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PRESENTED BY Kimberly Weston

ACCEPTED BY THE DEPARTMENT OF Mathematical Sciences

Dmitry Kramkov	April 2016
MAJOR PROFESSOR	DATE
Thomas Bohman	April 2016
DEPARTMENT HEAD	DATE

APPROVED BY THE COLLEGE COUNCIL

Frederick J. Gilman	April 2016
DEAN	DATE

Market Stability in Nonequivalent Markets and the Martingale Property of the Dual Optimizer

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Kim Weston

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Contents

Introduction

This thesis consists of two parts. Chapter I is based on the single-authored work, "Stability of utility maximization in nonequivalent markets," which will appear in *Finance & Stochastics*. Stability of the utility maximization problem with random endowment and indifference prices is studied for a sequence of financial markets in an incomplete Brownian setting. The novelty lies in the *nonequivalence* of markets, in which the volatility of asset prices (as well as the drift) varies. Degeneracies arise from the presence of nonequivalence. In the positive real line utility framework, a counterexample is presented showing that the expected utility maximization problem can be unstable. A positive stability result is proven for utility functions on the entire real line.

Chapter II is based on the joint work with Dmitry Kramkov, "Muckenhoupt's (A_p) condition and the existence of the optimal martingale measure." The chapter seeks to answer the question, "When is the dual optimizer a martingale?" In the problem of optimal investment with a utility function defined on $(0, \infty)$, we formulate sufficient conditions for the dual optimizer to be a uniformly integrable martingale. Our key requirement consists of the existence of a martingale measure whose density process satisfies the probabilistic (A_p) condition for the power $p = 1/(1-a)$, where $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. In Section II.6, we construct a counterexample showing that this (A_p) condition is sharp.

Chapter I

Stability of Utility Maximization in Nonequivalent Markets

I.1 Introduction

As part of Hadamard's well-posedness criteria, stability of the utility maximization problem with random endowment is studied with respect to perturbations in both volatility and drift. Specifically, we seek to answer the question:

What conditions on the utility function and modes of convergence on the sequence of volatilities and drifts guarantee convergence of the corresponding value functions and indifference prices?

Perhaps surprisingly, convergence can fail even in the tamest of settings when the utility function is finite only on $\mathbb{R}$ and the volatility can vary. We present a simple counterexample in a stochastic volatility setting with power utility. When the utility function is finite only on $\mathbb{R}_+$, the admissibility criterion is harsh: negative values in terminal wealth plus random endowment equate to minus infinity in utility. When volatility can vary, a contingent claim that is replicable only in the limiting market requires strictly more initial capital in every pre-limiting market in order to avoid a minus infinity contribution towards expected utility. As part of the counterexample, we prove a positive convergence result in which the limiting market adopts an additional admissibility condition that is implicitly present in each pre-limiting market.

When the investor's utility function is finite on the entire real line, the admissibility criterion is different. Our main result provides conditions on the utility function and on the sequence of markets so that we have convergence of the value functions and indifference prices. We consider a similar setup to

[41], and our main assumptions are analogous to theirs. The only non-standard assumption we require is an assumption on the limiting market. The significant difficulty stems from the growth of the dual utility function at infinity because in contrast to utilities on $\mathbb{R}_+$, the conjugate of a real line utility grows strictly faster than linearly at infinity. We provide two sufficient conditions. These conditions include:

1. The first condition applies to a contingent claim that is replicable in the limiting market yet not replicable in any pre-limiting market. The corresponding stability problem is relevant when a claim's underlying asset is not liquidly traded but is closely linked to a liquidly traded asset. This situation arises, e.g., when hedging weather derivatives by trading in related energy futures or when an executive wants to hedge his position in company stock options but is legally restricted from liquidly trading his own company's stock. Practical and computational aspects of this problem are considered by [11], [45], and in more generality by [20].

2. The second sufficient condition requires exponential preferences and additional regularity of the limiting market but places no restrictions on the claim's replicability. This case covers a general incomplete Brownian market structure under a mild BMO condition on the limiting market. The connection between BMO and exponential utility is long established; see, for example, [12] and [22].

The questions of existence and uniqueness for the optimal investment problem from terminal wealth are thoroughly studied. The surrounding literature is vast, and only a small subset of work is mentioned here. For general utility functions on $\mathbb{R}_+$ in a general semimartingale framework, [37] finish a long line of research on incomplete markets without random endowment. In [9], this work is extended to include bounded random endowment, while [26] study the unbounded random endowment case. For utility functions on $\mathbb{R}$ in a locally bounded semimartingale framework, [49] studies the case with no random endowment, while [47] handle the unbounded random endowment case. In [6], the authors study the non-locally bounded semimartingale setting without random endowment and unify the framework for utilities on $\mathbb{R}$ and $\mathbb{R}_+$.

Stability with respect to perturbations in the market price of risk for fixed volatility is first studied in [41] for utility on $\mathbb{R}_+$ and later in [5] for exponential utility. Both works consider risky assets with continuous price processes and no random endowment. For a locally bounded asset and an investor with random endowment, [32] study a market stability problem in which the financial market and random endowment stay fixed while the subjective probability measure and utility function vary. A BSDE stability result is used in [19] to study a specific stability problem for an exponential investor related to the indifference price formulas derived in [20]. Using this BSDE stability result, [19]'s market stability result extends to a case with a fixed market price of risk and a varying underlying correlation factor between the traded and nontraded securities. In contrast to these previous works, we seek to prove a stability result for a general utility function on $\mathbb{R}$ allowing for varying both volatility and market price of risk in the presence of random endowment.

Stability is related to the concept of robustness with respect to a collection of probability measures. Robustness in option pricing dates back to the uncertain volatility models (UVM) of [4] and [43], who consider a range of possible volatilities and determine the best- and worst-case option prices. In contrast to UVM, which seek to price claims in a *complete* yet uncertain market, we seek to determine stability properties using indifference prices in an *incomplete* market. With utility maximization, both the volatility and the drift impact investors' optimal trading decisions. In [17], [44], and [50], the authors consider robust utility maximization problems, in which both the volatility and drift vary within a class of subjective probability measures. Robust optimization seeks the best trading strategy in the worst possible model, whereas our investor firmly believes in the specified subjective model, and we seek to determine which of these models are stable.

The structure of the paper is as follows. Section I.2 presents a counterexample for a power investor with unspanned stochastic volatility. Section I.3 lays out the model assumptions and states the main result. The proofs are presented in Section I.4. Finally, Section I.5 provides a counterexample showing the necessity of a nondegeneracy assumption and provides sufficient conditions on the structure of the dual problem for this assumption to hold.

I.2 Stability Counterexample for Power Utility

When an investor's preferences are described by utility on the positive real line and random endowment is present, the admissibility condition provides an additional implicit constraint. As we will prove, this constraint can create a discontinuity in the value function and indifference prices for markets with varying martingale drivers. The following are simple incomplete Brownian models with a contingent claim that can only be replicated in the limiting market.

I.2.1 Market Model

We let B and W be independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ is the natural filtration of (B, W) completed with $\mathbb{P}$-null sets and $\mathcal{F} = \mathcal{F}_T$. We consider stock market models, S^ρ, with stochastic volatility indexed by correlation parameter $\rho \in (-1, 1)$ where

$$\begin{aligned} dS_t^\rho &= \mu\nu_t dt + \sqrt{\nu_t}\left(\sqrt{1-\rho^2}dB_t + \rho dW_t\right), \quad S_0^\rho := 0, \\ d\nu_t &= \kappa\left(\theta - \nu_t\right)dt + \sigma\sqrt{\nu_t}dB_t, \quad \nu_0 := 1. \end{aligned} \tag{I.2.1}$$

The constants $\kappa, \theta, \sigma > 0$ satisfy Feller's condition, $2\kappa\theta \ge \sigma^2$, which guarantees that there exists a unique strong solution for ν that is strictly positive for all $\rho \in (-1, 1)$. The risky asset S^ρ is traded, whereas the stochastic volatility ν is not traded. The dynamics of S^ρ are written in an arithmetic fashion, which can be viewed as the returns of a positive asset. For our purposes, the outcome of trading is unchanged whether we consider arithmetic or geometric specifications of the dynamics. For a fixed ρ, [36] studies the utility maximization problem in the context of this model. Each ρ market also has a bank account with zero interest rate.

A contingent claim f is defined by $f := \phi(B_T)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is a bounded, continuous function. The claim f is replicable in the $\rho = 0$ market; however, it is not replicable in any other market. We define $\phi_{\min} := \inf \phi$, which corresponds to the subreplication price of f in the $\rho \neq 0$ markets (see Proposition I.2.1 below). We allow for the possibility that ϕ is a constant function, in which case the endowment f can be viewed as a deterministic initial endowment.

Remark I.2.1. Our model assumes that all markets share the same probability space and filtration. In particular, we assume that both Brownian motions, B and W, are observable in each ρ market. However, suppose an investor in the ρ market can only observe the path of the risky asset, S^ρ. Then such an investor can also observe both B and W.[1] Since the quadratic variation of S^ρ is observable and there exists a unique (positive) strong solution to the SDE for ν, we can observe ν and B from $\langle S^\rho \rangle$. Also from the observation of S^ρ and μ, we can determine $(\sqrt{1-\rho^2}B + \rho W)$, which allows the investor to observe both B and W separately (for $\rho \neq 0$).

I.2.2 Optimal Investment Problem

An investor is modeled by power utility $U(x) = x^\gamma/\gamma$ for $x > 0$ with $\gamma < 0$. As a convention, $U(x) = -\infty$ for $x \le 0$. The investor begins with initial

[1] Many thanks to an anonymous reviewer for making this keen observation.

capital $x > -\phi_{\min}$. A progressively measurable process H is *integrable* if $\int_0^T \nu_t H_t^2 dt < \infty$, a.s. An integrable H is called *ρ-admissible* if there exists a finite constant $K = K(H)$ such that $(H \cdot S^\rho)_t \geq -K$ for all $t \in [0, T]$. We define the primal optimization set by

$$\mathcal{C}(\rho) := \left\{ (H \cdot S^\rho)_T : H \text{ is } \rho\text{-admissible} \right\}.$$

For $\rho \in (-1, 1)$, the primal value function is defined by

$$u(x, \rho) := \sup_{X \in \mathcal{C}(\rho)} \mathbb{E}\left[U\left(x + X + f \right) \right], \quad x > -\phi_{\min}. \tag{I.2.2}$$

Remark I.2.2. For $\rho = 0$, $u(\cdot, 0)$ is well-defined for a larger x-domain than $(-\phi_{\min}, \infty)$. Yet the x-domain is tight for every $\rho \neq 0$. This discontinuity in the domains at $\rho = 0$ hints at the issue of (dis)continuity with respect to ρ in the primal problem. See [9] for more details on the primal domain definition.

For each $\rho \in (-1, 1)$, we define the dual domain by

$$\mathcal{D}(\rho) := \left\{ \text{measures } \mathbb{Q} \sim \mathbb{P} : \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] = 1 \text{ and } \mathbb{E}^{\mathbb{Q}}\left[X \right] \leq 0 \ \forall X \in \mathcal{C}(\rho) \right\}.$$

Lemma 5.2 in [8] shows that $\mathcal{D}(\rho) \neq \emptyset$. Similar to [33], we have the following result, which will be proven in Section I.4.

Proposition I.2.1. *Let $\rho \neq 0$ be given. The subreplication price of f is ϕ_{min}; that is,*

$$\inf_{\mathbb{Q} \in \mathcal{D}(\rho)} \mathbb{E}^{\mathbb{Q}}\left[\phi(B_T) \right] = \phi_{min}.$$

Moreover, for all $x \in \mathbb{R}$ and $(H \cdot S^\rho)_T \in \mathcal{C}(\rho)$ such that $x + (H \cdot S^\rho)_T + f \geq 0$, we have

$$x + (H \cdot S^\rho)_T \geq -\phi_{min}. \tag{I.2.3}$$

We consider a different optimization problem for $\rho = 0$ with an additional admissibility constraint motivated by (I.2.3). For any $x > -\phi_{\min}$, we define the admissibly-constrained primal optimization sets in the $\rho = 0$ market by

$$\mathcal{C}_c(x) := \left\{ X \in \mathcal{C}(0) : x + X \geq -\phi_{\min} \right\}.$$

The corresponding admissibly-constrained primal value function is defined by

$$u_c(x) := \sup_{X \in \mathcal{C}_c(x)} \mathbb{E}\left[U\left(x + X + f \right) \right], \quad x > -\phi_{\min}. \tag{I.2.4}$$

The following is the main result of the section. We note that when $\phi(z) = 0$ for all $z \in \mathbb{R}$, we have that $\mathcal{C}(\rho = 0)$ for $u(x, 0)$ corresponds to $\mathcal{C}_c(x)$, and $u(x, 0) = u_c(x)$ for $x > -\phi_{\min}$. In this case, the next theorem provides a stability result in the spirit of [41].

Theorem I.2.2. *Assume the market dynamics* (I.2.1) *and utility function* $U(x) = x^\gamma/\gamma$*, for* $x > 0$*, with* $\gamma < 0$*. Assume the random endowment function* ϕ *is continuous and bounded, and the initial endowment is* $x > -\phi_{min}$*. Let* u *and* u_c *be as in* (I.2.2) *and* (I.2.4)*, respectively. Then,*

$$\lim_{\rho \to 0} u(x, \rho) = u_c(x) .$$

The proofs of Theorem I.2.2 and its Corollary I.2.4 (below) will follow in Section I.4. The corollary says that when ϕ is not constant, indifference prices for f do not converge to the unique arbitrage-free price in the $\rho = 0$ market as $\rho \to 0$. For any $\rho \in (-1, 1)$, we define the value function without random endowment by

$$w(x, \rho) := \sup_{X \in \mathcal{C}(\rho)} \mathbb{E} \left[U \left(x + X \right) \right], \quad x > 0. \tag{I.2.5}$$

Definition I.2.3. Given $x > -\phi_{\min}$ and $\rho \in (-1, 1)$, $p = p(x, \rho) \in \mathbb{R}$ is called the *indifference price for* f *at* x *in the* ρ *market* if $w(x + p, \rho) = u(x, \rho)$.

Of course, for $\rho = 0$, the indifference price corresponds to the unique arbitrage-free price for the bounded replicable claim, f. Also notice that since indifference prices are arbitrage-free prices, then $p(x, \rho) \geq \phi_{\min}$ for every $x > -\phi_{\min}$.

Corollary I.2.4. *Under the assumptions of Theorem I.2.2 and for* ϕ *not constant: For* $x > -\phi_{min}$*, the indifference prices for* f *do not converge to the arbitrage-free price in the* $\rho = 0$ *market. Indeed,* $\limsup_{\rho \to 0} p(x, \rho) < p(x, 0)$.

Remark I.2.3. For the sake of clarity, emphasis is placed on the simplicity of the power investor's problem. Some aspects can be generalized at the expense of more lengthy proofs and set-ups, e.g., a more general utility function or more general asset dynamics. For the special case when $f = 0$, Theorem I.2.2 can be extended to the more general market models of Section I.3 in order to generalize the value function convergence of Theorem 2.12 in [41] in the varying volatility setting. The difficulty in generalizing beyond $f = 0$ stems from the need for a dual conjugacy result for u_c, which is not available in the literature due to the Inada condition not being satisfied at $x = 0$ for the (ω-dependent) function $x \mapsto U(x + f - \phi_{\min})$.

I.3 Utility Functions on $\mathbb{R}$

Modeling investor preferences on the entire real line removes the fixed admissibility lower bound, which prevents the degeneracy of Theorem I.2.2 from occurring. The remainder of this work is devoted to studying conditions that guarantee stability for real line utility functions.

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a filtered probability space with the filtration generated by d-dimensional Brownian motion $B = (B^1, \dots, B^d)$. We assume that $\mathbb{F}$ is completed with all $\mathbb{P}$ null sets and $\mathcal{F} = \mathcal{F}_T$, for a fixed time horizon $T \in (0, \infty)$.

We consider a sequence of financial market models with stocks S^n valued in $\mathbb{R}$, for $1 \le n \le \infty$,

$$dS^n_t = \lambda^n_t \, |\sigma^n_t|^2 \, dt + \sigma^n_t dB_t \, , \quad S^n_0 = 0. \tag{I.3.1}$$

Letting $\mathcal{L}^p := \{\text{progressively measurable } \theta : \int_0^T |\theta_t|^p dt < \infty, \text{ a.s}\}$, $p = 1, 2$, we require that $\sigma^n = (\sigma^{n,1}, \dots, \sigma^{n,d})$ satisfies $\sigma^{n,i} \in \mathcal{L}^2$ for $1 \le n \le \infty$, $1 \le i \le d$, and $\lambda^n |\sigma^n|^2 \in \mathcal{L}^1$ for $1 \le n \le \infty$. For $1 \le n \le \infty$, we define the local martingales M^n by

$$M^n := (\sigma^{n,1} \cdot B^1) + \ldots + (\sigma^{n,d} \cdot B^d), \tag{I.3.2}$$

so that the dynamics of S^n are of the form

$$dS^n_t = \lambda^n_t d\langle M^n \rangle_t + dM^n_t \, , \quad S^n_0 = 0.$$

Additionally, we assume that $\lambda^n \sigma^{n,i} \in \mathcal{L}^2$ for $1 \le n \le \infty$, $1 \le i \le d$, so that $(\lambda^n \cdot M^n)$ is well-defined. We let $Z^n_t := \mathcal{E}(-\lambda^n \cdot M^n)_t$, $t \in [0, T]$, denote each market's minimal martingale density process, where $\mathcal{E}(\cdot)$ refers to the stochastic exponential. Each market is assumed to have a bank account with a zero interest rate.

