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Numerical Approximations of Problems That Arise in Elasticity

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Abstract

Two problems arising from elasticity are investigated in this report. The first one involves the nonstandard mixed finite element formulations of linear elasticity problems for which we demonstrate a necessary and sufficient condition for a subspace where existence and uniqueness of solutions are guaranteed. In a numerical setting, a stable boundary finite element is constructed that improves the approximation of boundary conditions. A numerical example is conducted to show its efficacy.

The second problem is a mathematical model that simulates ground motion during an earthquake where dislocation occurs in a thin fault region. We illustrate that, under appropriate scaling, solutions of this problem can be approximated by solutions of a limit problem where the fault region reduces to a surface. In a numerical context, the reduced model eliminates the need to resolve the large deformation in the fault region. A numerical example is presented to exhibit the effectiveness of this strategy.[1]

[1] Numerical examples were implemented using "NjwFem" a C++ finite element package developed by Noel Walkington. ParaView was used for graphing.

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Chapter 1

Overview

The classical elasticity equation takes the form

$$\rho u_{tt} - \mathrm{div}(\mathbb{C}(\nabla u)) = f, \tag{1.1}$$

where $u(t,x) \in \mathbb{R}^d$ and $\rho(t,x) > 0$ are the displacement and density of the medium, $f(t,x) \in \mathbb{R}^d$ the force per unit mass. The equation of plasticity, on the other hand, is a more complex problem in the sense that it involves a nonlinear term through the deformation tensor. A class of plasticity models take the following well-defined form [1, 2, 3]

$$\begin{gathered} \rho u_{tt} - \mathrm{div}(\mathbb{C}(U)) = f \\ U_t - \nabla u_t + \mathrm{curl}(U) \times v_d = 0, \end{gathered}$$

where v_d is the dislocation velocity vector and $\mathrm{curl}(U) \times v_d$ is highly nonlinear. Contemporarily, there is no satisfactory existence and uniqueness results to this nonlinear problem [2]. Nevertheless, the linear part of these equations has the structural form of a mixed formulation

$$\begin{gathered} \rho u_{tt} - \mathrm{div}(\mathbb{C}(U)) = f \\ U_t - \nabla u_t = 0, \end{gathered}$$

and for the stationary case

$$\begin{gathered} -\mathrm{div}(\mathbb{C}(U)) = f \\ U - \nabla u = 0. \end{gathered}$$

This is a nonstandard mixed formulation of the elasticity equations for which a standard finite element approach in general fails. Reference [5] shows that the failure can be extremely subtle. Convergence to smooth solutions sometimes occurs, but if the exact solution is not smooth (as for the plasticity problem) then it diverges. In this context, we are motivated to explore the following two elasticity problems in this report.

The first problem we study is the finite element method for mixed and saddle point problems in the linear elasticity setting (Chapter 2). Reference [6] and [19] contain an analysis of the existence, uniqueness, stability and approximations of the saddle point problem: find $(u, w) \in U \times V$ such that

$$a(w, z) - b_1(u, z) = (f, z)_{U' \times U}, \quad \forall z \in U \tag{1.2}$$

$$b_2(v, w) = (g, v)_{V' \times V}, \quad \forall v \in V \tag{1.3}$$

where U, V are Hilbert spaces; $a : V \times U \to \mathbb{R}$, $b_1 : U \times U \to \mathbb{R}$ and $b_2 : V \times V \to \mathbb{R}$ are bilinear forms; $f \in U'$ and $g \in V'$ are given data. Here U' and V' are the dual spaces of U and V respectively. In Section 2.1, we show that the weak statement of linear elasticity (2.3) has essentially the same structure as the saddle point problem (1.2) and (1.3). In addition, we construct a subspace in which a solution exists and is unique. In Section 2.1.2, we state and prove a theorem clarifying the necessary and sufficient conditions for the Babuška-Brezzi criterion. Moreover, stable elements enabling higher order of convergence and numerical experiments are produced in Section 2.2.

The second problem we investigate is the modeling and prediction of material failure which has been a challenging problem from both theoretical and experimental point of view [10]. Generally speaking, the onset of failure is modeled by crack formation (brittle failure) or modifications of elasticity theory to admit plastic deformation (ductile failure). A major difficulty with modeling of brittle failure is predicting propagation paths of cracks [15]. The well-known mathematical approach to this issue is the variational techniques initiated by Ambrosio and Braides [4], which uses the spaces of bounded variation and extension to evolution problems by Francfort and Marigo [11] using the concept of quasi-static formulations to construct minimizing movements [9, 18, 8, 25]. The multi-layer problem that will be studied in Chapter 3 is a model that simulates the subsurface interaction between elastic materials and plastic regions (Figure 1.2). Since displacements and deformations are small in the elastic bulk Ω_ϵ, small deformation linear elasticity equation dictates the motion in this region. In the region S_ϵ, however, because of the presence of defects, fault and dislocations, plasticity theory governs the dynamics in this layer [27]. We are interested in the limiting behavior of these interactive layers when the size of the fault region ϵ turns to zero. One natural question arising from the asymptotic analysis is the relative sizes of the shear strain $u_{2x} + u_{1y}$ and γ, where γ satisfies a constitutive law (Section 1.2.2). This important scaling question is addressed in Section 3.1. In addition, gamma convergence of the energy at ϵ-scale to the one at the limiting scale is established in Section 3.2. By studying the limiting problems, one can get around the numerical difficulties encountered with direct simulation of the ϵ-scale problems which in general require extremely fine mesh to resolve the large deformations in the fault region S_ϵ. Moreover, we also exhibit the fact that the jump of stress in the normal direction being zero and the average stress in the normal direction being proportional to $[u_1] - \gamma$ where u_1 is the horizontal displacement. In Section 3.3, we illustrate existence and convergence for evolution equations. Finally, numerical examples, in both uncoupled and coupled cases, are presented in Section 3.4 demonstrating that the limiting sharp interface problem has higher accuracy and lower computational costs compared to the ϵ-problem.

1.1 Notation and Function Spaces

Standard notation is adopted for the Lebesgue spaces, $L^p(\Omega)$, and the Sobolev space $H^1(\Omega)$. Solutions of the evolution equations will be viewed as functions from $[0, T]$ into these spaces, and we use the usual notation, $L^2[0, T; H^1(\Omega)]$ and $C[0, T; H^1(\Omega)]$, etc. to indicate the temporal regularity. Strong and weak convergence in these spaces is denoted as $u^\epsilon \to u$ and $u^\epsilon \rightharpoonup u$ respectively. Divergences of vector and matrix valued functions are denoted by

$$\operatorname{div}(u) = \sum_i \frac{\partial u_i}{\partial x_i} = u_{i,i},$$

$\operatorname{div}(T) : \Omega \to \mathbb{R}^d$ is defined as

$$\operatorname{div}(T)_i = \sum_{j=1}^{d} \frac{\partial T_{ij}}{\partial x_j} = T_{ij,j},$$

and $\operatorname{curl}(\cdot)$ operator is defined by

$$\operatorname{curl}(U)_{mn} = \epsilon_{ijn} U_{mj,i},$$

where summation convention is used for repeated indices. ϵ_{ijn} is 1 if (i, j, n) is an even permutation of $(1, 2, 3)$, -1 if it is an odd permutation, and 0 if any index is repeated. Gradients of vector valued quantities are interpreted as matrices, $(\nabla u)_{ij} = u_{i,j}$, and the symmetric part of the gradient is written as $D(u) = (1/2)\big(\nabla u + (\nabla u)^\top\big)$, with the skew symmetric part expressed as $(\nabla u)^{skew} = (1/2)\big(\nabla u - (\nabla u)^\top\big)$. Inner products are typically denoted as pairings $(\cdot, \cdot)$ or, for clarity, the dot product of two vectors $v, w \in \mathbb{R}^d$ may be written as $v \cdot w = v_i w_i$ and Frobenious inner product of two matrices $A, B \in \mathbb{R}^{d \times d}$ as $A : B = A_{ij} B_{ij}$. Throughout this report, unless otherwise specified, C and c denote a generic constant depending on the context. Let's begin with recalling basic mathematical settings for linear elasticity.

1.2 Linear Elasticity

1.2.1 Basic Mathematical Settings

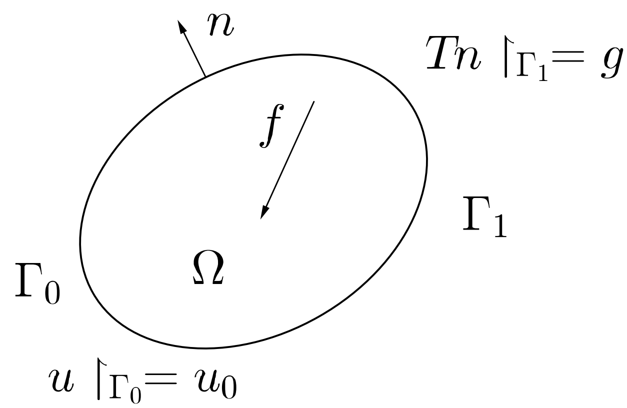

Figure 1.1: *Domain and boundary values for linear elasticity equations*

The equations of linear elasticity are used ubiquitously by engineers to design essentially every manufactured item ranging from machine parts to buildings and structures [26]. Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$. Ω represents the stress free position of an isotropic linearly elastic material subject to a body force, $f : \Omega \to \mathbb{R}^d$, and surface forces, $g : \Gamma_1 \to \mathbb{R}^d$, on Γ_1. The displacement $u : \Omega \to \mathbb{R}^d$ of Ω satisfies $u \in H^1(\Omega)^d$ and

$$\begin{aligned} -\mathrm{div}(T) &= f, \quad \text{in } \Omega \\ u\restriction_{\Gamma_0} = u_0, &\quad Tn\restriction_{\Gamma_1} = g \end{aligned} \tag{1.4}$$

where $u_0 : \Gamma_0 \to \mathbb{R}^d$ is a prescribed displacement and $T : \Omega \to \mathbb{R}^{d\times d}$ is the stress tensor which can be written as

$$T = \lambda\,\mathrm{div}(u) I + 2\mu D(u).$$

Here λ, $\mu : \Omega \to \mathbb{R}$ characterize material properties at $x \in \Omega$. The symmetric part of ∇u is denoted by $D(u) = (1/2)\big(\nabla u + (\nabla u)^\top\big)$. Let $U = \{v \in H^1(\Omega)^d \mid v\restriction_{\Gamma_0} = 0\}$ and $u_0 + U = \{u \in H^1(\Omega)^d \mid u\restriction_{\Gamma_0} = u_0\}$. Multiplying equation (1.4) by a test function $v \in U$ and integrating by parts, we arrive at the following weak statements: find $u \in u_0 + U$ such that

$$a(u, v) = F(v), \qquad \text{for all } v \in U,$$

where

$$a(u, v) = \int_\Omega \lambda \operatorname{div}(u) \operatorname{div}(v) + 2\,\mu D(u) : D(v), \qquad F(v) = \int_\Omega f \cdot v + \int_{\Gamma_1} g \cdot v.$$

1.2.2 Derivation of Governing Equations from a Plasticity Model

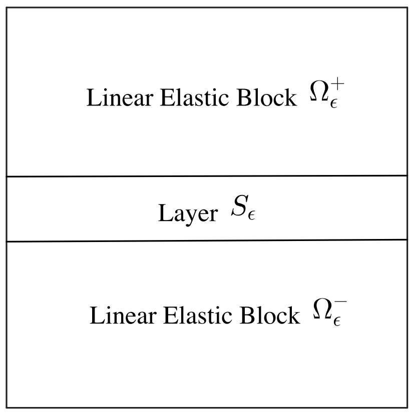

Figure 1.2: *The layer problem with elastic bulk* $\Omega_\epsilon = \Omega_\epsilon^+ \cup \Omega_\epsilon^-$ *and fault layer* S_ϵ

We introduce the following layer problem (Figure 1.2), which is one of the major problems we are going to solve. It will be studied in details in Chapter 3. Let $\Omega = (-1,1)^2$ be a subsurface domain consisting of a fault region $S_\epsilon = (-1,1) \times (-\epsilon/2, \epsilon/2)$ and an elastic bulk $\Omega_\epsilon = \Omega \backslash \bar{S}_\epsilon$. Displacement and gradients are assumed to be small in the elastic bulk Ω_ϵ so that the motion is governed by the equations of linear elasticity,

$$\rho u_{tt} - \operatorname{div}(\mathbb{C}(\nabla u)) = \rho f, \qquad \text{in } \Omega_\epsilon.$$

Small displacement plasticity theory models the motion in the fault region S_ϵ. In this theory the elastic deformation tensor, U, deviates from ∇u due to slips, dislocations and motion of defects. The balance of linear momentum becomes

$$\rho u_{tt} - \operatorname{div}(\mathbb{C}(U)) = \rho f, \qquad \text{in } S_\epsilon,$$

and the evolution of U is governed by an equation of the form

$$U_t - \nabla u_t + \operatorname{curl}(U) \times v_d = 0, \tag{1.5}$$

where v_d is a constitutively specified defect velocity. In this equation the curl$(\cdot)$ and cross product of a matrix act row-wise:

$$\mathrm{curl}(U)_{mn} = \epsilon_{ijn} U_{mj,i}, \qquad \text{and} \qquad (A \times v)_{mn} = \epsilon_{ijn} A_{mi} v_j.$$

The defect velocity v_d is chosen to model the typically large dissipation due to defect motion, and local energy changes due to distortion in the material during passage of a defect. The axial vector of the skew-symmetric part of a matrix A is denoted by $X(A)$, that is

$$X(A)_i = \epsilon_{ijk} A_{jk}, \qquad \text{or} \qquad A^{skew} v = X(A) \times v, \ \forall v \in \mathbb{R}^3,$$

where $A^{skew} = (1/2)\big(A - A^\top\big)$ is the skew-symmetric part of A. The following lemma establishes a representation of the governing equation for the plastic shear strain $\gamma : (0,T) \times S_\epsilon \to \mathbb{R}$ in terms of the defect velocity v_d, namely

$$\gamma_t + \gamma_x (v_d \cdot e_1) = 0,$$

where $e_1 = (1,0,0)^\top$ is the unit vector along the x-axis. This is another way of writing equation (1.5).

Lemma 1. *Let $\eta : \mathbb{R} \to \mathbb{R}$ and $\beta, \gamma, T_{12} : (0,T) \times S_\epsilon \to \mathbb{R}$ be smooth, $\nu \in \mathbb{R}$, and suppose that*

$$\gamma_t + \beta \gamma_x^2 \big(\eta'(\gamma) - \nu \gamma_{xx} - T_{12}\big) = 0, \quad \text{on } (0,T) \times S_\epsilon.$$

Let

$$u : (0,T) \times S_\epsilon \to \mathbb{R}^2 \hookrightarrow \mathbb{R}^3, \qquad \text{and} \qquad T : (0,T) \times S_\epsilon \to \mathbb{R}^{2\times 2}_{sym} \hookrightarrow \mathbb{R}^{3\times 3}_{sym}$$

be smooth with T_{12} as above, and let

$$U = \nabla u - \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad v_d = \left(I - \frac{\omega}{|\omega|} \otimes \frac{\omega}{|\omega|} \right) X\left(S\,\mathrm{curl}(U)\right),$$

where

$$S = \left(T - \begin{bmatrix} 0 & 2\eta'(\gamma) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\nu\, \mathrm{curl}(\mathrm{curl}(U)) \right)^{sym} \qquad \text{and} \qquad \omega = X(\mathrm{curl}(U)).$$

Then the triple (u, U, v_d) satisfies equation (1.5).

Under the anzatz of the lemma the matrices curl(U) and curl(curl(U)) become

$$\mathrm{curl}(U) = \begin{bmatrix} 0 & 0 & -\gamma_x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathrm{curl}(\mathrm{curl}(U)) = \begin{bmatrix} 0 & \gamma_{xx} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and vectors $X(\mathrm{curl}(U))$ and v_d are

$$X(\mathrm{curl}(U)) = \begin{pmatrix} 0 \\ \gamma_x \\ 0 \end{pmatrix}, \qquad v_d = \begin{pmatrix} \beta\gamma_x\big(\eta'(\gamma) - \nu\gamma_{xx} - T_{12}\big) \\ 0 \\ 0 \end{pmatrix},$$

In Chapter 3, we will demonstrate Γ-convergence of the stationary operator and interesting properties satisfied by the limiting stress tensor. In the following section, we review Lax-Milgram type theorems and Korn Poincaré type inequalities which will be cited repeatedly for well-posedness purposes.

1.3 Lax-Milgram Theorems and Basic Inequalities

We begin with some fundamental existence and uniqueness theorems that would be used later in this report. These well-known results are taken from [26, Theorem 4.2.2].

1.3.1 The Generalized Lax-Milgram Theorem

Theorem 2. *Let U be a Banach space, V be a Hilbert space and $a : U \times V \to \mathbb{R}$ be bilinear and continuous; namely, there exists $C > 0$ such that $|a(u,v)| \leq C\|u\|_U\|v\|_V$ for all $u \in U$ and $v \in V$. For $c > 0$, the following are equivalent:*

- *(Coercivity) For each $u \in U$,*

  $$\sup_{0 \neq v \in V} \frac{a(u,v)}{\|v\|_V} \geq c\|u\|_U,$$

  and for each $v \in V\backslash\{0\}, \sup_{u\in U} a(u,v) > 0$.

- *(Existence of Solutions) For each $f \in V'$, there exists a unique $u \in U$ such that*

  $$a(u,v) = f(v), \quad \forall v \in V,$$

  and $\|u\|_U \leq (1/c)\|f\|_{V'}$.

- *(Existence of Solutions for the Adjoint Problem) For each $g \in U'$, there exists a unique $v \in V$ such that*

  $$a(u,v) = g(u), \quad \forall u \in U,$$

  and $\|v\|_V \leq (1/c)\|g\|_{U'}$.

Corollary 3. *If U is also a Hilbert space then the coercivity property in Theorem 2 is equivalent to*

- *(Adjoint Coercivity) For each $u \in V$,*

$$\sup_{0 \neq u \in U} \frac{a(u, v)}{\|u\|_U} \geq c\|v\|_V,$$

for each $u \in U \backslash \{0\}$, $\sup_{v \in V} a(u, v) > 0$.

Corollary 4. *(Lax Milgram Lemma, 1954) Let U be a Hilbert space and $a : U \times U \to \mathbb{R}$ be bilinear and $f : U \to \mathbb{R}$ be linear and satisfy*

- *(Continuity) There exists $C > 0$ such that*

$$|a(u, v)| \leq C\|u\|_U \|v\|_U, \quad u, v \in U,$$

and there exists a constant $M > 0$ such that $f(v) \leq M\|v\|_U$ for $v \in U$.

- *(Coercivity) There exists $c > 0$ such that $|a(u, u)| \geq c\|u\|_U^2$ for all $u \in U$.*

Then there exists a unique $u \in U$ such that

$$a(u, u) = f(v), \quad \forall v \in U;$$

moreover, $\|u\|_U \leq M/c$.

1.3.2 Korn and Poincaré Inequalities

In linear elasticity theory, Korn's inequality is a very important tool as an a priori estimate. The symmetric part of the gradient is a measure of the strain that an elastic body experiences when it is deformed by a given vector-valued function. References [20, 13, 14, 21, 17] contain more detailed analysis on inequalities of Korn and Poincaré type and their applications to linear elasticity. The following version of Korn's inequality taken from [26] is the most convenient in our setting.

