \le t < \infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 \, d\mu(r,x) dr.$$

Moreover if we have two such solutions $\mu^i(\cdot)$ with initial data μ_0^i satisfying for $i=1,2$, $\mathrm{supp}(\mu_0^i)$ are compact and $\mathrm{supp}(\mu^i(t)) \subset B(r(t))$ for all $t>0$, then for all $k \in \mathbb{N}$ such that $B(r(t)) \subset K_k$ we have

(1.17)
$$d_W\left(\mu^1(t),\mu^2(t)\right) \le \exp\left(\left(-\lambda_{W,k}^- - \lambda_{V,k} + \frac{\|\nabla W\|_{L^\infty(\Omega_k-\Omega_k)} + \|\nabla V\|_{L^\infty(\Omega_k)}}{\eta}\right)t\right) d_W(\mu_0^1,\mu_0^2).$$

where $\lambda_{W,k},\lambda_{V,k}$ are the geodesic convexity constants of W and V in K_k and $\Omega_k = \Omega \cap K_k$.

Let us point out that we are not able to get the system of Evolution Variational Inequalities in its whole generality although they hold for compactly supported reference measures.

Remark 1.11. Here we illustrate on an example that well-posedness of weak measure solutions cannot hold on domains which have an inside corner. Let $\Omega = \{(r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 : 0 \le r \le 1, \frac{\pi}{4} \le \theta \le \frac{7\pi}{4}\}$ be as in Figure 2. Let $V(x) = -2x_1$ be the external potential and W be any C^2 convex interaction potential with $\nabla W(0) = 0$. Define $\gamma_1(s) = (1,-1)s$ and $\gamma_2(s) = (1,1)s$ for $0 \le s \le 1$. Then for initial datum $\mu_0 = \delta_0$ both $\mu_1(t) = \delta_{\gamma_1(t)}$ and $\mu_2(t) = \delta_{\gamma_2(t)}$ are weak measure solutions. Thus uniqueness and hence stability of solutions cannot hold.

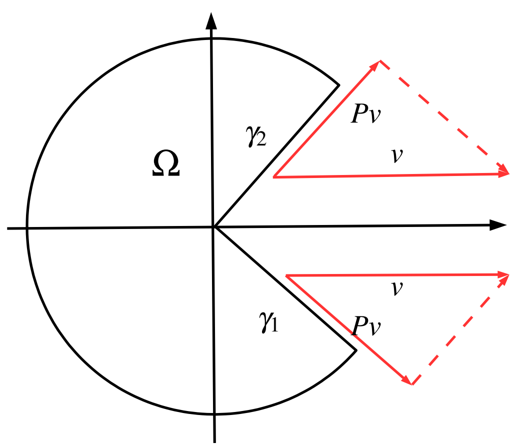

FIGURE 2. The red arrows show the projected velocity field Pv on γ_1 and γ_2, which are driving the particles apart from each other.

1.4. **Strategy of the proof.** The strategy to construct weak measure solutions to (1.1) is to show the existence of gradient flow with respect to $\mathscr{E}$. We approximate the initial data μ_0 in Wasserstein metric by $\mu_0^n = \sum_{i=1}^{k(n)} m_i^n \delta_{x_i^n}$ for $x_i^n \in \Omega \bigcap B(n)$, and solve (1.1) with $\mu^n(0) = \mu_0^n$. Then (1.1) becomes a discrete projected system, for $1 \le i \le k(n)$

$$\begin{cases} \dot{x}_i^n(t) = P_{x_i^n(t)}\left(-\sum_j m_j^n \nabla W\left(x_i^n - x_j^n\right) - \nabla V(x_i^n)\right) \text{ a.e. } t \ge 0, \\ x_i^n(0) = x_i^n \in \Omega, \end{cases} \tag{1.18}$$

which we show based on the well-posedness theory from non-convex sweeping process differential inclusions with perturbations. For the general theory of sweeping processes we refer to [8, 9, 19] and references therein. To be precise, based on [9] there exists a locally absolutely continuous function $[0,\infty) \ni t \mapsto x(t) = (x_1^n(t), \cdots, x_{k(n)}^n(t)) \in \Omega^n = \Omega \times \ldots \times \Omega$, such that for a.e. $t > 0$,

$$-\dot{x}(t) \in N(\Omega^n, x(t)) - v(t, x(t)), \tag{1.19}$$

where $v(t,x(t)) = -\sum_j m_j^n \nabla W\left(x_i^n - x_j^n\right) - \nabla V(x_i^n)$ in our case. We then show that the solution to (1.19) is actually a solution to (1.18).

Next we explore the properties of the sequence of solutions $\{\mu^n(\cdot)\}_n$. In particular,

- When Ω is bounded or Ω is unbounded but convex, we first prove the stability of $\mu^n(t)$

$$d_W(\mu^n(t), \mu^m(t)) \leq \exp(Ct) d_W(\mu_0^n, \mu_0^m), \tag{1.20}$$

  where $C = C(W,V)$ is a constant depending only on W, V. Thus $\mu^n(t)$ converges to some $\mu(t)$ in $\mathscr{P}_2(\Omega)$ as $n \to \infty$. Since μ^n satisfies the energy dissipation inequality,

$$\mathscr{E}(\mu^n(s)) \geq \mathscr{E}(\mu^n(t)) + \frac{1}{2}\int_s^t |(\mu^n)'|^2(r)dr + \frac{1}{2}\int_s^t \int_\Omega |P_x(-\nabla W * \mu^n(r)(x) - \nabla V(x))|^2 d\mu^n(r,x)dr,$$

  by the lower semicontinuity property, we are able to show that $\mu(\cdot)$ also satisfies the desired energy dissipation inequality

$$\mathscr{E}(\mu(s)) \geq \mathscr{E}(\mu(t)) + \frac{1}{2}\int_s^t |\mu'|^2(r)dr + \frac{1}{2}\int_s^t \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 d\mu(r,x)dr. \tag{1.21}$$

  We then show the chain rule, for $\tilde{v}(t)$ is the tangent velocity of $\mu(\cdot)$ at time t

$$\frac{d}{dt}\mathscr{E}(\mu(t)) = \int_\Omega \langle -P_x(-\nabla W * \mu(t)(x) - \nabla V(x)), \tilde{v}(t,x)\rangle d\mu(t,x), \tag{1.22}$$

  which together with the energy dissipation inequality yields that $\mu(\cdot)$ is a gradient flow with respect to $\mathscr{E}$ and a weak measure solution to (1.1).
- When Ω is unbounded and only η-prox-regular, we first show that the support of the solutions $\mu^n(t)$ grows at most exponentially, i.e.

$$\operatorname{supp}(\mu^n(t)) \subset B(r(t)), \tag{1.23}$$

  for $r(t) = (r_0+1)\exp(Ct)$ given that $\operatorname{supp}(\mu_0) \subset B(r_0)$. We then show that, given $\operatorname{supp}(\mu^n(t))$ has the same growth condition for all $n \in \mathbb{N}$, $\mu^n(\cdot)$ still converges to a locally absolutely continuous curve $\mu(\cdot)$ satisfying (1.21) and (1.22). Thus $\mu(\cdot)$ is a weak measure solution to (1.1).

1.5. **Outline.** The paper is organized as follows.

In Section 2, we show the properties of the projection P and then give the existence results for the discrete projected systems (1.18).

In Section 3, under the assumption that Ω is bounded, we prove the stability of solutions to the discrete projected systems $\mu^n(\cdot)$, i.e. (1.20). Thus $\mu^n(\cdot)$ converge to an absolutely continuous curve $\mu(\cdot)$. We show that $\mu(\cdot)$ is curve of maximum slope for the energy $\mathscr{E}$ and moreover a gradient flow solution of (1.1). We then show that $\mu(\cdot)$ is also a weak measure solution and that weak measure solutions satisfy the stability property (1.12). At the end of the section, we show that solutions are characterized by the system of Evolution Variational Inequalities (1.11).

Section 4 addresses the case of unbounded, convex Ω and general initial data $\mu_0 \in \mathscr{P}_2(\Omega)$, that is Theorem 1.9. The proof of Theorem 1.9 is similar to Theorem 1.5 and Theorem 1.6, we only concentrate on the key differences.

Section 5 is devoted to the case when Ω is unbounded and only η-prox-regular with $\operatorname{supp}(\mu_0)$ compact. We show that the support of the solutions to the discrete projected systems (1.18) satisfy exponential growth condition (1.23). By similar stability results as in Section 3, $\mu^n(\cdot)$ still converges to a locally absolutely continuous curve $\mu(\cdot)$ and $\mu(\cdot)$ is a solution to (1.1) with the desired energy dissipation (1.21). We then give the proof of the stability result (1.17) for solutions with control on growth of supports. We end the section by making a remark about well-posedness of (1.1) with time-dependent potentials W, V.

In the last Section 6, we prove Proposition 1.7 and discuss the conditions on the shape of the domain Ω such that there exist interaction potentials W for which solutions $\mu(\cdot)$ of (1.1) aggregate to a singleton (a single delta mass).

2. Existence of solutions to discrete systems

In this section, we first show properties of the projection P, in particular the lower semicontinuity and convexity property of P. Then we give the existence result of solutions to the discrete projected systems (1.18).

Recall that the tangent and normal cones $T(\Omega, x)$ and $N(\Omega, x)$ are closed convex cones by Definition 1.4.

Proposition 2.1. *Suppose Ω satisfies (M1) and $x \in \partial\Omega$. Then for any $v \in \mathbb{R}^d$, there exist a unique orthogonal decomposition $(v_T, v_N) \in T(\Omega, x) \times N(\Omega, x)$ of v:*

$$\langle v_T, v_N \rangle = 0 \ \text{ and } \ v = v_T + v_N.$$

Moreover, $v_T = \operatorname{proj}_{T(\Omega,x)}(v) = P_x(v), v_N = \operatorname{proj}_{N(\Omega,x)}(v)$.

Proposition 2.1 is a direct consequence of Moreau's decomposition theorem, see [14, 16] for the proof.

Proposition 2.2. *Assume Ω satisfies (M1), then the map $\Omega \times \mathbb{R}^d \ni (x, v) \mapsto |P_x(v)|^2$ is lower semicontinuous and for any fixed $x \in \Omega$, $\mathbb{R}^d \ni v \mapsto |P_x(v)|^2$ is convex.*

Proof. We first show the lower semicontinuity property. Let $\{x_n\}_n \subset \Omega, \{v^n\}_n \subset \mathbb{R}^d$ be such that $\lim_{n\to\infty} x_n = x \in \Omega, \lim_{n\to\infty} v^n = v$. If $x_n \in \mathring{\Omega}$ for all n sufficiently large, then $P_{x_n}(v^n) = v^n$ and we have $|P_x(v)|^2 \le |v|^2 = \lim_{n\to\infty} |v^n|^2$. And for any $x \in \mathring{\Omega}$, we have $x_n \in \mathring{\Omega}$ for n sufficiently large, thus

$$\liminf_{n\to\infty} |P_{x_n}(v^n)|^2 \ge |P_x(v)|^2.$$

So we only need to check for $x \in \partial\Omega$ and $\{x_n\}_n \subset \partial\Omega$ such that $\lim_{n\to\infty} x_n = x$. Denote the decomposition of v^n as in Proposition 2.1 by

$$v^n = v^n_T + v^n_N$$

where $v^n_T \in T(\Omega, x_n), v^n_N \in N(\Omega, x_n)$ and $\langle v^n_T, v^n_N \rangle = 0$. For any subsequence, which we do not relabel, such that there exists $w_N \in \mathbb{R}^d$ and $\lim_{n\to\infty} v^n_N = w_N$, we claim that $w_N \in N(\Omega, x)$ and $\langle v - w_N, w_N \rangle = 0$. Indeed, since Ω is η-prox-regular,

$$B_\eta\left(x_n + \eta \frac{v^n_N}{|v^n_N|}\right) \cap \Omega = \emptyset.$$

Taking $n \to \infty$ implies

$$B_\eta\left(x + \eta\frac{w_N}{|w_N|}\right) \cap \Omega = \emptyset,$$

which then implies $w_N \in N(\Omega, x)$. Also by taking $n \to \infty$ in $\langle v^n - v_N^n, v_N^n\rangle = 0$ we get $\langle v - w_N, w_N\rangle = 0$. We then know

$$\begin{aligned}
|P_x(v)|^2 &= |v_T|^2 \\
&= |v - v_N|^2 \\
&\le |v - w_N|^2 \\
&= \lim_{n\to\infty} |v^n - v_N^n|^2 \\
&= \lim_{n\to\infty} |P_{x_n}(v^n)|^2
\end{aligned}$$

So

$$\liminf_{n\to\infty} |P_{x_n}(v^n)|^2 \ge |P_x(v)|^2.$$

We turn to the convexity property. For any fixed $x \in \Omega$, if $x \in \mathring{\Omega}$ then $P_x(v) = v$ for all $v \in \mathbb{R}^d$ and $v \mapsto |v|^2$ is convex. Now for fixed $x \in \partial\Omega$, and any $v^1, v^2 \in \mathbb{R}^d, 0 \le \theta \le 1$, denote the unique projection of v^1, v^2 defined in Proposition 2.1 by

$$v^i = v_N^i + v_T^i$$

for $i = 1, 2$. Then

$$(1-\theta)v^1 + \theta v^2 = \left((1-\theta)v_T^1 + \theta v_T^2\right) + \left((1-\theta)v_N^1 + \theta v_N^2\right).$$

Note that $(1-\theta)v_T^1 + \theta v_T^2 \in T(\Omega, x)$ and $(1-\theta)v_N^1 + \theta v_N^2 \in N(\Omega, x)$, by Proposition 2.1 we have

$$\begin{aligned}
|P_x\left((1-\theta)v^1 + \theta v^2\right)|^2 &\le |(1-\theta)v_T^1 + \theta v_T^2|^2 \\
&\le (1-\theta)|v_T^1|^2 + \theta|v_T^2|^2 \\
&= (1-\theta)|P_x\left(v^1\right)|^2 + \theta|P_x\left(v^2\right)|^2.
\end{aligned}$$

Convexity is verified. □

We cite the following result from [8, 9] about the existence of differential inclusions

Theorem 2.3. *Assume that S is η-prox-regular as defined in Definition 1.3 and $F : \mathbb{R}^d \ni x \mapsto F(x) \in \mathbb{R}^d$ is a continuous function with at most linear growth, i.e., there exists some constant $C > 0$ such that*

$$|F(x)| \le C(1 + |x|).$$

Then the differential inclusion

$$\begin{cases} -\dot{x}(t) \in N(S, x(t)) + F(x(t)) \text{ a.e. } t \ge 0, \\ x(0) = x_0 \in S. \end{cases} \tag{2.1}$$

has at least one locally absolutely continuous solution.