A sequence $\{X_n\}_{n\ge 1}$ of semimartingales is said to converge to X in the *semimartingale topology* provided that

$$\sup_{|\theta|\le 1} \mathbb{E}\left[|(\theta \cdot (X^n - X))_T| \wedge 1 \right] \longrightarrow 0 \quad \text{as } n \to \infty.$$

Here, the supremum is taken over progressively measurable θ, which are bounded uniformly by 1 in t and ω. The following assumptions capture the necessary market regularity and the convergence of a sequence of markets.

Assumption I.3.1. The collections $\{M^n\}_{1\le n\le\infty}$ and $\{(\lambda^n \cdot M^n)\}_{1\le n\le\infty}$ satisfy the convergence relations:

$$M^n \longrightarrow M^\infty \quad \text{and} \quad (\lambda^n \cdot M^n) \longrightarrow (\lambda^\infty \cdot M^\infty)$$

in the semimartingale topology as $n \to \infty$.

The assumption that $(\lambda^n \cdot M^n) \longrightarrow (\lambda^\infty \cdot M^\infty)$ is similar to the appropriate topology assumption of [41], whereas the convergence assumption on M^n is new since the previous market stability work required the martingale components to remain constant.

Assumption I.3.2. Each minimal martingale density process, Z^n, for $1 \leq n \leq \infty$, is a $\mathbb{P}$-martingale.

Under the minimal martingale measure $\mathbb{Q}^n$, where $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = Z^n_T$, S^n is a local martingale and any $\mathbb{P}$-local martingale N such that $\langle N, M^n \rangle_t = 0$ for $t \in [0,T]$ remains a local martingale under $\mathbb{Q}^n$. We refer to [18] for a survey on minimal martingale measures and their use in mathematical finance.

Under Assumption I.3.1, we have that $(\lambda^n \cdot M^n)_T \longrightarrow (\lambda^\infty \cdot M^\infty)_T$ and $\langle \lambda^n \cdot M^n \rangle_T \longrightarrow \langle \lambda^\infty \cdot M^\infty \rangle_T$ in probability as $n \to \infty$. Hence, $Z^n_T \longrightarrow Z^\infty_T$ in probability as $n \to \infty$. Under Assumption I.3.2, each Z^n is a martingale, and so Scheffe's Lemma implies the seemingly stronger fact that $Z^n_T \longrightarrow Z^\infty_T$ in $L^1(\mathbb{P})$ as $n \to \infty$.

A further non-degeneracy assumption is needed on the limiting market in order to ensure that randomness does not disappear in a degenerate way. A counterexample showing that this condition is in some sense necessary is provided in Section I.5.

Assumption I.3.3. The dynamics of $\langle M^\infty \rangle$ satisfy the nondegeneracy condition $\sum_{i=1}^d (\sigma_t^{\infty,i})^2 \neq 0$ for all $t \in [0,T]$, $\mathbb{P}$-a.s.

Remark I.3.1. The markets $\{S^{\rho_n}\}_{1 \leq n \leq \infty}$ of Section I.2 satisfy Assumptions I.3.1, I.3.2, and I.3.3 for any $\rho_n \longrightarrow \rho \in [-1,1]$ as $n \to \infty$.

Finally, a contingent claim $f \in L^\infty(\mathbb{P})$ is given and is independent of $n \in \mathbb{N}$. We make no assumption on the replicability of f at this time.

I.3.1 Optimal Investment Problem

An investor is modeled by preferences $U : \mathbb{R} \to \mathbb{R}$, which are finite on the entire real line. U is assumed to be continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions at $-\infty$ and $+\infty$:

$$U'(-\infty) := \lim_{x \to -\infty} U'(x) = \infty \quad \text{and} \quad U'(+\infty) := \lim_{x \to \infty} U'(x) = 0.$$

Additionally, we assume that U satisfies the reasonable asymptotic elasticity conditions of [37] and [49]:

$$AE_{-\infty}(U) := \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1 \quad \text{and} \quad AE_{+\infty}(U) := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1\,. \tag{I.3.3}$$

The utility function's Fenchel conjugate is defined by for $y > 0$ by $V(y) := \sup_{x \in \mathbb{R}} \{U(x) - xy\}$. V is strictly convex and continuously differentiable. Without loss of generality, we assume that $U(0) > 0$. When $U(0) > 0$, we have $V(y) > 0$ for all $y > 0$.

Similar to [41], [32], and [5], we make the following assumption:

Assumption I.3.4. The collection of random variables $\{V(Z_T^n)\}_{1\le n\le\infty}$, where Z_T^n is the minimal martingale density for the S^n market, is uniformly integrable.

In [41], the authors show that Assumption I.3.4 is both necessary and sufficient for stability of the value function in the case of complete markets. They study the stability problem with a utility function defined on the positive real line, no random endowment, fixed volatility, and varying market price of risk; see [41] Proposition 2.13. In an incomplete setting, they provide a counterexample to the value function stability showing that in some sense Assumption I.3.4 is necessary.

For $1 \le n \le \infty$, a process H is S^n*-integrable* if $H\sigma^{n,i} \in \mathcal{L}^2$ for $1 \le i \le d$. Cauchy-Schwartz's inequality produces $H\lambda^n(\sigma^{n,i})^2 \in \mathcal{L}^1$ for $1 \le i \le d$. The S^n market's admissible strategies are defined by

$$\mathcal{H}^n_{\text{adm}} := \{H : H \text{ is } S^n\text{-integrable, } \exists K = K(H),\ (H\cdot S^n)_t \ge -K,\ \forall t\}\,.$$

The primal value function is defined by

$$u_n(x) := \sup_{H\in\mathcal{H}^n_{\text{adm}}} \mathbb{E}\left[U\left(x + (H\cdot S^n)_T + f\right)\right], \quad x \in \mathbb{R}. \tag{I.3.4}$$

Let $\mathcal{M}^n$ denote the set of probability measures $\mathbb{Q}$ such that $\mathbb{Q} \sim \mathbb{P}$ and S^n is a local martingale under $\mathbb{Q}$. We are primarily interested in such measures that have finite V-entropy: $\mathbb{E}[V(\frac{d\mathbb{Q}}{d\mathbb{P}})] < \infty$. Let $\mathcal{M}^n_V$ denote those measures $\mathbb{Q} \in \mathcal{M}^n$ having finite V-entropy. For $1 \le n \le \infty$, the dual value function is defined for the S^n market by

$$v_n(y) := \inf_{\mathbb{Q}\in\mathcal{M}^n_V} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) + y\frac{d\mathbb{Q}}{d\mathbb{P}}f\right], \quad y > 0. \tag{I.3.5}$$

At first glance, our definition of the dual value function differs from that of [47], who, for $1 \le n \le \infty$, consider the infimum over $\mathbb{Q} \ll \mathbb{P}$ such that S^n is a $\mathbb{Q}$-local martingale and $\mathbb{E}[V(\frac{d\mathbb{Q}}{d\mathbb{P}})] < \infty$. Assumptions I.3.2 and I.3.4 plus $Z_T^n > 0$ imply that $\mathcal{M}^n_V \neq \emptyset$. In this case, Theorem 1.1(iii) of [47] shows that the optimal dual element lies in the set $\mathcal{M}^n_V$, and thus the two dual value function definitions agree.

The primal admissible class of strategies is too small to attain a solution to the optimal investment problem. However, the behavior of the value function is our primary interest, rather than the behavior (or even attainability) of the optimizer. Using that $f \in L^\infty(\mathbb{P})$ and $\mathcal{M}^n_V \neq \emptyset$, Theorem 1.2(i) of [47] implies that our definition of the primal value function agrees with the definition of $u_{\mathcal{E}}$ of [47]. Here, $\mathcal{E} = x_n + f$ and $\mathcal{E}$ refers to the notation of the aforementioned work.

By using [1] and [2], for $1 \leq n \leq \infty$, we can rewrite any $\mathbb{Q} \in \mathcal{M}_V^n$ as $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T^n \mathcal{E}(L)_T$, where L is a local martingale null at 0 such that $\langle L, M^n \rangle_t = 0$ for all $t \in [0,T]$. We need to make a further assumption in order to ensure a "nice" structure of the limiting market's dual domain. For $n = \infty$, let $\mathcal{B}$ be defined by

$$\begin{aligned}\mathcal{B} := \{&\text{local martingales } L : L_0 = 0, \langle L, M^\infty \rangle_t = 0,\, \forall t \in [0,T], \\ &\qquad \exists \text{ constant } C = C(L),\, \mathcal{E}(L)_t \leq C,\, \forall t \in [0,T]\}.\end{aligned}$$

Assumption I.3.5. For $n = \infty$, the dual problem, (I.3.5), can be expressed as

$$v_\infty(y) = \inf_{L \in \mathcal{B}} \mathbb{E}\left[V\left(yZ_T^\infty \mathcal{E}(L)_T\right) + yZ_T^\infty \mathcal{E}(L)_T f\right], \quad y > 0,$$

where Z_T^∞ is the minimal martingale density in the S^∞ market.

This assumption is non-trivial to verify in general due to the fact that V is increasing strictly faster than linearly as $y \longrightarrow +\infty$. It is mathematical in nature and ensures that the dual optimizer does not vary "too much". Section I.5 provides two sufficient conditions. The first condition covers the original motivation for our stability problem, where the contingent claim is replicable in the (incomplete) limiting market but not replicable in any pre-limiting market. In this case, the limiting market consists of a driving Brownian motion, a replicable claim, and additional independent Brownian noise. The second condition makes no assumptions on the claim's replicability; however, it requires exponential preferences and imposes a mild BMO condition on the limiting market. Indeed, a BMO condition on the limiting market's minimal martingale density ensures that the dual optimizer has controlled oscillations, which implies Assumption I.3.5. Similarly, [12] make use of a form of BMO regularity of *some* dual element in order to establish BMO regularity of the *optimal* dual element.

The following is our main result.

Theorem I.3.6. *Suppose that the sequence of markets satisfies Assumptions I.3.1, I.3.2, and I.3.4. Suppose that the limiting market satisfies Assumptions I.3.3 and I.3.5. Then, for $x_n \longrightarrow x$ as $n \to \infty$,*

$$\lim_{n \to \infty} u_n(x_n) = u_\infty(x).$$

Moreover, for $y_n \longrightarrow y > 0$ as $n \to \infty$,

$$\lim_{n \to \infty} v_n(y_n) = v_\infty(y).$$

For $1 \leq n \leq \infty$, the value function without random endowment is defined by

$$w_n(x) := \sup_{H \in \mathcal{H}_{\mathrm{adm}}^n} \mathbb{E}\left[U\left(x + (H \cdot S^n)_T\right)\right], \quad x \in \mathbb{R}.$$

Definition I.3.7. Given $1 \le n \le \infty$ and $x \in \mathbb{R}$, $p_n = p_n(x)$ is called the *indifference price for f at x in the S^n market* if $w_n(x + p_n) = u_n(x)$.

The indifference price, p_n, exists by the continuity of w_n and boundedness of f. Its uniqueness is guaranteed since w_n is strictly increasing.

Corollary I.3.8. *Let the assumptions be as in Theorem I.3.6. Then for $x \in \mathbb{R}$, the indifference prices for f converge; that is, $\lim_{n\to\infty} p_n(x) = p_\infty(x)$.*

Remark I.3.2. The results in Theorem I.3.6 and Corollary I.3.8 remain true (with only minor notational changes to the proofs) in the case with varying random endowment. Specifically, the random endowments $\{f_n\}_{1\le n\le\infty}$ corresponding to the $\{S^n\}_{1\le n\le\infty}$ markets need to satisfy

$$\sup_n \|f_n\|_{L^\infty} < \infty \quad \text{and} \quad f_n \longrightarrow f_\infty \text{ in probability as } n \to \infty \tag{I.3.6}$$

in order for the results to hold. This additional flexibility allows us to consider the case of a varying quantity of contingent claims and also contingent claims that depend on the individual markets. For example, if $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous, then $f_n := g(S^n_T)$ will satisfy (I.3.6).

Remark I.3.3. The study of the optimal terminal wealths and the optimal dual elements is typical in utility maximization in addition to properties of the value functions; however, it is absent in the present work. When varying both the volatility and drift of the risky assets, a major hurdle to stability is handling the change in the primal and dual feasible elements from market to market. Here, we use the varying volatility as a tool for pricing financial securities via a "nearby" models with good properties, rather than using it for investment advice. Because the study of optimal strategies is rather involved, it is beyond the scope of the present work and would be an interesting question to address in future research.

Remark I.3.4. A special case of stability with varying volatility is considered in [32] (see their Remark 2.5). The authors consider a fixed risky asset with varying *equivalent* subjective probability measures. However, this approach relies on the invertibility of the volatility process in every market, which in particular implies completeness for all markets. In our model, such measures would correspond to the risky asset laws, $\mathbb{P}^n := \mathbb{P} \circ (S^n)^{-1}$. Due to the changes in the volatility structure with n, the laws $\mathbb{P}^n$ are *nonequivalent* in the present work. Moreover, our results do not rely on completeness.

I.4 Proofs

We begin by proving the results from Section I.2 for the power investor. As in Sections I.2 and I.3, we assume a (completed) Brownian filtration.

I.4.1 Dual Problems and Power Investor Proofs

We begin by proving Proposition I.2.1. Example 1 of [33] uses the duality between $L^\infty(\mathbb{P})$ and $L^1(\mathbb{P})$ in order to establish a similar result when the contingent claim is independent of the traded assets. Without independence, we cannot apply the duality result directly, and instead we explicitly construct a sequence of martingale measures realizing the subreplication price.

Proof of Proposition I.2.1. Let $\rho \neq 0$ be given. We first seek to show that for all $0 < t' < T$,

$$\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{D}(\rho)} \mathbb{E}^{\mathbb{Q}}[\phi(B_T)|\mathcal{F}_{t'}] = \phi_{\min},$$

which implies that the subreplication price is $\phi_{\min}$. Subsequently, we will show (I.2.3).

We fix $t' < T$ and let $T' \in (t', T)$ and $x \in \mathbb{R}$ be given. Then $B^\rho := \sqrt{1-\rho^2}B + \rho W$ and $W^\rho := \sqrt{1-\rho^2}W - \rho B$ are orthogonal $\mathbb{P}$-Brownian motions. Equivalently, we have that $B = \sqrt{1-\rho^2}B^\rho - \rho W^\rho$ and $W = \rho B^\rho + \sqrt{1-\rho^2}W^\rho$. Consider the local martingale Z defined for $t \in [0,T]$ by

$$Z_t := \mathcal{E}\left(-\mu\sqrt{\nu}\cdot B^\rho\right)_t \mathcal{E}\left(\frac{1}{\rho}\left(-\mu\sqrt{1-\rho^2}\sqrt{\nu} - \frac{x}{T} + \frac{\eta\mathbb{I}_{[T',T]}}{T-T'}\right)\cdot W^\rho\right)_t,$$

where $\eta := B_{T'} - xT'/T \in \mathcal{F}_{T'}$. In fact, Z is a martingale, which we verify by applying Novikov's condition locally. The following procedure is standard; see, e.g., Section 6.2 Example 3(a) in [42]. By Corollary 5.14 of [31], it suffices to find $\Delta > 0$ and $t_n := n\Delta$ such that for each $n \geq 1$,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_n}^{t_{n+1}} d\left\langle M\right\rangle_u\right)\right] < \infty, \tag{I.4.1}$$

where for $t \in [0,T]$,

$$M_t := -\mu\left(\sqrt{\nu}\cdot B^\rho\right)_t + \frac{1}{\rho}\left(\left(-\mu\sqrt{1-\rho^2}\sqrt{\nu} - \frac{x}{T} + \frac{\eta\mathbb{I}_{[T',T]}}{T-T'}\right)\cdot W^\rho\right)_t.$$

By applying Cauchy-Schwartz to (I.4.1), it suffices to choose $\Delta > 0$ such that for each $n \geq 1$, we have

$$\mathbb{E}\left[\exp\left(\int_{t_n}^{t_{n+1}} \frac{\mu^2}{\rho^2}\nu_u du\right)\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\exp\left(\Delta\frac{\eta^2}{\rho^2(T-T')}\right)\right] < \infty. \tag{I.4.2}$$

Jensen's Inequality and Tonelli's Theorem imply that

$$\mathbb{E}\left[\exp\left(\int_{t_n}^{t_{n+1}} \frac{\mu^2}{\rho^2}\nu_u du\right)\right] \leq \mathbb{E}\left[\int_{t_n}^{t_{n+1}} \exp\left(\frac{\Delta\mu^2}{\rho^2}\nu_u\right)\frac{du}{\Delta}\right].$$

By defining $\tilde{\nu}_t$ through the equality $\nu_t = \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\tilde{\nu}_t$, we have that $\tilde{\nu}_t$ is noncentrally χ^2-distributed with $\frac{4\kappa\theta}{\sigma^2}$ degrees of freedom and noncentrality parameter $\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}$. Then $\mathbb{E}[\exp(\int_{t_n}^{t_{n+1}} \frac{\mu^2}{\rho^2}\nu_u du)] < \infty$ so long as $\Delta \leq \frac{\rho^2\kappa}{\mu^2\sigma^2(1-e^{-\kappa T})}$.