Lemma 5. *(Korn's Inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $\Gamma \subset \partial\Omega$ have positive $(d-1)$ measure. Let $u \in \{u \in H^1(\Omega)^d \mid u\restriction_\Gamma = 0\}$. Then there exists a constant C_k depending only upon Ω and Γ such that*

$$\|u\|_{H^1(\Omega)} \leq C_k \|D(u)\|_{L^2(\Omega)},$$

where $D(u) = (1/2)\big(\nabla u + (\nabla u)^\top\big)$ is the symmetric part of ∇u.

Lemma 6. *(Poincaré inequality) Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open Lipschitz domain. Assume $u \in H^1(\Omega)^d$. Then there exists a constant C, depending only upon Ω, such that*

$$\|u - u_\Omega\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)},$$

where $u_\Omega = \big(1/|\Omega|\big) \int_\Omega u$ denotes the average of u over Ω.

Chapter 2

Mixed Formulations of Elasticity Problems

The main purpose of this chapter is to establish Theorem 13 which characterizes a necessary and sufficient condition for a subspace in which existence and uniqueness of solutions are ensured. We start with rewriting the equations of linear elasticity in terms of mixed formulations. In addition, we illustrate that our numerical approximation gives higher order of accuracy for the interior elements than the boundary elements using a simple one dimensional example (Section 2.2.1). Moreover, analysis of the asymptotic expansion for both momentum and constraint equations are given. Finally, we introduce a new boundary element which gives improved order of convergence for the problem with boundary conditions.

2.1 Saddle Point Problems

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. Let $u \in H_0^1(\Omega)^d$ and $W = \nabla u$. The equations of linear elasticity can be written as the following system: find $W \in L^2(\Omega)^{d\times d}$ and $u \in H_0^1(\Omega)^d$ such that

$$\left\{ \begin{array}{rl} -\operatorname{div} \mathbb{C}(W) & = f \\ W - \nabla u & = 0 \end{array} \right. , \tag{2.1}$$

where $f \in L^2(\Omega)^d$ is a specified external force and $\mathbb{C} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ denotes the classical isotropic elasticity tensor with shear and bulk moduli μ and λ,

$$\mathbb{C}(D) = \mu(D + D^\top) + \lambda \operatorname{tr}(D) I, \tag{2.2}$$

for $\mu > 0$, $\lambda > 0$ and $D \in \mathbb{R}^{d\times d}$. It is straight forward to rewrite (2.1) as a saddle point problem

$$\left\{ \begin{array}{rll} a(W, Z) - b(u, Z) & = 0, & \forall Z \in L^2(\Omega)^{d\times d} \\ b(v, \mathbb{C}(W)) & = (f, v)_{L^2(\Omega)^d}, & \forall v \in H_0^1(\Omega)^d. \end{array} \right. \tag{2.3}$$

where

$$a(W, Z) = \int_\Omega W : Z, \qquad \text{for } W, Z \in L^2(\Omega)^{d\times d},$$

and

$$b(u, W) = \int_\Omega \nabla u : W, \qquad \text{for } (u, W) \in H_0^1(\Omega)^d \times L^2(\Omega)^{d\times d}.$$

Letting $\mathcal{V} = H_0^1(\Omega)^d \times L^2(\Omega)^{d\times d}$, with $\|(u, W)\|_{\mathcal{V}}^2 = \|u\|_{H^1(\Omega)^d}^2 + \|W\|_{L^2(\Omega)^{d\times d}}^2$ for $(u, W) \in \mathcal{V}$, it is convenient to introduce the bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined by

$$A((u, W), (v, Z)) = \int_\Omega \mathbb{C}(W) : \nabla v + W : Z - \nabla u : Z \tag{2.4}$$

$$= a(W, Z) - b(u, Z) + b(v, \mathbb{C}(W)), \tag{2.5}$$

where $(u, W) \in \mathcal{V}$, $(v, Z) \in \mathcal{V}$. Assuming that A is continuous and coercive, it follows that $\forall h \in \mathcal{V}'$, by the Lax-Milgram theorem (Theorem 2), there exists a unique $(u, W) \in \mathcal{V}$ such that

$$A((u, W), (v, Z)) = h((v, Z)), \quad \forall (v, Z) \in \mathcal{V}, \tag{2.6}$$

and $\|(u, W)\|_{\mathcal{V}} \le C\|g\|_{\mathcal{V}'}$ for some constant $C > 0$. In particular, define $h \in \mathcal{V}'$ by

$$h((v, Z)) = \int_\Omega f \cdot v, \qquad \forall (v, Z) \in \mathcal{V},$$

where $f \in H^{-1}(\Omega)^d$. Then we can find a unique $(u, W) \in \mathcal{V}$, such that (2.6) holds. Namely,

$$a(W, Z) - b(u, Z) + b(v, \mathbb{C}(W)) = \int_\Omega f \cdot v, \qquad \forall (v, Z) \in \mathcal{V}.$$

In particular, picking $Z = 0$, we have

$$b(v, \mathbb{C}(W)) = \int_\Omega f \cdot v, \quad \forall v \in H_0^1(\Omega)^d, \tag{2.7}$$

and choosing $v = 0$, we have

$$a(W, Z) - b(u, Z) = 0, \quad \forall Z \in L^2(\Omega)^{d\times d}. \tag{2.8}$$

Equations (2.7) and (2.8) are exactly the saddle point problem (2.3).

2.1.1 Continuity and Coercivity

The following section is dedicated to prove that the bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ defined in (2.4) satisfies the hypothesis of Lax-Milgram theorem. It is clear that the tensor $\mathbb{C}$ defined in (2.2) induces a semi-norm $|D|_{\mathbb{C}}^2 \equiv \mathbb{C}(D) : D$ for $D \in \mathbb{R}^{d\times d}$, and

$$|D|_{\mathbb{C}}^2 \ge 2\mu |D|^2, \qquad \text{for symmetric matrix } D \in \mathbb{R}^{d\times d}, \tag{2.9}$$

where $|D|$ is the Frobenius norm of D.

Continuity

Theorem 7. *Bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is continuous.*

Proof. We have

$$\begin{aligned}
\left|\int_\Omega \mathbb{C}(W) : \nabla v\right| &= \left|\int_\Omega (\mu(W + W^\top) + \lambda \mathrm{tr}(W) I) : \nabla v\right| \\
&\leq \mu \left|\int_\Omega W : \nabla v + \int_\Omega W^\top : \nabla v\right| + \lambda \left|\int_\Omega \mathrm{tr}(W) I : \nabla v\right| \\
&\leq 2\mu \|W\|_{L^2(\Omega)^{d\times d}} \|\nabla v\|_{L^2(\Omega)^{d\times d}} + \lambda \left|\int_\Omega \mathrm{tr}(W) \mathrm{div}(v)\right| \\
&\leq (2\mu + \sqrt{d}\lambda) \|W\|_{L^2(\Omega)^{d\times d}} \|\nabla v\|_{L^2(\Omega)^{d\times d}}, \qquad (2.10)
\end{aligned}$$

where we used the fact that

$$\|\mathrm{tr}(W)\|^2_{L^2(\Omega)} = \int_\Omega \Big(\sum_{i=1}^d w_{ii}\Big)^2 \leq \int_\Omega d \sum_{i=1}^d w_{ii}^2 \leq d \|W\|^2_{L^2(\Omega)^{d\times d}}.$$

Now, plugging (2.10) into (2.4), we get

$$\begin{aligned}
|A((u, W), (v, Z))| &\leq \left|\int_\Omega \mathbb{C}(W) : \nabla v\right| + \left|\int_\Omega W : Z\right| + \left|\int_\Omega \nabla u : Z\right| \\
&\leq (2\mu + \sqrt{d}\lambda) \|W\|_{L^2(\Omega)^{d\times d}} \|\nabla v\|_{L^2(\Omega)^{d\times d}} + \|W\|_{L^2(\Omega)^{d\times d}} \|Z\|_{L^2(\Omega)^{d\times d}} \\
&\quad + \|\nabla u\|_{L^2(\Omega)^{d\times d}} \|Z\|_{L^2(\Omega)^{d\times d}} \\
&\leq (2\mu + \sqrt{d}\lambda + 2) \|(u, W)\|_{\mathcal{V}} \|(v, Z)\|_{\mathcal{V}}.
\end{aligned}$$

□

Coercivity

To prove coercivity, let's prove some preliminary lemmas which will be used in the sequel.

Lemma 8. *Bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfies*

$$A((u, W), (u, \mathbb{C}(W))) = \int_\Omega |W^{sym}|^2_{\mathbb{C}}.$$

Proof. Notice that $\mathbb{C}(W) = \mathbb{C}(W^{sym})$ and $\mathbb{C}(W) : W = \mathbb{C}(W) : W^\top$, we have that

$$\begin{aligned}
|\mathbb{C}(W^{sym})|^2 &= \mathbb{C}(W^{sym}) : W^{sym} = \mathbb{C}(W) : (W + W^\top)/2 \\
&= \mathbb{C}(W) : W \\
&= |W|^2_{\mathbb{C}}
\end{aligned}$$

If follows that, $LHS = \int_\Omega \mathbb{C}(W) : \nabla u + W : \mathbb{C}(W) - \nabla u : \mathbb{C}(W) = \int_\Omega |W|^2_{\mathbb{C}} = \int_\Omega |W^{sym}|^2_{\mathbb{C}}$. □

Lemma 9. *Bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfies*

$$A((u, W), (0, -\mathbb{C}(\nabla u))) \geq \frac{1}{2} \int_\Omega \left(|(\nabla u)^{sym}|^2_{\mathbb{C}} - |W^{sym}|^2_{\mathbb{C}} \right).$$

Proof. Notice that

$$\begin{aligned} LHS &= \int_\Omega W : (-\mathbb{C}(\nabla u)) + \nabla u : \mathbb{C}(\nabla u) \\ &= \int_\Omega |\nabla u|^2_{\mathbb{C}} - \int_\Omega W : \mathbb{C}(\nabla u). \end{aligned}$$

Since $(W, \nabla u)_{\mathbb{C}} \leq \frac{1}{2}(W, W)_{\mathbb{C}} + \frac{1}{2}(\nabla u, \nabla u)_{\mathbb{C}}$, we know that

$$(1/2)(\nabla u, \nabla u)_{\mathbb{C}} - (W, \nabla u)_{\mathbb{C}} + (W, W)_{\mathbb{C}} \geq 0.$$

This is equivalent to

$$(1/2)|\nabla u|^2_{\mathbb{C}} - W : \mathbb{C}(\nabla u) + |W|^2_{\mathbb{C}} \geq 0.$$

Rearanging and integrating both sides, we get

$$\int_\Omega |\nabla u|^2_{\mathbb{C}} - \int_\Omega W : \mathbb{C}(\nabla u) \geq \frac{1}{2} \int_\Omega |\nabla u|^2_{\mathbb{C}} - |W|^2_{\mathbb{C}},$$

which is what we want to show. □

Lemma 10. *Bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ satisfies*

$$A((u, W), (0, W^{skew})) \geq \frac{1}{2} \int_\Omega \left| W^{skew} \right|^2 - K \left| \nabla u^{sym} \right|^2,$$

for some constant $K > 0$.

Proof.

$$\begin{aligned} LHS &= \int_\Omega \mathbb{C}(W) : 0 + W : W^{skew} - \nabla u : W^{skew} \\ &= \int_\Omega (W^{sym} + W^{skew}) : W^{skew} - \nabla u : W^{skew} \\ &= \int_\Omega |W^{skew}|^2 - \nabla u : W^{skew} \\ &\geq \frac{1}{2} \int_\Omega |W^{skew}|^2 - |\nabla u|^2 \qquad \text{since } \nabla u : W^{skew} \leq (1/2)(|\nabla u|^2 + |W^{skew}|^2) \\ &\geq \frac{1}{2} \int_\Omega |W^{skew}|^2 - K|\nabla u^{sym}|^2 \qquad \text{by Korn's inequality (Lemma 5).} \end{aligned}$$

□

Lemma 11. *Let $(v, Z) = (u, \lambda_0 \mathbb{C}(W) - \lambda_1 \mathbb{C}(\nabla u) + \lambda_2 W^{skew}))$ for some positive constant λ_0, λ_1 and λ_2, then there exists a constant $C > 0$, such that*

$$\|(v, Z)\|_{\mathcal{V}} \leq C \|(u, W)\|_{\mathcal{V}}.$$

Proof.

$$\begin{aligned}
\|(v,Z)\|_{\mathcal{V}}^2 &= \|(u, \lambda_0 \mathbb{C}(W) - \lambda_1 \mathbb{C}(\nabla u) + \lambda_2 W^{skew})\|_{\mathcal{V}}^2 \\
&= \|u\|_{H^1(\Omega)^d}^2 + \|\lambda_0 \mathbb{C}(W) - \lambda_1 \mathbb{C}(\nabla u) + \lambda_2 W^{skew}\|_{L^2(\Omega)^{d\times d}}^2 \\
&\leq \|u\|_{H^1(\Omega)^d}^2 + \big(\lambda_0 \|\mathbb{C}(W)\|_{L^2(\Omega)^{d\times d}} + \lambda_1 \|\mathbb{C}(\nabla u)\|_{L^2(\Omega)^{d\times d}} + \lambda_2 \|W^{skew}\|_{L^2(\Omega)^{d\times d}}\big)^2 \\
&\leq \|u\|_{H^1(\Omega)^d}^2 + (\lambda_0^2 + \lambda_1^2 + \lambda_2^2)\big(\|\mathbb{C}(W)\|_{L^2(\Omega)^{d\times d}}^2 + \|\mathbb{C}(\nabla u)\|_{L^2(\Omega)^{d\times d}}^2 \\
&\quad + \|W^{skew}\|_{L^2(\Omega)^{d\times d}}^2\big) \\
&\leq \|u\|_{H^1(\Omega)^d}^2 + (\lambda_0^2 + \lambda_1^2 + \lambda_2^2)\big((2\mu+\lambda)^2 \|W\|_{L^2(\Omega)^{d\times d}}^2 + (2\mu+\lambda)^2 \|\nabla u\|_{L^2(\Omega)^{d\times d}}^2 \\
&\quad + \|W\|_{L^2(\Omega)^{d\times d}}^2\big) \\
&\leq \big(1 + (\lambda_0^2 + \lambda_1^2 + \lambda_2^2)(2\mu+\lambda)^2\big) \|u\|_{H^1(\Omega)^d}^2 \\
&\quad + (\lambda_0^2 + \lambda_1^2 + \lambda_2^2)\big((2\mu+\lambda)^2 + 1\big) \|W\|_{L^2(\Omega)^{d\times d}}^2 \\
&\leq C^2 \|(u, W)\|_{\mathcal{V}}^2
\end{aligned}$$

where we used the inequality

$$\|W^{skew}\|^2 \leq (1/4)\|W - W^\top\|^2 \leq (1/4) \cdot 2(\|W\|^2 + \|W^\top\|^2) = \|W\|^2,$$

and

$$\begin{aligned}
\|\mathbb{C}(W)\| &= \|\mu(W + W^\top) + \lambda \mathrm{tr}(W) I\| \\
&\leq \mu\|W + W^\top\| + \lambda \|\mathrm{tr}(W) I\| \\
&\leq \mu(\|W\| + \|W^\top\|) + \lambda \|W\| \\
&= (2\mu + \lambda)\|W\|.
\end{aligned}$$

Similarly, $\|\mathbb{C}(\nabla u)\| \leq (2\mu + \lambda)\|\nabla u\|$. Here the subscripts of the norms are omitted when the context makes it possible. □

Theorem 12. *Bilinear form $A : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is coercive.*

Proof. To show coercivity, it is enough to show

$$\sup_{(v,Z)\in\mathcal{V}} \frac{A((u,W),(v,Z))}{\|(v,Z)\|_{\mathcal{V}}} \geq C\|(u,W)\|_{\mathcal{V}}, \tag{2.11}$$

and show for each $(v, Z) \neq 0$, we have that

$$\sup_{(u,W)\in\mathcal{V}} A((u,W),(v,Z)) > 0. \tag{2.12}$$

Applying Lemma 8, Lemma 9 and Lemma 10, we have

$$
\begin{aligned}
& A((u,W),(u,\lambda_0\mathbb{C}(W)-\lambda_1\mathbb{C}(\nabla u)+\lambda_2 W^{skew})) \\
= \; & \lambda_0 A((u,W),(u,\mathbb{C}(W)))+\lambda_1 A((u,W),(0,-\mathbb{C}(\nabla u)))+\lambda_2 A((u,W),(0,W^{skew})) \\
\geq \; & \lambda_0\int_\Omega |W^{sym}|^2_{\mathbb{C}}+\frac{\lambda_1}{2}\int_\Omega (|\nabla u^{sym}|^2_{\mathbb{C}}-|W^{sym}|^2_{\mathbb{C}})+\frac{\lambda_2}{2}\int_\Omega |W^{skew}|^2-K|\nabla u^{sym}|^2 \\
= \; & (\lambda_0-\lambda_1/2)\int_\Omega |W^{sym}|^2_{\mathbb{C}}+\frac{\lambda_1}{2}\int_\Omega |\nabla u^{sym}|^2_{\mathbb{C}}-\frac{\lambda_2 K}{2}\int_\Omega |\nabla u^{sym}|^2+\frac{\lambda_2}{2}\int_\Omega |W^{skew}|^2 \\
\geq \; & \left(\lambda_0-\frac{\lambda_1}{2}\right)\int_\Omega |W^{sym}|^2_{\mathbb{C}}+\left(\frac{\lambda_1}{2}-\frac{\lambda_2 K}{4\mu}\right)\int_\Omega |\nabla u^{sym}|^2_{\mathbb{C}}+\frac{\lambda_2}{2}\int_\Omega |W^{skew}|^2 \\
\geq \; & \left(\lambda_0-\frac{\lambda_1}{2}\right)\int_\Omega 2\mu|W^{sym}|^2+\left(\frac{\lambda_1}{2}-\frac{\lambda_2 K}{4\mu}\right)\int_\Omega |\nabla u^{sym}|^2_{\mathbb{C}}+\frac{\lambda_2}{2}\int_\Omega |W^{skew}|^2 \\
\geq \; & c\big(\|u\|^2_{H^1_0(\Omega)^d}+\|W\|^2_{L^2(\Omega)^{d\times d}}\big) \\
= \; & c\|(u,W)\|^2_{\mathcal{V}}
\end{aligned}
$$

for some positive constant c, where we used Korn's inequality and inequality (2.9), $|A|^2_{\mathbb{C}} \geq 2\mu|A|^2$ for $A \in \mathbb{R}^{d\times d}$ symmetric. Let

$$(v,Z)=(u,\lambda_0\mathbb{C}(W)-\lambda_1\mathbb{C}(\nabla u)+\lambda_2 W^{skew}).$$

Pick $\lambda_2=1, \lambda_1=1+\frac{K}{2\mu}, \lambda_0=1+\frac{K}{4\mu}$, we have that

$$A((u,W),(v,Z)) \geq (1/2)\|(u,W)\|^2_{\mathcal{V}}. \tag{2.13}$$

Note that by Lemma 11, we know that $\|(u,W)\|_{\mathcal{V}} \geq \frac{1}{C}\|(v,Z)\|_{\mathcal{V}}$. Therefore, we have

$$A((u,W),(v,Z)) \geq (1/(2C))\|(u,W)\|_{\mathcal{V}}\|(v,Z)\|_{\mathcal{V}}.$$

This implies that equation (2.11) holds. □

2.1.2 The Inf-Sup Condition

In this subsection, we state and prove the necessary and sufficient conditions for the existence and uniqueness of solutions in subspaces. The following theorem is the main result in this chapter.