Note that the theorems, for example Theorem 5.1 from [9], are more general than Theorem 2.3. However, we only need the simplified version for our purpose. We also notice that (2.1) implies that $x(t) \in S$ for all $t \geq 0$. Indeed, since $N(S,x) = \emptyset$ for all $x \notin S$ we know $x(t) \in S$ for a.e. $t \geq 0$. Then the continuity of $x(t)$ and the fact that S is closed imply that $x(t) \in S$ for all $t \geq 0$. For completeness, we give a sketch of proof here.

Proof. For $T < \frac{1}{2C}$ where C is constant in the growth condition of F. For $n \in \mathbb{N}$, take the partition $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ and define $\delta_i^n = t_{i+1}^n - t_i^n, x_0^n = x_0, Z_0^n = F(x_0^n)$. Then define iteratively for $0 \leq i \leq n-1$

$$x_{i+1}^n = \mathrm{proj}_S(x_i^n - \delta_i^n Z_i^n)$$

and

$$Z_{i+1}^n = F(x_{i+1}^n).$$

Note that we have then

$$\|x_{i+1}^n\| \leq \|x_i^n\| + 2\delta_i^n \|Z_i^n\|$$

and

$$\|Z_i^n\| \leq C(1 + \|x_i^n\|).$$

Thus

$$\begin{aligned} \|x_{i+1}^n\| &\leq \|x_0\| + \sum_{j=0}^{i} 2\delta_j^n C(1 + \|x_j^n\|) \\ &\leq \|x_0\| + 2CT(1 + \max_{0 \leq j \leq i} \|x_j^n\|), \end{aligned}$$

which implies

$$\max_{0 \leq i \leq n} \|x_i^n\| \leq \|x_0\| + 2CT(1 + \max_{0 \leq i \leq n} \|x_i^n\|).$$

Since $2CT < 1$ we have uniformly in n

$$\max_{0 \leq i \leq n} \|x_i^n\| \leq \frac{\|x_0\| + 2CT}{1 - 2CT} < \infty,$$

and

$$\max_{0 \leq i \leq n} \|Z_i^n\| \leq C(1 + \max_{0 \leq i \leq n} \|x_i^n\|) < \infty.$$

We now define the approximation solution by

$$x_n(t) = u_i^n + \frac{x_{i+1}^n - x_i^n + \delta_i^n Z_i^n}{\delta_i^n} - (t - t_i^n) Z_i^n,$$

for $t_i^n \leq t < t_{i+1}^n$. Notice that x_n can also be written as

$$x_n(t) = x_0 + \int_0^t [\Pi_n(s) - Z_n(s)] ds \tag{2.2}$$

where

$$\Pi_n(t) = \sum_{i=0}^{n} \frac{x_{i+1}^n - x_i^n + \delta_i^n Z_i^n}{\delta_i^n} \chi_{(t_i^n, t_{i+1}^n]}(t)$$

and $Z_n(t) = Z_i^n$ for $t_i^n \leq t < t_{i+1}^n$. We have for a.e. $t \in [t_i^n, t_{i+1}^n)$

$$\dot{x}_n(t) + Z_n(t) = \Pi_n(t) \in N(S, x_n(t_i^n)).$$

Since $\|\Pi_n(t)\| \leq \|Z_i^n\|$ for $t \in (t_i^n, t_{i+1}^n]$, we know there exists a subsequence of n, which we do not relabel, such that

$$\Pi_n \rightharpoonup \Pi, \quad Z_n \rightharpoonup Z \qquad \text{as } n \to \infty$$

weakly in $L^2[0,T]$. We then have by (2.2) that x_n converges locally uniformly to x with

$$x(t) = x_0 + \int_0^t [\Pi(s) - Z(s)]ds.$$

We now claim that $x(t)$ is a solution to the differential inclusion on $[0,T]$. First we check that $x(t) \in S$ for all $t \in [0,T]$. Since

$$\|x_n(t_i^n) - x(t)\| \leq \|x_n(t) - x(t)\| + c|t_i^n - t|,$$

$x(t) = \lim_{n\to\infty} x_n(t_i^n) \in S$. We then verify that $\dot{x}(t) + Z(t) \in -N(S, x(t))$ for a.e. $t \in [0,T]$. Since $\dot{x}_n + Z_n = \Pi_n \rightharpoonup \Pi$ weakly in $L^2([0,T])$ and $\Pi_n(t) \in N(S, x_n(t_i^n))$ for $t_i^n < t \leq t_{i+1}^n$, by Mazur's lemma, for a.e. $t \in [0,T]$

$$\dot{x}(t) + Z(t) \in \bigcap_n \{\dot{x}_k(t) + Z_k(t) : k \geq n\}.$$

Then by Proposition 2.1 from [9], we know for a.e. $t \in [0,T]$,

$$\dot{x}(t) + Z(t) \in N(S, x(t)).$$

Now we only need to check that $Z(t) = F(x(t))$. We know that $Z_n(t) = F(x^n(t_i^n)$ for $t_i^n \leq t < t_{i+1}^n$. Define $\tilde{u}_n$ by $\tilde{x}_n(t) = x^n(t_i^n)$ for $t_i^n \leq t < t_{i+1}^n$ and note $Z_n(t) = F(x^n(t_i^n) = F(\tilde{x}_n(t))$. Then $\tilde{x}_n$ converges locally uniformly to x. Together with the fact that F is continuous, $F(\tilde{x}_n)$ converges to $F(x)$ in $L^2([0,T])$. Since it is direct to check Z_n converges weakly to Z in $L^2([0,T])$, we get $Z(t) = F(x(t))$ for a.e. $t \in [0,T]$. The claim is proved. □

We now show that the solutions for the differential inclusions are actually solutions for the projected systems.

Lemma 2.4. *Assume that S is η-prox-regular by Definition 1.3 and $x(t)$ is a locally absolutely continuous solution to the differential inclusion (2.1). Then*

$$\dot{x}(t) = P_{x(t)}\left(-F\left(x(t)\right)\right) \textit{ a.e. } t \geq 0. \tag{2.3}$$

Proof. Since S is η-prox-regular, it is tangentially regular, that is

$$T(S,x) = K(S,x)$$

where $T(S,x)$ is defined in Definition 1.4 and $K(S,x)$ is the contingent cone defined as

$$K(S,x) = \{v \in \mathbb{R}^d : \exists t_n \searrow 0 \; \exists v_n \to v \text{ s.t. } (\forall n)\, x + t_n v_n \in S\}.$$

We refer to [4] for the details. Now note that for a.e. t

$$\dot{x}(t) = \lim_{h\to 0^+} \frac{x(t+h) - x(t)}{h} \in K(S, x(t))$$

and

$$\dot{x}(t) = \lim_{h\to 0^-} \frac{x(t+h) - x(t)}{h} \in -K(S, x(t)).$$

Thus $\langle \dot{x}(t), n(x(t)) \rangle = 0$ for any $n(x(t)) \in N(S, x(t))$. From the differential inclusion (2.1),we know that $-F(x(t)) = \dot{x}(t) + n(x(t))$ for some $n(x(t)) \in N(S, x(t))$. Together with fact that $\dot{x}(t) \in T(S, x(t))$ and $\langle \dot{x}(t), n(x(t)) \rangle = 0$, by Proposition 2.1

$$\dot{x}(t) = P_{x(t)}\left(-F\left(x(t)\right)\right),$$

as claimed. $\square$

We turn to the existence of solutions to the discrete projected system (1.18), which we write as

$$\begin{cases} \dot{x}_i(t) = P_{x_i(t)}\left(v(x^n(t))\right), \\ x_i(0) = x_i \in S. \end{cases} \tag{2.4}$$

for $i = 1, \cdots, n$. For that purpose we apply Theorem 2.3 and Lemma 2.4 for $S = \Omega^n$ and $x_0 = (x_1, \cdots, x_n)$ with $F(x) = (-v_1(x(t)), \cdots, -v_n(x(t)))$, where $v_i(x(t)) = -\nabla W * \mu(t)(x_i(t)) - \nabla V(x_i(t)) = -\sum_{j=1}^n m_j \nabla W\left(x_i(t) - x_j(t)\right) - \nabla V(x_i(t))$. To do that, we first check that Ω^n is η-prox-regular.

Proposition 2.5. *If $\Omega \subset \mathbb{R}^d$ is η-prox-regular by Definition 1.3, then*

$$\Omega^n = \{(x_1, \cdots, x_n) : x_i \in \Omega,\ i = 1, \ldots, n\}$$

is η-prox-regular; Also for any $x = (x_1, \cdots, x_n) \in \Omega^n$ we have

$$N(\Omega^n, x) = N(\Omega, x_1) \times \cdots \times N(\Omega, x_n).$$

Proof. To see Ω^n is also η-prox-regular, first it is direct that Ω^n is a closed set. Now for any $x = (x_1, \cdots, x_n) \in \partial\Omega^n$ and $v = (v^1, \cdots, v^n) \in N(\Omega^n, x)$, by Definition 1.4 there exists $\alpha > 0$ such that

$$x \in P_{\Omega^n}\left(x + \alpha v\right),$$

which implies

$$x_i \in P_\Omega\left(x_i + \alpha v^i\right)$$

for $1 \le i \le n$. By the equivalent definition of η-prox-regularity of Ω (1.5), we then have

$$\langle v^i, y_i - x_i \rangle \le \frac{|v^i|}{2\eta} |y_i - x_i|^2$$

for any $y_i \in \Omega$. Thus

$$\begin{aligned} \langle v, y - x \rangle &= \sum_{i=1}^n \langle v^i, y_i - x_i \rangle \\ &\le \sum_{i=1}^n \frac{|v^i|}{2\eta} |y_i - x_i|^2 \\ &\le \frac{|v|}{2\eta} |y - x|^2, \end{aligned}$$

for any $y = (y_1, \cdots, y_n) \in \Omega^n$. Thus Ω^n is η-prox-regular by (1.5). We now turn to the relations between the normal cones. For $x = (x_1, \cdots, x_n) \in \Omega^n$ and $v = (v^1, \cdots, v^n)$

$$\begin{aligned} v \in N(\Omega^n, x) &\Leftrightarrow \exists \alpha > 0 \text{ s.t. } x \in P_{\Omega^n}\left(x + \alpha v\right) \\ &\Leftrightarrow x_i \in P_\Omega\left(x_i + \alpha v^i\right),\ i = 1, \ldots, n \\ &\Leftrightarrow v_i \in N(\Omega, x_i),\ i = 1, \ldots, n. \end{aligned}$$

Thus $N(\Omega^n, x) = N(\Omega, x_1) \times \cdots \times N(\Omega, x_n)$. $\square$

Now we give the main result regarding the existence of solutions to projected discrete systems.