By Cauchy-Schwartz, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\Delta\frac{\eta^2}{\rho^2(T-T')}\right)\right] \\
&\quad = \mathbb{E}\left[\exp\left(\frac{\Delta}{\rho^2(T-T')}\left(B_{T'}^2 - \frac{2xT'B_{T'}}{T} + \left(\frac{xT'}{T}\right)^2\right)\right)\right] \\
&\quad \leq e^{\frac{\Delta x^2(T')^2}{\rho^2 T^2(T-T')}}\sqrt{\mathbb{E}\left[\exp\left(\frac{2\Delta B_{T'}^2}{\rho^2(T-T')}\right)\right]\mathbb{E}\left[\exp\left(\frac{-4\Delta x T' B_{T'}}{\rho^2 T(T-T')}\right)\right]},
\end{aligned}$$

which is finite provided that $\Delta \leq \frac{\rho^2(T-T')}{8T'}$.

Thus, taking $\Delta := \rho^2\min(\frac{\kappa}{\mu^2\sigma^2(1-e^{-\kappa T})}, \frac{T-T'}{8T'})$ yields (I.4.2).

We define $\overline{\mathbb{Q}} \in \mathcal{D}(\rho)$ by $\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} := Z_T$ and the processes $\overline{B}^\rho$ and $\overline{W}^\rho$ by

$$\overline{B}_t^\rho := B_t^\rho + \mu\int_0^t \sqrt{\nu_u}\,du$$

and

$$\overline{W}_t^\rho := W_t^\rho + \frac{\mu\sqrt{1-\rho^2}}{\rho}\int_0^t \sqrt{\nu_u}\,du + \frac{xt}{\rho T} - \frac{\eta\int_0^t \mathbb{I}_{[T',T]}}{\rho(T-T')}.$$

By Girsanov's Theorem, $\overline{B}^\rho$ and $\overline{W}^\rho$ are orthogonal $\overline{\mathbb{Q}}$-Brownian motions. Moreover, $\eta = \sqrt{1-\rho^2}\overline{B}_{T'}^\rho - \rho\overline{W}_{T'}^\rho$, which implies that

$$B_T = \left(\sqrt{1-\rho^2}\overline{B}_T^\rho - \rho\overline{W}_T^\rho\right) + x - \left(\sqrt{1-\rho^2}\overline{B}_{T'}^\rho - \rho\overline{W}_{T'}^\rho\right).$$

Then, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{Q}}}\left[\phi(B_T) \mid \mathcal{F}_{t'}\right] \\
&\quad = \mathbb{E}^{\overline{\mathbb{Q}}}\left[\phi\left(\left(\sqrt{1-\rho^2}\overline{B}_T^\rho - \rho\overline{W}_T^\rho\right) - \left(\sqrt{1-\rho^2}\overline{B}_{T'}^\rho - \rho\overline{W}_{T'}^\rho\right) + x\right) \mid \mathcal{F}_{t'}\right] \\
&\quad = \mathbb{E}^{\overline{\mathbb{Q}}}\left[\phi\left(\left(\sqrt{1-\rho^2}\overline{B}_T^\rho - \rho\overline{W}_T^\rho\right) - \left(\sqrt{1-\rho^2}\overline{B}_{T'}^\rho - \rho\overline{W}_{T'}^\rho\right) + x\right)\right] \\
&\quad = \mathbb{E}^{\mathbb{P}}\left[\phi\left(B_T - B_{T'} + x\right)\right].
\end{aligned}$$

The choice of $T' \in (t', T)$ and $x \in \mathbb{R}$ is arbitrary, and therefore,

$$\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{D}(\rho)} \mathbb{E}^{\mathbb{Q}}\left[\phi(B_T) \mid \mathcal{F}_{t'}\right] = \phi_{\min}. \tag{I.4.3}$$

Finally, we suppose that $x \in \mathbb{R}$ and $(H \cdot S^\rho)_T \in \mathcal{C}(\rho)$ such that $x + (H \cdot S^\rho)_T + \phi(B_T) \geq 0$. Then for all $\mathbb{Q} \in \mathcal{D}(\rho)$, we have that $(H \cdot S^\rho)$ is a lower-bounded $\mathbb{Q}$-local martingale, and hence a $\mathbb{Q}$-supermartingale. For all $t' < T$, we have $0 \leq x + (H \cdot S^\rho)_{t'} + \mathbb{E}^{\mathbb{Q}}[\phi(B_T)|\mathcal{F}_{t'}]$. By (I.4.3) above, we have $0 \leq x + (H \cdot S^\rho)_{t'} + \phi_{\min}$. Continuity with respect to time and taking $t' \to T$ yields (I.2.3). $\square$

As is typical in convex optimization, we introduce the dual problem as tool for proving Theorem I.2.2 and Corollary I.2.4. For $y > 0$, define $V(y) := \sup_{x>0}\{U(x) - xy\}$. For $U(x) = x^\gamma/\gamma$, we have $V(y) = \frac{1-\gamma}{\gamma} y^{\gamma/(\gamma-1)}$. For $y > 0$ and $z \geq \phi_{\min}$, we define

$$\begin{aligned} V_c(y,z) &:= \sup_{x > -\phi_{\min}} \{U(x+z) - xy\} \\ &= \begin{cases} V(y) + yz, & \text{for } y < U'(z - \phi_{\min}), \\ U(z - \phi_{\min}) + y\phi_{\min}, & \text{otherwise.} \end{cases} \end{aligned}$$

We can then define a constrained form of the dual value function for $\rho \in (-1,1)$ by,

$$v_c(y,\rho) := \inf_{\mathbb{Q}\in\mathcal{D}(\rho)} \mathbb{E}\left[V_c\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}, f\right)\right], \quad y > 0. \tag{I.4.4}$$

For $Z^\rho_t := \mathcal{E}(-\mu\sqrt{\nu} \cdot B)_t$, $t \in [0,T]$, the random variable Z^ρ_T is the minimal martingale density corresponding to the S^ρ market. The martingale property of Z^ρ is shown in Lemma 5.2 of [8]. In particular, $v_c(y,\rho) < \infty$ for all $y > 0$ and $\rho \in (-1,1)$.

The constrained dual problem arises naturally from the endogenous primal admissibility constraint (I.2.3). For $\rho \neq 0$, we could define a constrained primal problem, $u^\rho_c = u^\rho_c(x)$ for $x > -\phi_{\min}$, and a corresponding constrained optimization set, $\mathcal{C}^\rho_c(x)$, analogously to u_c and $\mathcal{C}_c(x)$ in the $\rho = 0$ case. In that case, we would have $u^\rho_c(x) = u(x,\rho)$ for all $x > -\phi_{\min}$ by (I.2.3), and (I.4.4) would be the natural candidate for its dual conjugate. Indeed, for $\rho \neq 0$, [40] prove that the constrained form of the dual value function, (I.4.4), is in fact equal to the dual value function as it is defined in [9], Equation (3.1). (See [40] Theorem 4.2.)

Remark I.4.1. In [40], the authors consider the problem of *facelifting*, in which the primal and dual value functions in the presence of unspanned random endowment are not continuous with respect to time to maturity as the maturity decreases to 0. At first glance, our stability problem differs from that of varying maturity. However, both problems have the property that the random endowment is non-replicable in every pre-limiting market yet replicable in the limit. This property allows for the admissibility constraint, (I.2.3), to appear endogenously in the pre-limiting models, whereas (I.2.3) must be exogenously applied in the limiting model.

Lemma I.4.1. *Let the assumptions of the model be as in Theorem I.2.2. For $y > 0$,*

$$\limsup_{\rho\to 0} v_c(y,\rho) \leq \mathbb{E}\left[V_c(yZ_T^0, f)\right],$$

where Z_T^0 is the minimal martingale density for the S^0 market.

Proof. One can show that $\{V(yZ_T^\rho)\}_\rho$ is uniformly integrable, e.g., using the proof of Lemma 5.2 in [8]. For each $\rho \in (-1,1)$, Z^ρ is a martingale, and hence convergence in probability along with Scheffe's Lemma implies that $Z_T^\rho \longrightarrow Z_T^0$ in $L^1(\mathbb{P})$ as $\rho \to 0$. Convergence in $L^1(\mathbb{P})$ plus $f \in L^\infty(\mathbb{P})$ implies that $\{Z_T^\rho f\}_\rho$ is uniformly integrable. Since $V_c(yZ_T^\rho, f) \longrightarrow V_c(yZ_T^0, f)$ in probability as $\rho \to 0$ and

$$V_c(yZ_T^\rho, f) \leq V(yZ_T^\rho) + yZ_T^\rho f$$

for all $\rho \in (-1,1)$, Fatou's Lemma implies

$$\begin{aligned}\mathbb{E}\left[V_c\left(yZ_T^0, f\right)\right] &\geq \limsup_{\rho\to 0}\mathbb{E}\left[V_c\left(yZ_T^\rho, f\right)\right]\\ &\geq \limsup_{\rho\to 0}\ v_c(y,\rho).\end{aligned}$$

□

Lemma I.4.2. *Let the assumptions of the model be as in Theorem I.2.2. Let u and u_c be as defined in* (I.2.2) *and* (I.2.4)*, respectively. For any $x > -\phi_{min}$, $u_c(x) \leq \liminf_{\rho\to 0} u(x,\rho)$.*

Before proving Lemma I.4.2, we need two technical lemmas, which will again be used in the proof of Theorem I.3.6. While the notions of integrability are defined separately for Sections I.2 and I.3, the notions agree and are not referred to separately in Lemmas I.4.3 and I.4.4 below.

Lemma I.4.3. *Let X be a semimartingale and H be X-integrable. Suppose that there exists $K > 0$ such that $(H \cdot X)_t \geq -K$ for all $t \in [0,T]$. Then for any $\delta > 0$ there exists a sequence of progressively measurable integrands $\{H^n\}_{n\geq 1}$ such that for each $n \geq 1$, H^n is uniformly bounded in t and ω, while for all $t \in [0,T]$,*

$$(H^n \cdot X)_t \geq -K - \delta,$$

and $(H^n \cdot X)_T \longrightarrow (H \cdot X)_T$ in probability as $n \to \infty$.

Proof. For $n \geq 1$, we define the integrands $H^n := H\mathbb{I}_{\{|H|\leq n\}}$, where $\mathbb{I}_A$ denotes the indicator function of a set $A \subset \Omega \times [0,T]$. We define the stopping times

$$\sigma_n := \inf\left\{t \leq T : (H^n \cdot X)_t \leq -K - \delta\right\}.$$

Then $(H^n\mathbb{I}_{[\![0,\sigma_n]\!]} \cdot X)_t \geq -K - \delta$ for all $t \in [0,T]$. Moreover, we have that $\sup_t |((H^n - H) \cdot X)_t| \longrightarrow 0$ in probability as $n \to \infty$ by Lemma 4.11 and

Remark (ii) following Definition 4.8 both in [7]. This convergence implies that $\mathbb{P}(\sigma_n = T) \longrightarrow 1$ and hence $(H^n \cdot X)_{\sigma_n} \longrightarrow (H \cdot X)_T$ in probability as $n \to \infty$. Considering the sequence $\{H^n \mathbb{I}_{[\![0,\sigma^n]\!]}\}_{n\geq 1}$ yields the result. $\square$

Lemma I.4.4. *Let $\{X^n\}_{n\geq 1}$ be a sequence of semimartingales such that $X^n \longrightarrow X$ in the semimartingale topology as $n \to \infty$. Suppose that H is progressively measurable and uniformly bounded in t and ω and there exists a $K > 0$ such that $(H \cdot X)_t \geq -K$ for all $t \in [0,T]$. Then for any $\delta > 0$, there exists a sequence $\{H^n\}_{n\geq 1}$ such that for all $n \geq 1$, H^n is uniformly bounded in t and ω, for all $t \in [0,T]$,*

$$(H^n \cdot X^n)_t \geq -K - \delta,$$

and $(H^n \cdot X^n)_T \longrightarrow (H \cdot X)_T$ in probability as $n \to \infty$.

Proof. Since H is uniformly bounded and progressively measurable, it is X- and X^n-integrable for all $n \geq 1$. Moreover, the definition of semimartingale convergence implies that

$$(H \cdot X^n) \longrightarrow (H \cdot X) \text{ in the semimartingale topology as } n \to \infty. \tag{I.4.5}$$

For $n \geq 1$, we define the stopping times τ_n by

$$\tau_n := \inf\{t \leq T : (H \cdot X^n)_n < -K - \delta\}$$

and let $H^n := H\mathbb{I}_{[\![0,\tau_n]\!]}$. By definition of τ_n, we have $(H^n \cdot X^n)_t \geq -K - \delta$ for all $t \in [0,T]$. Using (I.4.5),

$$\begin{aligned}
\mathbb{P}(\tau_n < T) &= \mathbb{P}\left(\exists\, t' < T : (H \cdot X^n)_{t'} < -K - \delta\right) \\
&\leq \mathbb{P}\left(\sup_{t\leq T} |(H \cdot (X^n - X))_t| > \delta\right) + \mathbb{P}\left(\exists\, t' \leq T : (H \cdot X)_{t'} < -K\right) \\
&= \mathbb{P}\left(\sup_{t\leq T} |(H \cdot (X^n - X))_t| > \delta\right) + 0 \\
&\longrightarrow 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus, $(H^n \cdot X^n)_T = (H \cdot X^n)_{\tau_n} \longrightarrow (H \cdot X)_T$ in probability as $n \to \infty$. $\square$

Proof of Lemma I.4.2. Let $\varepsilon > 0$ be given. Since u_c is concave, it is continuous on the interior of its domain, and hence we may choose $x' < x$ such that $u_c(x) < u_c(x') + \varepsilon$. We then choose $(H \cdot S^0)_T \in \mathcal{C}_c(x')$ such that $u_c(x') \leq \mathbb{E}[U(x' + (H \cdot S^0)_T + f)] + \varepsilon$.

We define $\delta := \frac{x-x'}{4} > 0$. Since H is $(\rho = 0)$-admissible and we have $x' + (H \cdot S^0)_T \geq -\phi_{\min}$, Lemma I.4.3 provides us with a sequence of integrands $\{H^n\}_{n\geq 1}$ such that for each $n \geq 1$, H^n is bounded uniformly in t and ω while

$x' + \delta + (H^n \cdot S^0)_t \geq -\phi_{\min}$ for all $t \in [0,T]$. In particular, for all $n \geq 1$, $(H^n \cdot S^0)_T \in \mathcal{C}_c(x'+\delta) \subseteq \mathcal{C}_c(x)$, and we have the uniform lower bound

$$\begin{aligned} U\big(x + (H^n \cdot S^0)_T + f\big) &\geq U\left(x - x' - \delta + f - \phi_{\min}\right) \\ &\geq U\left(\frac{3}{4}(x - x')\right) > -\infty. \end{aligned}$$

Fatou's Lemma implies that

$$\begin{aligned} u_c(x) &\leq u_c(x') + \varepsilon \\ &\leq \mathbb{E}\left[U\left(x + (H \cdot S^0)_T + f\right)\right] + 2\varepsilon \\ &\leq \liminf_{n\to\infty} \mathbb{E}\left[U\left(x + (H^n \cdot S^0)_T + f\right)\right] + 2\varepsilon, \end{aligned}$$

which allows us to choose a sufficiently large n such that $\tilde{H} := H^n$ is uniformly bounded in t and ω, $(\tilde{H} \cdot S^0)_T \in \mathcal{C}_c(x'+\delta)$ and

$$u_c(x) \leq \mathbb{E}\left[U\left(x + (\tilde{H} \cdot S^0)_T + f\right)\right] + 3\varepsilon. \tag{I.4.6}$$

Now that we have achieved sufficiently nice regularity of a nearly-optimal strategy, $\tilde{H}$, we proceed by varying the parameter ρ. Let $\rho_k \longrightarrow 0$ be a sequence realizing the $\liminf$ in $\liminf_{\rho\to 0} u(x,\rho)$. Since $S^{\rho_k} \longrightarrow S^0$ in the semimartingale topology as $k \to \infty$, Lemma I.4.4 allows us to choose $\{\tilde{H}^k\}_{k\geq 1}$ such that for each $k \geq 1$, $\tilde{H}^k$ is bounded uniformly in t and ω while $x' + 2\delta + (\tilde{H}^k \cdot S^{\rho_k})_t \geq -\phi_{\min}$. Moreover, $(\tilde{H}^k \cdot S^{\rho_k})_T \longrightarrow (\tilde{H} \cdot S^0)_T$ in probability as $k \to \infty$. For every $k \geq 1$, we have the uniform lower bound

$$U(x + (\tilde{H}^k \cdot S^{\rho_k})_T + f) \geq U(x - x' - 2\delta + f - \phi_{\min}) \geq U(\frac{1}{2}(x - x')) > -\infty.$$