Theorem 13. *Let $\mathbb{W} \subset L^2(\Omega)^{d\times d}$ be a subspace which is closed under transpose and $tr(W)I \in \mathbb{W}$ for all $W \in \mathbb{W}$. Let $\mathbb{U} \subset H^1(\Omega)^d$ with boundary conditions satisfying Korn and Poincaré hypothesis. Let the bilinear form $A : (H^1(\Omega)^d \times L^2(\Omega)^{d\times d}) \times (H^1(\Omega)^d \times L^2(\Omega)^{d\times d}) \to \mathbb{R}$ be defined by*

$$A((u,W),(v,Z))=\int_\Omega \mathbb{C}(W):\nabla v+W:Z-\nabla u:Z.$$

Then the restriction $A : (\mathbb{U} \times \mathbb{W}) \times (\mathbb{U} \times \mathbb{W}) \to \mathbb{R}$ *is coercive if and only if there exists a constant* $c_b > 0$*, such that*

$$\sup_{Z \in \mathbb{W}^{sym}} \frac{\int_\Omega \nabla u : Z}{\|Z\|_{L^2(\Omega)^{d\times d}}} \geq c_b \|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}} \tag{2.14}$$

for each $u \in \mathbb{U}$.

Proof. Assume (2.14) holds for every $u \in \mathbb{U}$. We want to show that $A : (\mathbb{U} \times \mathbb{W}) \times (\mathbb{U} \times \mathbb{W}) \to \mathbb{R}$ is coercive. The idea of proof is similar to Theorem 12. To prove the coercivity of A in this setting, it suffices to prove Lemma 9, 10 and 11 hold when restricted to subspaces $\mathbb{U}$, $\mathbb{W}$. Notice that Lemma 9 and 11 hold automatically. We only need to prove Lemma 10. By assumption, we know for each $u \in \mathbb{U}$,

$$\sup_{Z \in \mathbb{W}^{sym}} \frac{\int_\Omega \nabla u : Z}{\|Z\|_{L^2(\Omega)^{d\times d}}} \geq c_b \|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}.$$

By rescaling, we have that there exists $Z \in \mathbb{W}^{sym}$, with $\|Z\|_{L^2(\Omega)^{d\times d}} = \|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}$, and $\delta > 0$, such that

$$-\int_\Omega \nabla u : Z \geq (c_b - \delta)\|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}}.$$

For notational simplification, call $c_b = c_b - \delta$, so we have

$$-\int_\Omega \nabla u : Z \geq c_b \|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}}$$

which implies

$$-\int_\Omega (\nabla u)^{sym} : Z \geq c_b \|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}}. \tag{2.15}$$

Now, pick $Z \in \mathbb{W}^{sym}$ as above, then we have

$$\begin{aligned}
&A((u, W), (0, Z)) \\
&= \int_\Omega W : Z - \nabla u : Z \\
&= \int_\Omega W^{sym} : Z - (\nabla u)^{sym} : Z \\
&\geq \int_\Omega W^{sym} : Z \, dy \, dx + c_b \|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}} \qquad \text{by (2.15)} \\
&\geq -(1/(2\epsilon))\|W^{sym}\|^2_{L^2(\Omega)^{d\times d}} - (\epsilon/2)\|Z\|^2_{L^2(\Omega)^{d\times d}} + c_b \|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}} \\
&= -(1/(2\epsilon))\|W^{sym}\|^2_{L^2(\Omega)^{d\times d}} + (c_b - \epsilon/2)\|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}} \\
&\geq \int_\Omega c_1 \|(\nabla u)^{sym}\|^2_{\mathbb{C}} - c_2 \|W^{sym}\|^2_{\mathbb{C}}
\end{aligned}$$

where we used the fact that $\|\cdot\|_{L^2}$ and $\|\cdot\|_{\mathbb{C}}$ are equivalent norms in the last inequality. Then use the same idea in the proof of Theorem 12, we complete the proof of one direction.

To prove the other direction, assume the restriction $A: (\mathbb{U}\times\mathbb{W})\times(\mathbb{U}\times\mathbb{W})\to\mathbb{R}$ is coercive. Fix $u\in\mathbb{U}$, then for $(\nabla u)^{sym}$ and $F=0$, there exists $\tilde{u},\tilde{W}$ satisfying (2.7) and (2.8), namely

$$\begin{aligned}(\mathbb{C}(\tilde{W}),\nabla v) &= ((\nabla u)^{sym},\nabla v) \\ (\tilde{W}-\nabla\tilde{u},Z) &= 0, \qquad \forall v\in\mathbb{U},\forall Z\in\mathbb{W},\end{aligned} \tag{2.16}$$

where $(\cdot,\cdot)$ denotes the inner product defined in (2.7) and (2.8). And the solution pair $(\tilde{u},\tilde{W})$ satisfies

$$\|(\tilde{u},\tilde{W})\|_{\mathcal{V}} \le C\|(0,(\nabla u)^{sym})\|_{\mathcal{V}} = C\|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}. \tag{2.17}$$

In particular, put $v=u$, $Z=\mathbb{C}(\tilde{W})$, then by (2.16)

$$\begin{aligned}\|(\nabla u)^{sym}\|^2_{L^2(\Omega)^{d\times d}} &= (\mathbb{C}(\tilde{W}),(\nabla u)^{sym}) \\ &= (\mathbb{C}(\tilde{W}),\nabla u) \\ &= (Z,\nabla u).\end{aligned}$$

So

$$\|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}} = \frac{(Z,\nabla u)}{\|Z\|_{L^2(\Omega)^{d\times d}}}\frac{\|Z\|_{L^2(\Omega)^{d\times d}}}{\|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}} \le C\frac{(Z,\nabla u)}{\|Z\|_{L^2(\Omega)^{d\times d}}},$$

where we used the fact that $\frac{\|Z\|_{L^2(\Omega)^{d\times d}}}{\|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}} \le C$ for some constant C. This is because

$$\|Z\|_{L^2(\Omega)^{d\times d}} = \|\mathbb{C}(\tilde{W})\|_{L^2(\Omega)^{d\times d}} \le (2\mu+\lambda d)\|(\tilde{u},\tilde{W})\|_{\mathcal{V}} \le C\|(\nabla u)^{sym}\|_{L^2(\Omega)^{d\times d}}.$$

where we used (2.17) in the last inequality and d is the dimension constant. □

2.2 Construction of Stable Elements

2.2.1 A One-Dimensional Example

Notation 14. *In one dimension $\Omega=(0,1)$. The following notations will be frequently used in this subsection*

1. $\mathcal{U}=\{u\in H^1(0,1)\mid u(0)=0\}$.
2. *$\mathcal{T}_h$ represents a uniform partition of $[0,1]$ with partition size $h=1/N$.*
3. *$\mathcal{P}^1(\mathcal{T}_h)$ and $\mathcal{P}^2(\mathcal{T}_h)$ denote piecewise linear and piecewise quadratic functions defined on $\mathcal{T}_h$ respectively.*

4. $\mathcal{U}_h = \{u_h \in \mathcal{P}^1(\mathcal{T}_h) \cap C(0,1) \ \mid \ u_h(0) = 0\}$, *with standard hat basis functions* $\{\phi_i^u\}_{i=1}^N$.

5. $\mathcal{W}_h = \mathcal{P}^2(\mathcal{T}_h) \cap C(0,1)$, *with standard quadratic basis functions* $\{\phi_i^w\}_{i=0}^{2N}$.

The problem with boundary conditions is illustrated in the following one dimensional example, $-u'' = f$, with boundary conditions $u(0) = 0$ and $u'(0) = g$, where g is a prescribed boundary data. The plasticity equation introduces the auxiliary variable $w = u'$. The weak statement is: find $(u, w) \in \mathcal{U} \times L^2(0,1)$, such that

$$\begin{cases} \int_0^1 wv' = \int_0^1 fv + gv(1) \\ \int_0^1 (w - u')z = 0, \end{cases}$$

for all $(v, z) \in \mathcal{U} \times L^2(0,1)$. Consider the following Galerkin approximations, find $(u_h, w_h) \in \mathcal{U}_h \times \mathcal{W}_h$, such that

$$\begin{cases} \int_0^1 w_h v_h' = \int_0^1 fv_h + gv_h(1) \\ \int_0^1 (w_h - u_h')z_h = 0, \end{cases} \tag{2.18}$$

for all $(v_h, w_h) \in \mathcal{U}_h \times \mathcal{W}_h$. Clearly, $\mathcal{U}_h \times \mathcal{W}_h \subset \mathcal{U} \times L^2(0,1)$.

In order to show that the above problem is well-posed, let bilinear form A : $\big(\mathcal{U} \times L^2(0,1)\big) \times \big(\mathcal{U} \times L^2(0,1)\big) \to \mathbb{R}$ be defined by

$$A((u,w),(v,z)) := \int_0^1 wv' + wz - u'z.$$

We are exactly in the settings of Theorem 13. The following lemma establishes the well-posedness of problem (2.18).

Existence of Solution, the Inf-Sup Condition

Lemma 15. *For $u \in \mathcal{U}_h$, there exists a constant $c > 0$ independent of u, such that*

$$\sup_{z \in \mathcal{W}_h} \frac{\int_0^1 u'z}{\|z\|_{L^2(0,1)}} \geq c\|u'\|_{L^2(0,1)}. \tag{2.19}$$

Proof. Since $u \in \mathcal{U}_h$, we know u' is piecewise constant in $(0,1)$. Let $u' = \sum_{i=1}^n c_i \chi_{I_i}$ where $(0,1) = \cup_{i=1}^n I_i$, then

$$\int_0^1 u'z = \int_0^1 \sum_{i=1}^n c_i \chi_{I_i} z = \sum_i c_i \int_{I_i} z. \tag{2.20}$$

Using the fact that the degrees of freedom of $z \in \mathcal{W}_h$ is strictly more than 2 on I_i, we can pick $z \in \mathcal{W}_h$, where $\mathcal{W}_h$ is the space of piecewise quadratic continuous functions in $(0,1)$, such that

$$\int_{I_i} z = \int_{I_i} c_i = c_i |I_i| \tag{2.21}$$

and

$$\int_{I_i} z^2 \le 2\int_{I_i} c_i^2 = 2c_i^2|I_i|. \tag{2.22}$$

By equations (2.20) and (2.21), we have

$$\int_0^1 u'z = \sum_i c_i \int_{I_i} z = \sum_i c_i^2|I_i| = \sum_i \int_{I_i} (u')^2 = \|u'\|^2_{L^2(0,1)}. \tag{2.23}$$

Using equation (2.22) we have

$$\|z\|^2_{L^2(0,1)} = \int_0^1 z^2 = \sum_i \int_{I_i} z^2 \le 2\sum_i c_i^2|I_i| = 2\sum_i \int_{I_i}(u')^2 = 2\|u'\|^2_{L^2(0,1)}. \tag{2.24}$$

Therefore, combining equations (2.23) and (2.24), it follows that

$$\frac{\int u'z}{\|z\|_{L^2(0,1)}} \ge \frac{\|u'\|^2_{L^2(0,1)}}{\sqrt{2}\|u'\|_{L^2(0,1)}} = (1/\sqrt{2})\|u'\|_{L^2(0,1)}.$$

□

Finite Element Schemes

In order to demonstrate the problem with boundary conditions. We explicitly solve this simple one dimensional example showing that the boundary element gives us one order less accurate approximations than expected (Figure 2.1). Using Notation 14, let $\{\phi_i^u\}_{i=1}^N$ be standard basis functions for $\mathcal{U}_h$ and $\{\phi_i^w\}_{i=0}^{2N}$ be standard basis functions for $\mathcal{W}_h$. Let's denote

$$u = \sum_{i=1}^N u_i\phi_i^u, \quad w = \sum_{i=0}^{2N} w_i\phi_i^w, \quad v = \sum_{i=1}^N v_i\phi_i^u, \quad z = \sum_{i=0}^{2N} z_i\phi_i^w. \tag{2.25}$$

Then

$$\begin{aligned}
&A((u,w),(v,z))\\
&= \int_0^1 wv' + wz - u'z\\
&= \int_0^1 \left(\sum_{j=0}^{2N} w_j\phi_j^w\right)\left(\sum_{i=1}^N v_i\phi_i^u\right)' + \left(\sum_{j=0}^{2N} w_j\phi_j^w\right)\left(\sum_{i=0}^{2N} z_i\phi_i^w\right) - \left(\sum_{j=1}^N u_j\phi_j^u\right)'\left(\sum_{i=0}^{2N} z_i\phi_i^w\right)\\
&= \int_0^1 \sum_{j=0}^{2N}\sum_{i=1}^N v_i(\phi_i^u)'\phi_j^w w_j + \sum_{i,j=0}^{2N} z_i\phi_i^w\phi_j^w w_j - \sum_{i=0}^{2N}\sum_{j=1}^N z_i\phi_i^w(\phi_j^u)'u_j\\
&= \begin{pmatrix} \mathbf{v}^\top, & \mathbf{z}^\top \end{pmatrix}\begin{pmatrix} 0 & B\\ -B^\top & C\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{w}\end{pmatrix},
\end{aligned}$$

where $\mathbf{u}, \mathbf{w}, \mathbf{v}, \mathbf{z}$ are column vectors composed of the coefficients of corresponding functions in (2.25). B is a $N \times (2N+1)$ matrix with $B_{ij} := \int_0^1 (\phi_i^u)' \phi_j^w$. And C is an $(2N+1) \times (2N+1)$ matrix with $C_{ij} := \int_0^1 \phi_i^w \phi_j^w$. In this simple one-dimensional case, we can find matrcies B and C as follows:

$$B = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{6} & 0 & \cdots & & & 0 \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{6} & 0 & & \vdots \\ & & & & \ddots & & & & & \\ 0 & & \cdots & & 0 & \frac{1}{6} & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{6} \\ 0 & & & \cdots & & & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix}_{N\times(2N+1)}$$

$$C = \begin{bmatrix} \frac{2h}{15} & \frac{h}{15} & -\frac{h}{30} & 0 & & & \cdots & & & 0 \\ \frac{h}{15} & \frac{8h}{15} & \frac{h}{15} & 0 & & & & & & 0 \\ & & & & & & \ddots & & & \\ -\frac{h}{30} & \frac{h}{15} & \frac{4h}{15} & \frac{h}{15} & -\frac{h}{30} & 0 & & & & 0 \\ 0 & 0 & \frac{h}{15} & \frac{8h}{15} & \frac{h}{15} & 0 & & & & 0 \\ & & & & & \ddots & & & & \\ 0 & & & \cdots & & & 0 & \frac{h}{15} & \frac{8h}{15} & \frac{h}{15} \\ 0 & & & \cdots & & & 0 & -\frac{h}{30} & \frac{h}{15} & \frac{2h}{15} \end{bmatrix}_{(2N+1)\times(2N+1)}.$$

Note that

$$A((u,w),(v,z)) = \int_0^1 fv, \quad \forall\, (v,z) \in \mathcal{U}_h \times \mathcal{W}_h$$

can be written as

$$\begin{pmatrix} \mathbf{v}^\top, & \mathbf{z}^\top \end{pmatrix} \begin{pmatrix} 0 & B \\ -B^\top & C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{v}^\top, & \mathbf{z}^\top \end{pmatrix} \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix} \tag{2.26}$$

for all $\mathbf{v} \in \mathbb{R}^N$, $\mathbf{z} \in \mathbb{R}^{2N+1}$, where $\mathbf{F}$ is a $N \times 1$ vector with $F_i := \int_0^1 f\phi_i^u$. Equation (2.26) implies that

$$\begin{pmatrix} 0 & B \\ -B^\top & C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ 0 \end{pmatrix}. \tag{2.27}$$

Note that equation (2.27) can be expressed as

$$\begin{cases} B\mathbf{w} &= \mathbf{F} \\ -B^\top\mathbf{u} + C\mathbf{w} &= 0 \end{cases}, \tag{2.28}$$

where the first equation in (2.28) is called the momentum equation, and the second equation in (2.28) is called the constraint equation.

The Problem with Boundary Layer: A Numerical Example

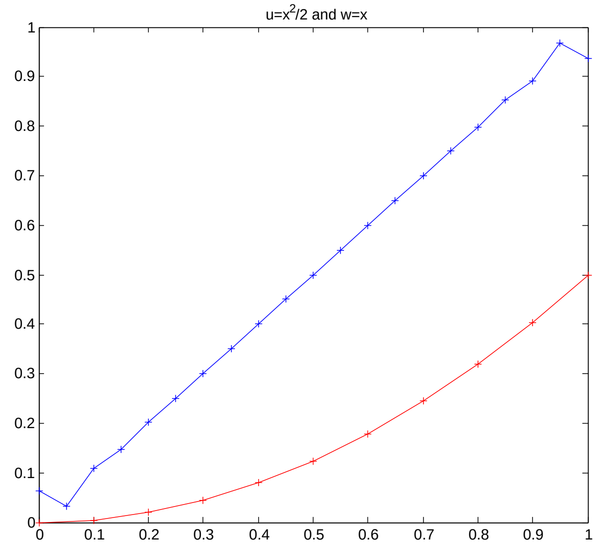

Figure 2.1: *Numerical solutions with bad boundary behavior*

To test the accuracy of the above scheme, let $u = \frac{x^2}{2}$ and $w = x$ be the exact solution in $(0, 1)$. Note that the boundary condition $u(0) = 0$ is automatically satisfied. Define $f(x) = -w' = -1$ and $g = w(1) = 1$. Solving linear system (2.27) with $N = 10$, we get solutions shown in Figure 2.1. We can see from this picture that w deviates tremendously from the true solution at the boundaries. However, both u and w agree with the exact solutions fairly well at the interior nodes. It is intuitively clear, at least from symmetry point of view, that the equations in the interior should give rise to at least second order accurate approximations. Consequently, we focus on the last interval where boundary conditions enter. It turns out that the approximation of the momentum equation enjoys higher order of accuracy (up to order h^3). It is the constraint equation where the approximation is poor at the boundary. In the following analysis, for notational convenience, we reindex the basis functions $\{\phi_i^w\}_{i=0}^{2N}$ as $\{\phi_k^w\}_{k=0}^{N}$ where $i = 2k$.

1. **The momentum equation:** In the last interval (x_{N-1}, x_N), the basis function for the displacement u is

$$\phi_N^u(x) = (x - x_{N-1})/h,$$

whose derivate is $1/h$. Introducing the change of variable $x = x_{N-1} + hs$, we calculate

$$\begin{aligned}\int_{x_{N-1}}^{x_N} w_h(\phi_N^u)'\,dx = (1/h)\int_{x_{N-1}}^{x_N} & w_{N-1}\phi_{N-1}^w + w_{N-1/2}\phi_{N-1/2}^w + w_N\phi_N^w\,dx \\ &= \int_0^1 2s(s-1/2)(w_{N-1}+w_N) + 4s(1-s)w_{N-1/2}\,ds \\ &= (1/6)(w_{N-1}+w_N) + (2/3)w_{N-1/2} \\ &= w(1) - (h/2)w'(1) + (h^2/6)w''(1) - (h^3/24)w'''(1) + \cdots\end{aligned}$$

where the last equation is the Taylor expansion of w at the right boundary $x = 1$. Similarly,

$$\begin{aligned}\int_{x_{N-1}}^{x_N} f\phi_N^u &= \int_{x_{N-1}}^{x_N} (f(1) + f'(1)(x-x_N) + (1/2)f''(1)(x-x_N)^2 + \cdots)\phi_N^u\,dx \\ &= \int_0^1 (f(1) + f'(1)h(s-1) + (1/2)f''(1)h^2(s-1)^2 + \cdots)sh\,ds \\ &= (h/2)f(1) - (h^2/6)f'(1) + (h^3/24)f''(1) + \cdots\end{aligned}$$

By weak formulation (2.18), $\int_0^1 w_h v_h' = \int_0^1 f v_h + g v_h(1)$ for all $v_h \in \mathcal{U}_h$. It follows from the above expansions that the last equation for the momentum is

$$\begin{aligned}w(1) - (h/2)w'(1) + & (h^2/6)w''(1) - (h^3/24)w'''(1) + \cdots \\ &= (h/2)f(1) - (h^2/6)f'(1) + (h^3/24)f''(1) + \cdots + g.\end{aligned}$$

Recalling that $w = u'$, so

$$w(1) = u'(1) = g, \quad -w'(1) = -u''(1) = f(1), \quad w''(1) = u'''(1) = -f'(1)$$

It is clear that the last equation is accurate up to the order of h^3.