Theorem 2.6. *Assume that Ω is η-prox-regular by Definition 1.3. If either Ω is bounded and W,V satisfy (A1)-(A2) or Ω is unbounded and W,V satisfy (GA2) and (GA4) i.e. (LA2) and (LA4), then for any $n \in \mathbb{N}$ and any $(x_1,\cdots,x_n) \in \Omega^n, (m_1,\cdots,m_n) \in \mathbb{R}^n$ with $m_i \geq 0, \sum_{i=1}^n m_i = 1$, the projected discrete system*

$$
\begin{cases} \dot{x}_i(t) = P_{x_i(t)}\left(v_i(x(t))\right), \\ x_i(0) = x_i \in \Omega, \end{cases} \tag{2.5}
$$

for $i = 1,\cdots,n$, where $v_i(x(t)) = -\nabla W * \mu(t)(x_i(t)) - \nabla V(x_i(t)) = -\sum_{j=1}^n m_j \nabla W(x_i(t)-x_j(t)) - \nabla V(x_i(t))$, has a locally absolutely continuous solution.

Proof. We just need to check the conditions for Theorem 2.3 to apply. We already know that Ω^n is η-prox-regular. If Ω is bounded and W,V satisfy (A1)-(A2), then the mapping $\Omega^n \ni y = (y_1,\cdots,y_n) \mapsto F(y) = (\nabla W * \mu(y_1) + \nabla V(y_1),\cdots,\nabla W * \mu(y_n) + \nabla V(y_n))$ where $\mu = \sum_{i=1}^n m_i \delta_{y_i}$, is continuous and bounded. Extend F to $\mathbb{R}^{dn}$ so that F is still continuous and bounded. Then by Theorem 2.3 there exists an absolutely continuous solution to the differential inclusion

$$
\begin{cases} -\dot{x}(t) \in N(\Omega^n, x(t)) + F(x(t)), \\ \ \ x(0) = (x_1,\cdots,x_n) \in \Omega^n. \end{cases} \tag{2.6}
$$

Similarly, if Ω is unbounded and $\nabla W, \nabla V$ satisfy liner growth conditions (GA2) and (GA4), then the mapping $\mathbb{R}^{dn} \ni y = (y_1,\cdots,y_n) \mapsto F(y) = (\nabla W * \mu(y_1) + \nabla(y_1),\cdots,\nabla W * \mu(y_n) + \nabla V(y_n))$ where $\mu = \sum_{i=1}^n m_i \delta_{y_i}$, is continuous and has linear growth on $\mathbb{R}^{dn}$. By Theorem 2.3, we still have an absolutely continuous solution to (2.6).
Now consider (2.6) in components yields for $1 \leq i \leq n$ and $v_i(x) = -\sum_{j=1}^n \nabla W(x_i - x_j)m_j - \nabla V(x_i)$,

$$
\begin{cases} -\dot{x}_i(t) \in N(\Omega, x_i(t)) - v_i(x(t)), \\ \ \ x_i(0) = x_i \in \Omega. \end{cases}
$$

Then similar argument as in Lemma 2.4 gives

$$
\begin{cases} \dot{x}_i(t) = P_{x_i(t)}\left(v_i(x(t))\right), \\ x_i(0) = x_i \in \Omega. \end{cases}
$$

□

3. Existence and stability of solutions with Ω bounded

In this section, we show the existence and stability of solutions to (1.1) for the case when Ω is bounded, prox-regular and W,V satisfy (A1)-(A2).

We approximate $\mu_0 \in \mathscr{P}_2(\Omega)$ by $\mu_0^n = \sum_{i=1}^{k(n)} m_i^n \delta_{x_i^n}$ such that $x_i^n \in \Omega$ and $\lim_{n\to\infty} d_W\left(\mu_0, \mu_0^n\right) = 0$. By Theorem 2.6, for each $n \in \mathbb{N}$ there exists a a locally absolutely continuous solution to

$$
\begin{cases} \dot{x}_i^n(t) = P_{x_i^n(t)}\left(v_i^n(x(t))\right), \quad 1 \leq i \leq k(n) \\ x_i^n(0) = x_i^n \in \Omega, \end{cases} \tag{3.1}
$$

for $t \geq 0$, where

$$
v_i^n\left(x(t)\right) = -\nabla W * \mu^n(t)(x_i^n(t)) - \nabla V(x_i^n(t)) = -\sum_{j=1}^{k(n)} m_j^n \nabla W\left(x_i^n(t) - x_j^n(t)\right) - \nabla V\left(x_i^n(t)\right)
$$

and $\mu^n(t)=\sum_{j=1}^{k(n)} m_j^n \delta_{x_j^n(t)}$. It is a straightforward calculation to see that for any $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\phi(x)d\mu^n(t,x)=\int_{\mathbb{R}^d}\langle\nabla\phi(x),P_x(v^n(t,x))\rangle\,d\mu^n(t,x).$$

Thus $\mu^n(t)$ satisfies

$$\partial_t\mu^n(t,x)+\operatorname{div}(\mu^n(t,x)P_x(v^n(t,x)))=0,$$

in the sense of distributions for $v^n(t,x)=-\nabla W*\mu^n(t)(x)-\nabla V(x)$.

The following proposition contains the key estimate on the stability of solutions in the discrete case. In particular it shows how the stability in Wasserstein metric d_W defined in (1.3) is affected by the lack of convexity of the domain.

Proposition 3.1. *Assume that Ω is bounded and satisfies (M1), W,V satisfy (A1) and (A2). Then for two solutions $\mu^n(\cdot)$ and $\mu^m(\cdot)$ to the discrete system with different initial data μ_0^n,μ_0^m, we have for all $t\geq 0$*

$$d_W(\mu^n(t),\mu^m(t))\leq \exp\left(\left(-\lambda_W^- -\lambda_V+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right)t\right)d_W(\mu_0^n,\mu_0^m). \tag{3.2}$$

Proof. Note that $\mu^n(\cdot)$ is solution to the continuity equation

$$\partial_t\mu^n(t,x)+\operatorname{div}(\mu^n(t,x)P_x(v^n(t,x)))=0, \tag{3.3}$$

for $v^n(t,x)=-\nabla W*\mu^n(t)(x)-\nabla V(x)$. Since the discrete solutions may have different numbers of particles we use a transportation plan to relate them. Let $\gamma_t\in\Gamma_o(\mu^n(t),\mu^m(t))$ be the optimal plan between μ^m and μ^n defined in (1.4). By Theorem 8.4.7 and Lemma 4.3.4 from [2]

$$\frac{1}{2}\frac{d}{dt}d_W^2(\mu^n(t),\mu^m(t))=\int_\Omega\langle P_x(v^n(t,x))-P_y(v^m(t,y)),x-y\rangle\,d\gamma_t(x,y). \tag{3.4}$$

We first establish the contractivity the solutions would have if the boundary conditions were not present and then account for the change due to velocity projection at the boundary. For v^n,v^m, by (A1) and (A2), that is the convexity of W and V,

$$\begin{aligned}
&\int_{\Omega\times\Omega}\langle v^n(t,x)-v^m(t,y),x-y\rangle d\gamma_t(x,y)\\
&=\int_{\Omega\times\Omega}\langle-\nabla W*\mu^n(t)(x)-\nabla V(x)+\nabla W*\mu^m(t)(y)-\nabla V(y),x-y\rangle d\gamma_t(x,y)\\
&=\frac{1}{2}\int_{\Omega\times\Omega}\int_{\Omega\times\Omega}\langle-\nabla W(x-z)+\nabla W(y-w),x-y-z+w\rangle d\gamma_t(z,w)d\gamma_t(x,y)\\
&+\int_{\Omega\times\Omega}\langle-\nabla V(x)+\nabla V(y),x-y\rangle d\gamma_t(x,y)\\
&\leq-\frac{1}{2}\lambda_W\int_{\Omega\times\Omega}\int_{\Omega\times\Omega}|x-z-y+w|^2d\gamma_t(z,w)d\gamma_t(x,y)-\lambda_V\int_{\Omega\times\Omega}|x-y|^2d\gamma_t(x,y) \qquad (3.5)\\
&\leq\left(-\lambda_W^- -\lambda_V\right)\int_{\Omega\times\Omega}|x-y|^2d\gamma_t(x,y)\\
&=\left(-\lambda_W^- -\lambda_V\right)d_W^2(\mu^n(t),\mu^m(t)).
\end{aligned}$$

For the boundary effect, by the fact that Ω is η-prox-regular we have (1.5), thus

$$\begin{aligned}
&\int_{\Omega\times\Omega}\langle P_x(v^n(t,x))-v^n(t,x)-P_y(v^m(t,y))+v^m(t,y),x-y\rangle d\gamma_t(x,y)\\
&\le\int_{\Omega\times\Omega}\frac{\|v^n(t)\|_{L^\infty(\Omega)}+\|v^m(t)\|_{L^\infty(\Omega)}}{2\eta}|y-x|^2d\gamma_t(x,y)\\
&=\frac{\|v^n(t)\|_{L^\infty(\Omega)}+\|v^m(t)\|_{L^\infty(\Omega)}}{2\eta}d_W^2(\mu^n(t),\mu^m(t)).
\end{aligned}\tag{3.6}$$

Notice that $v^i(x)=-\nabla W(x)*\mu^i(t)(x)-\nabla V(x)$ implies that for $i=n,m$

$$\|v^i\|_{L^\infty(\Omega)}\le\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}<\infty.$$

Plugging back into (3.4) we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}d_W^2(\mu^n(t),\mu^m(t))&=\int_\Omega\langle P_x(v^n(t,x))-P_y(v^m(t,y)),x-y\rangle d\gamma_t(x,y)\\
&=\int_{\Omega\times\Omega}\langle v^n(t,x)-v^m(t,y),x-y\rangle d\gamma_t(x,y)\\
&+\int_{\Omega\times\Omega}\langle P_x(v^n(t,x))-v^n(t,x)-P_y(v^m(t,y))+v^m(t,y),x-y\rangle d\gamma_t(x,y)\\
&\le\left(-\lambda_W^--\lambda_V+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right)d_W^2(\mu^n(t),\mu^m(t)).
\end{aligned}$$

By Gronwall's inequality, we know (3.2) for all $t\ge0$. □

Since $n\to\infty, d_W(\mu_0^n,\mu_0)\to0$, by Proposition 3.1 the solutions μ_n of (3.1) form a Cauchy sequence in n, with respect to Wasserstein metric. Thus

$$\mu^n(t)\xrightarrow{d_W}\mu(t)\quad\text{as } n\to\infty,\tag{3.7}$$

for some $\mu(t)\in\mathscr{P}_2(\Omega)$.

Remark 3.2. Our goal is to show that $\mu(\cdot)$ is a weak measure solution of (1.1). The most immediate idea would be to try to pass to limit directly in Definition 1.1. However note that since P_x is not continuous in x and thus the velocity field governing the dynamics is not continuous (at the boundary of Ω). Given that μ_n converge to μ only in the weak topology of measures, the lack of continuity of velocities prevents us to directly pass to limit in the integral formulation given in Definition 1.1. To show that $\mu(\cdot)$ is a weak measure solution of (1.1) we use the theory of gradient flows in the spaces of probability measures $\mathscr{P}_2(\Omega)$. Namely, we establish that $\mu(\cdot)$ satisfies the steepest descent property with respect to the total energy $\mathscr{E}$ defined in (1.2) by showing $\mu^n(\cdot)$ satisfies such property and the property is stable under the weak topology of measures (convergence in the Wasserstein metric d_W).

We turn to the introducing the elements of the theory of gradient flows in the space of probability measures.

Definition 3.3. Let $\mu\in\mathscr{P}_2(\Omega)$, a vector field ξ on Ω is said to be in the *subdifferential* of $\mathscr{E}$ at μ if $\xi\in L^2(\mu)$, i.e.

$$\int_\Omega|\xi(x)|^2d\mu(x)<\infty,$$

and

$$\mathscr{E}(\nu)-\mathscr{E}(\mu)\geq \inf_{\gamma\in\Gamma_o(\mu,\nu)}\int_{\Omega\times\Omega}\langle\xi(x),y-x\rangle d\gamma(x,y)+o\left(d_W\left(\mu,\nu\right)\right)$$

for any $\nu\in\mathscr{P}_2(\Omega)$.

Given a locally absolutely continuous curve $[0,\infty)\ni t\mapsto\mu(t)\in\mathscr{P}_2(\Omega)$, for the metric derivative $|\mu'|$ defined in (1.7) we have the following

Theorem 3.4. *There exists a unique Borel vector field $v(t)\in L^2(\mu(t))$ such that $\mu(\cdot)$ satisfies*

$$\partial_t\mu(t)+\operatorname{div}\left(\mu(t)v(t)\right)=0$$

in the sense of distributions and $v(t)$ satisfies

$$|\mu'|^2(t)=\int_\Omega|v(t,x)|^2d\mu(t,x),$$

for a.e. $t>0$.

For the proof of the theorem, refer to Theorem 8.3.1 from [2]. We call the unique vector field $v(t)$ the tangent velocity field and define

Definition 3.5. A locally absolutely continuous curve $[0,\infty)\ni t\mapsto\mu(t)\in\mathscr{P}_2(\Omega)$ is a *gradient flow* with respect to the energy functional $\mathscr{E}$ if for a.e. $t>0$

$$v(t)\in-\partial\mathscr{E}(\mu(t)),$$

where $v(t)$ is the tangent velocity field for $\mu(t)$.