Therefore by Fatou's Lemma and (I.4.6) above,

$$\begin{aligned} u_c(x) &\leq \mathbb{E}\left[U\left(x + (\tilde{H} \cdot S^0)_T + f\right)\right] + 3\varepsilon \\ &\leq \liminf_{k\to\infty} \mathbb{E}\left[U\left(x' + (\tilde{H}^k \cdot S^{\rho_k})_T + f\right)\right] + 3\varepsilon \\ &\leq \liminf_{k\to\infty} \mathbb{E}\left[U\left(x + (\tilde{H}^k \cdot S^{\rho_k})_T + f\right)\right] + 3\varepsilon \\ &\leq \liminf_{k\to\infty} u(x,\rho_k) + 3\varepsilon \\ &= \liminf_{\rho\to 0} u(x,\rho) + 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the desired result holds. $\square$

Proof of Theorem I.2.2. Fix $\rho \neq 0$. For $x > -\phi_{\min}$, $X \in \mathcal{C}(\rho)$ such that $x + X \geq -\phi_{\min}$, $y > 0$, and $\mathbb{Q} \in \mathcal{D}(\rho)$, we have

$$\begin{aligned}\mathbb{E}\left[U(x+X+f)\right] &\leq \mathbb{E}\left[V_c\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}, f\right) + y\frac{d\mathbb{Q}}{d\mathbb{P}}(x+X)\right] \\ &\leq \mathbb{E}\left[V_c\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}, f\right)\right] + xy.\end{aligned}$$

This strengthening of Fenchel's inequality relies on the bound $x + X \geq -\phi_{\min}$ in order to replace V with $V_c(\cdot, f)$. Next, we take the supremum over all $X \in \mathcal{C}(\rho)$ with $x + X \geq -\phi_{\min}$ and the infimum over all $\mathbb{Q} \in \mathcal{D}(\rho)$, which yields that for any $x > -\phi_{\min}$ and $y > 0$,

$$u(x, \rho) \leq v_c(y, \rho) + xy.$$

This inequality along with Lemmas I.4.1 and I.4.2 shows that for any $x > -\phi_{\min}$ and $y > 0$,

$$u_c(x) \leq \liminf_{\rho \to 0} u(x, \rho) \leq \limsup_{\rho \to 0} v_c(y, \rho) + xy \leq \mathbb{E}[V_c(yZ_T^0, f)] + xy\,. \tag{I.4.7}$$

Next, we show that $u_c(\cdot)$ and $v_c(\cdot, 0)$ are conjugates. We let $y > 0$ be given and define the candidate optimal terminal wealth $\hat{X}$ by

$$\hat{X} := \begin{cases} -V'(yZ_T^0) - f\,, & \text{if } yZ_T^0 \leq U'(f - \phi_{\min}), \\ -\phi_{\min}\,, & \text{otherwise.} \end{cases}$$

For $\frac{d\mathbb{Q}^0}{d\mathbb{P}} := Z_T^0 = \mathcal{E}(-\mu\sqrt{\nu} \cdot B)_T$, we have that $\hat{X} \in L^1(\mathbb{Q}^0)$. By martingale representation and the strict positivity of $\sqrt{\nu}$ by the Feller condition, we may write $\hat{X} = \mathbb{E}^{\mathbb{Q}^0}[\hat{X}] + (H \cdot S^0)_T$ for some integrable H. Since $\hat{X} \geq -\phi_{\min}$ and $(H \cdot S^0)$ is a $\mathbb{Q}^0$-martingale, we know that $(H \cdot S^0)_t \geq -\phi_{\min} - \mathbb{E}^{\mathbb{Q}^0}[\hat{X}] > -\infty$ for all $t \in [0, T]$. Thus, H is S^0-admissible.

We define $\hat{x} := \mathbb{E}^{\mathbb{Q}^0}[\hat{X}] > -\phi_{\min}$ so that $\hat{X} - \hat{x} \in \mathcal{C}_c(\hat{x})$. For any $y > 0$,

$$\begin{aligned}\mathbb{E}\left[V_c\left(yZ_T^0, f\right)\right] &= \mathbb{E}\left[U\left(\hat{X} + f\right) - yZ_T^0\hat{X}\right] \\ &= \mathbb{E}\left[U\left(\hat{X} + f\right)\right] - y\hat{x} \\ &\leq \sup_{x > -\phi_{\min}} \left\{\sup_{X \in \mathcal{C}_c(x)} \mathbb{E}\left[U(x + X + f)\right] - yx\right\} \\ &= \sup_{x > -\phi_{\min}} \left\{u_c(x) - xy\right\}\,.\end{aligned}$$

The other direction of the inequality holds by (I.4.7), and so we obtain that for any $y > 0$,

$$\mathbb{E}[V_c(yZ_T^0, f)] = \sup_{x > -\phi_{\min}} \left\{u_c(x) - xy\right\}.$$

Since $u_c(\cdot)$ is concave and upper semicontinuous on $(-\phi_{\min}, \infty)$, we have $u_c(x) = \inf_{y>0}\{\mathbb{E}[V_c(yZ_T^0, f)] + xy\}$ for $x > -\phi_{\min}$. Strict convexity of $y \mapsto \mathbb{E}[V_c(yZ_T^0, f)]$ implies the differentiability of $u_c(\cdot)$. (See, e.g., Proposition 6.2.1 on page 40 of [24].) Now for any $x > -\phi_{\min}$, choosing $y = \frac{d}{dx}u_c(x)$ yields equality in (I.4.7). $\square$

Finally, we show that indifference prices do not converge as $\rho \to 0$.

Proof of Corollary I.2.4. Let $x > -\phi_{\min}$ be given. Suppose that for $\rho_n \longrightarrow 0$, we have $p(x, \rho_n) \longrightarrow \bar{p}$ as $n \to \infty$. By being the limit of arbitrage-free prices in the $\{\rho_n\}_n$ models, we must have $\bar{p} \in [\inf \phi, \sup \phi]$.

Using that ϕ is not constant, for $x > -\phi_{\min}$, we have that $u_c(x) < u(x, 0)$. This result can be obtained, for example, by Theorem 2.2 of [37] and f's replicability in the S^0 market, which imply that $u(x, 0) = \mathbb{E}[U(I(\frac{\partial}{\partial x}u(x, 0)Z_T^0))]$ where $\mathbb{P}(I(\frac{\partial}{\partial x}u(x, 0)Z_T^0) < f - \phi_{\min}) > 0$. By Theorem I.2.2,

$$\lim_n u(x, \rho_n) = u_c(x) < u(x, 0) = w(x + p(x, 0), 0).$$

Taking $f = 0$ in Theorem I.2.2 and using the concavity of $w(\cdot, \rho)$ for every $\rho \in (-1, 1)$, we have that $w(\cdot, \rho) \longrightarrow w(\cdot, 0)$ uniformly on compacts in $(-\phi_{\min}, \infty)$ as $\rho \to 0$. Thus,

$$\lim_n w(x + p(x, \rho_n), \rho_n) = w(x + \bar{p}, 0),$$

which implies that $w(x + \bar{p}, 0) < w(x + p(x, 0), 0)$. Since $w(\cdot, 0)$ is strictly increasing, we conclude that $\bar{p} < p(x, 0)$. $\square$

I.4.2 Proof of the Main Result

The proof of the main result, Theorem I.3.6, follows Lemmas I.4.5 and I.4.7 (below), which establish lower and upper semicontinuity-type results for the sequence of primal and dual value functions, respectively.

Lemma I.4.5. *Suppose that the sequence of markets satisfies Assumption I.3.1, and $\mathcal{M}_V^\infty \neq \emptyset$. Then for $x \in \mathbb{R}$ and $x_n \longrightarrow x$ as $n \to \infty$,*

$$u_\infty(x) \le \liminf_{n\to\infty} u_n(x_n)\,.$$

Significant difficulty in proving Lemma I.4.5 stems from the nonequivalence of markets (the martingale drivers, M^n, differ). The idea behind the proof of Lemma I.4.5 is that since the pre-limiting markets are "close" to the S^∞ market, strategies in the S^∞ market are "close" to being strategies in the pre-limiting markets. This idea will be made precise by appropriate approximation and stopping. First, we need a helper lemma.

Lemma I.4.6. *Under Assumption I.3.1, $S^n \longrightarrow S^\infty$ in the semimartingale topology as $n \to \infty$.*

Proof. Since $M^n \longrightarrow M^\infty$ in the semimartingale topology as $n \to \infty$, it remains to show that $(\lambda^n \cdot \langle M^n \rangle) \longrightarrow (\lambda^\infty \cdot \langle M^\infty \rangle)$ in the semimartingale topology. We seek to show

$$A_n := \sum_{i=1}^d \int_0^T \left| \lambda^n (\sigma^{n,i})^2 - \lambda^\infty (\sigma^{\infty,i})^2 \right| dt \longrightarrow 0 \text{ in probability as } n \to \infty, \tag{I.4.8}$$

which will then imply the desired result.

The mapping $X \mapsto \langle X \rangle$ is continuous in the space of semimartingales with respect to semimartingale convergence, and so Assumption I.3.1 implies:

$$\sum_{i=1}^d \int_0^T \left(\sigma^{n,i} - \sigma^{\infty,i}\right)^2 dt \longrightarrow 0 \text{ in probability as } n \to \infty, \tag{I.4.9}$$

$$\sum_{i=1}^d \int_0^T \left(\lambda^n \sigma^{n,i} - \lambda^\infty \sigma^{\infty,i}\right)^2 dt \longrightarrow 0 \text{ in probability as } n \to \infty. \tag{I.4.10}$$

Let $\{A_n\}_{n \in N}$ be a subsequence of $\{A_n\}_{n \in \mathbb{N}}$, where for notational convenience we denote the subsequence index N as an infinite subset of $\mathbb{N}$. We choose a further subsequence $\{A_n\}_{n \in N'}$, where $N' \subset N$, such that the convergence in (I.4.9) and (I.4.10) occurs $\mathbb{P}$-a.s. as $n \to \infty$ for $n \in N'$.

We define the random variable

$$q := \sup_{n \in N'} \sum_{i=1}^d \int_0^T \left(\left(\sigma^{n,i}\right)^2 + \left(\lambda^n \sigma^{n,i}\right)^2 \right) dt.$$

The almost-sure convergence along the subsequence N' implies that $q < \infty$, $\mathbb{P}$-a.s., which allows us to define the equivalent probability measure $\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{e^{-q}}{\mathbb{E}^{\mathbb{P}}[e^{-q}]}$. Under $\mathbb{Q}$, we have more regularity of elements in N'; in particular,

$$\sum_{i=1}^d \left(\sigma^{n,i} - \sigma^{\infty,i}\right)^2 + \left(\lambda^n \sigma^{n,i} - \lambda^\infty \sigma^{\infty,i}\right)^2 \longrightarrow 0 \tag{I.4.11}$$

in $L^1(\mathbb{Q} \times \text{Leb})$ as $N' \ni n \to \infty$, where Leb denotes the Lebesgue measure on $[0, T]$. Hence, we have that $\{\sum_{i=1}^d (1 + (\lambda^n)^2)(\sigma^{n,i})^2\}_{n \in N'}$ is $(\mathbb{Q} \times \text{Leb})$-uniformly integrable. Since

$$\sum_{i=1}^d \left| \lambda^n (\sigma^{n,i})^2 - \lambda^\infty (\sigma^{\infty,i})^2 \right| \leq \sum_{i=1}^d \left[(1 + (\lambda^n)^2)(\sigma^{n,i})^2 + (1 + (\lambda^\infty)^2)(\sigma^{\infty,i})^2 \right]$$

for all $n \geq 1$ and by (I.4.11), $\sum_{i=1}^{d} |\lambda^n(\sigma^{n,i})^2 - \lambda^\infty(\sigma^{\infty,i})^2| \longrightarrow 0$ in $(\mathbb{Q} \times \text{Leb})$-measure as $n \to \infty$, we have that

$$\sum_{i=1}^{d} \left|\lambda^n(\sigma^{n,i})^2 - \lambda^\infty(\sigma^{\infty,i})^2\right| \longrightarrow 0 \text{ in } L^1(\mathbb{Q} \times \text{Leb}) \text{ as } N' \ni n \to \infty.$$

Now we choose a further subsequence $\{A_n\}_{n \in N''}$, where $N'' \subseteq N'$, such that

$$\sum_{i=1}^{d} \int_0^T \left|\lambda^n(\sigma^{n,i})^2 - \lambda^\infty(\sigma^{\infty,i})^2\right| \longrightarrow 0 \ \ \mathbb{Q}\text{-a.s. as } N'' \ni n \to \infty,$$

and note that this convergence also holds $\mathbb{P}$-a.s. by the equivalence of $\mathbb{P}$ and $\mathbb{Q}$. Thus, we have shown that for all subsequences $\{A_n\}_{n \in N}$, $N \subseteq \mathbb{N}$, there exists a further subsequence $\{A_n\}_{n \in N''}$, $N'' \subseteq N$, such that we have $\sum_{i=1}^{d} \int_0^T |\lambda^n(\sigma^{n,i})^2 - \lambda^\infty(\sigma^{\infty,i})^2| \longrightarrow 0$ $\mathbb{P}$-a.s. for $n \in N''$ as $n \to \infty$. Therefore, (I.4.8) holds, which completes the proof. $\square$

Proof of Lemma I.4.5. First, we show that the supremum in the limiting primal optimization problem, (I.3.4), can be taken over all admissible wealth processes whose integrands are bounded. Let $H \in \mathcal{H}^\infty_{\text{adm}}$ be given, and let $K \in (0, \infty)$ be such that $(H \cdot S^\infty)_t \geq -K$ for all $t \in [0, T]$. Lemma I.4.3 provides us with a sequence of integrands $\{H^n\}_{n \geq 1}$ such that for each $n \geq 1$, H^n is bounded uniformly in t and ω while $(H^n \cdot S^\infty)_t \geq -2K$ for all $t \in [0, T]$ and $(H^n \cdot S^\infty)_T \longrightarrow (H \cdot S^\infty)_T$ in probability as $n \to \infty$. In particular, $(H^n \cdot S^\infty) \in \mathcal{H}^\infty_{\text{adm}}$ with $\{(H^n \cdot S^\infty)\}_{n \geq 1}$ sharing the same lower admissibility bound, $-2K$. By Fatou's Lemma,

$$\mathbb{E}\left[U\left(x + (H \cdot S^\infty)_T + f\right)\right] \leq \liminf_{n \to \infty} \mathbb{E}\left[U\left(x + (H^n \cdot S^\infty)_T + f\right)\right].$$

Therefore, it suffices to take the supremum in (I.3.4) over all $\tilde{H} \in \mathcal{H}^\infty_{\text{adm}}$ such that $\tilde{H}$ is uniformly bounded in t and ω. That is,

$$u_\infty(x) = \sup_{\tilde{H} \in \mathcal{H}^\infty_{\text{adm}}, \tilde{H} \text{ bdd}} \mathbb{E}\left[U\left(x + (\tilde{H} \cdot S^\infty)_T + f\right)\right]. \tag{I.4.12}$$

Now let $\tilde{H} \in \mathcal{H}^\infty_{\text{adm}}$ be given such that $\tilde{H}$ is uniformly bounded in t and ω by a constant $K \in (0, \infty)$. Even though $\tilde{H}$ is S^∞-admissible and S^n-integrable for every n, it is not necessarily admissible for each S^n market. Using Lemma I.4.4, we mitigate this issue by choosing $\{\tilde{H}^n\}_{n \geq 1}$ such that for each $n \geq 1$, $\tilde{H}^n$ is bounded uniformly in t and ω while $(\tilde{H}^n \cdot S^n)_t \geq -3K$ for all $t \in [0, T]$ and $(\tilde{H}^n \cdot S^n)_T \longrightarrow (\tilde{H} \cdot S^\infty)_T$ in probability as $n \to \infty$.

Applying Fatou's Lemma gives us that

$$\begin{aligned}\mathbb{E}\left[U(x+(\tilde{H}\cdot S^{\infty})_T+f)\right] &\le \liminf_{n\to\infty}\mathbb{E}\left[U(x_n+(\tilde{H}^n\cdot S^n)_T+f)\right]\\ &\le \liminf_{n\to\infty} u_n(x_n).\end{aligned}$$

Taking the supremum over all uniformly bounded $\tilde{H}\in\mathcal{H}^{\infty}_{\mathrm{adm}}$, as in (I.4.12), yields the result. □

We next proceed to the second main lemma, which establishes an upper-semicontinuity result for the dual problem.