2. **The constraint equation:** Consider next the equations on the final interval of the constraint equation. Taking the change of variable $x = x_{N-1} + hs$, the last basis function $\phi_N^w = (2/h^2)(x-1+h/2)(x-1+h)$ gives the equation

$$\begin{aligned}& \int_{x_{N-1}}^{x_N} (w_h - u_h')\phi_N^w\,dx \\ = \; & \int_{x_{N-1}}^{x_N} (w_{N-1}\phi_{N-1}^w + w_{N-1/2}\phi_{N-1/2}^w + w_N\phi_N^w - u_h')\phi_N^w\,dx \\ = \; & \int_0^1 (w_{N-1}2(s-1)(s-1/2) + w_{N-1/2}4s(1-s) \\ & + w_N 2s(s-1/2) - u_h')2s(s-1/2)h\,ds \\ = \; & (-h/30)w_{N-1} + (h/15)w_{N-1/2} + (2h/15)w_N - (1/6)(u_N - u_{N-1}) \\ = \; & (h/6)(w(1) - u'(1)) + (h^2/12)u''(1) + \cdots\end{aligned}$$

Rescaling by $1/h$, it follows that there is an error of order h.

This motivates us to correct the lower order term on the boundary. To fix this boundary problem, it turns out that it suffices to modify only one basis function on the final element $[x_{N-1}, x_N]$ for test function z. That is, we introduce a special cubic function

$$\phi_N^z(\xi) := 10\xi(\xi - 1/2)(\xi - 4/5) \quad \text{for } \xi \in (0, 1),$$

for the last element $[x_{N-1}, x_N]$. The basis functions for u, v, w and z on the parent element $(-1, 1)$ are defined in the following figures (see Figure 2.2).

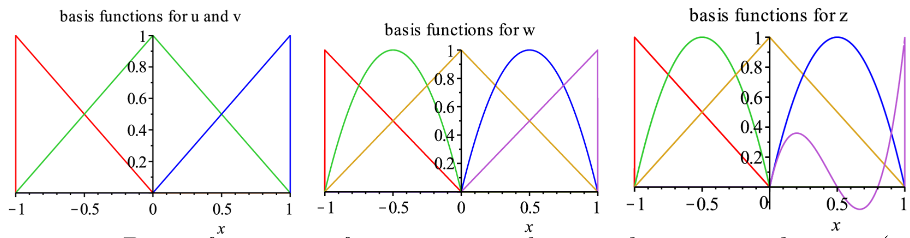

Figure 2.2: *Basis functions for u, v, w and z on the parent element* $(-1, 1)$

The last basis function ϕ_N^z was chosen in such a way that the error terms from w_h and u_h' cancel out at the boundary $x = 1$. We demonstrate that this cubic basis function ϕ_N^z indeed improves the boundary problem to a higher order of accuracy. In fact, the finite difference equation for the relation $u' = w$ at $x = 1$ now becomes

$$\int_{x_{N-1}}^{x_N} (w_h - u_h')\phi_N^z \, dx = (h/6)(w(1) - u'(1)) + (h^2/12)(u''(1) - w'(1)) + \cdots,$$

from which we see that the error is of order h^2 upon rescaling. With this new basis function, the solutions of (2.18) are given in the following picture (Figure 2.3).

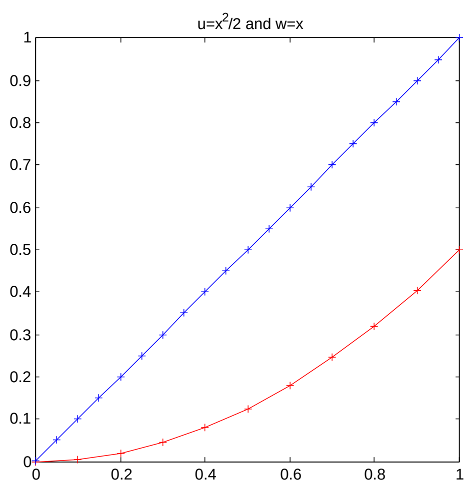

Figure 2.3: *Numerical solutions with improved accuracy*

2.2.2 The Two-Dimensional Case

The analysis for two dimensions is essentially similar. Let $\Omega = (-1,1)^2$, $\mathbb{U} \subset H^1(\Omega)^2$ be specified. We would like to determine a subspace $\mathbb{W} \subset L^2(\Omega)^{2\times 2}$ that is the most economical in the sense that it has the smallest degrees of freedom for which a solution exists and is unique. Recall that the necessary condition for existence is given by the inf-sup condition (2.14) in Theorem 13:

$$\sup_{Z \in \mathbb{W}^{sym}} \frac{\int_\Omega \nabla u : Z}{\|Z\|_{L^2(\Omega)^{2\times 2}}} \geq c_b \|(\nabla u)^{sym}\|_{L^2(\Omega)^{2\times 2}}. \tag{2.29}$$

More specifically, let $\mathbb{U} = \mathcal{Q}^1(\Omega)^2$ where $\mathcal{Q}^1(\Omega) = \mathcal{P}^1(\Omega) \otimes \mathcal{P}^1(\Omega)$, with $\mathcal{P}^1(\Omega)$ being the space of piecewise linear continuous functions defined in Ω. Taking $\mathbb{W}$ to be the entire space, namely, $\mathbb{W} = L^2(\Omega)^{2\times 2}$, then (2.29) is trivially satisfied; of course this space is far from being the "cheapest"; in fact, it is the most expensive one. On the other hand, taking $\mathbb{W} = \mathcal{Q}^1(\Omega)^{2\times 2}$, the inf-sup condition (2.29) then fails. We need to add as few as possible quadratic (or cubic) bubble functions to the space $\mathcal{Q}^1(\Omega)^{2\times 2}$ to ensure the inf-sup condition holds. It turns out that the following three bubble functions together with $\mathcal{Q}^1(\Omega)^{2\times 2}$ is a good candidate for $\mathbb{W}$.

Three Bubble Functions

Let ξ and η be two independent variables in $\Omega = (-1,1)^2$. Define

$$b(\xi,\eta) = (1-\xi^2)(1-\eta^2) \qquad \text{for } (\xi,\eta) \in \Omega.$$

We claim that if we add the following 3 bubble functions to $\mathbb{W}$,

$$bb_1(\xi,\eta) = b(\xi,\eta)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.30}$$

$$bb_2(\xi,\eta) = b(\xi,\eta)(1+\xi+\eta)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.31}$$

$$bb_3(\xi,\eta) = b(\xi,\eta)(1+\xi+\eta)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.32}$$

then the inf-sup condition (2.29) is satisfied.

The Inf-Sup Statement

In order to show the inf-sup condition holds, we need the idea of macroelements. We consider discretization of Ω into cells each of which is affine equivalent to a master cell $\hat{K}$. Isoparametric elements may be accommodated provided the Jacobians scale appropriately with the mesh size. The definition of a macroelement is as follows [26].

Definition 16. *A macroelement is a connected set of cells each affine equivalent to a reference cell $\hat{K}$. Two macroelements M and M' are equivalent if*

1. *There exists a continuous bijection $\chi : M \to M'$ which preserves the simplicial structure (i.e. χ is a cell-complex isomorphism).*

2. *If $K \subset M$ and $K' = \chi(K) \subset M'$ are cells in M and M' respectively, then*

$$\chi\restriction_K = F_{K'} \circ F_K^{-1},$$

where $F_K : \hat{K} \to K$ is an affine cell-diffeomorphism from the reference cell $\hat{K}$ to K.

Lemma 17. *(The Inf-Sup Condition) Let $\mathcal{Q}^1(\Omega) = \mathcal{P}^1(\Omega) \otimes \mathcal{P}^1(\Omega)$. Let $\mathbb{W}$ be the linear space spanned by functions in $\mathcal{Q}^1(\Omega)^{2\times 2}$ and the three bubble functions given in (2.30), (2.31) and (2.32). Then for each $u \in \mathcal{Q}^1(\Omega)^2$, for a uniform mesh of rectangles with at least 3 elements in each direction, there exists a constant $c > 0$ which depends only upon the aspect ratio of the elements, such that*

$$\sup_{Z \in \mathbb{W}^{sym}} \frac{\int_\Omega \nabla u : Z}{\|Z\|_{L^2(\Omega)^{2\times 2}}} \geq c\, \|(\nabla u)^{sym}\|_{L^2(\Omega)^{2\times 2}}. \tag{2.33}$$

Proof. We borrow the idea from [16, 24] for the proof of this lemma. Let $\mathcal{T}_h$ be a discretization of Ω. Let $\Omega = \cup_{M \in \mathcal{T}_h} M$ be a covering of Ω by macroelements which are a 3×3 patch of squares. Each $x \in \Omega$ belongs to at most n macroelements. The attached Maple calculation at the end of this chapter shows that the inf-sup condition (2.33) holds on a local patch M, namely for fixed $u_h \in \mathcal{Q}^1(\Omega)^2$, for each M, there exists a $Z_M \in \mathbb{W}^{sym}\restriction_M$ satisfying $\|Z_M\|_{L^2(M)^{2\times 2}} = \|(\nabla u_h)^{sym}\|_{L^2(M)^{2\times 2}}$ and

$$\int_M \nabla u_h : Z_M \geq c\|(\nabla u_h)^{sym}\|^2_{L^2(M)^{2\times 2}}.$$

The next step is to show that a global inf-sup condition in $\mathcal{T}_h$ will follow from the local inf-sup condition. To see this, define $Z_h = \sum_M Z_M$ and each Z_M has support in M. Use the property that $\Omega = \cup_{M \in \mathcal{T}_h} M$ to conclude

$$\int_\Omega \nabla u_h : Z_h \geq c\, \|(\nabla u_h)^{sym}\|^2_{L^2(\Omega)^{2\times 2}}. \tag{2.34}$$

We next show that $\|Z_h\|_{L^2(\Omega)^{2\times 2}}$ is dominated by $\|(\nabla u_h)^{sym}\|_{L^2(\Omega)^{2\times 2}}$. Using the property that Z_M has support in M, we have

$$\begin{aligned}
\int_\Omega |Z_h|^2 &= \int_\Omega \Big(\sum_M Z_M\Big)\Big(\sum_N Z_N\Big) \\
&= \sum_{M,N} \int_{M\cap N} Z_M Z_N \\
&\leq \sum_{M,N} \int_{M\cap N} (1/2)\left(|Z_M|^2 + |Z_N|^2\right) \\
&= \sum_{M,N} \int_{M\cap N} |Z_M|^2 \\
&\leq n \sum_M \int_M |Z_M|^2 \\
&= n \sum_M \int_M |(\nabla u_h)^{sym}|^2 \\
&\leq n^2 \int_\Omega |(\nabla u_h)^{sym}|^2
\end{aligned}$$

That is

$$\|Z_h\|_{L^2(\Omega)^{2\times 2}} \leq n\|(\nabla u_h)^{sym}\|_{L^2(\Omega)^{2\times 2}}. \tag{2.35}$$

From (2.34) and (2.35), it follows

$$\frac{\int_\Omega \nabla u_h : Z_h}{\|Z_h\|_{L^2(\Omega)^{2\times 2}}} \geq c\,\|(\nabla u_h)^{sym}\|_{L^2(\Omega)^{2\times 2}}$$

for some constant $c > 0$. $\square$

A Numerical Example

In this subsection, we present a numerical example illustrating the effectiveness of the above scheme. A solution of the problem: find $W \in L^2(\Omega)^{2\times 2}$ and $u \in H^1(\Omega)^2$ with boundary conditions satisfying Korn and Poincaré hypothesis, such that

$$\begin{cases} -\operatorname{div}\mathbb{C}(W) &= f \\ \quad W - \nabla u &= 0 \end{cases}$$

is constructed by setting

$$u(x,y) = \begin{pmatrix} \psi_y \\ -\psi_x \end{pmatrix},$$

$$W = \nabla u = \begin{bmatrix} \psi_{yx} & \psi_{yy} \\ -\psi_{xx} & -\psi_{xy} \end{bmatrix},$$

where $\psi(x,y) = y^2 + (1/100)e^{-\pi y}\cos(\pi x)$ for $(x,y) \in \Omega = (-1,1)^2$; and the right hand side data f for this stationary problem is manufactured so that the equation holds.

Galerkin approximations are conducted on subspaces $\mathbb{U} \subset H^1(\Omega)^2$ and $\mathbb{W} \subset L^2(\Omega)^{2\times 2}$ under the hypothesis of Lemma 17. Table 2.1 demonstrates the errors in the numerical approximations. Clearly we have second order convergence rate in $L^2(\Omega)^2$ and first order convergence rate for the derivative. Figure 2.4 illustrates the approximation errors of ∇u in Ω.

h	$\Vert u - u_h\Vert_{L^2(\Omega)^2}$	$\Vert\nabla(u - u_h)\Vert_{L^2(\Omega)^{2\times 2}}$	$\Vert W - W_h\Vert_{L^2(\Omega)^{2\times 2}}$
1/8	2.697084e-02	4.194267e-01	2.387249e-01
1/16	6.798636e-03	2.090260e-01	9.285764e-02
1/32	1.714489e-03	1.041137e-01	3.793997e-02
1/64	4.312588e-04	5.188712e-02	1.650732e-02
1/128	1.081863e-04	2.588814e-02	7.579534e-03
1/256	2.709492e-05	1.292830e-02	3.612130e-03
norms	2.345541	4.395533	4.395533

Table 2.1: *Nonstandard mixed method approximation errors*

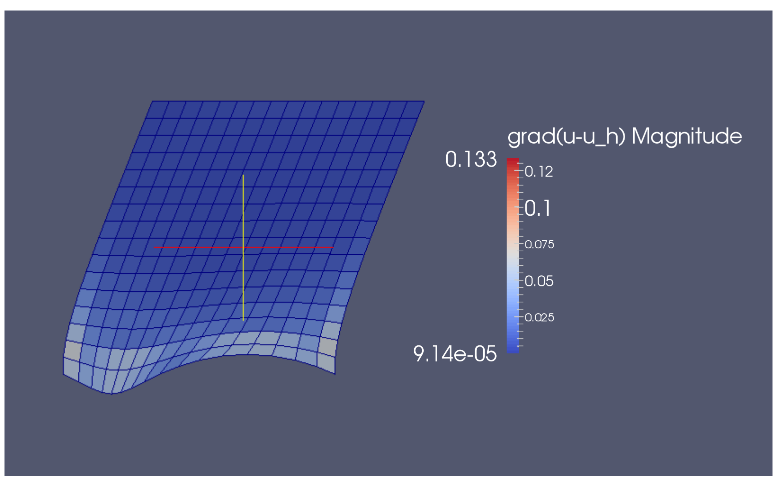

Figure 2.4: *Nonstandard mixed method approximation errors for the displacement gradient* ($h = 1/16$).

Maple Calculations

The following maple calculation shows that the inf-sup condition holds on a local patch, which is the first part of the proof of Lemma 17.

```
# Two Dimensions, N^2 squares, Symmetric matrix
restart:
with(LinearAlgebra):
N := 3:
for i from 0 to N do
for j from 0 to N do
  for m from 0 to 1 do
  for n from m to 1 do
    eqns[i][j][m][n] := 0:
  end: end:
end: end:

eqnsB := NULL:

for i from 0 to N-1 do
for j from 0 to N-1 do

  uu[0] := 0: uu[1] := 0;

  for ii from 0 to 1 do
  for jj from 0 to 1 do
  for m  from 0 to 1 do
    uu[m] :=  uu[m] + (1-(-1)^ii * xi) * (1-(-1)^jj * eta) * u[i+ii][j+jj][m]:
  end: end: end:

  du[0][0] := diff(uu[0],xi):
  du[0][1] := (diff(uu[0],eta) + diff(uu[1],xi))/2:
  du[1][1] := diff(uu[1],eta):

  bb := (1-xi^2) * (1-eta^2):

  # Three Bubble Functions
 eqnsB := eqnsB,
     int(int(du[0][1]*bb, xi=-1..1), eta=-1..1),
     int(int(du[0][0]*bb*(1+xi+eta), xi=-1..1), eta=-1..1),
     int(int(du[1][1]*bb*(1+xi+eta), xi=-1..1), eta=-1..1):

  for ii from 0 to 1 do
  for jj from 0 to 1 do
    phi := (1 - (-1)^ii * xi) * (1 - (-1)^jj * eta):

   for m from 0 to 1 do
   for n from m to 1 do
      eqns[i+ii][j+jj][m][n] := eqns[i+ii][j+jj][m][n]
                              + int(int(du[m][n]*phi, xi=-1..1), eta=-1..1):
```

```
    end: end:
  end: end:
end: end:

vars := seq(seq(seq(u[i][j][m], i=0..N), j=0..N), m = 0..1):

eqnsV := seq(seq(seq(seq(eqns[i][j][m][n],n=m..1),m=0..1),i=1..N-1),j=1..N-1):

eqnsV := eqnsV, u[0][0][0], u[0][0][1], u[N][N][1]: # Remove rigid body motion

ss := solve({eqnsV, eqnsB}, {vars}):

A,f := GenerateMatrix([eqnsV,eqnsB],[vars]):
          nops([eqnsV,eqnsB] ), nops([vars]), Rank(A);

Matrix(N+1,N+1, (i,j) -> subs(ss, u[i-1][j-1][0]) ),
Matrix(N+1,N+1, (i,j) -> subs(ss, u[i-1][j-1][1]) ):
map(factor, subs(ss,[du[0][0], du[0][1], du[1][1]]));

# output:
     42, 32, 32  # the rank of the final matrix is equal to the # of variables
     [0, 0, 0]   # symmetric part of grad u is 0
```

Chapter 3

Layer Problems

The content of this chapter is mostly from a joint paper with Noel Walkington published in quarterly of applied mathematics.[1]

3.1 Introduction

Models used to simulate ground motion during an earthquake frequently represent the subsurface as a union of linearly elastic materials separated by thin (fault) regions within which large deformations (rupture) occur (Figure 1.2 Chapter 1). Below we analyze limiting models which circumvent the numerical difficulties encountered with direct simulation of these models which arise when very fine meshes are required to resolve the large deformations in the fault region. The fault region in reduced models is represented as a surface and the rupture is realized as discontinuities in certain components of the solution. Figures 3.1 and 3.2 illustrate these issues; the fine mesh in Figure 3.1 is unnecessary when the large shear across the fault is represented as the discontinuity in the horizontal displacement shown in Figure 3.2. Theorems 20 and 22 justify this approach by establishing that solutions of a certain class of these models converge to the solution of a reduced problem as the width tends to zero.