We now show that $\mu(t)$ is a curve of maximal slope with respect to $\mathscr{E}$.

Theorem 3.6. *$\mu(t)$ satisfies for any $0\leq s<t<\infty$*

$$\mathscr{E}\left(\mu(s)\right)\geq\mathscr{E}\left(\mu(t)\right)+\frac{1}{2}\int_s^t|\mu'|^2(r)dr+\frac{1}{2}\int_s^t\int_\Omega\left|P_x\left(v(r,x)\right)\right|^2d\mu(r,x)dr, \tag{3.8}$$

where $v(r,x)=-\int_\Omega\nabla W(x-y)d\mu(r,y)-\nabla V(x)$.

Before proving the theorem, we need the following

Lemma 3.7. *Assume (M1) holds for Ω and $\nu^n\in\mathscr{P}_2(\Omega)$ converges narrowly to $\nu\in\mathscr{P}_2(\Omega)$ with $\sup_n\int_\Omega|x|^2d\nu^n(x)<\infty$, then*

$$\int_\Omega\left|P_x\left(v(x)\right)\right|^2d\nu(x)\leq\liminf_{n\to\infty}\int_\Omega\left|P_x\left(v^n(x)\right)\right|^2d\nu^n(x), \tag{3.9}$$

where $v^n(x)=-\int_\Omega\nabla W(x-y)d\nu^n(y)-\nabla V(x)$ and $v(x)=-\int_\Omega\nabla W(x-y)d\nu(y)-\nabla V(x)$.

Proof. Similar argument as in Lemma 2.7 from [5] yields that $\nabla W*\nu^n$ converges weakly to $\nabla W*\nu$, i.e., for any $\phi\in C_b^0\left(\mathbb{R}^d\right)$

$$\lim_{n\to\infty}\int_\Omega\nabla W*\nu^n(x)\cdot\phi(x)d\nu^n(x)=\int_\Omega\nabla W*\nu(x)\cdot\phi(x)d\nu(x).$$

Then by Proposition 2.2 we proved in Section 2 and Proposition 6.42 from [10], we know that there exist two sequences of bounded continuous functions a_i,b_i such that for all $x\in\Omega, v\in\mathbb{R}^d$

$$\left|P_x\left(v\right)\right|^2=\sup_{i\in\mathbb{N}}\left\{a_i(x)+b_i(x)\cdot v\right\}.$$

Thus

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega |P_x(v^n(x))|^2 d\nu^n(x) &= \liminf_{n\to\infty}\int_\Omega \sup_i \{a_i(x)+b_i(x)\cdot v^n(x)\}\,d\nu^n(x)\\ &\geq \liminf_{n\to\infty}\int_\Omega (a_i(x)+b_i(x)(-\nabla W*\nu^n(x)-\nabla V(x)))\,d\nu^n(x)\\ &= \int_\Omega (a_i(x)+b_i(x)(-\nabla W*\nu(x)-\nabla V(x)))\,d\nu(x).\end{aligned}$$

Taking supremum over $i\in\mathbb{N}$ and using Lebesgue's monotone convergence theorem then gives

$$\begin{aligned}\liminf_{n\to\infty}\int_\Omega |P_x(v^n(x))|^2 d\nu^n(x) &\geq \sup_{i\in\mathbb{N}}\int_\Omega (a_i(x)+b_i(x)(\nabla W*\nu(x)+\nabla V(x)))\,d\nu(x)\\ &= \int_\Omega |P_x(v(x))|^2 d\nu(x).\end{aligned}$$

□

We now start to prove the theorem

Proof of Theorem 3.6. We first show that the map $t\mapsto \mathscr{E}(\mu^n(t))$ is locally absolutely continuous. Indeed, for $0\leq s<t<\infty$

$$\begin{aligned}&|\mathscr{E}(\mu(t))-\mathscr{E}(\mu(s))|\\ &=\left|\sum_{i=1}^{k(n)} m_i^n\left(V(x_i^n(t))-V(x_i^n(s))\right)+\frac{1}{2}\sum_{i,j=1}^{k(n)} m_i^n m_j^n\left(W(x_i^n(t)-x_j^n(t))-W(x_i^n(s)-x_j^n(s))\right)\right|\\ &\leq \sum_{i=1}^{k(n)} m_i^n\left|V(x_i^n)-V(x_j^n)\right|+\frac{1}{2}\sum_{i,j=1}^{k(n)} m_i^n m_j^n\left|W(x_i^n(t)-x_j^n(t))-W(x_i^n(s)-x_j^n(s))\right|\\ &\leq \sum_{i=1}^{k(n)} m_i^n\|\nabla V\|_{L^\infty(\mathrm{Conv}(\Omega))}|x_i^n(t)-x_i^n(s)|+\sum_{i=1}^{k(n)} m_i^n\|\nabla W\|_{L^\infty(\mathrm{Conv}(\Omega-\Omega))}|x_i^n(t)-x_i^n(s)|\\ &\leq \left(\|\nabla V\|_{L^\infty(\mathrm{Conv}(\Omega))}+\|\nabla W\|_{L^\infty(\mathrm{Conv}(\Omega-\Omega))}\right)\sum_{i=1}^{k(n)} m_i^n|x_i^n(t)-x_i^n(s)|.\end{aligned}\tag{3.10}$$

Thus $t\mapsto\mathscr{E}(\mu(t))$ is locally absolutely continuous since $t\mapsto x_i^n(t)$ is locally absolutely continuous.

Since $\mu^n(\cdot)$ are solutions to the discrete systems, it is direct to calculate that

$$\frac{d}{dt}\mathscr{E}(\mu^n(t))=-\int_\Omega |P_x(v^n(t,x))|^2 d\mu^n(t,x),$$

and $|(\mu^n)'|^2(t)\leq\int_\Omega |P_x(v^n(t,x))|^2 d\mu^n(t,x)$ for a.e. $t>0$. Combining with the fact that $t\mapsto\mathscr{E}(\mu^n(t))$ is locally absolutely continuous then gives,

$$\mathscr{E}(\mu^n(s))\geq\mathscr{E}(\mu^n(t))+\frac{1}{2}\int_s^t|(\mu^n)'|^2(r)dr+\frac{1}{2}\int_s^t\int_\Omega|P_x(v^n(r,x))|^2 d\mu^n(r,x)dr.\tag{3.11}$$

Note that Ω is bounded, $W,V\in C^1(\mathbb{R}^d)$ and $\lim_{n\to\infty} d_W(\mu^n(r),\mu(r))=0$ for any $0\leq r<\infty$, we get

$$\lim_{n\to\infty}\mathscr{E}(\mu^n(r))=\mathscr{E}(\mu(r)).$$

Also by Lemma 3.7, for any $0 \le r < \infty$

$$\liminf_{n\to\infty} \int_\Omega |P_x(v^n(r,x))|^2 d\mu^n(r,x) \ge \int_\Omega |P_x(v(r,x))|^2 d\mu(r,x).$$

By Fatou's lemma, we then have

$$\liminf_{n\to\infty} \int_s^t \int_\Omega |P_x(v^n(r,x))|^2 d\mu^n(r,x)dr \ge \int_s^t \int_\Omega |P_x(v(r,x))|^2 d\mu(r,x)dr. \tag{3.12}$$

We now claim that

$$\liminf_{n\to\infty} \int_s^t |(\mu^n)'|^2(r)dr \ge \int_s^t |\mu'|^2(r)dr. \tag{3.13}$$

To see that, first notice that $\sup_n \int_s^t |(\mu^n)'|^2(r)dr < \infty$, so $|(\mu^n)'| \in L^2([s,t])$ and converges weakly in $L^2([s,t])$ to some function A as $n \to \infty$. We then have for any $0 \le s \le S \le T \le t < \infty$

$$\begin{aligned} d_W(\mu(S),\mu(T)) &= \lim_{n\to\infty} d_W(\mu^n(S),\mu^n(T)) \\ &\le \liminf_{n\to\infty} \int_S^T |(\mu^n)'|(r)dr \\ &= \int_S^T A(r)dr. \end{aligned}$$

Thus we have

$$|\mu'|(r) \le A(r)$$

for $s \le r \le t$, which then implies

$$\begin{aligned} \int_s^t |\mu'|^2(r)dr &\le \int_s^t A^2(r)dr \\ &\le \liminf_{n\to\infty} \int_s^t |(\mu^n)'|^2(r)dr. \end{aligned}$$

The claim is proved. Now take $n \to \infty$ in (3.11)gives

$$\mathscr{E}(\mu(s)) \ge \mathscr{E}(\mu(t)) + \frac{1}{2}\int_s^t |\mu'|^2(r)dr + \frac{1}{2}\int_s^t \int_\Omega |P_x(v(r,x))|^2 d\mu(r,x)dr,$$

as desired. $\square$

Note that as a byproduct of the proof, we obtain that $\mu(\cdot)$ is a locally absolutely continuous curve in $\mathscr{P}_2(\Omega)$. We now show the proof of the main Theorem 1.5

Proof of Theorem 1.5. Since $\mu(\cdot) \in \mathscr{P}_2(\Omega)$ is locally absolutely continuous, by Theorem 8.31 from [2], there exists a unique Borel vector field $\tilde{v}$ such that the continuity equation

$$\partial_t \mu(t) + \operatorname{div}(\mu(t)\tilde{v}(t)) = 0, \tag{3.14}$$

in the sense of distributions, i.e., tested against all $\phi \in C_c^\infty\left([0,\infty)\times\mathbb{R}^d\right)$, and

$$\int_\Omega |\tilde{v}(t,x)|^2 d\mu(t,x) = |\mu'|^2(t),$$

for a.e. $t \ge 0$. Then by Proposition 8.4.6 from [2], for a.e. $t > 0$

$$\lim_{h\to 0}\left(\pi^1, \frac{1}{h}\left(\pi^2 - \pi^1\right)\right)_\sharp \gamma_t^h = (\mathrm{Id}\times\tilde{v}(t))_\sharp \mu(t), \tag{3.15}$$

in $(\mathscr{P}_2(\Omega), d_W)$ for any $\gamma_t^h \in \Gamma_o(\mu(t), \mu(t+h))$. Here we also need the following stronger convergence: Denote the disintegration of γ_t^h with respect to $\mu(t)$ by ν_x^h, then as $h \to 0$, $\int_\Omega \frac{y-\cdot}{h} d\nu_\cdot^h(y)$ converges to the vector field $\tilde{v}(t,\cdot)$ weakly in $L^2(\mu(t))$. The observation is that

$$\begin{aligned}
\lim_{h\to 0} \left\| \int_\Omega \frac{y-\cdot}{h} d\nu_\cdot^h(y) \right\|^2_{L^2(\mu(t))} &= \lim_{h\to 0} \int_\Omega \left| \int_\Omega \frac{y-x}{h} d\nu_x^h(y) \right|^2 d\mu(t,x) \\
&\le \lim_{h\to 0} \int_{\Omega\times\Omega} \frac{|y-x|^2}{h^2} d\gamma_t^h(x,y) \\
&= \lim_{h\to 0} \frac{d_W^2(\mu(t), \mu(t+h))}{h^2} \\
&< \infty.
\end{aligned}$$

Thus $\int_\Omega \frac{y-\cdot}{h} d\nu_\cdot^h(y)$ converges weakly in $L^2(\mu(t))$ to some vector field $\hat{v}(t,\cdot)$. This together with (3.15) implies $\hat{v} = \tilde{v}$ and we have the weak $L^2(\mu(t))$ convergence of $\int_\Omega \frac{y-\cdot}{h} d\nu_\cdot^h(y)$ to $\tilde{v}(t)$ as stated.