Lemma I.4.7. *Let the assumptions of the model be as in Theorem I.3.6. Then for* $\{y_n\}_{1\le n<\infty}\subseteq(0,\infty)$ *such that* $y_n\longrightarrow y>0$ *as* $n\to\infty$,

$$v_{\infty}(y)\ge\limsup_{n\to\infty} v_n(y_n)\,.$$

Using Assumption I.3.5, the following lemma will further refine the collection $\mathcal{B}$ over which the infimum is taken in the limiting market's dual problem. We define $\mathcal{B}'$ by

$$\begin{aligned}\mathcal{B}' := \{L\in\mathcal{B} :\exists \text{ constants } c=c(L), d=d(L),&\\ 0<c\le\mathcal{E}(L)_t\le d<\infty,\ \forall t\in[0,T],\ \text{and}\ \langle L\rangle_T\le d\}&\end{aligned}\tag{I.4.13}$$

The following lemma builds on Corollary 3.4 in [41].

Lemma I.4.8. *Suppose that the limiting market's dual problem satisfies Assumption I.3.5 and that* $\mathbb{E}[V(Z^{\infty}_T)]<\infty$*, where* Z^{∞}_T *is the minimal martingale density for* S^{∞}*. Let* $\mathcal{B}'$ *be defined as in* (I.4.13)*. Then for* $y>0$,

$$v_{\infty}(y)=\inf_{L\in\mathcal{B}'}\mathbb{E}\left[V\left(yZ^{\infty}_T\mathcal{E}(L)_T\right)+yZ^{\infty}_T\mathcal{E}(L)_T f\right].$$

Proof. The first part of the proof is based on the proof of Corollary 3.4 of [41]. Let $L\in\mathcal{B}$ be given. By the convexity of V, we have

$$\begin{aligned}&\mathbb{E}\left[V\left(yZ^{\infty}_T\left(\frac{1}{n}+\frac{n-1}{n}\mathcal{E}(L)_T\right)\right)+yZ^{\infty}_T\left(\frac{1}{n}+\frac{n-1}{n}\mathcal{E}(L)_T\right)f\right]\\ &\le\frac{1}{n}\mathbb{E}\left[V\left(yZ^{\infty}_T\right)+yZ^{\infty}_T f\right]+\frac{n-1}{n}\mathbb{E}\left[V\left(yZ^{\infty}_T\mathcal{E}(L)_T\right)+yZ^{\infty}_T\mathcal{E}(L)_T f\right]\\ &\longrightarrow\mathbb{E}\left[V\left(yZ^{\infty}_T\mathcal{E}(L)_T\right)+yZ^{\infty}_T\mathcal{E}(L)_T f\right]\quad\text{as } n\to\infty,\end{aligned}$$

because $V(yZ^{\infty}_T)\in L^1(\mathbb{P})$ by the assumption that $\mathbb{E}[V(Z^{\infty}_T)]<\infty$ and reasonable asymptotic elasticity, (I.3.3). For each $n\ge1$, we let L^n denote the element $L^n\in\mathcal{B}$ such that $\frac{1}{n}+\frac{n-1}{n}\mathcal{E}(L)=\mathcal{E}(L^n)$.

Let $\varepsilon > 0$ be given, and choose N sufficiently large such that

$$\mathbb{E}\left[V\left(yZ_T^\infty \mathcal{E}(L^N)_T\right) + yZ_T^\infty \mathcal{E}(L^N)_T f\right] \\ \leq \mathbb{E}\left[V\left(yZ_T^\infty \mathcal{E}(L)_T\right) + yZ_T^\infty \mathcal{E}(L)_T f\right] + \varepsilon.$$

We define the sequence of stopping times $\{\tau_k\}_{1\leq k<\infty}$ by $\tau_k := \inf\{t \leq T : \langle L^N\rangle_t \geq k\}$. Then $(L^N)^{\tau_k} \in \mathcal{B}'$ for each k. By continuity of L^N and finiteness of $\langle L^N\rangle_T$, we have that $\mathcal{E}(L^N)_{\tau_k} \longrightarrow \mathcal{E}(L^N)_T$ in probability as $k \to \infty$. Scheffe's Lemma implies that the $L^1(\mathbb{P}) - \lim_k Z_T^\infty \mathcal{E}(L^N)_{\tau_k} = Z_T^\infty \mathcal{E}(L^N)_T$, which implies that $\lim_k \mathbb{E}[yZ_T^\infty \mathcal{E}(L^N)_{\tau_k} f] = \mathbb{E}[yZ_T^\infty \mathcal{E}(L^N)_T f]$.

Convergence in probability of $\{\mathcal{E}(L^N)_{\tau_k}\}_{1\leq k<\infty}$ also implies that

$$V\left(yZ_T^\infty \mathcal{E}(L^N)_{\tau_k}\right) \longrightarrow V\left(yZ_T^\infty \mathcal{E}(L^N)_T\right) \quad \text{in probability as } k \to \infty.$$

Let C be the bound on $\mathcal{E}(L^N)$ from above given to us in definition of $\mathcal{B}$. Since $\frac{1}{N} \leq \mathcal{E}(L^N)_t \leq C$ for all t, we have for all k that $V(yZ_T^\infty \mathcal{E}(L^N)_{\tau_k}) \leq \max(V(\frac{1}{N}Z_T^\infty), V(CZ_T^\infty))$, where $\max(V(\frac{1}{N}Z_T^\infty), V(CZ_T^\infty))$ is in $L^1(\mathbb{P})$ by reasonable asymptotic elasticity, (I.3.3). Thus,

$$V(yZ_T^\infty \mathcal{E}(L^N)_{\tau_k}) \longrightarrow V(yZ_T^\infty \mathcal{E}(L^N)_T) \quad \text{in } L^1(\mathbb{P}) \text{ as } k \to \infty.$$

Finally, we may choose K sufficiently large so that $\mathbb{E}[V(yZ_T^\infty \mathcal{E}(L^N)_{\tau_K}) + yZ_T^\infty \mathcal{E}(L^N)_{\tau_K} f] \leq \mathbb{E}[V(yZ_T^\infty \mathcal{E}(L^N)_T) + yZ_T^\infty \mathcal{E}(L^N)_T f] + \varepsilon$, which then implies that

$$\mathbb{E}\left[V\left(yZ_T^\infty \mathcal{E}(L^N)_{\tau_K}\right) + yZ_T^\infty \mathcal{E}(L^N)_{\tau_K} f\right] \\ \leq \mathbb{E}\left[V\left(yZ_T^\infty \mathcal{E}(L)_T\right) + yZ_T^\infty \mathcal{E}(L)_T f\right] + 2\varepsilon.$$

Since $\varepsilon > 0$ and $L \in \mathcal{B}$ are arbitrary, Assumption I.3.5 allows us to conclude the desired result. $\square$

Establishing an upper-semicontinuity property for the dual problem is difficult because with small changes in the limiting market, we must produce a dual element of a pre-limiting market with appropriately small changes. Using the Kunita-Watanabe decomposition, we decompose elements $L \in \mathcal{B}'$ in terms of strongly orthogonal components based on the varying martingale drivers, M^n.

A $\mathbb{P}$-local martingale, N, is said to be in $H_0^2(\mathbb{P})$ provided $N_0 = 0$ and $\mathbb{E}[\langle N\rangle_T] < \infty$, in which case N is a martingale. A sequence of martingales $\{N^n\}_{1\leq n<\infty} \subseteq H_0^2(\mathbb{P})$ converges to N in $H_0^2(\mathbb{P})$ if $\mathbb{E}[\langle N^n - N\rangle_T] \longrightarrow 0$ as $n \to \infty$. We say that two (locally square integrable) local martingales, M and N, are *strongly orthogonal* if $\langle M, N\rangle_t = 0$ for all $t \in [0, T]$. Note that as our filtration is Brownian, all local martingales are continuous and hence locally square integrable.

Lemma I.4.9. *Let $\{M^n\}_{1\leq n\leq\infty}$ be local martingales such that $M^n \longrightarrow M^\infty$ in the semimartingale topology as $n\to\infty$, and suppose that M^∞ satisfies Assumption I.3.3. Let $L\in H_0^2(\mathbb{P})$ be strongly orthogonal to M^∞ and decompose L into its (unique) Kunita-Watanabe decomposition for $1\leq n<\infty$ by*

$$L = L^n + (H^n\cdot M^n),$$

where L^n and $(H^n\cdot M^n)$ are in $H_0^2(\mathbb{P})$ and L^n is strongly orthogonal to M^n. Then $L^n\longrightarrow L$ in $H_0^2(\mathbb{P})$ as $n\to\infty$.

Proof. The filtration $\mathbb{F}=(\mathcal{F}_t)_{0\leq t\leq T}$ is the ($\mathbb{P}$-completed) filtration generated by the d-dimensional Brownian motion $(B^1,\dots,B^d)$ on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ with $\mathcal{F}=\mathcal{F}_T$. For notational concreteness, we denote

$$L=(\nu^1\cdot B^1)+\ldots+(\nu^d\cdot B^d),$$

for $\nu^k\in\mathcal{L}^2$, $1\leq k\leq d$. Recall that for $1\leq n\leq\infty$, M^n has the form (I.3.2). For $\mathbf{x}=(x_1,\dots,x_d),\mathbf{y}=(y_1,\dots,y_d)\in\mathbb{R}^d$, we let $|\mathbf{x}|$ denote the Euclidean norm, $|\mathbf{x}|:=\sqrt{x_1^2+\cdots+x_d^2}$, and let the inner product be $\mathbf{x}\boldsymbol{\cdot}\mathbf{y}:=x_1y_1+\ldots+x_dy_d$. We define the vector $\nu:=(\nu^1,\dots,\nu^d)$.

For $1\leq n<\infty$, we define

$$H^n:=\frac{\nu\boldsymbol{\cdot}\sigma^n}{|\sigma^n|^2}\,\mathbb{I}_{\{|\sigma^n|\neq 0\}}.$$

Then H^n is progressively measurable and M^n-integrable with $(H^n\cdot M^n)\in H_0^2(\mathbb{P})$. We define $L^n:=L-(H^n\cdot M^n)\in H_0^2(\mathbb{P})$. L^n and M^n are strongly orthogonal, and thus $L=L^n+(H^n\cdot M^n)$ is the Kunita-Watanabe decomposition for L with respect to M^n. Since L^n and M^n are strongly orthogonal, $L^n\longrightarrow L$ in $H_0^2(\mathbb{P})$ if and only if $(H^n\cdot M^n)\longrightarrow 0$ in $H_0^2(\mathbb{P})$ as $n\to\infty$. Hence, we seek to show that $\mathbb{E}[\langle H^n\cdot M^n\rangle_T]=\mathbb{E}[\int_0^T\frac{(\nu\boldsymbol{\cdot}\sigma^n)^2}{|\sigma^n|^2}\mathbb{I}_{\{|\sigma^n|\neq 0\}}dt]\longrightarrow 0$ as $n\to\infty$.

Since $L\in H_0^2(\mathbb{P})$, we have for $1\leq n<\infty$,

$$\frac{(\nu\boldsymbol{\cdot}\sigma^n)^2}{|\sigma^n|^2}\,\mathbb{I}_{\{|\sigma^n|\neq 0\}}\leq|\nu|^2\in L^1(\mathbb{P}\times\mathrm{Leb}).$$

The assumption that $M^n\longrightarrow M^\infty$ in the semimartingale topology as $n\to\infty$ implies that for $1\leq k\leq d$, $\sigma^{n,k}\longrightarrow\sigma^{\infty,k}$ in ($\mathbb{P}\times$ Leb)-measure as $n\to\infty$. Since $\langle L,M^\infty\rangle=0$, we have that $\nu\boldsymbol{\cdot}\sigma^\infty=0$ ($\mathbb{P}\times$ Leb)-a.e. Assumption I.3.3 ensures that $|\sigma^\infty|\neq 0$ ($\mathbb{P}\times$ Leb)-a.e., and hence,

$$\frac{(\nu\boldsymbol{\cdot}\sigma^n)^2}{|\sigma^n|^2}\,\mathbb{I}_{\{|\sigma^n|\neq 0\}}\longrightarrow 0\quad\text{in } (\mathbb{P}\times\mathrm{Leb})\text{-measure as } n\to\infty.$$

Thus dominated convergence implies that $\mathbb{E}[\langle H^n\cdot M^n\rangle_T]\longrightarrow 0$ as $n\to\infty$, which completes the proof of the claim. □

Proof of Lemma I.4.7. We let $\mathcal{B}'$ be defined as in (I.4.13) and let $L \in \mathcal{B}'$ be given. Let $K \in (0, \infty)$ be the constant given in the definition of $\mathcal{B}'$ such that $|L_t| \leq K$ for all t and $\langle L \rangle_T \leq K$.

We let L^n be given as in Lemma I.4.9. Then $L^n \longrightarrow L$ in H_0^2 as $n \to \infty$. For $1 \leq n < \infty$, define stopping times

$$\tau_n := \inf\left\{t \leq T : |L_t^n - L_t| \geq 1 \text{ or } \langle L^n \rangle_t \geq K + 1\right\}.$$

The $H_0^2(\mathbb{P})$ convergence of $\{L^n\}_{1 \leq n < \infty}$ implies that $\langle L^n \rangle_T \longrightarrow \langle L \rangle_T$ in $L^1(\mathbb{P})$ as $n \to \infty$, while the Burkholder-Davis-Gundy inequalities additionally give us that $\mathbb{P}(\sup_t |L_t^n - L_t| \geq 1) \longrightarrow 0$ as $n \to \infty$. Hence, $\mathbb{P}(\tau_n = T) \longrightarrow 1$ as $n \to \infty$. We conclude that $L_{\tau_n}^n \longrightarrow L_T$ and $\langle L^n \rangle_{\tau_n} \longrightarrow \langle L \rangle_T$ in probability as $n \to \infty$, which yields

$$\mathcal{E}(L^n)_{\tau_n} \longrightarrow \mathcal{E}(L)_T \quad \text{in probability as } n \to \infty.$$

Furthermore, the definition of τ_n provides upper and lower bounds on $\mathcal{E}(L^n)_{\tau_n}$, which are independent of n:

$$e^{-2K-2} \leq \mathcal{E}(L^n)_{\tau_n} \leq e^{K+1}. \tag{I.4.14}$$

Such uniform bounds and the choice of the L^ns are made possible by the choice of $L \in \mathcal{B}'$.

For $1 \leq n \leq \infty$, Z^n is a martingale by Assumption I.3.2, and by Fatou's Lemma, $1 = \mathbb{E}[Z_T^\infty] \leq \liminf_{n\to\infty} \mathbb{E}[Z_T^n] = 1$. Hence, $\lim_{n\to\infty} \mathbb{E}[Z_T^n] = \mathbb{E}[Z_T^\infty]$. Scheffe's Lemma then implies that $Z_T^n \longrightarrow Z_T^\infty$ in $L^1(\mathbb{P})$ as $n \to \infty$, and in particular, $\{Z_T^n\}_{1 \leq n \leq \infty}$ is uniformly integrable. By (I.4.14) and using that $f \in L^\infty(\mathbb{P})$, we have that

$$0 \leq y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} f \leq \left(\sup_m y_m\right) e^{K+1} \|f\|_\infty Z_T^n,$$

which implies that $\{y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} f\}_{1 \leq n \leq \infty}$ is uniformly integrable. Convergence in probability and uniform integrability imply that

$$y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} f \longrightarrow y Z_T^\infty \mathcal{E}(L)_T f \;\; \text{in } L^1(\mathbb{P}) \text{ as } n \to \infty.$$

We use (I.4.14) again in order to obtain uniform integrability of the remaining term in the dual value function. As mentioned in Assumption 1.2(i) of [47], the reasonable asymptotic elasticity condition (I.3.3) along with the $U(0) > 0$ is equivalent to the following: for all $\lambda > 0$ there exists $C > 0$ such that $V(\lambda y) \leq CV(y)$ for all $y \geq 0$. Then for $1 \leq n < \infty$,

$$
\begin{aligned}
0 &\leq V\left(y_n Z_T^n \mathcal{E}(L^n)_{\tau_n}\right) \\
&\leq V\left(y_n Z_T^n e^{K+1}\right) \mathbb{I}_{\{y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} \geq U'(0)\}} \\
&\qquad + V\left(y_n Z_T^n e^{-2K-2}\right) \mathbb{I}_{\{y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} < U'(0)\}} \\
&\leq V\left(\left(\sup_m y_m\right) e^{K+1} Z_T^n\right) + V\left(\left(\inf_m y_m\right) e^{-2K-2} Z_T^n\right) \\
&\leq (C_1 + C_2) V(Z_T^n),
\end{aligned}
$$

where C_1, C_2 are the constants produced by the reasonable asymptotic elasticity of U. The constants C_1, C_2 depend on the choice of L, K, $\inf_m y_m$, and $\sup_m y_m$ but not on n. Assumption I.3.4 now guarantees the uniform integrability of $\{V(y_n Z_T^n \mathcal{E}(L^n)_{\tau_n})\}_{1 \leq n < \infty}$. Convergence in probability and uniform integrability imply that $V(y_n Z_T^n \mathcal{E}(L^n)_{\tau_n}) \longrightarrow V(y Z_T^\infty \mathcal{E}(L)_T)$ in $L^1(\mathbb{P})$ as $n \to \infty$. Finally, we have

$$
\begin{aligned}
&\mathbb{E}\left[V(y Z_T^\infty \mathcal{E}(L)_T) + y Z_T^\infty \mathcal{E}(L)_T f\right] \\
&\quad = \lim_n \mathbb{E}\left[V(y_n Z_T^n \mathcal{E}(L^n)_{\tau_n}) + y_n Z_T^n \mathcal{E}(L^n)_{\tau_n} f\right] \\
&\quad \geq \limsup_n v_n(y_n).
\end{aligned}
$$

Taking the infimum over all $L \in \mathcal{B}'$ and applying Lemma I.4.8 yields that $v_\infty(y) \geq \limsup_n v_n(y_n)$. $\square$

Proof of Theorem I.3.6. We first note that the assumption that $\mathcal{M}_V^\infty \neq \emptyset$ of Lemma I.4.5 is satisfied by Assumption I.3.4. For $x_n \longrightarrow x \in \mathbb{R}$ and $y = y(x)$, Lemmas I.4.5 and I.4.7 imply

$$
u_\infty(x) \leq \liminf_{n \to \infty} u_n(x_n) \leq \limsup_{n \to \infty} v_n(y) + x_n y \leq v_\infty(y) + xy = u_\infty(x). \quad \text{(I.4.15)}
$$

The last equality can be shown by Theorem 1.1 of [47] by taking $\mathcal{E} = x + f$ and $y = \mathbb{E}[\frac{d\hat{\mu}(x)}{d\mathbb{P}}]$. Here, $\mathcal{E}$ and $\hat{\mu}(x)$ refer to the notation used in [47].