In Section 1.2.2 Chapter 1, we showed that the two dimensional layer problem derived in [3, 27], is a consistent ansatz with the three dimensional rupture model initially proposed in [1]. We consider a cross section of a subsurface region $\Omega = (-1,1)^2$ containing a horizontal fault $S_\epsilon = (-1,1) \times (-\epsilon/2, \epsilon/2)$. The balance of momentum equation takes the classical form

$$\rho u_{tt} - \operatorname{div}(T) = \rho f \qquad \text{on } (0,T) \times \Omega, \tag{3.1}$$

where $u(t,x) \in \mathbb{R}^2$ and $\rho(t,x) > 0$ represent the displacement and density of the medium, $f(t,x) \in \mathbb{R}^2$ the force per unit mass, and T is the (Cauchy) stress tensor

[1] J. Liu, X. Lu, and N. J. Walkington, *Analysis of a dislocation model for earthquakes*, Quarterly of Applied Mathematics, April 2016

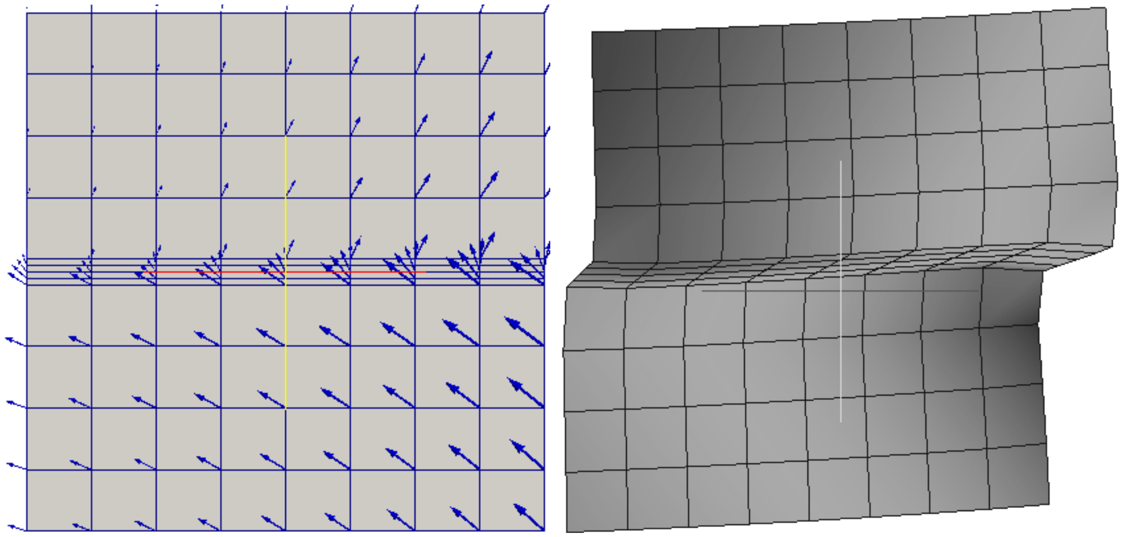

Figure 3.1: *Solution with deformation resolved.*

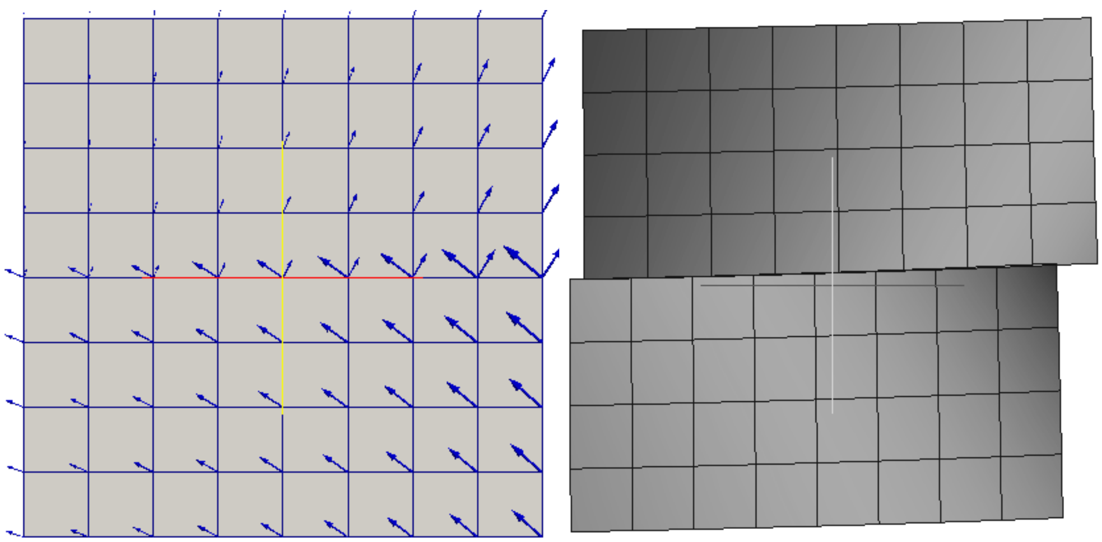

Figure 3.2: *Solution of the limit problem.*

and with

$$T = \begin{cases} \mathbb{C}(\nabla u), & \text{in } \Omega \setminus \bar{S}_\epsilon \\ \mathbb{C}\left(\begin{bmatrix} u_{1x} & \epsilon u_{1y} - \gamma \\ \epsilon u_{2x} & u_{2y} \end{bmatrix}\right), & \text{in } S_\epsilon \end{cases} \tag{3.2}$$

where $\mathbb{C} : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is the classical linear isotropic elasticity tensor defined by

$\mathbb{C}(W) = \mu(W + W^T) + \lambda \mathrm{tr}(W)I$ for $W \in \mathbb{R}^{2\times 2}$ and $\mu, \lambda > 0$. In other words,

$$T = \begin{cases} 2\mu D(u) + \lambda \operatorname{div}(u) I, & \text{in } \Omega \setminus \bar{S}_\epsilon \\ \begin{bmatrix} (2\mu+\lambda)u_{1x} + \lambda u_{2y} & \mu\left(\epsilon(u_{2x}+u_{1y}) - \gamma\right) \\ \mu\left(\epsilon(u_{2x}+u_{1y}) - \gamma\right) & (2\mu+\lambda)u_{2y} + \lambda u_{1x} \end{bmatrix}, & \text{in } S_\epsilon \end{cases} \tag{3.3}$$

where $D(u) = (1/2)(\nabla u + (\nabla u)^\top)$ and $\mu = \mu(x)$ and $\lambda = \lambda(x)$ are the shear and bulk moduli. Here $\gamma = \gamma(t,x)$ models the permanent deformation due to damage, defects and healing in a fault [27] and evolves according to

$$(1/\beta)\gamma_t + \left(\hat{\eta}\gamma - T_{12} - \nu\gamma_{xx}\right) = 0, \qquad \text{in } S_\epsilon. \tag{3.4}$$

This model of rupture was inspired by the plasticity theories developed in [3, 1] where the coefficients β, $\hat{\eta}$ and ν are typically nonlinear functions of γ and its derivatives. To date, there is no satisfactory mathematical theory for these models of nonlinear elasticity [2] so for the analysis below we assume that the coefficients in (3.4) are specified which, for example, would be the case for one step of a linearly implicit time stepping scheme for the fully nonlinear problem. In this context we address the following problems.

Strain Energy: If $\gamma(x)$ is specified (or more generally $\gamma_\epsilon \to \gamma$ sufficiently strongly) we verify that the strain energies for the stationary problem (3.1)–(3.3) converge to the limiting energy

$$I(u) = \int_{\Omega\setminus S_0} 2\mu|D(u)|^2 + \lambda \operatorname{div}(u)^2 + \int_{S_0} \mu([u_1] - \gamma)^2,$$

where $[u_1(x)]$ denotes the jump in the horizontal component of u across the line $S_0 = (-1,1)\times\{0\}$. The corresponding Euler-Lagrange operator is then

$$-\operatorname{div}(T) \quad \text{on } \Omega\setminus S_0 \qquad \text{with} \qquad [Tn] = 0 \text{ and } T_{12} = \mu([u_1]-\gamma) \quad \text{on } S_0,$$

where $T = 2\mu D(u) + \lambda \operatorname{div}(u) I$ and $[Tn]$ is the jump of the traction across S_0 with $n = (0,1)^\top$ denoting the normal.

This shows that the scaling introduced in (3.3) is the "mathematically interesting" case for which a nontrivial limit exists. With different scalings the equations for u and γ either decouple (the last term in the energy vanishes) or lock, $[u_1] = \gamma$, in the limit. The limiting energy for the coupled stationary problem (3.1)–(3.4) is

$$I(u,\gamma) = \int_{\Omega\setminus S_0} 2\mu|D(u)|^2 + \lambda \operatorname{div}(u)^2 + \int_{S_0} \mu([u_1]-\gamma)^2 + (1/2)\left(\hat{\eta}\gamma^2 + \nu\gamma_x^2\right). \tag{3.5}$$

Examples of numerical solutions to both the uncoupled and coupled problems are presented in Section 3.4.

Evolutionary Problem: In Section 3.3 we show that the solutions of the coupled system (3.1)–(3.4) converge to a limit which satisfies the reduced system,

$$\rho u_{tt} - \operatorname{div}(T) = \rho f \qquad \text{on } \Omega \setminus S_0,$$

with $T = 2\mu D(u) + \lambda \operatorname{div}(u) I$, and

$$[Tn] = 0, \quad T_{12} = \mu([u_1] - \gamma), \text{ and } (1/\beta)\gamma_t + (\hat{\eta}\gamma - T_{12} - \nu\gamma_{xx}) = 0 \qquad \text{on } S_0.$$

For definiteness, we consider displacement boundary conditions $u(\cdot, \pm 1) = 0$ on the top and bottom of Ω and traction free boundary data on the sides $T(\pm 1, \cdot)n = 0$, and $\nu\gamma_x(\pm 1, \cdot) = 0$. We omit analogous results for other boundary conditions which are routine technical extensions of the proof techniques presented below. The same energy and limiting problem are obtained with the "engineering approximations" utilized in [27] where the shear stress $T_{12}(x, y)$ in the equation (3.4) is approximated by its average $\bar{T}_{12}(x)$ across S_ϵ so that γ depends only upon x,

$$(1/\beta)\gamma_t + \left(\hat{\eta}\gamma - \bar{T}_{12} - \nu\gamma_{xx}\right) = 0, \qquad \text{where} \qquad \bar{T}_{12}(t, x) = \frac{1}{\epsilon}\int_{-\epsilon/2}^{\epsilon/2} T_{12}(t, x, y)\, dy.$$

The ideas presented below extend directly to the analysis of this variation of the problem.

3.1.1 Notation and Function Spaces

In addition to the standard notations introduced in Section 1.1, the following notation is used to characterize the dependence upon ϵ of the elastic and fault regions.

Notation 18. *Let $\Omega = (-1, 1)^2$ and $0 < \epsilon < 1/2$.*

1. *The fault regions are denoted by $S_\epsilon = (-1, 1) \times (-\epsilon/2, \epsilon/2)$ and $S_0 = (-1, 1) \times \{0\}$ and their complements, the elastic regions, denoted as $\Omega_\epsilon = \Omega \setminus \bar{S}_\epsilon$ and $\Omega_0 = \Omega \setminus S_0$.*

2. *The sub-spaces of functions on the elastic region which vanish on the top and bottom boundaries are*

$$\begin{aligned} U &= \{u \in H^1(\Omega) \mid u(x, \pm 1) = 0,\ -1 < x < 1\}, \\ U_\epsilon &= \{u \in H^1(\Omega_\epsilon) \mid u(x, \pm 1) = 0,\ -1 < x < 1\}, \\ U_0 &= \{u \in H^1(\Omega_0) \mid u(x, \pm 1) = 0,\ -1 < x < 1\}. \end{aligned}$$

3. *The restriction $u \mapsto u|_{\Omega_\epsilon}$ is identified as an embedding of the spaces $H^1(\Omega) \hookrightarrow H^1(\Omega_\epsilon)$ and $U \hookrightarrow U_\epsilon$; similarly $H^1(\Omega) \hookrightarrow H^1(\Omega_0)$ and $U \hookrightarrow U_0$.*

4. *Below χ_A denotes the characteristic function of $A \subset \Omega$; $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.*

The following lemma quantifies the dependence upon ϵ of embedding constants and properties of the function spaces for which the energy is continuous and coercive. Here C and c denote generic constants which may vary from instance to instance but will always be independent of ϵ.

Lemma 19. *Denote the domains and spaces as in Notation 18, and if $u^\epsilon \in H^1(\Omega_\epsilon)$ and $u \in H^1(\Omega_0)$ denote by $[u^\epsilon]$ and $[u]$ the jump in their traces across the fault regions;*

$$[u^\epsilon] = u^\epsilon(\cdot, \epsilon/2) - u^\epsilon(\cdot, -\epsilon/2) \qquad \text{and} \qquad [u] = u(\cdot, 0^+) - u(\cdot, 0^-).$$

1. *The constant in Korn's inquality on U_ϵ is independent of ϵ.*

2. *The following Poincaré inequality holds for functions in U.*

$$(1/2)\|u\|_{L^2(S_\epsilon)} \leq \left(\epsilon^2 \|u_y\|^2_{L^2(S_\epsilon)} + (\epsilon/2)\|u\|^2_{H^1(\Omega_\epsilon)}\right)^{1/2}.$$

3. *If $u^\epsilon \in H^1(\Omega)$ and*

$$u^\epsilon \rightharpoonup u, \qquad \chi_{\Omega_\epsilon} u^\epsilon_x \rightharpoonup g_0, \qquad u^\epsilon_y \rightharpoonup g_1, \qquad \text{in } L^2(\Omega),$$

then $u \in H^1(\Omega)$ and $\nabla u = (g_0, g_1)^\top$.

4. *If $u^\epsilon \in H^1(\Omega)$ and*

$$u^\epsilon \rightharpoonup u, \qquad u^\epsilon_x \rightharpoonup g_0, \qquad \chi_{\Omega_\epsilon} u^\epsilon_y \rightharpoonup g_1 \qquad \text{in } L^2(\Omega), \tag{3.6}$$

then $u \in H^1(\Omega_0)$ and $\nabla u = (g_0, g_1)^\top$. In addition $[u^\epsilon] \rightharpoonup [u]$ in $L^2(-1,1)$.

5. *Let $\phi_\epsilon : \Omega_\epsilon \to \Omega_0$ be the mapping*

$$\phi_\epsilon(x,y) = \begin{cases} \left(x, \frac{y-\epsilon/2}{1-\epsilon/2}\right), & \text{for } y \in (\epsilon/2, 1), \\ \left(x, \frac{y+\epsilon/2}{1-\epsilon/2}\right), & \text{for } y \in (-1, -\epsilon/2). \end{cases}$$

Then the linear functions $E_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega_0)$ given by

$$E_\epsilon(u^\epsilon) = u^\epsilon \circ \phi_\epsilon^{-1} \tag{3.7}$$

are isomorphisms and their norms and the norms of their inverses converge to one as $\epsilon \to 0$. The restriction of E_ϵ to U_ϵ is an isomorphisims onto U_0.

6. *If $u_1 \in H^1(\Omega_0)$ then there exists $u^\epsilon_1 \in H^1(\Omega)$ such that*

   (a) $\|E_\epsilon(u^\epsilon_1) - u_1\|_{H^1(\Omega_0)} \to 0$.

   (b) $\|(u^\epsilon_1)_x\|_{L^2(S_\epsilon)} \to 0$.

(c) $u^\epsilon_{1y}(x,.)$ is independent of y in S_ϵ and $\int_{-\epsilon/2}^{\epsilon/2} u^\epsilon_{1y}(.,y)\,dy = [u^\epsilon_1] \to [u_1]$ in $L^2(-1,1)$.

In addition, if $u_1 \in U_0$ then $u^\epsilon_1 \in U$.

Proof. 1. For the first assertion,

- Extend functions on $(-1,1)\times(\epsilon,\epsilon+1)$ to $(-1,1)\times(0,1)$: Translate $(-1,1)\times(\epsilon,\epsilon+1/2)$ to the fixed domain $(-1,1)\times(0,1/2)$, extend $u(.,.+\epsilon)$ to $(-1,1)\times(-1/2,1/2)$ and then translate back to $(-1,1)\times(\epsilon,\epsilon+1/2)$. This gives an extension of $H^1(-1,1)\times(\epsilon,1)$ to $H^1(-1,1)\times(0,1)$ with constant independent of ϵ.
- Apply Korn's inequality on the fixed domains $H^1(-1,1)\times(0,1)$ and $H^1(-1,1)\times(-1,0)$

2. The second assertion is essentially a one-dimensional calculation; if $u(1)=0$,

$$\begin{aligned}(1/2)u(y)^2 &= (-1/2)\int_y^1 \frac{d}{\xi}u(\xi)^2\,d\xi \\ &= -\int_y^\epsilon u(\xi)u'(\xi)d\xi - \int_\epsilon^1 u(\xi)u'(\xi)d\xi \\ &\le \|u\|_{L^2(0,\epsilon)}\|u'\|_{L^2(0,\epsilon)} + \|u\|_{L^2(\epsilon,1)}\|u'\|_{L^2(\epsilon,1)}.\end{aligned}$$

Integrate both sides over $y\in(0,\epsilon)$ and use the inequality $ab \le a^2/4+b^2$ to get

$$(1/4)\|u\|^2_{L^2(0,\epsilon)} \le \epsilon^2\|u'\|^2_{L^2(0,\epsilon)} + (\epsilon/2)\|u\|^2_{H^1(\epsilon,1)}.$$

3. To verify that $g_0 = u_x$, let $\phi\in C_c^\infty(\Omega)$ then

$$\int_\Omega g_0\phi = \lim_\epsilon \int_\Omega \chi_{\Omega_\epsilon} u^\epsilon_x \phi = -\lim_\epsilon \int_\Omega u^\epsilon \chi_{\Omega_\epsilon}\phi_x = -\int_\Omega u\phi_x,$$

since $\chi_{\Omega_\epsilon}\phi_x \to \phi_x$ in $L^2(\Omega)$. Similarly, we can show that $g_1 = u_y$. It follows that $u\in H^1(\Omega)$.

4. To verify that $g_1 = u_y$ on Ω_0, let $\phi\in C_c^\infty(\Omega_0)$. Using the property that for ϵ sufficiently small $\mathrm{supp}(\phi)\subset\Omega_\epsilon$ so that,

$$\int_\Omega g_1\phi = \lim_\epsilon \int_\Omega \chi_{\Omega_\epsilon} u^\epsilon_y \phi = -\lim_\epsilon \int_\Omega \chi_{\Omega_\epsilon} u^\epsilon \phi_y = -\int_\Omega u\phi_y,$$

since $\chi_{\Omega_\epsilon}\phi_y \to \phi_y$ in $L^2(\Omega)$. Using the same argument as above, we get $g_0 = u_x$. It follows that $u\in H^1(\Omega_0)$. Next we show that $[u^\epsilon] \rightharpoonup [u]$ in $L^2(-1,1)$. It suffices to show that $u^\epsilon(\cdot,\pm\epsilon) \rightharpoonup u(\cdot,0^\pm)$ in $L^2(-1,1)$. We show that $u^\epsilon(\cdot,\epsilon) \rightharpoonup u(\cdot,0^+)$.