We now claim the following chain rule: for a.e. $t > 0$

$$\frac{d}{dt}\mathscr{E}(\mu(t)) = \int_\Omega \langle \nabla W * \mu(t)(x) + \nabla V(x), \tilde{v}(t,x) \rangle \, d\mu(t,x). \tag{3.16}$$

Indeed, we first notice that since $\mu(\cdot)$ is locally absolutely continuous, $\mathscr{E}(\mu(\cdot))$ is also locally absolutely continuous. To see that we have

$$\begin{aligned}
|\mathscr{E}(\mu(t)) - \mathscr{E}(\mu(s))| &\le \frac{1}{2} \left| \int_{\Omega\times\Omega} W(x-y) d\mu(t,x) d\mu(t,y) - \int_{\Omega\times\Omega} W(z-w) d\mu(s,z) d\mu(s,w) \right| \\
&\quad + \left| \int_\Omega V(x) d\mu(t,x) - \int_\Omega V(z) d\mu(s,z) \right| \\
&\le \int_{\Omega\times\Omega} \left(\|\nabla V\|_{L^\infty(\mathrm{Conv}(\Omega))} + \|\nabla W\|_{L^\infty(\mathrm{Conv}(\Omega-\Omega))} \right) |x-z| d\gamma(x,z) \\
&\le \left(\|\nabla V\|_{L^\infty(\mathrm{Conv}(\Omega))} + \|\nabla W\|_{L^\infty(\mathrm{Conv}(\Omega-\Omega))} \right) d_W(\mu(t), \mu(s)).
\end{aligned}$$

Thus by the locally absolute continuity of $\mu(\cdot)$, $\mathscr{E}(\mu(\cdot))$ is also locally absolutely continuous. Now for any fixed $\mu, \nu \in \mathscr{P}_2(\Omega)$ and $\gamma \in \Gamma_o(\mu, \nu)$, consider the function

$$\begin{aligned}
f(t) = &\frac{W(t(x_1-x_2) - (1-t)(y_1-y_2)) - W(x_1-x_2)}{2t} \\
&+ \frac{2V(tx_2 + (1-t)y_2) - 2V(x_2)}{2t} - \frac{\lambda_V}{2} t |x_2-y_2|^2 - \frac{\lambda_W}{2} t \left(|x_1-y_1|^2 + |x_2-y_2|^2 \right).
\end{aligned} \tag{3.17}$$

Due to (A1) and (A2), the λ-geodesic convexity of W,V, we know f is non-decreasing on $[0,1]$. So $f(1) \geq \liminf_{t\to 0^+} f(t)$. Integrating over $d\gamma(x_1,y_1)d\gamma(x_2,y_2)$ gives

$$\begin{aligned}\mathscr{E}(\nu)-\mathscr{E}(\mu) &= \int_{\Omega\times\Omega}\int_{\Omega\times\Omega}\frac{W(y_1-y_2)+2V(y_2)-W(x_1-x_2)-2V(x_2)}{2}d\gamma(x_1,y_1)d\gamma(x_2,y_2)\\ &\geq \int_{\Omega\times\Omega}\int_{\Omega\times\Omega}\langle\nabla W(x_2-x_1)+\nabla V(x_2),y_2-x_2\rangle\, d\gamma(x_1,y_1)d\gamma(x_2,y_2)+o\left(d_W(\mu,\nu)\right)\\ &= \int_{\Omega\times\Omega}\left\langle\int_\Omega\nabla W(x_2-x_1)d\mu(x_1)+\nabla V(x_2),y_2-x_2\right\rangle d\gamma(x_2,y_2)+o\left(d_W(\mu,\nu)\right)\\ &= \int_{\Omega\times\Omega}\langle\nabla W*\mu(x_2)+\nabla V(x_2),y_2-x_2\rangle\, d\gamma(x_2,y_2)+o\left(d_W(\mu,\nu)\right).\end{aligned}$$

Denote $v(t,x)=-\nabla W*\mu(t,x)-\nabla V(x)$, we notice that

$$\begin{aligned}\langle -v(t,x_2),y_2-x_2\rangle &= \langle -P_{x_2}\left(v(t,x_2)\right),y_2-x_2\rangle+\langle -v(t,x_2)+P_{x_2}\left(v(t,x_2)\right),y_2-x_2\rangle\\ &\geq \langle -P_{x_2}\left(v(t,x_2)\right),y_2-x_2\rangle-\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{2\eta}|y_2-x_2|^2,\end{aligned}$$

and

$$\int_{\Omega\times\Omega}|x_2-y_2|^2 d\gamma(x_2,y_2)=d_W^2(\mu,\nu).$$

Thus

$$\mathscr{E}(\nu)-\mathscr{E}(\mu)\geq\int_{\Omega\times\Omega}\langle -P_{x_2}\left(-\nabla W*\mu(x_2)-\nabla V(x_2)\right),y_2-x_2\rangle\, d\gamma(x_2,y_2)+o\left(d_W(\mu,\nu)\right),$$

which implies

$$-P\left(v(t)\right)=-P\left(-\nabla W*\mu(t)-\nabla V\right)\in\partial\mathscr{E}(\mu(t)).$$

Take $\mu=\mu(t),\nu=\mu(t+h)$ and $\gamma_t^h\in\Gamma_o(\mu(t),\mu(t+h))$ then gives

$$\begin{aligned}&\lim_{h\to 0^+}\frac{\mathscr{E}\left(\mu(t+h)\right)-\mathscr{E}\left(\mu(t)\right)}{h}\\ &\geq\limsup_{h\to 0^+}\left(\int_{\Omega\times\Omega}\left\langle\nabla W*\mu(t,x_2)+\nabla V(x_2),\frac{y_2-x_2}{h}\right\rangle d\gamma_t^h(x_2,y_2)+\frac{1}{h}o\left(d_W\left(\mu(t),\mu(t+h)\right)\right)\right)\\ &=\int_\Omega\langle\nabla W*\mu(t)(x_2)+\nabla V(x_2),\tilde{v}(t,x_2)\rangle\, d\mu(t,x_2),\end{aligned}$$

where the last equality comes from (3.15). Similarly, by taking $\mu=\mu(t),\nu=\mu(t-h)$, we have

$$\lim_{h\to 0^+}\frac{\mathscr{E}\left(\mu(t)\right)-\mathscr{E}\left(\mu(t-h)\right)}{h}\leq\int_\Omega\langle\nabla W*\mu(t)(x_2)+\nabla V(x_2),\tilde{v}(t,x_2)\rangle\, d\mu(t,x_2).$$

Together with the fact that $\mathscr{E}(\mu(\cdot)$ is locally absolutely continuous, we have for a.e. $t>0$

$$\frac{d}{dt}\mathscr{E}\left(\mu(t)\right)=\int_\Omega\langle\nabla W*\mu(t)(x)+\nabla V(x),\tilde{v}(t,x)\rangle\, d\mu(t,x).$$

The claim is proved. Now for $v_N(t,x) = v(t,x) - P_x(v(t,x))$, we have $v_N(t,x) \in N(\Omega,x)$ and $\|v_N(t)\|_{L^\infty(\Omega)} \le \|v(t)\|_{L^\infty(\Omega)} < \infty$. Thus

$$\begin{aligned}\lim_{h\to 0^+}\int_{\Omega\times\Omega}\left\langle v_N(t,x),\frac{y-x}{h}\right\rangle d\gamma_t^h(x,y) &\le \lim_{h\to 0^+}\int_{\Omega\times\Omega}\frac{\|v_N(t)\|_{L^\infty(\Omega)}}{2\eta}\frac{1}{h}|x-y|^2 d\gamma_t^h(x,y)\\ &\le \lim_{h\to 0^+}\frac{\|v(t)\|_{L^\infty(\Omega)}}{2\eta}\frac{d_W^2\left(\mu(t),\mu(t+h)\right)}{h}\\ &=0,\end{aligned}$$

which together with the weak $L^2(\mu(t))$-convergence of $\int_\Omega \frac{y-\cdot}{h}d\nu_\cdot(y)$ implies

$$\int_\Omega \langle v_N(t,x),\tilde{v}(t,x)\rangle\, d\mu(t,x) \le 0.$$

We then know that

$$\frac{d}{dt}\mathscr{E}\left(\mu(t)\right) \ge -\int_\Omega \langle P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right),\tilde{v}(t,x)\rangle\, d\mu(t,x). \tag{3.18}$$

Together with (3.8), we get for a.e $t>0$

$$\tilde{v}(t,x) = P_x\left(v(t,x)\right) = P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right) \in -\partial\mathscr{E}\left(\mu(t)\right),$$

$$|\mu'|^2(t) = \int_\Omega |P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right)|^2 d\mu(t,x)$$

and for any $0\le s\le t<\infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t\int_\Omega |P_x\left(-\nabla W * \mu(r)(x) - \nabla V(x)\right)|^2 d\mu(r,x)dr.$$

Thus $\mu(\cdot)$ is a gradient flow with respect to $\mathscr{E}$ and by (3.14), a weak measure solution to (1.1). $\square$

Remark 3.8. In [6], Carrillo, Lisini and Mainini showed weak $L^2(\mu(t))$ convergence of $\int_\Omega \frac{y-\cdot}{h}d\nu_\cdot^h(y)$ to $\tilde{v}(t,\cdot)$ in a more general setting than ours.

We turn to the proof of Theorem 1.6

Proof of Theorem 1.6. We show (1.12) first. Let $\mu^1(\cdot),\mu^2(\cdot)$ be two solutions to (1.1), by Theorem 8.4.7 and Lemma 4.3.4 from [2], we have

$$\frac{d}{dt}d_W^2\left(\mu^1(t),\mu^2(t)\right) = 2\int_{\Omega\times\Omega}\langle P_x\left(v^1(t,x)\right) - P_y\left(v^2(t,y)\right),x-y\rangle d\gamma_t(x,y), \tag{3.19}$$

where $\gamma_t\in\Gamma_o\left(\mu^1(t),\mu^2(t)\right)$ and $v^i(t,x) = -\nabla W*\mu^i(t)(x) - \nabla V(x)$ for $i=1,2$. For v^i, by (A1) and (A2) similar argument as in the proof of Proposition 3.1 gives

$$\int_{\Omega\times\Omega}\langle v^1(t,x)-v^2(t,y),x-y\rangle d\gamma_t(x,y) \le \left(-\lambda_W^- - \lambda_V\right)d_W^2\left(\mu^1(t),\mu^2(t)\right).$$

By the fact that Ω is η-prox-regular we have

$$\begin{aligned}&\int_{\Omega\times\Omega}\langle P_x\left(v^1(t,x)\right) - v^1(t,x) - P_y\left(v^2(t,y)\right) + v^2(t,y),x-y\rangle d\gamma_t(x,y)\\ &\le \frac{\|v^1(t)\|_{L^\infty(\Omega)} + \|v^2(t)\|_{L^\infty(\Omega)}}{2\eta}d_W^2\left(\mu^1(t),\mu^2(t)\right),\end{aligned}$$

where v^i satisfies

$$\|v^i\|_{L^\infty(\Omega)} \le \|\nabla W\|_{L^\infty(\Omega-\Omega)} + \|\nabla V\|_{L^\infty(\Omega)} < \infty.$$

Plugging back into (3.19) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^1(t),\mu^2(t)\right) &= \int_{\Omega\times\Omega}\langle P_x\left(v^1(t,x)\right)-P_y\left(v^2(t,y)\right),x-y\rangle d\gamma_t(x,y)\\ &= \int_{\Omega\times\Omega}\langle v^1(t,x)-v^2(t,y),x-y\rangle d\gamma_t(x,y)\\ &+\int_{\Omega\times\Omega}\langle P_x\left(v^1(t,x)\right)-v^1(t,x)-P_y\left(v^2(t,y)\right)+v^2(t,y),x-y\rangle d\gamma_t(x,y)\\ &\le\left(-\lambda_W^- -\lambda_V+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right)d_W^2\left(\mu^1(t),\mu^2(t)\right).\end{aligned}$$

Then by Gronwall's inequality we have for all $t\ge 0$

$$d_W\left(\mu^1(t),\mu^2(t)\right)\le\exp\left(\left(-\lambda_W^- -\lambda_V+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right)t\right)d_W\left(\mu_0^1,\mu_0^2\right). \tag{3.20}$$

(1.12) is proved. For (1.11), we have if $\mu(\cdot)$ is a weak measure solution to (1.1), then for any $\nu\in\mathscr{P}_2(\Omega)$ and $\gamma_t\in\Gamma_o(\mu(t),\nu)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu(t),\nu\right) &= \int_{\Omega\times\Omega}\langle P_x\left(v(t,x)\right),x-y\rangle\, d\gamma_t(x,y)\\ &= \int_{\Omega\times\Omega}\left(\langle v(t,x),x-y\rangle+\langle P_x\left(v(t,x)\right)-v(t,x),x-y\rangle\right)d\gamma_t(x,y)\\ &\le\mathscr{E}(\nu)-\mathscr{E}(\mu(t))+\int_{\Omega\times\Omega}\left(-\frac{\lambda_W^-}{2}-\frac{\lambda_V}{2}+\frac{\|v(t)\|_{L^\infty(\Omega)}}{2\eta}\right)|x-y|^2 d\gamma_t(x,y)\\ &\le\mathscr{E}(\nu)-\mathscr{E}(\mu(t))+\left(-\frac{\lambda_W^-}{2}-\frac{\lambda_V}{2}+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{2\eta}\right)d_W^2\left(\mu(t),\nu\right).\end{aligned}$$

On the other hand, if $\mu^1(\cdot)$ satisfies (1.11) and $\mu^2(\cdot)$ is the solution to (1.1) such that $\mu_0^1=\mu_0^2$,by Lemma 4.3.4 from [2] we get

$$\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^1(t),\mu^2(t)\right)\le\left(-\lambda_W^- -\lambda_V+\frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}+\|\nabla V\|_{L^\infty(\Omega)}}{\eta}\right)d_W^2\left(\mu^1(t),\mu^2(t)\right).$$

Again by Gronwall's inequality we have $\mu^1(t)=\mu^2(t)$ for all $t\ge 0$. Thus the weak measure solution is characterized by the system of evolution variational inequalities (1.11). ☐

4. Existence and stability of solutions with Ω unbounded: Global case

In this section we prove the existence and stability of (1.1) with Ω unbounded, convex and W,V satisfying (GA1)-(GA4).