Moreover, the inequality chain (I.4.15) shows for $y > 0$, $v_n(y) \longrightarrow v_\infty(y)$ as $n \to \infty$. For $y_n \longrightarrow y > 0$, we also have that $v_n(y_n) \longrightarrow v_\infty(y)$ as $n \to \infty$ because the convexity of each v_n implies that $v_n \longrightarrow v_\infty$ uniformly on compacts in $(0, \infty)$ as $n \to \infty$. $\square$

Proof of Corollary I.3.8. Let $\{p_{n_k}(x)\}_{1 \leq k < \infty}$ be a convergent subsequence of $\{p_n(x)\}_{1 \leq n < \infty}$ with $\lim_k p_{n_k}(x) = p \in \mathbb{R}$. By Theorem I.3.6,

$$
u_\infty(x) = \lim_k u_{n_k}(x),
$$

while $w_{n_k}(x + p_{n_k}(x)) = u_{n_k}(x)$ for each $k \geq 1$ by the definition of the indifference price. Next, we take the contingent claim to be 0 and note that $\lim_k x + p_{n_k}(x) = x + p$, which allows us to conclude from Theorem I.3.6 that

$$w_\infty(x + p) = \lim_k w_{n_k}(x + p_{n_k}(x)),$$

which implies that $p = p_\infty(x)$. Since $f \in L^\infty(\mathbb{P})$, $\{p_n(x)\}_n$ is bounded, hence any subsequence has a further subsequence that converges to $p_\infty(x)$. Therefore, $\lim_n p_n(x)$ exists and equals $p_\infty(x)$. □

I.5 Examples

The first example shows that Assumption I.3.3 is necessary in the sense that its absence can allow Theorem I.3.6's conclusion to fail. The example is constructed in a very simple setting, but the same idea can generate more complex counterexamples whenever Assumption I.3.3 fails.

Example I.5.1. Let $d = 1$, so that the probability space is generated by a 1-dimensional Brownian motion, B. We define the martingales $M^n := \frac{1}{n}B$ for $1 \leq n < \infty$ and $M^\infty := 0$. Let $\lambda^n := 0$ for all $1 \leq n \leq \infty$ so that $S^\infty_t = 0$ for all $t \in [0, T]$ and for $1 \leq n < \infty$, S^n has the dynamics

$$dS^n = \frac{1}{n}dB, \quad S^n_0 = 0.$$

The stock markets satisfy Assumptions I.3.1 and I.3.2, but the limiting market does not satisfy Assumption I.3.3.

Let the contingent claim be given by $f := \mathbb{I}_{\{B_T \geq 0\}}$. By Itô's representation theorem and the boundedness of f, there exists a progressively measurable H such that $\mathbb{E}[\int_0^T H_t^2 dt] < \infty$ and $f = \frac{1}{2} + (H \cdot B)_T$. Moreover, $(H \cdot B)$ is bounded since for all $t \in [0, T]$,

$$(H \cdot B)_t = \mathbb{E}\left[(H \cdot B)_T \mid \mathcal{F}_t\right] = \mathbb{E}\left[f - \frac{1}{2} \mid \mathcal{F}_t\right] \in \left[-\frac{1}{2}, \frac{1}{2}\right], \quad \mathbb{P}\text{-a.s..}$$

Hence, we can conclude by Theorem 2.1 of [49] that for all $1 \leq n < \infty$ and $x \in \mathbb{R}$,

$$u_n(x) = U\left(x + \frac{1}{2}\right).$$

Yet for all $x \in \mathbb{R}$, Jensen's inequality implies $u_\infty(x) = \mathbb{E}[U(x + f)] < U(x + \frac{1}{2})$.

The following two examples provide sufficient conditions on the limiting market for Assumption I.3.5 to hold.

Example I.5.2. This example covers the original motivation for this work, where the contingent claim is replicable in the (possibly incomplete) limiting market. In this case, the limiting market consists of a driving Brownian motion, a replicable claim, and additional independent Brownian noise.

Recall that $(B^1, \dots, B^d)$ is the d-dimensional Brownian motion generating the completed filtration, $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$. We let $(\mathcal{F}^1_t)_{0 \le t \le T}$ denote the filtration generated by B^1, completed with all $\mathbb{P}$-null sets. The risky asset, S^∞, has dynamics as in (I.3.1) and is $(\mathcal{F}^1_t)_{0 \le t \le T}$-adapted. The contingent claim, $f \in L^\infty(\Omega, \mathcal{F}^1_T, \mathbb{P})$, is replicable: there exists an S^∞-integrable H and constant c such that $f = c + (H \cdot S^\infty)_T$.

Proposition I.5.3. *Suppose that S^∞ is $(\mathcal{F}^1_t)_{0 \le t \le T}$-adapted with dynamics* (I.3.1) *and satisfies Assumption I.3.3. Suppose that $f \in L^\infty(\Omega, \mathcal{F}^1_T, \mathbb{P})$ is replicable. Then Assumption I.3.5 is satisfied.*

Proof. Let $y > 0$ and $\mathbb{Q} \in \mathcal{M}^\infty_V$ be given. Write $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z^\infty_T \mathcal{E}(L)_T$ for its Radon-Nikodym density. We have that $Z^\infty_T \in \mathcal{F}^1_T$, while Assumption I.3.3 implies that $\langle L, B^1 \rangle_t = 0$ for $t \in [0, T]$. By localization and Lemma 5.3 of [7],

$$\mathbb{E}[\mathcal{E}(L)_T | \mathcal{F}^1_T] \le \mathbb{E}[\mathcal{E}(L)_0 | \mathcal{F}^1_T] = 1.$$

Then $\mathbb{E}[\mathcal{E}(L)_T | \mathcal{F}^1_T] = 1$, $\mathbb{P}$-a.s., since

$$1 = \mathbb{E}[Z^\infty_T \mathcal{E}(L)_T] = \mathbb{E}[Z^\infty_T \mathbb{E}[\mathcal{E}(L)_T | \mathcal{F}^1_T]\,] \le \mathbb{E}[Z^\infty_T] = 1,$$

with equality holding if and only if $\mathbb{E}[\mathcal{E}(L)_T | \mathcal{F}^1_T] = 1$, $\mathbb{P}$-a.s. By Jensen's inequality,

$$\begin{aligned} \mathbb{E}[V(yZ^\infty_T \mathcal{E}(L)_T)] &= \mathbb{E}\left[\mathbb{E}\left[V(yz\mathcal{E}(L)_T) | \mathcal{F}^1_T\right] |_{z = Z^\infty_T}\right] \\ &\ge \mathbb{E}\left[V(yz\mathbb{E}\left[\mathcal{E}(L)_T | \mathcal{F}^1_T\right]) |_{z = Z^\infty_T}\right] \\ &= \mathbb{E}[V(yZ^\infty_T)]. \end{aligned}$$

Since f is bounded and replicable, $\mathbb{Q} \mapsto \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} f]$ is constant on $\mathcal{M}^\infty$. Hence, for all $\mathbb{Q} \in \mathcal{M}^\infty_V$, $\mathbb{E}[V(yZ^\infty_T) + yZ^\infty_T f] \le \mathbb{E}[V(y\frac{d\mathbb{Q}}{d\mathbb{P}}) + y\frac{d\mathbb{Q}}{d\mathbb{P}} f]$, which implies that Z^∞_T is the density of the dual minimizer, and so Assumption I.3.5 is satisfied. $\square$

Example I.5.4 (Exponential Investors)**.** For the exponential investor, Assumption I.3.5 is satisfied, under an easier-to-verify BMO assumption. We refer to [34] for additional details on BMO martingales.

Definition I.5.5. A $\mathbb{P}$-local martingale N is said to be in $BMO(\mathbb{P})$ if

$$\sup_\tau \left\| \mathbb{E}^{\mathbb{P}}\left[|N_T - N_\tau| \; |\mathcal{F}_\tau \right] \right\|_\infty < \infty,$$

where the supremum is taken over stopping times $\tau \le T$.

Assumption I.5.6. $(\lambda^\infty \cdot M^\infty) \in \mathrm{BMO}(\mathbb{P})$.

For the remainder of this section, we let $U(x) = -\exp(-\alpha x)$ for a positive constant α. The conjugate to U is $V(y) = \frac{y}{\alpha}(\log \frac{y}{\alpha} - 1)$, $y > 0$. We have the following relationships for $c \in \mathbb{R}$ and $y > 0$:

$$V'(cy) = V'(y) + \frac{1}{\alpha}\log c, \tag{I.5.1}$$

$$V(y) + yc = y\left(V'(ye^{\alpha c}) - \frac{1}{\alpha}\right). \tag{I.5.2}$$

For a set $A \in \mathcal{F}$ and random variable $X \in L^1(\mathbb{P})$, we adopt the notation $\mathbb{E}[X; A] := \mathbb{E}[X\mathbb{I}_A] = \int_A X d\mathbb{P}$.

Theorem I.5.7. *Let $U(x) = -\exp(-\alpha x)$ for a positive constant α and assume that Assumption I.5.6 holds. Let $\mathbb{Q}^\infty$ denote the minimal martingale measure, $\frac{d\mathbb{Q}^\infty}{d\mathbb{P}} := Z_T^\infty = \mathcal{E}(-\lambda^\infty \cdot M^\infty)_T$, and suppose that $\mathbb{Q}^\infty \in \mathcal{M}_V^\infty$. Then Assumption I.3.5 is satisfied.*

Proof. Let $x \in \mathbb{R}$ and $Z_T^\infty \mathcal{E}(L)_T = \mathcal{E}(-(\lambda^\infty \cdot M^\infty) + L)_T \in \mathcal{M}_V^\infty$ be the dual optimizer for the dual problem (I.3.5) with $n = \infty$ and $y := u'_\infty(x)$. For $1 \leq n < \infty$, we define the stopping times $\tau_n := \inf\{t \leq T : \mathcal{E}(L)_t \geq n\}$. Using that $V(0) = 0$ and the definition of τ_n, it is not difficult to verify that each probability density $Z_T^\infty \mathcal{E}(L)_{\tau_n}$ corresponds to a martingale measure in $\mathcal{M}_V^\infty$.

Theorem 2.1 of [29] implies that there exists an S^∞-integrable $\hat{H}$ such that $\hat{H}$ is optimal for (I.3.4) with $n = \infty$ and $(\hat{H} \cdot S^\infty)$ is a martingale with respect to every measure $\mathbb{Q} \in \mathcal{M}_V^\infty$. The process $\hat{H}$ is a *permissible* wealth process (in the S^∞ market), rather than an admissible wealth process; see [47] Definition 1.1 for details. Then Proposition 4.1 from [47] implies that $x + (\hat{H} \cdot S^\infty)_T + f = -V'(yZ_T^\infty \mathcal{E}(L)_T)$. Hence, for any $\mathbb{Q} \in \mathcal{M}_V^\infty$, (I.5.1) with $c = x$ implies that

$$\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} V'\left(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f}\right)\right] = \mathbb{E}\left[Z_T^\infty \mathcal{E}(L)_T V'\left(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f}\right)\right]. \tag{I.5.3}$$

Then,

$$\begin{aligned}
0 &\leq \mathbb{E}\left[V(yZ_T^\infty \mathcal{E}(L)_{\tau_n}) + yZ_T^\infty \mathcal{E}(L)_{\tau_n} f\right] - v_\infty(y) \\
&= \mathbb{E}\left[V(yZ_T^\infty \mathcal{E}(L)_{\tau_n}) + yZ_T^\infty \mathcal{E}(L)_{\tau_n} f\right] - \mathbb{E}\left[V(yZ_T^\infty \mathcal{E}(L)_T) + yZ_T^\infty \mathcal{E}(L)_T f\right] \\
&= \mathbb{E}\left[yZ_T^\infty \mathcal{E}(L)_{\tau_n} V'(yZ_T^\infty \mathcal{E}(L)_{\tau_n} e^{\alpha f}) \; - yZ_T^\infty \mathcal{E}(L)_T V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f})\right] \\
&\qquad \text{by (I.5.2)} \\
&= \mathbb{E}\left[yZ_T^\infty \mathcal{E}(L)_{\tau_n}\left(V'(yZ_T^\infty \mathcal{E}(L)_{\tau_n} e^{\alpha f}) - V'(yZ_T^\infty \mathcal{E}(L)_T e^{\alpha f})\right)\right] \\
&\qquad \text{by (I.5.3)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{y}{\alpha}\mathbb{E}\left[Z_T^\infty \mathcal{E}(L)_{\tau_n}\left(\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T\right)\right] \\
&\hspace{25em} \text{by (I.5.1)} \\
&= \frac{y}{\alpha}\mathbb{E}^{\mathbb{Q}^\infty}\left[n \log\left(\frac{n}{\mathcal{E}(L)_T}\right); \{\tau_n < T\}\right] \\
&= \frac{y}{\alpha}\left(n \log n\, \mathbb{Q}^\infty(\tau_n < T) - n\, \mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T; \{\tau_n < T\}]\right).
\end{aligned}
$$

In order to show Assumption I.3.5, it now suffices to show

$$
n \log n\, \mathbb{Q}^\infty(\tau_n < T) - n\mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T; \{\tau_n < T\}] \longrightarrow 0 \quad \text{as } n \to \infty. \quad \text{(I.5.4)}
$$

Showing $n \log n\, \mathbb{Q}^\infty(\tau_n < T) \longrightarrow 0$ as $n \to \infty$ will employ Doob's submartingale inequality, whereas $n\, \mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T; \{\tau_n < T\}] \longrightarrow 0$ relies on the assumption that $(\lambda^\infty \cdot M^\infty) \in \mathrm{BMO}(\mathbb{P})$.

Let $\phi(y) := y \log y$. We have that ϕ is convex, $\phi \geq -1/e$, and ϕ is increasing on $[1/e, \infty)$. Using that $Z_T^\infty \mathcal{E}(L)_T$ is the dual optimizer, it is not difficult to check that $\phi(\mathcal{E}(L)_t) \in L^1(\mathbb{Q}^\infty)$ for each $t \in [0, T]$. Convexity of ϕ implies that $\phi(\mathcal{E}(L))$ is a $\mathbb{Q}^\infty$-submartingale. (Note that $\mathcal{E}(L)$ is a $\mathbb{Q}^\infty$-martingale since $\mathbb{E}^{\mathbb{Q}^\infty}[\mathcal{E}(L)_T] = \mathbb{E}^{\mathbb{P}}[Z_T^\infty \mathcal{E}(L)_T] = 1$.)

For a process Y, we let $Y^* := \sup_{0 \leq t \leq T} Y_t$. For any $n > 1$,

$$
\mathcal{E}(L)^* \geq n \ \text{ if and only if } \ \phi(\mathcal{E}(L))^* = (\mathcal{E}(L) \log \mathcal{E}(L))^* \geq n \log n.
$$

Doob's submartingale inequality implies that for $n > 1$,

$$
\begin{aligned}
n \log n\, \mathbb{Q}^\infty(\mathcal{E}(L)^* \geq n) &= n \log n\, \mathbb{Q}^\infty\left(\phi(\mathcal{E}(L))^* \geq n \log n\right) \\
&\leq \mathbb{E}^{\mathbb{Q}^\infty}\left[\phi(\mathcal{E}(L)_T)^+; \{\phi(\mathcal{E}(L))^* \geq n \log n\}\right] \\
&= \mathbb{E}^{\mathbb{Q}^\infty}\left[\phi(\mathcal{E}(L)_T)^+; \{\mathcal{E}(L)^* \geq n\}\right].
\end{aligned}
$$

Since $\phi(\mathcal{E}(L)_T) \in L^1(\mathbb{Q}^\infty)$, we have that

$$
\begin{aligned}
\limsup_{n \to \infty}\ n \log n\ \mathbb{Q}^\infty(\tau_n < T) &\leq \limsup_{n \to \infty}\ n \log n\ \mathbb{Q}^\infty(\mathcal{E}(L)^* \geq n) \\
&\leq \limsup_{n \to \infty}\ \mathbb{E}^{\mathbb{Q}^\infty}[\phi(\mathcal{E}(L)_T)^+; \{\mathcal{E}(L)^* \geq n\}] \\
&= 0.
\end{aligned}
$$

Now suppose that Assumption I.5.6 holds. Then by Lemma 3.1 of [12] the density of the dual optimizer, $Z^\infty \mathcal{E}(L)$, satisfies $\mathcal{R}_{L \log L}(\mathbb{P})$; that is, $Z^\infty \mathcal{E}(L)$ is a $\mathbb{P}$-martingale and

$$
\sup_\tau \left\| \mathbb{E}^{\mathbb{P}}\left[\frac{Z_T^\infty \mathcal{E}(L)_T}{Z_\tau^\infty \mathcal{E}(L)_\tau} \log\left(\frac{Z_T^\infty \mathcal{E}(L)_T}{Z_\tau^\infty \mathcal{E}(L)_\tau}\right) \middle| \mathcal{F}_\tau\right]\right\|_\infty < \infty,
$$

where the supremum is taken over all stopping times $\tau \leq T$. Lemma 2.2 of [22] shows that $-(\lambda^\infty \cdot M^\infty) + L \in \mathrm{BMO}(\mathbb{P})$, which then implies that $L \in \mathrm{BMO}(\mathbb{P})$.