The proof that $u^\epsilon(\cdot,-\epsilon) \rightharpoonup u(\cdot,0^-)$ being identical. Letting $\phi \in L^2(-1,1)$, we compute

$$\begin{aligned}\int_{-1}^{1} \big(u^\epsilon(x,\epsilon) - u(x,\epsilon)\big)\phi\,dx &= \frac{1}{1-\epsilon}\int_{-1}^{1}\int_{\epsilon}^{1} \frac{d}{dy}\big((1-y)(u-u^\epsilon)\big)\,dy\,\phi\,dx \\ &= \frac{1}{1-\epsilon}\int_{-1}^{1}\int_{\epsilon}^{1} \big((1-y)(u_y-u^\epsilon_y) - (u-u^\epsilon)\big)\,\phi\,dy\,dx \\ &\to 0,\end{aligned}$$

by assumptions in (3.6). Also $u_y\phi \in L^1(\Omega)$ so

$$\int_{-1}^{1} \big(u(x,\epsilon) - u(x,0^+)\big)\phi(x)\,dx = \int_{-1}^{1}\int_{0}^{\epsilon} u_y\phi\,dy\,dx \to 0$$

It follows that $u^\epsilon(\cdot,\pm\epsilon) \rightharpoonup u(\cdot,0^\pm)$ in $L^2(-1,1)$.

5. $E_\epsilon u = u\circ\phi_\epsilon^{-1}$ where $\phi_\epsilon : \Omega_\epsilon \to \Omega_0$ is the mapping appearing in the definition of E_ϵ. The Jacobian of this map is $\mathrm{diag}(1, 1/(1-\epsilon/2))$ which is clearly bounded and has bounded inverse when $0<\epsilon<1/2$.

6. For $u_1 \in H^1(\Omega_0)$, define $u_1^\epsilon : \Omega \to \mathbb{R}$ by

$$u_1^\epsilon(x,y) = \begin{cases} \eta_\epsilon * (u_1\circ\phi_\epsilon), & \text{for } (x,y)\in\Omega_\epsilon, \\ (1/2 - y/\epsilon)u_1^\epsilon(x,-\epsilon/2) + (1/2+y/\epsilon)u_1^\epsilon(x,\epsilon/2), & \text{for } (x,y)\in\bar{S}_\epsilon. \end{cases}$$

(Figure 3.3), where η_ϵ are standard mollifiers such that the traces $u_1^\epsilon(\cdot,\pm\epsilon/2)$ satisfy

$$\|u^\epsilon_{1x}(\cdot,\pm\epsilon/2)\|_{L^1(-1,1)} \le \epsilon^{-1/3}. \tag{3.8}$$

We first show that $\|u^\epsilon_{1x}\|_{L^2(S_\epsilon)} \to 0$ as $\epsilon\to 0$. To see this,

$$\begin{aligned}\int_{S_\epsilon} |u^\epsilon_{1x}|^2 dy\,dx &\le 2\int_{S_\epsilon} (1/2-y/\epsilon)^2 u^\epsilon_{1x}(x,-\epsilon/2)^2 + (1/2+y/\epsilon)^2 u^\epsilon_{1x}(x,\epsilon/2)^2\,dy\,dx \\ &= 2\,\|u^\epsilon_{1x}(\cdot,-\epsilon/2)\|^2_{L^2(-1,1)}\int_{-\epsilon/2}^{\epsilon/2}(1/2-y/\epsilon)^2\,dy \\ &\quad +2\,\|u^\epsilon_{1x}(\cdot,\epsilon/2)\|^2_{L^2(-1,1)}\int_{-\epsilon/2}^{\epsilon/2}(1/2+y/\epsilon)^2\,dy \\ &\le 4\,\epsilon^{-2/3}\int_{-\epsilon/2}^{\epsilon/2}(1/4+y^2/\epsilon^2)\,dy \\ &\le 4\,\epsilon^{1/3} \to 0\end{aligned}$$

where we used inequality (3.8) in the above estimate. Secondly, we prove that $[u_1^\epsilon] \to [u_1]$ in $L^2(-1,1)$. Indeed

$$\begin{aligned}
& \|[u_1^\epsilon] - [u_1]\|_{L^2(-1,1)} \\
= \; & \|u_1^\epsilon(\cdot, \epsilon/2) - u_1^\epsilon(\cdot, -\epsilon/2) - u_1(\cdot, 0^+) + u_1(\cdot, 0^-)\|_{L^2(-1,1)} \\
\leq \; & \|u_1^\epsilon(\cdot, \epsilon/2) - u_1(\cdot, 0^+)\|_{L^2(-1,1)} + \|u_1^\epsilon(\cdot, -\epsilon/2) - u_1(\cdot, 0^-)\|_{L^2(-1,1)} \\
\to \; & 0
\end{aligned}$$

which follows directly from the construction of u_1^ϵ and the properties of mollifiers. Finally, to see that $\|E_\epsilon(u_1^\epsilon) - u_1\|^2_{H^1(\Omega_0)} \to 0$, intuitively, note that as shown in Figure 3.3, for given $u_1 \in H^1(\Omega_0)$, u_1^ϵ is constructed by rescaling the domain from Ω_0 to Ω_ϵ by composing u_1 with ϕ_ϵ, and the values in the strip S_ϵ are linearly interpolated using the traces u_1^ϵ from above and below. On the other hand, the mappings $E_\epsilon : H^1(\Omega_\epsilon) \to H^1(\Omega_0)$ exactly undo this process by composing u_1^ϵ with the inverse map ϕ_ϵ^{-1}.

□

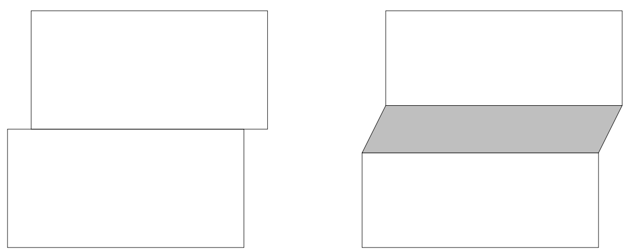

Figure 3.3: *Approximation of horizontal displacement $u \in U_0$ by a function $u^\epsilon \in U \subset H^1(\Omega)$*

3.2 Gamma Convergence of the Stationary Operator

Letting $\mathbb{C} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ denote the classical isotropic elasticity tensor with shear and bulk moduli μ and λ, the associated strain energy function will be denoted as

$$W(A) \equiv |A|^2_{\mathbb{C}} \equiv \mathbb{C}(A) : A = 2\mu(A_{11}^2 + A_{22}^2) + \mu(A_{12} + A_{21})^2 + \lambda(A_{11} + A_{22})^2.$$

Define $I_\epsilon : H^1(\Omega)^2 \to \mathbb{R}$ to be the energy,

$$\begin{aligned}
I_\epsilon(u) &= \int_{\Omega_\epsilon} W(\nabla u) + \int_{S_\epsilon} W\left(\begin{bmatrix} u_{1x} & \sqrt{\epsilon} u_{1y} - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x} & u_{2y} \end{bmatrix}\right) \qquad (3.9) \\
&= \int_{\Omega} W\left(\begin{bmatrix} u_{1x} & u_{1y}\chi_{\Omega_\epsilon} \\ u_{2x}\chi_{\Omega_\epsilon} & u_{2y} \end{bmatrix}\right) + \int_{S_\epsilon} W\left(\begin{bmatrix} 0 & \sqrt{\epsilon} u_{1y} - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x} & 0 \end{bmatrix}\right)
\end{aligned}$$

In this section, we set up the following theorem, which establishes the gamma convergence of the energies in the following sense [7].

- (lim-inf inequality) If $\{u_\epsilon\} \subset U \times U$ with $\{I(u_\epsilon)\}_{\epsilon>0} \subset \mathbb{R}$ bounded then there exists $u \in U_0 \times U$ and a sub-sequence for which $u^\epsilon \rightharpoonup u$ in $H^1_{loc}(\Omega_0) \times H^1(\Omega)$ and $I(u) \leq \liminf_{\epsilon\to 0} I_\epsilon(u^\epsilon)$.

- (lim-sup inequality) For each $u \in U_0 \times U$ there exists a sequence $\{u_\epsilon\}_{\epsilon>0} \subset U \times U$ such that $u^\epsilon \rightharpoonup u$ in $H^1_{loc}(\Omega_0) \times H^1(\Omega)$ and $I(u) \geq \limsup_{\epsilon\to 0} I_\epsilon(u_\epsilon)$.

Theorem 20. *Denote the domains and spaces as in Notation 18 and let $I_\epsilon : U \times U \to \mathbb{R}$ be as in equation (3.9) with $\gamma \in L^2(-1,1)$ fixed. Assume that the shear and bulk moduli μ and λ are bounded above and below by positive constants and that there exists $\epsilon_0 > 0$ such that the shear modulus μ is independent of y on S_ϵ for $\epsilon < \epsilon_0$. Then $I_\epsilon \xrightarrow{\Gamma} I$ where $I : U_0 \times U \to \mathbb{R}$ is given by*

$$I(u) = \int_{\Omega_0} W(\nabla u) + \int_{-1}^{1} \mu([u_1] - \gamma)^2,$$

for which the strong form of the Euler Lagrange operator is

$$-\mathrm{div}(\mathbb{C}(\nabla u)) \text{ on } \Omega_0, \quad \text{with} \quad [\mathbb{C}(\nabla u)]n = 0 \text{ and } \{\mathbb{C}(\nabla u)\}n = \mu([u_1] - \gamma) \text{ on } S_0.$$

Here $[\cdot]$ and $\{\cdot\}$ denote the jump and average across the fault line $y = 0$.

The following lemma quantifies the coercivity properties of the energies I_ϵ and the corresponding bounds required for the proof of Theorem 20. In this lemma we use the property stated in (2.9), that in two dimensions the assumptions on the Lamé parameters μ, λ guarantee $W(A) \geq 2c_0|A^{sym}|^2$, where A^{sym} denotes the symmetric part of $A \in \mathbb{R}^{2\times 2}$.

Lemma 21. *Denote the domains and spaces as in Notation 18 and let $I_\epsilon : U \times U \to \mathbb{R}$ be as in equation (3.9) with $\gamma \in L^2(-1,1)$ fixed. Assume that the shear and bulk moduli are bounded above, $\mu \geq c_0 > 0$ and $2\mu + \lambda \geq c_0 > 0$. Then*

$$\|u_{1x}\|^2_{L^2(\Omega)} + \|u_{1y}\|^2_{L^2(\Omega_\epsilon)} + \|u_{2x}\|^2_{L^2(\Omega_\epsilon)} + \|u_{2y}\|^2_{L^2(\Omega)} \leq C I_\epsilon(u),$$

and

$$\|\sqrt{\epsilon} u_{2x}\|^2_{L^2(\Omega_0)} + \|\sqrt{\epsilon} u_{1y}\|^2_{L^2(\Omega_0)} \leq C \left(I_\epsilon(u) + \|\gamma\|^2_{L^2(-1,1)} \right).$$

In particular,

$$\|u_1\|^2_{L^2(\Omega)} + \|u_2\|^2_{L^2(\Omega)} \leq C \left(I_\epsilon(u) + \|\gamma\|^2_{L^2(-1,1)} \right).$$

Proof. It is immediate that

$$\|u_{1x}\|^2_{L^2(\Omega)} + \|u_{1y} + u_{2x}\|^2_{L^2(\Omega_\epsilon)} + \|u_{2y}\|^2_{L^2(\Omega)} + \|\sqrt{\epsilon}(u_{2x} + u_{1y}) - \gamma/\sqrt{\epsilon}\|^2_{L^2(S_\epsilon)} \leq C I_\epsilon(u),$$

and Korn's inequality on Ω_ϵ shows

$$\|u_{2x}\|^2_{L^2(\Omega_\epsilon)} + \|u_{1y}\|^2_{L^2(\Omega_\epsilon)} \leq C \left(\|u_{1y} + u_{2x}\|^2_{L^2(\Omega_\epsilon)} + \|u_{1x}\|^2_{L^2(\Omega_\epsilon)} + \|u_{2y}\|^2_{L^2(\Omega_\epsilon)} \right).$$

Next, use the triangle inequality and the identity $\|\gamma/\sqrt{\epsilon}\|_{L^2(S_\epsilon)} = \|\gamma\|_{L^2(-1,1)}$ to obtain

$$\|\sqrt{\epsilon}(u_{2x} + u_{1y})\|_{L^2(S_\epsilon)} \leq \|\sqrt{\epsilon}(u_{2x} + u_{1y}) - \gamma/\sqrt{\epsilon}\|_{L^2(S_\epsilon)} + \|\gamma\|_{L^2(-1,1)},$$

Korn's inequality for the vector field $\tilde{u} = \sqrt{\epsilon}(u_1, u_2)$ on Ω_0 shows

$$\begin{aligned} &\|\sqrt{\epsilon} u_{2x}\|^2_{L^2(\Omega_0)} + \|\sqrt{\epsilon} u_{1y}\|^2_{L^2(\Omega_0)} \\ &\qquad \leq C \left(\|\sqrt{\epsilon}(u_{2x} + u_{1y})\|^2_{L^2(\Omega_0)} + \epsilon\big(\|u_{1x}\|^2_{L^2(\Omega_0)} + \|u_{2y}\|^2_{L^2(\Omega_0)}\big) \right). \end{aligned}$$

□

Proof. (of Theorem 20) *Lim-Inf Inequality:* Let $\{u^\epsilon\}_{\epsilon>0} \subset H^1(\Omega)^2$ and suppose $I_\epsilon(u^\epsilon)$ is bounded. Lemma 21 then shows that the functions

$$u_1^\epsilon,\ u_{1x}^\epsilon,\ u_{1y}^\epsilon \chi_{\Omega_\epsilon}, \qquad \text{and} \qquad u_2^\epsilon,\ u_{2x}^\epsilon \chi_{\Omega_\epsilon},\ u_{2y}^\epsilon,$$

are all bounded in $L^2(\Omega)$. Upon passing to a subsequence we may then assume each of them converges weakly in $L^2(\Omega)$ to a limit $u = (u_1, u_2) \in L^2(\Omega)^2$ and from Lemma 19 it concludes $u \in U_0 \times U$; in particular,

$$\begin{bmatrix} u_{1x}^\epsilon & u_{1y}^\epsilon \chi_{\Omega_\epsilon} \\ u_{2x}^\epsilon \chi_{\Omega_\epsilon} & u_{2y}^\epsilon \end{bmatrix} \rightharpoonup \begin{bmatrix} u_{1x} & u_{1y} \\ u_{2x} & u_{2y} \end{bmatrix} \qquad \text{in } L^2(\Omega)^{2\times 2},$$

Since W is convex and continuous it is weakly lower semi-continuous; in particular, the limit of the first term in equation (3.9) is bounded as

$$\int_{\Omega_0} W(\nabla u) \leq \liminf_{\epsilon} \int_\Omega W\left(\begin{bmatrix} u_{1x}^\epsilon & u_{1y}^\epsilon \chi_{\Omega_\epsilon} \\ u_{2x}^\epsilon \chi_{\Omega_\epsilon} & u_{2y}^\epsilon \end{bmatrix} \right).$$

To compute the limit of the second term in equation (3.9), use Jensen's inequality and the quadratic homogeneity of $W(\cdot)$ to obtain

$$\begin{aligned} \int_{S_\epsilon} W\left(\begin{bmatrix} 0 & \sqrt{\epsilon} u_{1y}^\epsilon - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x}^\epsilon & 0 \end{bmatrix} \right) \ &\geq\ \int_{-1}^1 \epsilon W\left(\frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} \begin{bmatrix} 0 & \sqrt{\epsilon} u_{1y}^\epsilon - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x}^\epsilon & 0 \end{bmatrix} dy \right) dx \\ &=\ \int_{-1}^1 W\left(\begin{bmatrix} 0 & [u_1^\epsilon] - \gamma \\ \int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon \, dy & 0 \end{bmatrix} \right) dx, \end{aligned}$$

where $[u_1^\epsilon](x) = u_1(x,\epsilon/2) - u_1(x,-\epsilon/2)$. Lemma 19 shows $[u_1^\epsilon] \rightharpoonup [u_1]$ in $L^2(-1,1)$, so the lim-inf inequality will follow upon showing that $\int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon \, dy \rightharpoonup 0$ in $L^2(-1,1)$. To verify this, first use the Cauchy Schwarz inequality and Lemma 21 to bound this term in $L^2(-1,1)$,

$$\int_{-1}^{1} \left(\int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon \, dy \right)^2 \, dx \leq \int_{-1}^{1} \int_{-\epsilon/2}^{\epsilon/2} \epsilon (u_{2x}^\epsilon)^2 \, dy \, dx$$
$$= \|\sqrt{\epsilon} u_{2x}^\epsilon\|_{L^2(S_\epsilon)}^2 \leq C \left(I_\epsilon(u_1^\epsilon) + \|\gamma\|_{L^2(-1,1)}^2 \right)^{1/2}.$$

To show that this term converges weakly to zero let $\phi \in C_0^\infty(-1,1)$ and compute

$$\left| \int_{-1}^{1} \int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon \, dy \, \phi \, dx \right| = \left| \int_{-1}^{1} \int_{-\epsilon/2}^{\epsilon/2} u_2^\epsilon \, \phi' \, dy \, dx \right| \leq \|u_2^\epsilon\|_{L^2(S_\epsilon)} \sqrt{\epsilon} \|\phi'\|_{L^2(-1,1)}.$$

The sharp Poincaré inequality in Lemma 19 shows $\|u_2^\epsilon\|_{L^2(S_\epsilon)} \leq C\sqrt{\epsilon}$, and it follows that $\int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon \, dy \rightharpoonup 0$ in $L^2(-1,1)$.

Lim-Sup Inequality: To construct a recovery sequence for $u \in H^1(\Omega_0) \times H^1(\Omega)$ select $u^\epsilon = (u_1^\epsilon, u_2)$ where u_1^ϵ is the lifting of u_1 to $H^1(\Omega)$ guaranteed by item 6 of Lemma 19.

Lemma 19 implies that $u^\epsilon \circ \phi_\epsilon^{-1} \to u$ in $H^1(\Omega_0)$, and since $u \mapsto W(\nabla u)$ is continuous on this space it follows that the energy in the bulk converges,

$$\int_{\Omega_\epsilon} W(\nabla u^\epsilon) = \int_{\Omega_0} W\left(\nabla(u^\epsilon \circ \phi_\epsilon^{-1})\right)(1-\epsilon/2) \to \int_{\Omega_0} W(\nabla u).$$

The energy in the fault regions S_ϵ takes the form

$$\int_{S_\epsilon} W\left(\begin{bmatrix} u_{1x}^\epsilon & \sqrt{\epsilon} u_{1y}^\epsilon - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x} & u_{2y} \end{bmatrix} \right) = \int_{S_\epsilon} W\left(\begin{bmatrix} u_{1x}^\epsilon & ([u_1^\epsilon] - \gamma)/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x} & u_{2y} \end{bmatrix} \right)$$

Since u_{2x}, $u_{2y} \in L^2(\Omega)$ are independent of ϵ and $|S_\epsilon| \to 0$ it is immediate that $\|u_{2x}\|_{L^2(S_\epsilon)}$ and $\|u_{2y}\|_{L^2(S_\epsilon)}$ both converge to zero, and from Lemma 19 it follows that $\|u_{1x}^\epsilon\|_{L^2(S_\epsilon)}$ also converges to zero. In addition, $[u_1^\epsilon] - \gamma$ is independent of y and $[u_1^\epsilon] \to [u_1]$ in $L^2(-1,1)$ so

$$\frac{1}{\sqrt{\epsilon}} \|[u_1^\epsilon] - \gamma\|_{L^2(S_\epsilon)} = \|[u_1^\epsilon] - \gamma\|_{L^2(-1,1)} \to \|[u_1] - \gamma\|_{L^2(-1,1)}.$$

Since $W : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is continuous, non-negative, and has quadratic growth it follows that

$$\int_{S_\epsilon} W\left(\begin{bmatrix} u_{1x}^\epsilon & \sqrt{\epsilon} u_{1y}^\epsilon - \gamma/\sqrt{\epsilon} \\ \sqrt{\epsilon} u_{2x} & u_{2y} \end{bmatrix} \right) \to \int_{S_\epsilon} W\left(\begin{bmatrix} 0 & ([u_1] - \gamma) \\ 0 & 0 \end{bmatrix} \right),$$

and $\{u^\epsilon\}_{\epsilon>0} \subset H^1(\Omega)^2$ is a recovery sequence. $\square$

3.3 Evolution Equations

In this section, we show that solutions of the ϵ-problem with governing equations (3.1)–(3.4) converge as $\epsilon \to 0$ to the solution of a limiting problem with the spatial Euler Lagrange operator corresponding to the gamma limit obtained in the previous section.