For any initial data $\mu_0\in\mathscr{P}_2(\Omega)$ and fixed $x_0\in\Omega$, denote $B_n(x_0)=\{x\in\mathbb{R}^d:|x-x_0|<n\}$, we can take $\mu_0^n=\sum_{i=1}^{k(n)}m_j^n\delta_{x_i^n}$ for $x_i^n\in\Omega\cap\overline{B_n(x_0)}$ and $\lim_{n\to\infty}d_W\left(\mu_0^n,\mu_0\right)=0$. To see that, note $\int_\Omega|x-x_0|^2d\mu_0(x)<\infty$, thus $\lim_{n\to\infty}\int_{\Omega\setminus B_n(x_0)}|x-x_0|^2d\mu_0(x)=0$. For $\mu_0\lfloor_{\Omega\cap\overline{B_n(x_0)}}$, we can find $\tilde{\mu}_0^n$ composed of delta measures with the same total mass as $\mu_0\lfloor_{\Omega\cap\overline{B_n(x_0)}}$, such that $\operatorname{supp}(\mu_0^n)\subset\Omega\cap\overline{B_n(x_0)}$ and $\lim_{n\to\infty}d_W\left(\mu_0\lfloor_{\Omega\cap\overline{B_n(x_0)}},\tilde{\mu}_0^n\right)=0$. Then $\mu_0^n=\tilde{\mu}_0^n+\left(1-\mu_0\left(\Omega\cap\overline{B_n(x_0)}\right)\right)\delta_{x_0}$ satisfies the required conditions. Without loss of generality, we assume that $x_0=0\in\Omega$ and denote $B(n)=B_n(0)$.

As in Section 3, we first show the convergence of $\mu^n(\cdot)$ as $n \to \infty$.

Proposition 4.1. *Assume that Ω is unbounded and convex, W, V satisfy (GA1)-(GA4). Then for two solutions $\mu^m(\cdot), \mu^n(\cdot)$ to the discrete system with different initial data μ_0^m, μ_0^n, we have for all $t \geq 0$*

$$d_W\left(\mu^n(t), \mu^m(t)\right) \leq \exp\left(-\left(\lambda_W^- + \lambda_V\right)t\right) d_W\left(\mu_0^n, \mu_0^m\right). \tag{4.1}$$

Proof. The proof is similar to the proof of Proposition 3.1 once we notice that since Ω is ∞-prox-regular, by (1.5) for any $x, y \in \Omega$

$$\langle P_x\left(v^n(t,x)\right) - v^n(t,x), x - y\rangle \leq 0.$$

□

So as $n \to \infty$ we again know that $\mu^n(t)$ converges to some $\mu(t) \in \left(\mathscr{P}_2(\Omega), d_W\right)$. Before proving that $\mu(t)$ is a curve of maximal slope, we need

Proposition 4.2. *Let $\mu_n, \mu \in \mathscr{P}_2(\Omega)$ be such that $\lim_{n\to\infty} d_W\left(\mu_n, \mu\right) = 0$ then*

$$\lim_{n\to\infty} \mathscr{V}(\mu_n) = \mathscr{V}(\mu),$$

and

$$\lim_{n\to\infty} \mathscr{W}(\mu_n) = \mathscr{W}(\mu).$$

Proof. Since the arguments are similar, it is enough for us to show the property for $\mathscr{V}$. By (GA4), there exists a constant $C > 0$ such that $|V(x)| \leq C(1+|x|^2)$. By Lemma 5.1.7 from [2], since $V(x) + C|x|^2$ is lower semicontinuous and bounded from below,

$$\liminf_{n\to\infty} \int_\Omega \left(V(x) + C|x|^2\right) d\mu_n(x) \geq \int_\Omega \left(V(x) + C|x|^2\right) d\mu(x).$$

$\lim_{n\to\infty} d_W\left(\mu_n, \mu\right) = 0$, we know

$$\lim_{n\to\infty} \int_\Omega |x|^2 d\mu_n(x) = \int_\Omega |x|^2 d\mu(x).$$

Thus

$$\liminf_{n\to\infty} \int_\Omega V(x) d\mu_n(x) \geq \int_\Omega V(x) d\mu(x).$$

Similarly, the condition $C|x|^2 - V(x)$ is lower semicontinuous and bounded from below implies

$$\limsup_{n\to\infty} \int_\Omega V(x) d\mu_n(x) \leq \int_\Omega V(x) d\mu(x).$$

Thus

$$\lim_{n\to\infty} \mathscr{V}(\mu_n) = \mathscr{V}(\mu),$$

as claimed. □

We estimate the growth of support of the solutions $\mu^n(\cdot)$ to (1.18).

Lemma 4.3. *Let μ_0^n be the approximation of μ_0 such that $\operatorname{supp}\left(\mu_0^n\right) \subset \Omega \cap B(n)$. Then $\operatorname{supp}\left(\mu^n(t)\right) \subset \Omega \cap B(r(t))$ for $r(t) \leq (n+1)\exp(Ct)$ for some $C = C(W,V)$ independent of n.*

Proof. Define $r(t) = \sup_i |x_i^n(t)|$. For fixed $t > 0$, assume that $x_i^n(t)$ realizes $R(t)$ i.e., $r(t) = |x_i^n(t)|$, then

$$\begin{aligned}\left|\frac{d}{dt}|x_i^n|^2\right| &= 2\left|\left\langle x_i^n(t), P_{x_i^n}\left(-\sum_{j=1}^{k(n)} m_j \nabla W(x_i^n - x_j^n) - \nabla V(x_i^n)\right)\right\rangle\right| \\ &\le 2|x_i^n(t)|\left(\sum_{j=1}^{k(n)} m_j \left|\nabla W(x_i^n(t) - x_j^n(t))\right| + |\nabla V(x_i^n(t))|\right) \\ &\le 2|x_i^n(t)|\left(\sum_{j=1}^{k(n)} m_j C\left(1 + |x_i^n(t) + |x_j^n(t)|\right) + C\left(1 + |x_i^n(t)|\right)\right) \\ &\le C\left(1 + |x_i^n(t)|^2\right).\end{aligned}$$

Thus

$$r(t) \le r(0)\exp(Ct) + \exp(Ct) - 1$$

for $r(0) \le n$ and C depending only on W,V, in particular independent of the number of particles $k(n)$. $\square$

We can now show

Theorem 4.4. *Assume Ω is unbounded and convex, W,V satisfy (GA1)-(GA4), then $\mu(\cdot)$ satisfies for any $0 \le s < t < \infty$*

$$\mathscr{E}(\mu(s)) \ge \mathscr{E}(\mu(t)) + \frac{1}{2}\int_s^t |\mu'|^2(r)dr + \frac{1}{2}\int_s^t \int_\Omega |P_x(v(r,x))|^2 d\mu(r,x)dr, \tag{4.2}$$

where $v(r,x) = -\int_\Omega \nabla W(x-y)d\mu(r,y) - \nabla V(x)$.

Proof. We first check that for fixed $n \in \mathbb{N}$, the function $t \mapsto \mathscr{E}(\mu^n(t))$ is locally absolutely continuous. For fixed $0 \le s < t < \infty$, by Lemma 4.3, $\|\nabla V(x)\|_{L^\infty(\Omega \cap B(r(t)))} < \infty$ and $\|\nabla W\|_{L^\infty(\Omega \cap B(r(t)) - \Omega \cap B(r(t))} < \infty$. Then by the same argument as in (3.10), $t \mapsto \mathscr{E}(\mu(t))$ is locally absolutely continuous. Together with Proposition 4.2, the proof is now identical to the proof of Theorem 3.11. We omit it here. $\square$

We proceed to the proof of Theorem 1.9

Proof of Theorem 1.9. Let $\tilde{v}$ be the tangential velocity field for $\mu(\cdot)$, i.e. $\mu(\cdot)$ satisfies (3.14) and $\|\tilde{v}(t)\|_{L^2(\mu(t))} = |\mu'|(t)$. Similar arguments as in the proof of Theorem 1.5 still gives that for any $\mu, \nu \in \mathscr{P}_2(\Omega)$

$$\mathscr{E}(\nu) - \mathscr{E}(\mu) \ge \int_{\Omega\times\Omega} \langle \nabla W * \mu(x_2) + \nabla V(x_2), y_2 - x_2\rangle\, d\gamma(x_2,y_2) + o\left(d_W(\mu,\nu)\right),$$

and for a.e. $t > 0$

$$\frac{d}{dt}\mathscr{E}(\mu(t)) = \int_\Omega \langle \nabla W * \mu(t)(x) + \nabla V(x), \tilde{v}(t,x)\rangle\, d\mu(t,x).$$

Now since Ω is convex, we have $\langle v_N(t,x), y - x\rangle \le 0$, thus

$$\mathscr{E}(\nu) - \mathscr{E}(\mu) \ge \int_{\Omega\times\Omega} \langle -P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right), y - x\rangle\, d\gamma(x,y),$$

and

$$\lim_{h\to 0^+}\int_{\Omega\times\Omega}\left\langle v_N(t,x), \frac{y-x}{h}\right\rangle d\gamma_t^h(x,y) \le 0.$$

So we have $-P(v(t)) = -P(-\nabla W * \mu(t) - \nabla V) \in \partial\mathscr{E}(\mu(t))$ and

$$\int_\Omega \langle v_N(t,x), \tilde{v}(t,x)\rangle d\mu(t,x) \le 0.$$

Thus

$$\frac{d}{dt}\mathscr{E}(\mu(t)) \ge -\int_\Omega \langle P_x(-\nabla W * \mu(t)(x) - \nabla V(x)), \tilde{v}(t,x)\rangle d\mu(t,x),$$

which together with Theorem 5.1 implies for a.e. $t > 0$

$$\tilde{v}(t,x) = P_x(v(t,x)) = P_x(-\nabla W * \mu(t)(x) - \nabla V(x)) \in -\partial\mathscr{E}(\mu(t)), \tag{4.3}$$

$$|\mu'|^2(t) = \int_\Omega |P_x(-\nabla W * \mu(t)(x) - \nabla V(x))|^2 d\mu(t,x) \tag{4.4}$$

and for any $0 \le s \le t < \infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t \int_\Omega |P_x(-\nabla W * \mu(r)(x) - \nabla V(x))|^2 d\mu(r,x)dr. \tag{4.5}$$

Thus $\mu(\cdot)$ is a gradient flow with respect to $\mathscr{E}$ and by (3.14), a weak measure solution to (1.1).
For the stability result (1.15), we only need to notice that for any two solutions $\mu^1(\cdot), \mu^2(\cdot)$ to (1.1), since Ω is convex, $\left\langle v^i(t,x) - P_x\left(v^i(t,x)\right), y - x\right\rangle \le 0$ for $i = 1,2$. Thus

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^1(t),\mu^2(t)\right) &= \int_{\Omega\times\Omega} \langle P_x\left(v^1(t,x)\right) - P_y\left(v^2(t,y)\right), x-y\rangle d\gamma_t(x,y)\\
&= \int_{\Omega\times\Omega} \langle v^1(t,x) - v^2(t,y), x-y\rangle d\gamma_t(x,y)\\
&+ \int_{\Omega\times\Omega} \langle P_x\left(v^1(t,x)\right) - v^1(t,x) - P_y\left(v^2(t,y)\right) + v^2(t,y), x-y\rangle d\gamma_t(x,y)\\
&\le -\left(\lambda_W^- + \lambda_V\right) d_W^2\left(\mu^1(t),\mu^2(t)\right).
\end{aligned}$$

Then by Gronwall's inequality we get (1.15).
For evolution variational inequalities (1.16), if $\mu(\cdot)$ is a solution to (1.1) then for any $\nu \in \mathscr{P}_2(\Omega)$ and $\gamma \in \Gamma_o(\mu(t),\nu)$ an optimal plan

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}d_W^2(\mu(t),\nu) &= \int_{\Omega\times\Omega} \langle P_x(v(t,x)), x-y\rangle d\gamma_t(x,y)\\
&= \int_{\Omega\times\Omega} \left(\langle v(t,x), x-y\rangle + \langle P_x(v(t,x)) - v(t,x), x-y\rangle\right) d\gamma_t(x,y)\\
&\le \mathscr{E}(\nu) - \mathscr{E}(\mu(t)) - \int_{\Omega\times\Omega}\left(\frac{\lambda_W^-}{2} + \frac{\lambda_V}{2}\right)|x-y|^2 d\gamma_t(x,y)\\
&\le \mathscr{E}(\nu) - \mathscr{E}(\mu(t)) - \left(\frac{\lambda_W^-}{2} + \frac{\lambda_V}{2}\right) d_W^2(\mu(t),\nu).
\end{aligned}$$

On the other hand, if $\mu^1(\cdot)$ satisfies (1.16) and $\mu^2(\cdot)$ is the solution to (1.1) such that $\mu_0^1 = \mu_0^2$ we know that for any $\nu \in \mathscr{P}_2(\Omega)$ and $i = 1,2$

$$\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^i(t),\nu\right) + \left(\frac{\lambda_W^-}{2} + \frac{\lambda_V}{2}\right) d_W^2\left(\mu^i(t),\nu\right) \le \mathscr{E}(\nu) - \mathscr{E}\left(\mu^i(t)\right).$$

By Lemma 4.3.4 from [2] we then have

$$\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^1(t),\mu^2(t)\right) \le -\left(\lambda_W^- + \lambda_V\right)d_W^2\left(\mu^1(t),\mu^2(t)\right).$$

So by Gronwall's inequality we have $\mu^1(t) = \mu^2(t)$ for all $t \geq 0$. Thus the weak measure solution to (1.1) is characterized by the system of evolution variational inequalities (1.16). □

5. Existence and stability of solutions with Ω unbounded: Compactly supported initial data case

In this section, we show the existence and stability results in the case when Ω is unbounded and W,V satisfy (LA1)-(LA4). The novelty is that λ-geodesic convexity of energy is only assumed locally (which is automatically satisfied if V and W are C^2 functions).