Since $\langle -\lambda^\infty \cdot M^\infty, L\rangle_t = 0$ for all $t \in [0, T]$, then Theorem 3.6 of [34] implies that $L = L - \langle -\lambda^\infty \cdot M^\infty, L\rangle \in \mathrm{BMO}(\mathbb{Q}^\infty)$. Then by Theorem 2.4 of [34], L satisfies

$$\sup_\tau \left\| \mathbb{E}^{\mathbb{Q}^\infty}\left[\log^+ \left(\frac{\mathcal{E}(L)_\tau}{\mathcal{E}(L)_T} \right) \middle| \mathcal{F}_\tau \right] \right\|_\infty < \infty, \tag{I.5.5}$$

where the supremum is taken over all stopping times $\tau \leq T$. Re-writing (I.5.5), and considering only the stopping times τ_n for $n \geq 1$, we have

$$K := \sup_n \left\| \mathbb{E}^{\mathbb{Q}^\infty} \left[(\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T)\, \mathbb{I}_{\{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\}} | \mathcal{F}_{\tau_n} \right] \right\|_\infty < \infty.$$

For each $n \geq 1$, $\{\tau_n < T\} \in \mathcal{F}_{\tau_n}$ and $\mathcal{E}(L)_{\tau_n} = n$ on $\{\tau_n < T\}$. Then,

$$\begin{aligned}
&-\mathbb{E}^{\mathbb{Q}^\infty}\left[\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\} \cap \{\tau_n < T\}\right] \\
&\quad \leq \mathbb{E}^{\mathbb{Q}^\infty}\left[\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T; \{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\} \cap \{\tau_n < T\}\right] \\
&\quad = \mathbb{E}^{\mathbb{Q}^\infty}\left[\mathbb{E}^{\mathbb{Q}^\infty}\left[(\log \mathcal{E}(L)_{\tau_n} - \log \mathcal{E}(L)_T)\, \mathbb{I}_{\{\mathcal{E}(L)_{\tau_n} \geq \mathcal{E}(L)_T\}} | \mathcal{F}_{\tau_n}\right]; \{\tau_n < T\}\right] \\
&\quad \leq K\, \mathbb{Q}^\infty\left(\tau_n < T\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
-n\mathbb{E}^{\mathbb{Q}^\infty}&\left[\log \mathcal{E}(L)_T; \{\tau_n < T\}\right] \\
&= -n\mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_T > n\} \cap \{\tau_n < T\}] \\
&\quad - n\mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T; \{\mathcal{E}(L)_T \leq n\} \cap \{\tau_n < T\}] \\
&\leq 0 + nK\mathbb{Q}^\infty(\tau_n < T).
\end{aligned}$$

Equation (I.5.4) now follows from

$$\begin{aligned}
0 &\leq n \log n\, \mathbb{Q}^\infty(\tau_n < T) - n\, \mathbb{E}^{\mathbb{Q}^\infty}[\log \mathcal{E}(L)_T;\ \{\tau_n < T\}] \\
&\leq n \log n\, \mathbb{Q}^\infty(\tau_n < T) + nK\, \mathbb{Q}^\infty(\tau_n < T) \\
&\longrightarrow 0, \quad \text{as } n \to \infty.
\end{aligned}$$

□

Chapter II

When is the Dual Optimizer a Martingale?

II.1 Introduction

An unpleasant qualitative feature of the general theory of optimal investment with a utility function defined on $(0, \infty)$ is that the dual optimizer $\widehat{Y}$ may not be a uniformly integrable martingale. In the presence of jumps, it may even fail to be a local martingale. The corresponding counterexamples can be found in [37]. In this paper, we seek to provide conditions under which the uniform martingale property for $\widehat{Y}$ holds and thus, $\widehat{Y}/\widehat{Y}_0$ defines the density process of the *optimal* martingale measure $\widehat{\mathbb{Q}}$.

The question of whether $\widehat{Y}$ is a uniformly integrable martingale is of long-standing interest in mathematical finance and can be traced back to [23] and [30]. This problem naturally arises in situations involving utility-based arguments. For instance, it is relevant for pricing in incomplete markets, where according to [27] the existence of $\widehat{\mathbb{Q}}$ is equivalent to the fact that for every bounded contingent claim ψ its marginal utility-based price p is unique. In this case,

$$p = \mathbb{E}^{\widehat{\mathbb{Q}}}[\psi] = \mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\psi\right],$$

and thus $\widehat{\mathbb{Q}}$ plays the role of the pricing measure from the classical Black and Scholes theory of complete financial markets, see [48] and [10]. Notice that the nonexistence of $\widehat{\mathbb{Q}}$ is equivalent to $\mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\right] < 1$. Then for $\psi = 1$ the expression $\mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_0}\psi\right]$ fails to be even an arbitrage-free price!

Of course, if the dual minimizer $\widehat{Y}$ can be computed *explicitly* as in [35], then its uniform integrability property may be verified using either the sufficient conditions of Novikov and Kazamaki or the necessary and sufficient criteria based on Hellinger processes. We refer the reader to [34, Section 1.4] for the former and to [28, Section IV.2] for the latter. However, for a generic incomplete model there is little hope of obtaining an explicit representation for $\widehat{Y}$, and a different approach should be used.

Our key requirement consists of the existence of a dual supermartingale Z, which satisfies the probabilistic Muckenhoupt (A_p) condition for the power $p > 1$ such that

$$p = \frac{1}{1-a}. \tag{II.1.1}$$

Here $a \in (0, 1)$ is a lower bound on the relative risk-aversion of the utility function. As we prove in Theorem II.5.1, this condition, along with the existence of an upper bound for the relative risk-aversion, yields $(A_{p'})$ for $\widehat{Y}$ for some $p' > 1$. This property in turn implies that the dual minimizer $\widehat{Y}$ is of class $(\mathbf{D})$, that is, the family of its values evaluated at all stopping times is uniformly integrable. In Proposition II.6.1, we construct a counterexample showing that the bound (II.1.1) is the best possible for $\widehat{Y}$ to be of class $(\mathbf{D})$ even in the case of power utilities and continuous stock prices.

In the case of the power utility function

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

with the relative risk-aversion $a \in (0, 1)$ the dual optimizer $\widehat{Y}$ satisfies (A_p) with p given by (II.1.1) if and only if the (A_p) condition holds for some dual supermartingale Z. Moreover, $\widehat{Y}$ has the smallest (A_p)-constant among all such Z. This fact has been already established in [46]. For reader's convenience we shall restate it as Proposition II.3.3.

A similar idea of passing regularity from some dual element to the optimal one has been employed in [13], [21] and [12] for respectively, quadratic, power and exponential utility functions defined on the whole real line. These papers use appropriate versions of the Reverse Hölder (R_q) inequality. We recall that (A_p) and (R_q) conditions are dual in the sense that if Z is the density process of the equivalent probability measure $\mathbb{Q}$ and $1/p + 1/q = 1$, then (A_p) for Z (under $\mathbb{P}$) is equivalent to (R_q) for $1/Z$ under $\mathbb{Q}$. We also remind the reader that contrary to (A_p), the uniform integrability property is not implied but rather *required* by (R_q). While this requirement is not a problem for real-line utilities, where the optimal martingale measures always exist, it is clearly an issue for utility functions defined on $(0, \infty)$.

Observe also that for power and exponential utilities and for continuous stock prices one can characterize $\widehat{Y}$ in terms of a solution to a one-dimensional quadratic BSDE, see [25]. Under additional assumption of bounded market price of risk the theory of such equations yields that $\widehat{Y}$ is a stochastic exponential of a BMO-martingale and in particular is a uniformly integrable martingale.

Even if the dual minimizer $\widehat{Y}$ is of class $(\mathbf{D})$, it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [37, Example 5.1′]. In Proposition II.4.2 we prove that every *maximal* dual supermartingale (in particular, $\widehat{Y}$) is a local martingale if the ratio of any two positive wealth processes is σ-bounded.

Our main results, Theorems II.5.1 and II.5.3, are stated in Section II.5. They are accompanied by Corollaries II.5.5 and II.5.6, which exploit well known connections between the (A_p) condition and BMO martingales.

II.2 Setup

We use the same framework as in [37, 38] and refer to these papers for more details. There is a financial market with a bank account paying zero interest and d stocks. The process of stocks' prices $S = (S^i)$ is a semimartingale with values in $\mathbf{R}^d$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Here T is a finite maturity and $\mathcal{F} = \mathcal{F}_T$, but we remark that our results also hold for the case of infinite maturity.

A (self-financing) portfolio is defined by an initial capital $x \in \mathbf{R}$ and a predictable S-integrable process $H = (H^i)$ with values in $\mathbf{R}^d$ of the number of stocks. Its corresponding wealth process X evolves as

$$X_t = x + \int_0^t H_u dS_u, \quad t \in [0, T].$$

We denote by $\mathcal{X}$ the family of non-negative wealth processes:

$$\mathcal{X} \triangleq \{X \geq 0 : \ X \text{ is a wealth process}\}$$

and by $\mathcal{Q}$ the family of equivalent local martingale measures for $\mathcal{X}$:

$$\mathcal{Q} \triangleq \{\mathbb{Q} \sim \mathbb{P} : \ \text{every } X \in \mathcal{X} \text{ is a local martingale under } \mathbb{Q}\}.$$

We assume that

$$\mathcal{Q} \neq \emptyset, \tag{II.2.1}$$

which is equivalent to the absence of arbitrage; see [14, 16].

There is an economic agent whose preferences over terminal wealth are modeled by a utility function U defined on $(0, \infty)$. We assume that U is of *Inada type*, that is, it is strictly concave, strictly increasing, continuously differentiable on $(0, \infty)$, and

$$U'(0) = \lim_{x\to 0} U'(x) = \infty, \quad U'(\infty) = \lim_{x\to\infty} U'(x) = 0.$$

For a given initial capital $x > 0$, the goal of the agent is to maximize the expected utility of terminal wealth. The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X},\, X_0 = x} \mathbb{E}\left[U(X_T)\right]. \tag{II.2.2}$$

Following [37], we define the dual optimization problem to (II.2.2) as

$$v(y) = \inf_{Y \in \mathcal{Y}, Y_0 = y} \mathbb{E}\left[V(Y_T)\right], \quad y > 0, \tag{II.2.3}$$

where V is the convex conjugate to U:

$$V(y) = \sup_{x>0} \left\{U(x) - xy\right\}, \quad y > 0,$$

and $\mathcal{Y}$ is the family of "dual" supermartingales to $\mathcal{X}$:

$$\mathcal{Y} = \left\{Y \geq 0 : \ XY \text{ is a supermartingale for every } X \in \mathcal{X}\right\}.$$

Note that the set $\mathcal{Y}$ contains the density processes of all $\mathbb{Q} \in \mathcal{Q}$ and that, as $1 \in \mathcal{X}$, every element of $\mathcal{Y}$ is a supermartingale.

It is known, see [38, Theorem 2], that under (II.2.1) and

$$v(y) < \infty, \quad y > 0, \tag{II.2.4}$$

the value functions u and $-v$ are of Inada type, v is the convex conjugate to u, and

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right], \quad y > 0. \tag{II.2.5}$$

The solutions $X(x)$ to (II.2.2) and $Y(y)$ to (II.2.3) exist. If $y = u'(x)$ or, equivalently, $x = -v'(y)$, then

$$U'(X_T(x)) = Y_T(y),$$

and the product $X(x)Y(y)$ is a uniformly integrable martingale.

The last two properties actually characterize optimal $X(x)$ and $Y(y)$. For convenience of future references, we recall this "verification" result.

Lemma II.2.1. *Let $\widehat{X} \in \mathcal{X}$ and $\widehat{Y} \in \mathcal{Y}$ be such that*

$$U'(\widehat{X}_T) = \widehat{Y}_T, \quad \mathbb{E}\left[V(\widehat{Y}_T)\right] < \infty, \quad \mathbb{E}\left[\widehat{X}_T\widehat{Y}_T\right] = \widehat{X}_0\widehat{Y}_0.$$

Then $\widehat{X}$ solves (II.2.2) *for $x = \widehat{X}_0$ and $\widehat{Y}$ solves* (II.2.5) *for $y = \widehat{Y}_0$.*

Proof. The result follows immediately from the identity

$$U(\widehat{X}_T) = V(\widehat{Y}_T) + \widehat{X}_T \widehat{Y}_T$$

and the inequalities

$$\begin{aligned} U(X_T) &\le V(\widehat{Y}_T) + X_T \widehat{Y}_T, \quad X \in \mathcal{X}, \\ U(\widehat{X}_T) &\le V(Y_T) + \widehat{X}_T Y_T, \quad Y \in \mathcal{Y}, \end{aligned}$$

after we recall that XY is a supermartingale for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. □

The goal of the paper is to find sufficient conditions for the lower bound in (II.2.5) to be attained at some $\mathbb{Q}(y) \in \mathcal{Q}$ called the *optimal martingale measure* or, equivalently, for the dual minimizer $Y(y)$ to be a *uniformly integrable martingale*; in this case,

$$Y_T(y) = y \frac{d\mathbb{Q}(y)}{d\mathbb{P}}.$$

Our criteria are stated in Theorem II.5.1 below, where a key role is played by the probabilistic version of the classical Muckenhoupt (A_p) condition.

II.3 (A_p) condition for the dual minimizer

Following [34, Section 2.3], we recall the probabilistic (A_p) condition.

Definition II.3.1. Let $p > 1$. An optional process $R \ge 0$ *satisfies* (A_p) if $R_T > 0$ and there is a constant $C > 0$ such that for every stopping time τ

$$\mathbb{E}\left[\left(\frac{R_\tau}{R_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \le C.$$

Observe that if an optional process $R > 0$ satisfies (A_p), then by Hölder's inequality it satisfies (A_q) for every $q \ge p$. If in addition R is a continuous local martingale, then it also satisfies (A_r) for some $r < p$. The latter fact is delicate and follows from Gehring's inequality, see Corollary 3.3 in [34].

An important consequence of the (A_p) condition is a uniform integrability property. For continuous local martingales this fact is well known and can be found e.g., in [34, Section 2.3].

Lemma II.3.2. *If an optional process $R \ge 0$ satisfies (A_p) for some $p > 1$ and $\mathbb{E}[R_T] < \infty$, then R is of class* $(\mathbf{D})$*:*

$$\{R_\tau : \tau \text{ is a stopping time}\} \text{ is uniformly integrable.}$$

Proof. Let τ be a stopping time. As $p > 1$, the function $x \mapsto x^{-\frac{1}{p-1}}$ is convex. Hence, by Jensen's inequality,

$$\mathbb{E}\left[\left(\frac{R_\tau}{R_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] = R_\tau^{\frac{1}{p-1}}\mathbb{E}\left[R_T^{-\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] \geq R_\tau^{\frac{1}{p-1}}\left(\mathbb{E}\left[R_T|\,\mathcal{F}_\tau\right]\right)^{-\frac{1}{p-1}}.$$

Using the constant $C > 0$ from (A_p), we obtain that

$$R_\tau \leq C^{p-1}\mathbb{E}\left[R_T|\,\mathcal{F}_\tau\right],$$

and the result follows. $\square$

To motivate the use of the (A_p) condition in the study of the dual minimizers $Y(y)$, $y > 0$, we first consider the case of power utility with a positive power. The following result has been already established in [46].

Proposition II.3.3. *Let* (II.2.1) *hold. Assume that*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

with the relative risk-aversion $a \in (0,1)$ and denote $p \triangleq \frac{1}{1-a} > 1$. Then for $y > 0$, the solution $Y(y)$ to the dual problem (II.2.3) *exists if and only if*

$$\mathbb{E}\left[Y_T^{-\frac{1}{p-1}}\right] < \infty \quad \textit{for some} \quad Y \in \mathcal{Y} \tag{II.3.1}$$

and, in this case, for every $Y \in \mathcal{Y}$, $Y > 0$ and every stopping time τ,

$$\mathbb{E}\left[\left(\frac{Y_\tau(y)}{Y_T(y)}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] \leq \mathbb{E}\left[\left(\frac{Y_\tau}{Y_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right].$$

In particular, $Y(y)$ satisfies (A_p) if and only if there is $Y \in \mathcal{Y}$ satisfying (A_p).