Solutions of the ϵ-problem (3.1)–(3.4) satisfy $(u(t), \gamma(t)) \in U^2 \times G$ and

$$\int_\Omega (\rho u_{tt}, \hat{u}) + (\mathbb{C}_\epsilon(D(u)), D(\hat{u})) + \int_{S_\epsilon} \mu(\epsilon(u_{2x} + u_{1y}) - \gamma, \hat{u}_{2x} + \hat{u}_{1y}) = \int_\Omega (\rho f, \hat{u}), \tag{3.10}$$

$$\frac{1}{\epsilon} \int_{S_\epsilon} (1/\beta)(\gamma_t, \hat{\gamma}) + l(\gamma, \hat{\gamma}) - \mu(\epsilon(u_{2x} + u_{1y}) - \gamma, \hat{\gamma}) = 0 \tag{3.11}$$

for all $(\hat{u}, \hat{\gamma}) \in U^2 \times G_\epsilon$ where

$$U = \{u \in H^1(\Omega) \mid u(\cdot, \pm 1) = 0\} \quad \text{and} \quad G_\epsilon = \{\gamma \in L^2(S_\epsilon) \mid \gamma_x \in L^2(S_\epsilon)\}.$$

In this weak statement $D(u) = (1/2)\big(\nabla u + (\nabla u)^\top\big)$ is the symmetric part of the displacement gradient and

$$\mathbb{C}_\epsilon(D) = \mathbb{C}\left(\begin{bmatrix} D_{11} & D_{12}\chi_{\Omega_\epsilon} \\ D_{21}\chi_{\Omega_\epsilon} & D_{22} \end{bmatrix}\right), \qquad l(\gamma, \hat{\gamma}) = \nu \gamma_x \hat{\gamma}_x + \eta' \gamma \hat{\gamma}.$$

Solutions of the limiting system satisfy $(u(t), \gamma(t)) \in (U_0 \times U) \times G$ such that

$$\int_\Omega (\rho u_{tt}, \hat{v}) + (\mathbb{C}(D(u)), D(\hat{v})) + \int_{-1}^{1} \mu([u_1] - \gamma, [\hat{v}_1]]) = \int_\Omega (\rho f, \hat{v}), \tag{3.12}$$

$$\int_{-1}^{1} (1/\beta)(\gamma_t, \hat{\gamma}) + l(\gamma, \hat{\gamma}) - \mu([u_1] - \gamma, \hat{\gamma}) = 0, \tag{3.13}$$

for all $(\hat{v}, \hat{\gamma}) \in (U_0 \times U) \times G$ where

$$U_0 = \{u \in H^1(\Omega_0) \mid u(\cdot, \pm 1) = 0\} \quad \text{and} \quad G = H^1(-1, 1).$$

In this section we prove the following theorem which establishes convergence of solutions of equations (3.10)–(3.11) to solutions of (3.12)–(3.13).

Theorem 22. *Denote the domains and spaces as in Notation 18 and assume that the coefficients in equations (3.10)–(3.11) are independent of time and there exists constants C, c such that*

$$0 < c \le \rho(x),\ \mu(x),\ \beta(x),\ \nu(x),\ \mu(x) + \lambda(x) \le C, \quad \textit{and} \quad 0 < \hat{\eta}(x) < C,$$

and there exists $\epsilon_0 > 0$ such that the shear modulus μ is independent of y on S_ϵ for $\epsilon < \epsilon_0$. Fix $f \in L^1[0, T; L^2(\Omega)]$ and initial data $u_t(0) \in L^2(\Omega), \gamma(0) \in H^1(-1, 1)$ and

$u(0) \in U_0 \times U$ for the sharp interface problem and let the initial values for equations (3.10)–(3.11) be

$$u_t^\epsilon(0) = u_t(0), \quad u_2^\epsilon(0) = u_2(0), \quad \gamma^\epsilon(0) = \gamma(0),$$

and $u_1^\epsilon(0) = u_1$ if $u_1(0) \in U$; otherwise, select $\{u_1^\epsilon(0)\}_{\epsilon>0} \subset U$ such that

$$\|u_1^\epsilon(0) - u_1(0)\|_{H^1(\Omega_\epsilon)} \to 0 \quad \text{and} \quad \|u_{1x}^\epsilon(0)\|_{L^2(S_\epsilon)} + \sqrt{\epsilon}\|u_{1y}^\epsilon(0)\|_{L^2(S_\epsilon)} \leq C\|u_1(0)\|_{H^1(\Omega_0)}.$$

Let $(u^\epsilon, \gamma^\epsilon)$ denote the solution of (3.10)–(3.11) with this data and let $\bar{\gamma}^\epsilon(t,x) = (1/\epsilon)\int_{-\epsilon/2}^{\epsilon/2} \gamma^\epsilon(t,x,y)\,dy$. Then $\{(u^\epsilon, \bar{\gamma}^\epsilon)\}_{\epsilon>0}$ converges weakly in $H^1[0,T;L^2(\Omega)^2 \times L^2(-1,1)]$ and strongly in $L^2[0,T;L^2(\Omega)^2 \times L^2(-1,1)]$ to a limit

$$(u,\gamma) \in \mathcal{U} \equiv H^1[0,T;L^2(\Omega)^2 \times L^2(-1,1)] \cap L^2[0,T;(U_0 \times U) \times H^1(-1,1)]$$

with initial data $(u(0), \gamma(0))$ which satisfies

$$\begin{aligned}
\int_0^T \int_{\Omega_0} -(\rho u_t, \hat{u}_t) + (\mathbb{C}(D(u)), D(\hat{u})) + \int_0^T \int_{-1}^1 \mu([u_1] - \gamma, [\hat{u_1}]) &= \int_\Omega (\rho u_t(0), \hat{u}(0)) \\
&\quad + \int_0^T \int_\Omega (\rho f, \hat{u}), \\
\int_0^T \int_{-1}^1 (1/\beta)(\gamma_t, \hat{\gamma}) + l(\gamma, \hat{\gamma}) - \mu([u_1] - \gamma, \hat{\gamma}) &= 0,
\end{aligned}$$

for all $(\hat{u}, \hat{\gamma}) \in \mathcal{U}$ with $\hat{u}(T) = 0$.

3.3.1 Existence of Solutions and Bounds

Equations (3.10)–(3.11) and (3.12)–(3.13) both have the structure of a degenerate wave equation on a product space $U \times G$ in the form

$$C(u,\gamma)_{tt} + B(u,\gamma)_t + A(u,\gamma) = (\rho f, 0) \tag{3.14}$$

where

$$C(u,\gamma) = (\rho u, 0), \qquad \text{and} \qquad B(u,\gamma) = (0, \gamma/\beta), \tag{3.15}$$

(with the later scaled by $1/\epsilon$ for the ϵ equation), and $A : U \times G \to U' \times G'$ is the Riesz map for the product space $U \times G$. For the limit problem

$$A(u,\gamma)(u,\gamma) = \|(u,\gamma)\|_0^2 = \int_{\Omega_0} \mathbb{C}(\nabla u) : (\nabla u) + \int_{-1}^1 l(\gamma,\gamma) + \mu([u_1] - \gamma)^2,$$

and for the ϵ equation $A(u,\gamma)(u,\gamma) = \|(u,\gamma)\|_\epsilon^2$ with

$$\|(u,\gamma)\|_\epsilon^2 = \int_\Omega \mathbb{C}_\epsilon(\nabla u) : (\nabla u) + \int_{S_\epsilon} (1/\epsilon) l(\gamma,\gamma) + \mu(\sqrt{\epsilon}(u_{2x} + u_{1y}) - \gamma/\sqrt{\epsilon})^2.$$

The hypotheses on the initial data in Theorem 22 guarantee $\|(u^\epsilon(0), \gamma^\epsilon(0))\|_\epsilon \to \|(u(0), \gamma(0)\|_0$.

3.3.2 Existence of Solutions

The following theorem from [22, Corollary VI.4.2] establishes existence of strong solutions to equations which take the form shown in (3.14). In this theorem $\mathcal{L}(V, V')$ denotes the continuous linear operators from V to V' and $C \in \mathcal{L}(V, V')$ is monotone if $Cv(v) \geq 0$ for all $v \in V$.

Theorem 23. *Let A be the Riesz map of the Hilbert space V and let W be the semi–normed space obtained from the symmetric and monotone $C \in \mathcal{L}(V, V')$. Let $D(B) \subset V$ be the domain of a linear monotone operator $B : D(B) \to V'$. Assume that $B + C$ is strictly monotone and $A + B + C : D(B) \to V'$ is surjective. Then for every $f \in C^1[0, \infty, W')$ and every pair $v_0 \in V$ and $v_1 \in D(B)$ with $Av_0 + Bv_1 \in W'$, there exists a unique*

$$v \in C[0, \infty, V) \cap C^1(0, \infty, V) \cap C^1[0, \infty, W) \cap C^2(0, \infty, W),$$

with $v(0) = v_0$, $Cv'(0) = Cv_1$ and for each $t > 0$, $v' \in D(B)$, $Av(t) + Bv'(t) \in W'$ and

$$(Cv'(t))' + Bv'(t) + Av(t) = f(t). \tag{3.16}$$

When $V = U^2 \times G$ or $(U_0 \times U) \times G$ with operators as in equation (3.15) the state space is $W = L^2(\Omega)$ with weight ρ and $D(B) = V$ is the whole space. Then $(B + C)(u, \gamma) = (\rho u, \gamma/\beta)$ is strictly monotone, and $C + B + A : V \to V'$ is the sum of the Riesz map with a monotone map, so is surjective.

The existence of strong solutions guaranteed by Theorem 23 was obtained upon writing equation (3.16) as a first order system, $\mathcal{B}(v, v')' + \mathcal{A}(v, v') = \tilde{f}$ with

$$\mathcal{B} = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \qquad \mathcal{A} = \begin{bmatrix} 0 & -A \\ A & B \end{bmatrix} \qquad \text{and} \qquad \tilde{f} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

Classical semi-group theory then provides necessary and sufficient conditions upon the data for the existence of strong solutions. An alternative to the semi-group approach is to use [23, Proposition III.3.3] which establishes existence of weak solutions for a broader class of data and problems with time dependent coefficients. Weak solutions exist when $f \in L^1[0, t; W']$ and satisfy

$$\|v'\|_{L^\infty[0,T;W]} + \|v\|^2_{L^\infty[0,T;V]} + \int_0^T Bv'(v') \leq C \left(\|v'(0)\|^2_W + \|v(0)\|^2_V + \|f\|^2_{L^1[0,T;W']} \right).$$

The following corollary summarizes bounds available for solutions of (3.10)–(3.11) that results from this theory and the Korn and sharp Poincaré inequalities stated in Lemma 19.

Corollary 24. *Under the hypotheses of Theorem 22 there exists a constant $C > 0$ independent of ϵ for which solutions $(u^\epsilon, \gamma^\epsilon)$ of (3.10)–(3.11) satisfy*

$$\begin{aligned}
&\|u_t^\epsilon\|_{L^\infty[0,T;L^2(\Omega)]} + \|u^\epsilon\|_{L^\infty[0,T;H^1(\Omega_\epsilon)]} \\
&\quad + (1/\sqrt{\epsilon})\|\gamma_t^\epsilon\|_{L^2[0,T;L^2(S_\epsilon)]} + (1/\sqrt{\epsilon})\|\gamma^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]} + (1/\sqrt{\epsilon})\|\gamma_x^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]} \\
&\quad + \|u_{1x}\|_{L^\infty[0,T;L^2(S_\epsilon)]} + \|u_{2y}\|_{L^\infty[0,T;L^2(S_\epsilon)]} + \|\sqrt{\epsilon}(u_{2x}^\epsilon + u_{1y}^\epsilon) - \gamma^\epsilon/\sqrt{\epsilon}\|_{L^\infty[0,T;L^2(S_\epsilon)]} \\
&\qquad\qquad \leq C\left(\|u_t(0)\|_{L^2(\Omega)} + \|u(0)\|_{H^1(\Omega_0)} + \|\gamma(0)\|_{L^2(-1,1)} + \|f\|_{L^1[0,T;L^2(\Omega)]}\right).
\end{aligned}$$

In particular, $\|\mathbb{C}_\epsilon(\nabla u^\epsilon)\|_{L^\infty[0,T;L^2(\Omega)^{2\times 2}]}$ and $\|\bar\gamma_t^\epsilon\|_{L^2[0,T;L^2(-1,1)]}$ and $\|\bar\gamma^\epsilon\|_{L^\infty[0,T;H^1(-1,1)]}$ are bounded where $\bar\gamma^\epsilon(t,x) = (1/\epsilon)\int_{-\epsilon/2}^{\epsilon/2} \gamma^\epsilon(t,x,y)\,dy$ is the average of γ^ϵ over the fault region, and the Korn and sharp Poincaré inequality in Lemma 19 imply

$$\|u_{1y}^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]} + \|u_{2x}^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]} \leq C/\sqrt{\epsilon} \quad \text{and} \quad \|u^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]} \leq C\sqrt{\epsilon}.$$

3.3.3 Proof of Theorem 22

Fix test functions

$$\hat u \in \{\hat u \in H^1[0,T;H^2(\Omega_0)\times H^2(\Omega)] \mid \hat u(.,.,\pm 1) = 0 \text{ and } \hat u(T,.,.) = 0\}$$

and $\hat\gamma \in L^2[0,T;H^1(-1,1)]$, and note that test functions $\hat u$ with this regularity are dense in $\{\hat u \in H^1[0,T;U_0\times U] \mid \hat u(T) = 0\}$. Let $\hat u_1^\epsilon \in H^1(\Omega)$ be the function (see Figure 3.3)

$$\hat u_1^\epsilon(t,x,y) = \begin{cases} \hat u_1\left(t,x,\frac{y-\epsilon/2}{1-\epsilon/2}\right) & \epsilon/2 < y < 1, \\ \left(\frac12 + \frac{y}{\epsilon}\right)\hat u_1(t,x,0^+) + \left(\frac12 - \frac{y}{\epsilon}\right)\hat u(t,x,0^-) & -\epsilon/2 \leq y \leq \epsilon/2, \\ \hat u_1\left(t,x,\frac{y+\epsilon/2}{1-\epsilon/2}\right) & -1 < y < -\epsilon/2, \end{cases}$$

and set the test functions in equations (3.10)–(3.11) to be $\hat u^\epsilon = (\hat u_1^\epsilon, \hat u_2) \in U_\epsilon$ and $\hat\gamma(t,x,y) = \hat\gamma(t,x)$ and integrate the equation for u^ϵ by parts in time to get

$$\begin{aligned}
&\int_0^T\int_\Omega -(\rho u_t^\epsilon, \hat u_t^\epsilon) + (\mathbb{C}_\epsilon(D(u^\epsilon)), D(\hat u^\epsilon)) + \int_0^T\int_{-1}^1 \mu\left([u_1^\epsilon] - \bar\gamma^\epsilon, [\hat u_1]\right) \qquad (3.17)\\
&\quad + \int_0^T\int_{S_\epsilon} \mu\left(\epsilon(u_{2x}^\epsilon + u_{1y}^\epsilon) - \gamma^\epsilon, \hat u_{2x}\right) + \mu(u_{2x}^\epsilon, [\hat u_1]) = \int_\Omega (\rho u_t^\epsilon(0), \hat u^\epsilon(0)) + \int_0^T\int_\Omega (\rho f, \hat u^\epsilon),
\end{aligned}$$

and

$$\int_0^T\int_{-1}^1 (1/\beta)(\bar\gamma_t^\epsilon, \hat\gamma) + l\,(\bar\gamma^\epsilon, \hat\gamma) - \mu\left([u_1^\epsilon] - \bar\gamma^\epsilon, \hat\gamma\right) - \int_0^T\int_{S_\epsilon} \mu(u_{2x}^\epsilon, \hat\gamma) = 0, \qquad (3.18)$$

where $\bar\gamma^\epsilon(t,x) = (1/\epsilon)\int_{-\epsilon/2}^{\epsilon/2}\gamma^\epsilon(t,x,y)\,dy$ is the average shear in the fault region. The last terms on the left of these two equations represent the "consistency error" corresponding to approximating a fault region of finite width with a sharp interface. We verify that these terms vanish as $\epsilon \to 0$, and upon passing to a sub-sequence the remaining terms consist of weakly converging terms paired with a strongly converging test function, so the limits of these pairings are the pairings of their limits from which the theorem follows.

Using the bounds in Corollary 24 and Lemma 19 we may pass to a subsequence for which

$$\begin{array}{rll} u_t^\epsilon \rightharpoonup^* u_t, & \text{in} & L^\infty[0,T;L^2(\Omega)^2] \\ \mathbb{C}_\epsilon(Du^\epsilon) \rightharpoonup^* \mathbb{C}(Du) & \text{in} & L^\infty[0,T;\text{Ł}^2(\Omega)^{2\times 2}] \\ [u_1^\epsilon] \rightharpoonup^* [u_1], & \text{in} & L^\infty[0,T;L^2(-1,1)] \\ \bar{\gamma}^\epsilon \rightharpoonup^* \gamma, & \text{in} & L^\infty[0,T;L^2(-1,1)] \\ \bar{\gamma}_t^\epsilon \rightharpoonup \gamma_t, & \text{in} & L^2[0,T;L^2(-1,1)] \\ \bar{\gamma}_x^\epsilon \rightharpoonup^* \gamma_x, & \text{in} & L^\infty[0,T;L^2(-1,1)] \end{array}$$

The first two terms in equations (3.17) and (3.18) are paired with the test functions

$$\begin{array}{rll} \hat{u}_t^\epsilon \to \hat{u}_t, & \text{in} & L^1[0,T;L^2(\Omega)^2] \\ \begin{bmatrix} \hat{u}_{1x}^\epsilon & \hat{u}_{1y}^\epsilon \chi_{\Omega_\epsilon} \\ \hat{u}_{2x}\chi_{\Omega_\epsilon} & \hat{u}_{2y}^\epsilon \end{bmatrix} \to \nabla \hat{u} & \text{in} & L^\infty[0,T;L^2(\Omega_0)^{2\times 2}], \end{array}$$

and the terms involving $[u_1^\epsilon] - \bar{\gamma}^\epsilon$ are paired with test functions independent of ϵ, from which it follows that the first three terms on the left hand sides of equations (3.17) and (3.18) converge as claimed.