We start by giving the control the support of the solutions $\mu^n(t)$ to (1.18). Notice that when approximating μ_0 by $\mu_0^n = \sum_{i=1}^{k(n)} m_i^n \delta_{x_i^n}$, since $\operatorname{supp}(\mu_0) \subset \Omega \cap B(r_0)$, we can take $x_i^n \in \Omega \cap B(r_0+1)$ for all $n \in \mathbb{N}$ and $1 \leq i \leq k(n)$ such that we have

$$\lim_{n\to\infty} d_W\left(\mu_0^n, \mu_0\right) = 0.$$

So without loss of generality, we assume $\operatorname{supp}\left(\mu_0^n\right) \subset B(r_0)$ for all $n \in \mathbb{N}$. Then by Lemma 4.3, $\operatorname{supp}\left(\mu^n(t)\right) \subset \Omega \cap B(r(t))$ for $r(t) \leq (r_0+1)\exp(Ct)$ for some $C = C(W,V)$ independent of n.

Proposition 5.1. *There exists a locally absolutely continuous curve $\mu(\cdot)$ in $\mathscr{P}_2(\Omega)$ such that $\mu^n(t)$ converges to $\mu(t)$ in $\mathscr{P}_2(\Omega)$ for any $0 \leq t < \infty$.*

Proof. For any fixed $0 < T < \infty$ and any $0 \leq t \leq T$, we know that $\operatorname{supp}(\mu^n(t)) \subset B(r(T))$ for all $0 \leq t \leq T$ uniformly in n. Let K_k and $\lambda_{W,k}, \lambda_{V,k}$ be the sequences of compact convex sets and constants such that W,V are $\lambda_{W,k}$ and $\lambda_{V,k}$-geodesically convex on K_k. Take k_0 be such that $B(2r(T)) \subset K_k$ for all $k \geq k_0$. Still denote $\gamma_t \in \Gamma_o\left(\mu^n(t), \mu^m(t)\right)$ an optimal plan. Now notice that $\operatorname{supp}(\mu^n(t)), \operatorname{supp}(\mu^m(t)) \subset B(r(t)) \cap \Omega \subset K_k \cap \Omega = \Omega_k$, thus

$$\int_{\Omega\times\Omega} \langle v^n(t,x) - v^m(t,y), x-y\rangle \, d\gamma_t(x,y) \leq \int_{\Omega\times\Omega} \left(\lambda_{W,k}^- + \lambda_{V,k}\right)|x-y|^2,$$

and

$$\begin{aligned}&\int_{\Omega\times\Omega} \langle P_x\left(v^n(t,x)\right) - v^n(t,x) - P_y\left(v^m(t,y)\right) + v^m(t,y), x-y\rangle \, d\gamma_t(x,y) \\ &\leq \int_{\Omega\times\Omega} \left(\frac{\|v^n(t)\|_{L^\infty(\Omega_t)} + \|v^m(t)\|_{L^\infty(\Omega_t)}}{2\eta}\right)|x-y|^2 d\gamma_t(x,y),\end{aligned}$$

where $\Omega_t = \Omega \cap B(r(t))$. Since $v^n(t,x) = -\int_\Omega \nabla W(x-y)d\mu^n(t,y) - \nabla V(x)$ we know

$$\|v^n(t)\|_{L^\infty(\Omega_t)} \leq \|\nabla W\|_{L^\infty(\Omega_T-\Omega_T)} + \|\nabla V\|_{L^\infty(\Omega_T)} < \infty.$$

Thus as in the proof of Proposition 3.1, we have for $0 \le t \le T$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}d_W^2\left(\mu^n(t),\mu^m(t)\right) &= \int_\Omega \left\langle P_x\left(v^n(t,x)\right) - P_y\left(v^m(t,y)\right), x-y\right\rangle d\gamma_t(x,y) \\
&= \int_{\Omega\times\Omega} \langle v^n(t,x) - v^m(t,y), x-y\rangle\, d\gamma_t(x,y) \\
&+ \int_{\Omega\times\Omega} \langle P_x\left(v^n(t,x)\right) - v^n(t,x) - P_y\left(v^m(t,y)\right) + v^m(t,y), x-y\rangle\, d\gamma_t(x,y) \\
&\le \left(-\lambda_{W,k}^- - \lambda_{V,k} + \frac{\|\nabla W\|_{L^\infty(\Omega_T-\Omega_T)} + \|\nabla V\|_{L^\infty(\Omega_T)}}{\eta}\right) d_W^2\left(\mu^n(t),\mu^m(t)\right).
\end{aligned}$$

By Gronwall's inequality, we have for all $0 \le t \le T$

$$d_W\left(\mu^n(t),\mu^m(t)\right) \le \exp\left(\left(-\lambda_{W,k}^- - \lambda_{V,k} + \frac{\|\nabla W\|_{L^\infty(\Omega_T-\Omega_T)} + \|\nabla V\|_{L^\infty(\Omega_T)}}{\eta}\right) t\right) d_W(\mu_0^n,\mu_0^m).$$

Thus as $n \to \infty$, $\mu^n(t)$ converges in $\mathscr{P}_2(\Omega)$ to some $\mu(t)$. □

Theorem 5.2. *$\mu(\cdot)$ is a curve of maximal slope, for any $0 \le s < t < \infty$*

$$\mathscr{E}\left(\mu(s)\right) \ge \mathscr{E}\left(\mu(t)\right) + \frac{1}{2}\int_s^t |\mu'|^2(r)dr + \frac{1}{2}\int_s^t\int_\Omega |P_x\left(v(r,x)\right)|^2 d\mu(r,x)dr, \tag{5.1}$$

where $v(r,x) = -\int_\Omega \nabla W(x-y)d\mu(r,y) - \nabla V(x)$.

Proof. We use similar argument as in Theorem 3.6 and Theorem 4.4. For any fixed $n \in \mathbb{N}$, since $\mathrm{supp}(\mu^n(t)) \subset \Omega \cap B(r(t))$, we can still control the L^∞-norm of ∇V and ∇W. Then the same argument as in the proof of Theorem 4.4 shows that $t \mapsto \mathscr{E}\left(\mu(t)\right)$ is locally absolutely continuous. Thus the fact that μ^n are solutions to the discrete systems implies,

$$\mathscr{E}\left(\mu^n(s)\right) \ge \mathscr{E}\left(\mu^n(t)\right) + \frac{1}{2}\int_s^t |(\mu^n)'|^2(r)dr + \frac{1}{2}\int_s^t\int_\Omega |P_x\left(v^n(r,x)\right)|^2 d\mu^n(r,x)dr. \tag{5.2}$$

Note that $W,V \in C^1(\mathbb{R}^d)$ and $\lim_{n\to\infty} d_W\left(\mu^n(r),\mu(r)\right) = 0$ with $\mathrm{supp}(\mu^n(r)) \subset \Omega \cap B(r(T)$ for any $0 \le r < t \le T$, we get

$$\lim_{n\to\infty} \mathscr{E}(\mu^n(r)) = \mathscr{E}(\mu(r)).$$

By Lemma 3.7 and notice that $\nabla W * \mu^n(r) + \nabla V$ still converges weakly to $\nabla W * \mu(r) + \nabla V$ for any $0 \le r \le T$, then

$$\liminf_{n\to\infty}\int_\Omega |P_x\left(v^n(r,x)\right)|^2 d\mu^n(r,x) \ge \int_\Omega |P_x\left(v(r,x)\right)|^2 d\mu(r,x).$$

By Fatou's lemma,

$$\liminf_{n\to\infty}\int_s^t\int_\Omega |P_x\left(v^n(r,x)\right)|^2 d\mu^n(r,x)dr \ge \int_s^t\int_\Omega |P_x\left(v(r,x)\right)|^2 d\mu(r,x)dr.$$

Now by the same argument as in the proof of (3.13), we again obtain

$$\liminf_{n\to\infty}\int_s^t |(\mu^n)'|^2(r)dr \ge \int_s^t |\mu'|^2(r)dr.$$

Take $n \to \infty$ in (5.2) gives

$$\mathscr{E}\left(\mu(s)\right) \ge \mathscr{E}\left(\mu(t)\right) + \frac{1}{2}\int_s^t |\mu'|^2(r)dr + \frac{1}{2}\int_s^t\int_\Omega |P_x\left(v(r,x)\right)|^2 d\mu(r,x)dr.$$

□

We now start to prove Theorem 1.10

Proof of Theorem 1.10. Since $\mu(\cdot)$ is locally absolutely continuous, we know that there exists a unique Borel vector field $\tilde{v}$ such that

$$\partial_t \mu(t) + \mathrm{div}\,(\mu(t)\tilde{v}(t)) = 0$$

holds in the sense of distributions. For a fixed $T > 0$ and any $\mu, \nu \in \mathscr{P}_2(\Omega)$ with $\mathrm{supp}(\mu), \mathrm{supp}(\nu) \subset B(r(T))$, let $\gamma \in \Gamma_o(\mu, \nu)$. Since W, V are $\lambda_{W,k}$ and $\lambda_{V,k}$-geodesically convex on $K_k \supset B(r(t)) \cap \Omega$, we have that the function f we defined in (3.17) by taking $\lambda = \lambda_k$ is non-decreasing in t for any $(x_1, y_1), (x_2, y_2) \in \mathrm{supp}\,\gamma$. Thus we still have

$$\mathscr{E}(\nu) - \mathscr{E}(\mu) \geq \int_{\Omega\times\Omega} \langle \nabla W * \mu(x_2) + \nabla V(x_2), y_2 - x_2 \rangle d\gamma(x_2, y_2).$$

For any $0 < t < T$, and $h > 0$ such that $t - h \geq 0, t + h \leq T$, we take $\mu = \mu(t), \nu = \mu(t+h)$ to get

$$\lim_{h\to 0^+} \frac{\mathscr{E}(\mu(t+h)) - \mathscr{E}(\mu(t))}{h} \geq \int_\Omega \langle \nabla W * \mu(t)(x) + \nabla V(x), \tilde{v}(t,x) \rangle d\mu(t,x)$$

Again take $\mu = \mu(t), \nu = \mu(t-h)$ gives

$$\lim_{h\to 0^+} \frac{\mathscr{E}(\mu(t)) - \mathscr{E}(\mu(t-h))}{h} \leq \int_\Omega \langle \nabla W * \mu(t)(x) + \nabla V(x), \tilde{v}(t,x) d\mu(t,x).$$

Also $\mathscr{E}(\mu(t))$ is locally absolutely continuous, so for a.e. $t > 0$

$$\frac{d}{dt}\mathscr{E}(\mu(t)) = \int_\Omega \langle \nabla W * \mu(t)(x) + \nabla V(x), \tilde{v}(t,x) \rangle d\mu(t,x),$$

which again implies

$$\frac{d}{dt}\mathscr{E}(\mu(t)) \geq -\int_\Omega \langle P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right), \tilde{v}(t,x) \rangle d\mu(t,x).$$

Combine with (5.1) yields

$$\tilde{v}(t,x) = P_x\left(-\nabla W * \mu(t)(x) - \nabla V(x)\right),$$

and for any $0 \leq s \leq t < \infty$

$$\mathscr{E}(\mu(s)) = \mathscr{E}(\mu(t)) + \int_s^t \int_\Omega \left| P_x\left(-\nabla W * \mu(r)(x) - \nabla V(x)\right)\right|^2 d\mu(r,x)dr.$$