The Cauchy Schwarz inequality is used to estimate the first consistency error term in equation (3.17),

$$\begin{array}{rl} & \displaystyle\int_0^T \int_{S_\epsilon} \mu \left(\epsilon(u_{2x}^\epsilon + u_{1y}^\epsilon) - \bar{\gamma}^\epsilon, \hat{u}_{2x}\right) \\ \leq & C\sqrt{\epsilon}\,\|\sqrt{\epsilon}(u_{2x}^\epsilon + u_{1y}^\epsilon) - \gamma^\epsilon/\sqrt{\epsilon}\|_{L^\infty[0,T;L^2(S_\epsilon)]}\|\hat{u}_{2x}\|_{L^1[0,T;L^2(S_\epsilon)]} \\ \to & 0. \end{array}$$

The final terms on the left hand side of equations (3.17) and (3.18) involve u_{2x}^ϵ paired with test functions which are independent of y. It then suffices to show that $\bar{u}_{2x}^\epsilon \equiv \int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon(.,.,y)\,dy$ converges weakly star to zero in $L^\infty[0,T;L^2(-1,1)]$. To do this the Cauchy Schwarz inequality and Corollary 24 are used to first show that it is bounded,

$$\int_{-1}^1 \left(\int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon(t,x,y)\,dy \right)^2 dx \leq \int_{-1}^1 \epsilon \int_{-\epsilon/2}^{\epsilon/2} u_{2x}^\epsilon(t,x,y)^2\,dy\,dx = \epsilon \|u_{2x}^\epsilon(t)\|_{L^2(S_\epsilon)}^2 \leq C.$$

To establish weak star convergence to zero it then suffices to test against smooth functions $\hat{\phi}$ with compact support in $(0,T)\times\Omega$ since they are dense in $L^1[0,T;L^2(-1,1)]$,

$$\int_0^T \int_{-1}^1 \int_{-\epsilon/2}^{\epsilon/2} (\hat{u}_{2x}^\epsilon, \hat{\phi}) = \int_0^T \int_{-1}^1 \int_{-\epsilon/2}^{\epsilon/2} -(\hat{u}_2^\epsilon, \phi_x) \leq \|\hat{u}_2^\epsilon\|_{L^\infty[0,T;L^2(S_\epsilon)]}\|\hat{\phi}\|_{L^1[0,T;L^2(S_\epsilon)]} \leq C\sqrt{\epsilon}.$$

It follows that the limit (u, γ) is a solution of the sharp interface problem, and the theorem follows provided it takes the specified initial values. However, this is direct since $(u^\epsilon, \bar{\gamma}^\epsilon)$ converges weakly in $H^1[0, T; L^2(\Omega)^2 \times L^2(-1, 1)]$ from which it follows that the initial values of the limit (u, γ) are the limit of the initial values.

3.4 Numerical Examples

In this section, we present a numerical example to exhibit the contrast between direct numerical simulation of the stationary form of equations (3.1)–(3.4) and the limit problem considered in Section 3.2. A solution of the limit problem with constant Lamé parameters is constructed by setting

$$u(x, y) = \begin{cases} \frac{1}{2}\begin{pmatrix} e^{a(x-y)} \\ e^{a(x-\kappa y)} \end{pmatrix} + \begin{pmatrix} \phi_y(x, y) \\ -\phi_x(x, y) \end{pmatrix} & y > 0, \\ \frac{1}{2}\begin{pmatrix} -e^{a(x+y)} \\ e^{a(x+\kappa y)} \end{pmatrix} + \begin{pmatrix} \phi_y(x, y) \\ -\phi_x(x, y) \end{pmatrix} & y < 0, \end{cases} \tag{3.19}$$

where $\kappa = \lambda/(2\mu + \lambda)$ and $\phi(x, y) = e^{\ell y}\cos(\ell x)$. Then

$$[u_1(x)] = e^{ax}, \qquad \gamma(x) \equiv [u_1] - (1/\mu)\{\mathbb{C}(\nabla u)_{12}\} = e^{ax} - 2\ell^2\cos(\ell x), \tag{3.20}$$

and right hand sides for the stationary problem are manufactured so that the equations are satisfied,

$$f = -\text{div}(\mathbb{C}(\nabla u)).$$

In the numerical examples below the parameters are set to

$$\mu = 1, \quad \lambda = 2, \quad a = 1/2, \quad \ell = 1/4, \quad \hat{\eta} = 2, \quad \mu = 0, \quad \epsilon = 1/10.$$

and for the limit problem uniform rectangular elements of size $h = 1/n$ with $n \in \mathbb{N}$ are utilized. When $\epsilon > 0$ the fault region S_ϵ is meshed with rectangular elements of size $1/n \times \epsilon/n$; the mesh with $n = 4$ is illustrated in Figure 3.1. Galerkin approximations of the solution to the elasticity problems are computed using the piecewise quadratic finite element spaces on these meshes.

To demonstrate the differences between direct numerical approximations of (3.1)–(3.4) and numerical approximation of the limit problem we first tabulate the errors, $u^0 - u^0_h$, of the numerical approximation of the solution of (3.19), where u^0 is the limiting displacement. Numerical approximations u^ϵ_h of the stationary equations (3.1)–(3.4) are then computed using the same boundary data and body force f. While the exact solution, u^ϵ, of the problem with this data is not known, we tabulate (norms of) the differences $u^0 - u^\epsilon_h$ for ϵ fixed. As $h \to 0$ this difference converges to the "modeling" error $u^0 - u^\epsilon$ associated with approximating the fault region by a surface. An estimate of the mesh size required to resolve the deformation in the fault region is obtained by observing when the difference $u^0 - u^\epsilon_h$ stabilizes. Note that in general $\|u^0 - u^\epsilon\|_{L^2(\Omega)} \to 0$ but $\|u^0 - u^\epsilon\|_{H^1(\Omega_0)} \not\to 0$ as $\epsilon \to 0$.

3.4.1 Uncoupled Problems

Table 3.1 illustrates the errors in the numerical approximation of the solution (3.19) of the limit problem considered in Section 3.2 with γ the function specified in (3.20). It is clear that we have third order convergence rate in $L^2(\Omega)$ and second order rate for the derivative. Norms of the differences $u^0 - u_h^\epsilon$ are presented in Table 3.2. For this example it is clear that highly accurate solutions of the limit problem can be computed on very modest meshes while resolution of the deformation in the fault region requires significantly finer meshes. The norms computed on the finest meshes give an estimate of the modeling error $\|u^0 - u^\epsilon\|_{L^2(\Omega)} \approx 0.35$. Representative solutions for each of the problems are illustrated in Figures 3.4 and 3.5.

h	$\|u^0 - u_h^0\|_{L^2(\Omega)^2}$	$\|u^0 - u_h^0\|_{H^1(\Omega_0)^2}$	# unknowns
1/8	1.684735e-05	4.402589e-04	629
1/16	2.098734e-06	1.091350e-04	2277
1/32	2.617986e-07	2.718018e-05	8645
1/64	3.268957e-08	6.783206e-06	33669
1/128	4.084190e-09	1.694404e-06	132869
1/256	5.154866e-10	4.234319e-07	527877
Norms	1.435134	1.662724	

Table 3.1: *Errors for the uncoupled limit problem ($\epsilon = 0$).*

h	$\|u^0 - u_h^\epsilon\|_{L^2(\Omega)^2}$	$\|u^0 - u_h^\epsilon\|_{H^1(\Omega_0)^2}$	# unknowns
1/8	2.148338e-01	4.553696e+00	1666
1/16	3.029500e-01	5.114175e+00	6402
1/32	3.183221e-01	5.119497e+00	25090
1/64	3.358588e-01	5.177173e+00	99330
1/128	3.459277e-01	5.223399e+00	395266
1/256	3.433117e-01	5.218226e+00	1576962

Table 3.2: *Differences between the uncoupled ϵ-problem and limit problem with $\epsilon = 0.1$.*

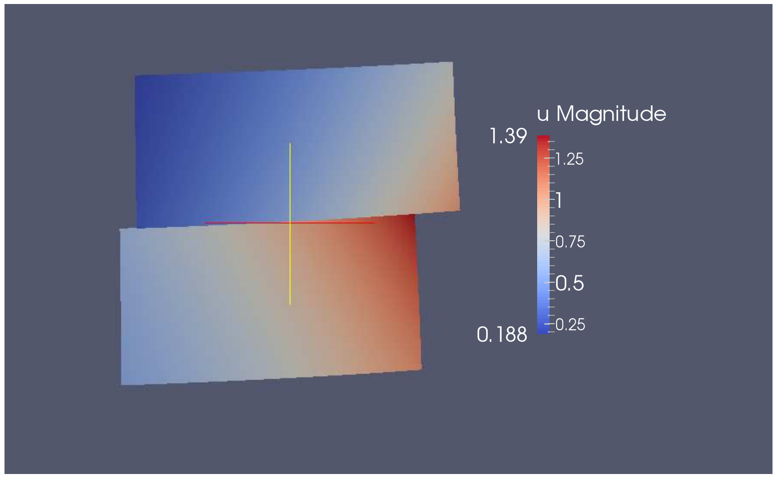

Figure 3.4: *Displacement for the uncoupled limit problem with* $h = 1/256$.

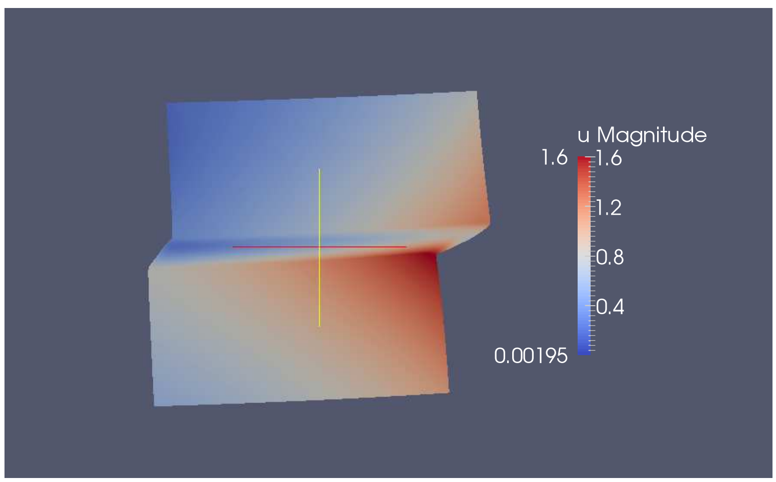

Figure 3.5: *Displacement for the uncoupled* ϵ*-problem with* $\epsilon = 0.1, h = 1/256$.

3.4.2 Coupled Problems

Table 3.3 exhibits the errors for the coupled problem when numerical approximations of both u and γ are calculated using the limit energy given in equation (3.5). Again the optimal third order rate in the $L^2(\Omega)$ norms for both u and γ and second order rate the derivatives of u is obtained. Norms of the differences $u^0 - u_h^\epsilon$ and $\gamma - \gamma_h^\epsilon$ are demonstrated in Table 3.4. As for the uncoupled case, extremely accurate solutions of the limit problem can be computed on inexpensive meshes while resolution of the deformation in the fault region requires much finer meshes. The modeling errors for this problem are $\|u^0 - u^\epsilon\|_{L^2(\Omega)} \approx 0.16$ and $\|\gamma - \gamma^\epsilon\|_{L^2} \approx 0.11$. Representative deformations are illustrated in Figures 3.6 and 3.7.

h	$\|u^0 - u_h^0\|_{L^2(\Omega)^2}$	$\|u^0 - u_h^0\|_{H^1(\Omega_0)^2}$	$\|\gamma - \gamma_h\|_{L^2(-1,1)}$	# unknowns
1/8	1.683317e-05	4.403035e-04	1.597969e-05	646
1/16	2.098047e-06	1.091376e-04	2.077018e-06	2310
1/32	2.617686e-07	2.718031e-05	2.646097e-07	8710
1/64	3.268836e-08	6.783211e-06	3.338419e-08	33798
1/128	4.084169e-09	1.694404e-06	4.191933e-09	133126
1/256	5.161811e-10	4.234319e-07	5.258066e-10	528390
Norms	1.435134	1.662724	1.365834	

Table 3.3: *Errors for the coupled limit problem ($\epsilon = 0$).*

h	$\|u^0 - u_h^\epsilon\|_{L^2(\Omega)^2}$	$\|u^0 - u_h^\epsilon\|_{H^1(\Omega_0)^2}$	$\|\gamma - \gamma_h^\epsilon\|_{L^2(S_\epsilon)}$	# unknowns
1/8	1.575165e-01	4.626232e+00	4.662827e-02	1955
1/16	1.582971e-01	4.628384e+00	5.548420e-02	7491
1/32	1.584715e-01	4.655150e+00	6.875543e-02	29315
1/64	1.594472e-01	4.670685e+00	7.885558e-02	115971
1/128	1.596179e-01	4.675027e+00	8.156652e-02	461315
1/256	1.593571e-01	4.676821e+00	8.152622e-02	1840131

Table 3.4: *Differences between coupled ϵ-problem and limit problem with $\epsilon = 0.1$.*

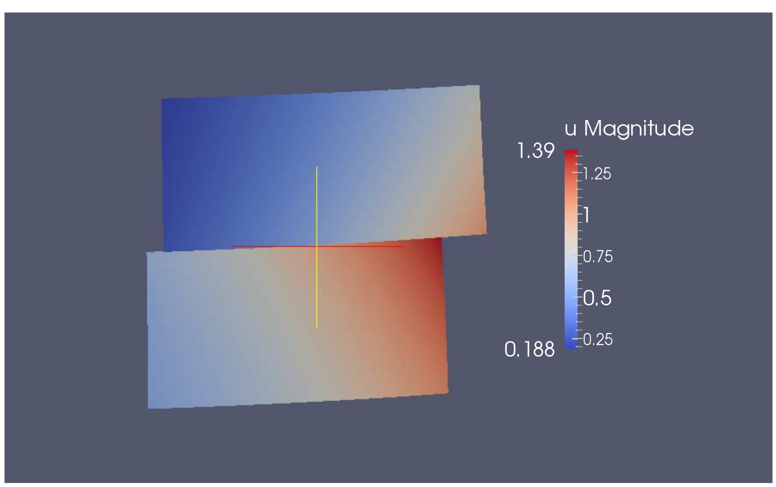

Figure 3.6: *Displacement for the coupled limit problem with $h = 1/256$.*

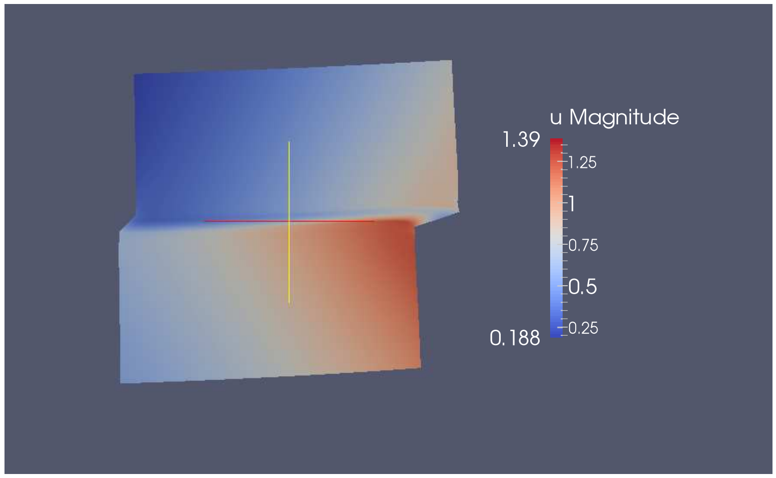

Figure 3.7: *Displacement for the coupled ϵ-problem with $\epsilon = 0.1$, and $h = 1/256$.*

3.4.3 Dislocation

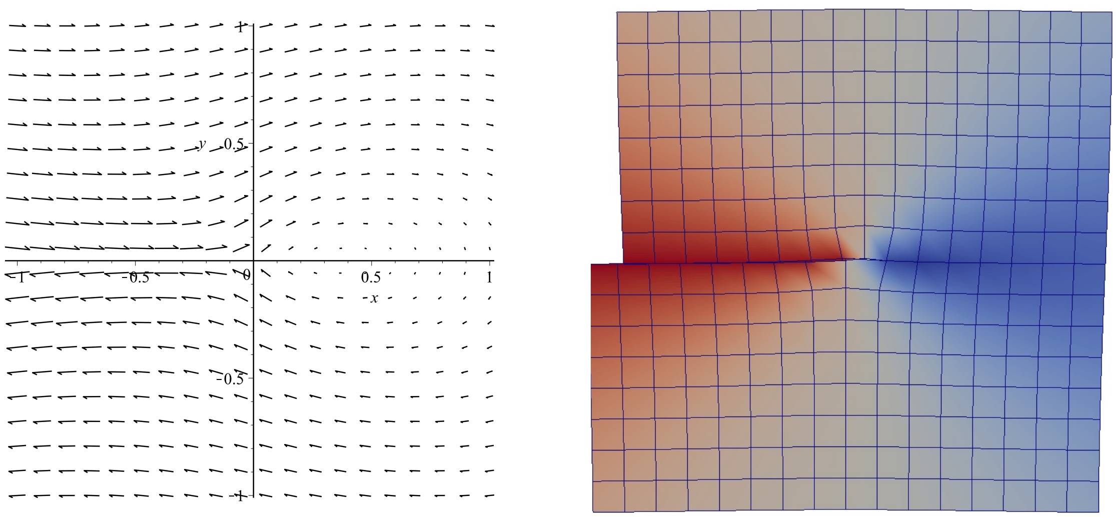

Figure 3.8: *Displacement field with dislocation at the origin and numerical approximation.*

An explicit solution for the linear elasticity problem with an edge dislocation along the z–axis is [12]

$$u(x,y) = \frac{1}{2\pi}\begin{pmatrix} \arctan(y/x) + \frac{xy}{2(1-\hat{\nu})(x^2+y^2)} \\ \frac{-1}{4(1-\hat{\nu})}\left((1-2\hat{\nu})\ln(x^2+y^2) + \frac{x^2-y^2}{x^2+y^2}\right) \end{pmatrix}, \text{ with Poisson ratio } \hat{\nu} = \frac{\lambda}{2(\mu+\lambda)}.$$

This solution, illustrated in Figure 3.8, represents the displacement that results when a dislocation, currently at the origin, has propagated along the negative x–axis so that $[u_1(x)] = 1$ for $x < 0$ and $[u_1(x)] = 0$ for $x > 0$. The stress has a singularity of order $O(1/r)$ at the origin and is otherwise continuous, and the displacement is square integrable but its derivatives are not. While the results of the prior sections are not applicable to singular solutions, almost singular solutions arise in engineering practice so it is important for the numerical schemes to be robust in this context.

h	$\|u - u_h\|_{L^2(\Omega)}$	$\|\gamma_h\|_{L^2(-1,1)}$
1/8	1.226581e-02	7.005017
1/16	6.141691e-03	9.835010
1/32	3.073164e-03	13.85832
1/64	1.537180e-03	19.56296
1/128	7.687419e-04	27.64101
1/256	3.844092e-04	39.07248

Table 3.5: *Errors in displacement and norm of shear for dislocation example.*

To illustrate the robustness properties of codes using the limit energy a singular solutions of the stationary limit problem is manufactured by setting $\gamma = [u_1] - (1/\mu)T_{12}$, so that the jump condition is satisfied, and non–homogeneous right hand side for the equation for γ,

$$f_0(x) = \hat{\eta}\gamma(x) - T_{12}(x, 0) = \hat{\eta}\,[u_1(x)] + \frac{(\lambda + \mu)(\hat{\eta} + \mu)}{\pi(2\mu + \lambda)x}.$$

(Since γ_{xx} does not exist we set the coefficient of this term to be zero.) Inner products of this (non–integrable) function with basis functions were approximated using Gaussian quadrature. Table 3.5 shows that the error $\|u - u_h\|_{L^2(\Omega)}$ converges linearly with h and $\|\gamma_h\|_{L^2(-1,1)} \approx O(1/\sqrt{h})$ diverges since the limit $\gamma(x) \approx O(1/x)$ is not integrable. Figure 3.8 illustrates the deformation computed with quadratic elements on a 16×16 grid.

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