For the contraction property (1.17), we notice that for any $0 \leq t \leq T < \infty$ and $k \in \mathbb{N}$ such that $B(r(T)) \subset K_k$

$$\frac{1}{2}\frac{d}{dt} d_W^2\left(\mu^1(t), \mu^2(t)\right) \leq \left(-\lambda_{W,k}^- - \lambda_{V,k} + \frac{\|\nabla W\|_{L^\infty(\Omega_k - \Omega_k)} + \|\nabla V\|_{L^\infty(\Omega_k)}}{\eta}\right) d_W^2\left(\mu^1(t), \mu^2(t)\right).$$

where $\Omega_k = \Omega \cap K_k$. Thus by Gronwall's inequality, we have for all $0 \leq t \leq T$

$$d_W\left(\mu^1(t), \mu^2(t)\right) \leq \exp\left(\left(-\lambda_{W,k}^- - \lambda_{V,k} + \frac{\|\nabla W\|_{L^\infty(\Omega_k - \Omega_k)} + \|\nabla V\|_{L^\infty(\Omega_k)}}{\eta}\right) t\right) d_W(\mu_0^1, \mu_0^2).$$

$\square$

Remark 5.3. When the external and interaction potentials are time-dependent $V = V(t,x), W = W(t,x)$, then with some modifications of the arguments we have before, we can still show the existence and stability results of the solutions to (1.1) in all the three different cases as in the time-independent settings before. For example, we assume that there are constants $\lambda \in \mathbb{R}, \eta > 0$ and a positive function $\beta \in L^1([0,\infty))$ such that

- **(M1)** Ω is bounded and η-prox-regular.
- **(TA1)** $W \in C^1([0,\infty)\times\mathbb{R}^d)$ is λ-geodesically convex on $\mathrm{Conv}\,(\Omega-\Omega)$ uniformly in t.
- **(TA2)** $V \in C^1([0,\infty)\times\mathbb{R}^d)$ is λ-geodesically convex on $\mathrm{Conv}\,(\Omega)$ uniformly in t.
- **(TA3)** $|\nabla V(t,x)| \leq \beta(t)(1+|x|)$ and $|\nabla W(t,x)| \leq \beta(t)(1+|x|)$ for all $x \in \mathbb{R}^d$.
- **(TA4)** $|\frac{\partial V}{\partial t}(t,x)| \leq \beta(t)(1+|x|^2)$ and $|\frac{\partial W}{\partial t}(t,x)| \leq \beta(t)(1+|x|^2)$ for all $x \in \mathbb{R}^d$.

Then we can show the existence of a weak measure solution $\mu(\cdot)$ to (1.1) satisfying (1.8), (1.9) and stability estimate

$$d_W(\mu^1(t),\mu^2(t)) \leq \exp\left(-2\lambda t + \frac{C(\Omega)}{\eta}\int_0^t \beta(s)ds\right) d_W\left(\mu_0^1,\mu_0^2\right), \tag{5.3}$$

where $C(\Omega) = \sup_{x\in\Omega} \mathrm{dist}(x,0)$. We sketch the proof and concentrate on the differences. Approximate the initial data μ_0 by a sequence of particle measures μ_0^n as before. Note that we can still show the existence of solutions to the projected ODE system by citing Theorem 5.1 from [9]. Thus for total energy defined as $\mathscr{E}(t,\mu) = \frac{1}{2}\int_{\Omega\times\Omega} W(t,x-y)d\mu(x)d\mu(y) + \int_\Omega V(t,x)d\mu(x)$, we have the following energy dissipation along the solutions $\mu^n(\cdot)$,

$$\begin{aligned}\mathscr{E}(s,\mu^n(s)) \geq \mathscr{E}(t,\mu^n(t)) &- \frac{1}{2}\int_s^t \int_{\Omega\times\Omega} \frac{\partial W}{\partial r}(r,x-y)d\mu(r,x)d\mu(r,y)dr \\ &- \int_s^t \int_\Omega \frac{\partial V}{\partial r}(r,x)d\mu(r,x)dr + \frac{1}{2}\left|(\mu^n)'\right|^2 + \frac{1}{2}\int_\Omega |P_x(v^n(t,x))|^2 d\mu^n(t,x).\end{aligned} \tag{5.4}$$

Similar stability argument as before shows that the sequence $\{\mu^n(\cdot)\}_n$ satisfies the stability estimate (5.3). Thus we know $\mu^n(\cdot)$ converges in d_W to a locally absolutely curve $\mu(\cdot)$ and $\mu(\cdot)$ satisfies the same energy dissipation (5.4) by similar lower semicontinuity arguments. By the λ-geodesic convexity and C^1 regularity of W and V, we can then show the following chain rule for $\mu(\cdot)$,

$$\begin{aligned}\frac{d}{dt}\mathscr{E}(t,\mu(t)) \geq \frac{1}{2}\int_{\Omega\times\Omega} \frac{\partial W}{\partial t}(t,x-y)d\mu(t,x)d\mu(t,y) &+ \int_\Omega \frac{\partial V}{\partial t}(t,x)d\mu(t,x) \\ &- \int_\Omega \langle P_x(v(t,x)), \tilde{v}(t,x)\rangle d\mu(t,x).\end{aligned} \tag{5.5}$$

Combining (5.4) with (5.5), we show that $\mu(\cdot)$ is a weak measure solution to (1.1) satisfying (1.8) and (1.9). Then (5.3) comes from the stability argument of the time-independent setting.

6. Aggregation on nonconvex domains

In this section, we consider the following question: what are the conditions on Ω to ensure the existence of an interaction potential W such that the solution $\mu(\cdot)$ to (1.1) aggregates to a singleton (delta mass) as time goes to infinity?

Let Ω be bounded and η-prox-regular, $V \equiv 0$ and W satisfy (A1) for some $\lambda_W > 0$, such that Theorem 1.5 holds and we have a weak measure solution $\mu(\cdot)$ to (1.1). We recall $\Xi = \{\delta_x : x \in \mathbb{R}^d\}$ the set of singletons, and start to estimate the evolution of $d_W(\mu(\cdot),\Xi)$, the distance of $\mu(\cdot)$ to Ξ. That is we prove Proposition 1.7.

Proof. It suffices to show that for all $t \geq 0$

$$\frac{1}{2}\frac{d^+}{dt}d_W^2\left(\mu(t),\Xi\right) \leq \left(-\lambda_W + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{2\eta}\right) d_W^2\left(\mu(t),\Xi\right)$$

since then by Gronwall's inequality the result follows.

By shifting time we can assume that $t = 0$. Denote the center of mass for μ_0 by $\bar{x}$, that is $\bar{x} = \int_\Omega x d\mu(0,x)$. It is direct computation to show that $d_W(\mu(0),\Xi) = d_W\left(\mu(0),\delta_{\bar{x}}\right)$, and for any $t > 0$, $d_W(\mu(t),\Xi) \leq d_W\left(\mu(t),\delta_{\bar{x}}\right)$. Thus

$$\begin{aligned}
\frac{1}{2}\frac{d^+}{dt}\bigg|_{t=0} d_W^2\left(\mu(t),\Xi\right) &\leq \frac{1}{2}\frac{d^+}{dt}\bigg|_0 d_W^2\left(\mu(t),\delta_{\bar{x}}\right) \\
&= \int_\Omega \langle P_x\left(v(0,x)\right), x-\bar{x}\rangle \, d\mu(0,x) \\
&= \int_\Omega \left(\langle v(0,x), x-\bar{x}\rangle + \langle P_x\left(v(0,x)\right) - v(0,x), x-\bar{x}\rangle\right) d\mu(0,x).
\end{aligned}$$

Now we follow similar argument as in the proof of Proposition 3.1. To be precise, by (3.5) with $\mu^m(t) = \delta_{\bar{x}}$, we have

$$\begin{aligned}
\int_\Omega \langle v(0,x), x-\bar{x}\rangle \, d\mu(0,x) &\leq -\frac{\lambda_W}{2}\int_{\Omega\times\Omega} |x-y|^2 d\mu(0,x)d\mu(0,y) \\
&= -\lambda_W \int_\Omega |x-\bar{x}|^2 d\mu(0,x) \\
&= -\lambda_W d_W^2\left(\mu(0),\delta_{\bar{x}}\right),
\end{aligned}$$

where we used the fact that $\int_\Omega (x-\bar{x})d\mu(0,x) = 0$ for the definition of center of mass.

Also by (3.6) with $\mu^m(t) = \delta_{\bar{x}}$,

$$\int_{\Omega\times\Omega} \langle P_x\left(v(0,x)\right) - v(0,x), x-\bar{x}\rangle d\mu(0,x) \leq \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{2\eta} d_W^2\left(\mu(0),\delta_{\bar{x}}\right).$$

Combine the estimates yields

$$\begin{aligned}
\frac{1}{2}\frac{d^+}{dt}\bigg|_{t=0} d_W^2\left(\mu(t),\Xi\right) &\leq \left(-\lambda_W + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{2\eta}\right) d_W^2\left(\mu(0),\delta_{\bar{x}}\right) \\
&= \left(-\lambda_W + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{2\eta}\right) d_W^2\left(\mu(0),\Xi\right).
\end{aligned}$$

□

We now prove Theorem 1.8.

Proof. It turns out that the quadratic interaction potential leads to the sharpest bound for general domains. Furthermore, since multiplying a potential by a positive constant only leads to a constant rescaling in time of the dynamics, we consider $W(x) = \frac{1}{2}|x|^2$. To verify the inequality (1.13) note that $\nabla W(x) = x, \operatorname{Hess} W(x) = \mathrm{Id}$ and $\lambda_W = 1$. Thus $\sup_{\Omega-\Omega}|\nabla W| \leq \sup_{x,y\in\Omega}|x-y| = \operatorname{diam}(\Omega)$ and

$$-\lambda_W + \frac{\|\nabla W\|_{L^\infty(\Omega-\Omega)}}{2\eta} \leq -1 + \frac{1}{2\eta}\operatorname{diam}(\Omega) =: C(\Omega) < 0$$

which via inequality (1.13) implies the desired result. □

Remark 6.1. We notice that (1.13) implies that $\lim_{t\to\infty} d_W\left(\mu(t),\delta_{\bar{x}(t)}\right)=0$ where $\bar{x}(t)=\int_\Omega x d\mu(t,x)$ is the center of mass for $\mu(t)$. Hence as $t\to\infty$, $\mu(t)$ converges in d_W to a singleton, i.e., all mass aggregates to one point to form a delta mass of size 1. Thus Theorem 1.8 gives a sufficient condition on the shape of the domain Ω on which there exists a radially symmetric interaction potential W so that solutions aggregate to a point. We note that the simple condition given in the theorem is also sharp in the following sense: for any $\varepsilon>0$ there exists Ω bounded and η-prox-regular with $0<\eta\le(\frac{1}{2}-\varepsilon)\operatorname{diam}(\Omega)$, and an initial condition μ_0 such that the solution starting from μ_0 does not aggregate to a point.

Let $\Omega=\{(r\cos\theta,r\sin\theta)\in\mathbb{R}^2 : 1-\varepsilon\le r\le 1, -\varepsilon\le\theta\le\pi+\varepsilon\}$ for $0<\varepsilon<\frac{1}{2}$ be as shown in Figure 3. Let $x^1=(-(1-\varepsilon)\cos\varepsilon,-(1-\varepsilon)\sin\varepsilon), x^2=((1-\varepsilon)\cos\varepsilon,-(1-\varepsilon)\sin\varepsilon)$ and set $\mu_0=\frac{1}{2}\delta_{x^1}+\frac{1}{2}\delta_{x^2}$. Then Ω is η-prox-regular with $\eta=|x^1-x^2|/2>1-2\varepsilon$. Since $\operatorname{diam}(\Omega)=2$, thus $(\frac{1}{2}-2\varepsilon)\operatorname{diam}(\Omega)<\eta<\frac{1}{2}\operatorname{diam}(\Omega)$. For any radially symmetric W which satisfies (A1) for some $\lambda_W>0$, a direct calculation yields that $v(0,x^1)=-\frac{1}{2}\nabla W(x^1-x^2)\in N(\Omega,x^1)$. Thus $P_{x^1}\left(v(0,x^1)\right)=0$ and similarly $P_{x^2}\left(v(0,x^2)\right)=0$. We then see that $\mu(t)\equiv\mu_0$ is the solution to (1.1), and hence aggregation to a singleton, (1.14), does not hold.

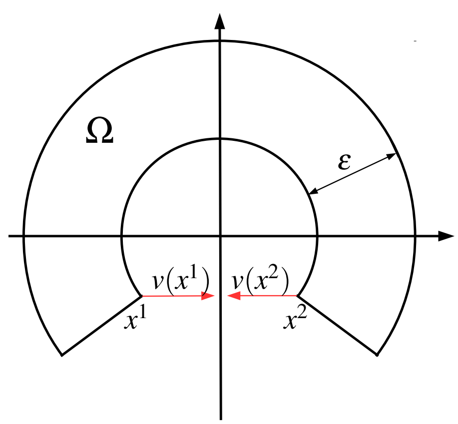

FIGURE 3. The velocity v at x^1 and x^2 are shown as the red arrows, which lie in the normal cones of the points respectively.

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