Proof. Observe that the convex conjugate to U is given by

$$V(y) = \frac{a}{1-a}y^{-\frac{1-a}{a}} = (p-1)y^{-\frac{1}{p-1}}, \quad y > 0.$$

Then (II.3.1) is equivalent to (II.2.4), which, in turn, is equivalent to the existence of the optimal $Y(y)$, $y > 0$. Denote $\widehat{Y} \triangleq Y(1)$. Clearly, $Y(y) = y\widehat{Y}$. Recall that $\widehat{Y} > 0$.

Let a stopping time τ and a process $Y \in \mathcal{Y}$, $Y > 0$, be such that

$$\mathbb{E}\left[\left(\frac{Y_\tau}{Y_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] < \infty.$$

We have to show that

$$\xi \triangleq \mathbb{E}\left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] - \mathbb{E}\left[\left(\frac{Y_\tau}{Y_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] \leq 0.$$

For a set $A \in \mathcal{F}_\tau$, the process

$$Z_t \triangleq \widehat{Y}_t 1_{\{t\leq\tau\}} + \widehat{Y}_\tau\left(\frac{Y_t}{Y_\tau}1_A + \frac{\widehat{Y}_t}{\widehat{Y}_\tau}(1-1_A)\right)1_{\{t>\tau\}}, \quad t \in [0,T],$$

belongs to $\mathcal{Y}$ and is such that $Z_0 = 1$ and $Z_\tau = \widehat{Y}_\tau$. We obtain that

$$\begin{aligned}\mathbb{E}\left[\left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] &= \mathbb{E}\left[\left(\frac{Y_\tau}{Y_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right]1_A + \mathbb{E}\left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right](1-1_A)\\ &= \mathbb{E}\left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\middle|\mathcal{F}_\tau\right] - \xi 1_A.\end{aligned}$$

Dividing both sides by $Z_\tau^{\frac{1}{p-1}} = \widehat{Y}_\tau^{\frac{1}{p-1}}$ and choosing $A = \{\xi \geq 0\}$, we deduce that

$$\mathbb{E}\left[\left(\frac{1}{Z_T}\right)^{\frac{1}{p-1}}\right] = \mathbb{E}\left[\left(\frac{1}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\right] - \mathbb{E}\left[\left(\frac{1}{\widehat{Y}_\tau}\right)^{\frac{1}{p-1}}\max(\xi,0)\right].$$

However, the optimality of $\widehat{Y} = Y(1)$ implies that

$$\mathbb{E}\left[\left(\frac{1}{\widehat{Y}_T}\right)^{\frac{1}{p-1}}\right] \leq \mathbb{E}\left[\left(\frac{1}{Z_T}\right)^{\frac{1}{p-1}}\right].$$

Hence $\xi \leq 0$. □

We now state the main result of the section.

Theorem II.3.4. *Let* (II.2.1) *hold. Suppose that there are constants* $0 < a < 1$*,* $b \geq a$ *and* $C > 0$ *such that*

$$\frac{1}{C}\left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)} \leq C\left(\frac{y}{x}\right)^b, \quad x \leq y, \tag{II.3.2}$$

and there is a supermartingale $Z \in \mathcal{Y}$ *satisfying* (A_p) *with*

$$p = \frac{1}{1-a}.$$

Then for every $y > 0$*, the solution* $Y(y)$ *to* (II.2.3) *exists and satisfies* $(A_{p'})$ *with*

$$p' = 1 + \frac{b}{1-a}.$$

Remark II.3.1. Notice that if the relative risk-aversion of U is well-defined and bounded away from 0 and ∞, then in (II.3.2) we can take $C = 1$ and choose a and b as lower and upper bounds:

$$0 < a \leq -\frac{xU''(x)}{U'(x)} \leq b < \infty, \quad x > 0.$$

In particular, if

$$1 \leq -\frac{xU''(x)}{U'(x)} \leq b, \quad x > 0,$$

then choosing $a \in (0,1)$ sufficiently close to 1 we fulfill the conditions of Theorem II.3.4 if there exists a supermartingale $Z \in \mathcal{Y}$ satisfying (A_p) for *some* $p > 1$.

Observe also that for the positive power utility function U with relative risk-aversion $a \in (0,1)$ we can select $b = a$ and then obtain same estimate as in Proposition II.3.3:

$$p' = 1 + \frac{a}{1-a} = \frac{1}{1-a} = p.$$

The proof of Theorem II.3.4 relies on the following lemma.

Lemma II.3.5. *Assume* (II.2.1) *and suppose that there are constants* $0 < a < 1$ *and* $C_1 > 0$ *such that*

$$\frac{1}{C_1}\left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)}, \quad x \leq y, \tag{II.3.3}$$

and there is a supermartingale $Z \in \mathcal{Y}$ *satisfying* (A_p) *with*

$$p = \frac{1}{1-a}.$$

Then for every $y > 0$ *the solution* $Y(y)$ *to* (II.2.3) *exists, and there is a constant* $C_2 > 0$ *such that for every stopping time* τ *and every* $y > 0$,

$$\mathbb{E}\left[I(Y_T(y))Y_T(y) \middle| \mathcal{F}_\tau\right] \leq C_2 I(Y_\tau(y))Y_\tau(y), \tag{II.3.4}$$

where $I = -V'$.

Remark II.3.2. Recall that for $x = -v'(y)$ the optimal wealth process $X(x)$ has the terminal value

$$X_T(x) = -V'(Y_T(y)) = I(Y_T(y))$$

and the product $X(x)Y(y)$ is a uniformly integrable martingale. It follows that for every stopping time τ

$$X_\tau(x) = \frac{1}{Y_\tau(y)}\mathbb{E}\left[I(Y_T(y))Y_T(y) \middle| \mathcal{F}_\tau\right]$$

and therefore, inequality (II.3.4) is equivalent to

$$X_\tau(x) \leq C_2 I(Y_\tau(y)).$$

Proof of Lemma II.3.5. To show the existence of $Y(y)$ we need to verify (II.2.4). As $I = -V'$ is the inverse function to U', condition (II.3.3) is equivalent to

$$\frac{I(x)}{I(y)} \leq C_3 \left(\frac{y}{x}\right)^{1/a}, \quad x \leq y, \tag{II.3.5}$$

where $C_3 = C_1^{1/a}$. From (II.3.5) we deduce that for $y \leq 1$

$$\begin{aligned} V(y) &= V(1) + \int_y^1 I(t)dt \leq V(1) + C_3 I(1) \int_y^1 t^{-1/a} dt \\ &= V(1) + C_3 I(1) \frac{a}{1-a} (y^{-\frac{1-a}{a}} - 1) \\ &= V(1) + C_3 I(1)(p-1)(y^{-\frac{1}{p-1}} - 1). \end{aligned}$$

Hence, there is a constant $C_4 > 0$ such that

$$V(y) \leq C_4(1 + y^{-\frac{1}{p-1}}), \quad y > 0.$$

As Z satisfies (A_p), we have

$$\mathbb{E}\left[Z_T^{-\frac{1}{p-1}}\right] < \infty.$$

It follows that

$$v(y) \leq \mathbb{E}\left[V(yZ_T/Z_0)\right] < \infty, \quad y > 0,$$

which completes the proof of the existence of $Y(y)$.

Let τ be a stopping time and let $y > 0$. We set $\widehat{Y} \triangleq Y(y)$ and define the process

$$Y_t \triangleq \widehat{Y}_t 1_{\{t \leq \tau\}} + \widehat{Y}_\tau \frac{Z_t}{Z_\tau} 1_{\{t > \tau\}}, \quad t \in [0, T].$$

Clearly, $Y \in \mathcal{Y}$ and $Y_0 = \widehat{Y}_0 = y$. We represent

$$I(\widehat{Y}_T)\widehat{Y}_T = \xi_1 + \xi_2 + \xi_3,$$

by multiplying the left-side on the elements of the unity decomposition:

$$1 = 1_{\left\{\widehat{Y}_\tau \leq \widehat{Y}_T\right\}} + 1_{\left\{Y_T \leq \widehat{Y}_T < \widehat{Y}_\tau\right\}} + 1_{\left\{\widehat{Y}_T < Y_T, \widehat{Y}_T < \widehat{Y}_\tau\right\}}.$$

For the first term, since $I = -V'$ is a decreasing function, we have that

$$\xi_1 = I(\widehat{Y}_T)\widehat{Y}_T 1_{\left\{\widehat{Y}_\tau \leq \widehat{Y}_T\right\}} \leq I(\widehat{Y}_\tau)\widehat{Y}_T.$$

Using the supermartingale property of $\widehat{Y}$, we obtain that

$$\mathbb{E}\left[\xi_1 \middle| \mathcal{F}_\tau\right] \le I(\widehat{Y}_\tau)\widehat{Y}_\tau.$$

For the second term, we deduce from (II.3.5) that

$$\begin{aligned}\xi_2 &= I(\widehat{Y}_T)\widehat{Y}_T 1_{\left\{Y_T \le \widehat{Y}_T < \widehat{Y}_\tau\right\}} = I(\widehat{Y}_T)\widehat{Y}_T^{\frac{1}{a}}\widehat{Y}_T^{-\frac{1-a}{a}} 1_{\left\{Y_T \le \widehat{Y}_T < \widehat{Y}_\tau\right\}} \\ &\le C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} 1_{\{\tau<T\}} \le C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T}\right)^{\frac{1-a}{a}} \\ &= C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{p-1}}\end{aligned}$$

and the (A_p) condition for Z yields the existence of a constant $C_5 > 0$ such that

$$\mathbb{E}\left[\xi_2 \middle| \mathcal{F}_\tau\right] \le C_5 I(\widehat{Y}_\tau)\widehat{Y}_\tau.$$

For the third term, we deduce from (II.3.5) that

$$\begin{aligned}\xi_3 &= I(\widehat{Y}_T)\widehat{Y}_T 1_{\left\{\widehat{Y}_T < Y_T, \widehat{Y}_T < \widehat{Y}_\tau\right\}} \le I(\widehat{Y}_T)\widehat{Y}_T 1_{\left\{\widehat{Y}_T < Y_T\right\}} \\ &= I(\widehat{Y}_T)^a \widehat{Y}_T I(\widehat{Y}_T)^{1-a} 1_{\left\{\widehat{Y}_T < Y_T\right\}} \le C_1 I(Y_T)^a Y_T I(\widehat{Y}_T)^{1-a} \\ &= C_1 (I(Y_T)Y_T)^a (I(\widehat{Y}_T)Y_T)^{1-a}\end{aligned}$$

and then from Hölder's inequality that

$$\mathbb{E}\left[\xi_3 \middle| \mathcal{F}_\tau\right] \le C_1 (\mathbb{E}\left[I(Y_T)Y_T \middle| \mathcal{F}_\tau\right])^a \left(\mathbb{E}\left[I(\widehat{Y}_T)Y_T \middle| \mathcal{F}_\tau\right]\right)^{1-a}.$$

We recall that the terminal wealth of the optimal investment strategy with $\widehat{X}_0 = -v'(y)$ is given by

$$I(\widehat{Y}_T) = \widehat{X}_T.$$

It follows that

$$\begin{aligned}\mathbb{E}\left[I(\widehat{Y}_T)Y_T \middle| \mathcal{F}_\tau\right] &= \mathbb{E}\left[\widehat{X}_T Y_T \middle| \mathcal{F}_\tau\right] \le \widehat{X}_\tau Y_\tau = \widehat{X}_\tau \widehat{Y}_\tau \\ &= \mathbb{E}\left[\widehat{X}_T \widehat{Y}_T \middle| \mathcal{F}_\tau\right] = \mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau\right].\end{aligned}$$

To estimate $\mathbb{E}\left[I(Y_T)Y_T \middle| \mathcal{F}_\tau\right]$ we decompose

$$I(Y_T)Y_T = I(Y_T)Y_T 1_{\left\{\widehat{Y}_\tau \le Y_T\right\}} + I(Y_T)Y_T 1_{\left\{\widehat{Y}_\tau > Y_T\right\}}.$$

Since I is decreasing, we have that

$$I(Y_T)Y_T 1_{\left\{\widehat{Y}_\tau \le Y_T\right\}} \le I(\widehat{Y}_\tau)Y_T.$$

As Y is a supermartingale and $Y_\tau = \widehat{Y}_\tau$, we obtain that

$$\mathbb{E}\left[I(Y_T)Y_T 1_{\{\widehat{Y}_\tau \leq Y_T\}} \middle| \mathcal{F}_\tau \right] \leq I(\widehat{Y}_\tau)Y_\tau = I(\widehat{Y}_\tau)\widehat{Y}_\tau.$$

For the second term, using (II.3.5) we deduce that

$$\begin{aligned} I(Y_T)Y_T 1_{\{\widehat{Y}_\tau > Y_T\}} &= I(Y_T)Y_T^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} 1_{\{\widehat{Y}_\tau > Y_T\}} \leq C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau^{\frac{1}{a}} Y_T^{-\frac{1-a}{a}} \\ &= C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau \left(\frac{\widehat{Y}_\tau}{Y_T} \right)^{\frac{1-a}{a}} = C_3 I(\widehat{Y}_\tau)\widehat{Y}_\tau \left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \end{aligned}$$

and the (A_p) condition for Z implies that

$$\mathbb{E}\left[I(Y_T)Y_T 1_{\{\widehat{Y}_\tau > Y_T\}} \middle| \mathcal{F}_\tau \right] \leq C_5 I(\widehat{Y}_\tau)\widehat{Y}_\tau.$$

Thus we have

$$\mathbb{E}\left[I(Y_T)Y_T \middle| \mathcal{F}_\tau \right] \leq \eta \triangleq (1 + C_5) I(\widehat{Y}_\tau)\widehat{Y}_\tau$$

and then

$$\mathbb{E}\left[\xi_3 \middle| \mathcal{F}_\tau \right] \leq C_1 \eta^a \left(\mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau \right] \right)^{1-a}.$$

Adding together the estimates for $\mathbb{E}\left[\xi_i \middle| \mathcal{F}_\tau \right]$ we obtain that

$$\mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau \right] \leq \eta + C_1 \eta^a \left(\mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau \right] \right)^{1-a}$$

or equivalently that

$$\zeta \triangleq \frac{1}{\eta} \mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau \right] \leq 1 + C_1 \zeta^{1-a}.$$

Observe now that

$$0 \leq x \leq 1 + C_1 x^{1-a} \quad \text{if and only if} \quad 0 \leq x \leq x^*,$$

where x^* is the only root of

$$x = 1 + C_1 x^{1-a}, \quad x > 0.$$

It follows that $\zeta \leq x^*$ and

$$\mathbb{E}\left[I(\widehat{Y}_T)\widehat{Y}_T \middle| \mathcal{F}_\tau \right] = \zeta\eta \leq x^*\eta = x^*(1 + C_5) I(\widehat{Y}_\tau)\widehat{Y}_\tau,$$

We thus have proved inequality (II.3.4) with $C_2 = (1 + C_5)x^*$. □

Proof of Theorem II.3.4. Fix $y > 0$. In view of Lemma II.3.5, we only have to verify that $\widehat{Y} \triangleq Y(y)$ satisfies $(A_{p'})$.

Denote $\widehat{X} \triangleq X(-v'(y))$ and recall that by Lemma II.3.5 and Remark II.3.2, there is $C_2 > 0$ such that, for every stopping time τ,

$$\widehat{X}_\tau \leq C_2 I(\widehat{Y}_\tau).$$

Observe also that as $I = -V'$ is the inverse function to U', the second inequality in (II.3.2) is equivalent to

$$\frac{y}{x} \leq C \left(\frac{I(x)}{I(y)} \right)^b, \quad x \leq y.$$

We fix a stopping time τ. Since $I(\widehat{Y}_T) = \widehat{X}_T$, we deduce from the inequalities above that

$$\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{1/b} \leq \max \left(1, C^{1/b} \frac{I(\widehat{Y}_T)}{I(\widehat{Y}_\tau)} \right) \leq \max \left(1, C_3 \frac{\widehat{X}_T}{\widehat{X}_\tau} \right),$$

where $C_3 = C^{1/b} C_2$. It follows that

$$\begin{aligned} \left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p'-1}} = \left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1-a}{b}} &\leq \max \left(1, C_3^{1-a} \left(\frac{\widehat{X}_T}{\widehat{X}_\tau} \right)^{1-a} \right) \\ &\leq 1 + C_3^{1-a} \left(\frac{\widehat{X}_T Z_T}{\widehat{X}_\tau Z_\tau} \right)^{1-a} \left(\frac{Z_\tau}{Z_T} \right)^{1-a}. \end{aligned}$$

Denoting by $C_1 > 0$ the constant in the (A_p) condition for Z, we deduce from Hölder's inequality and the supermartingale property of $\widehat{X}Z$ that

$$\begin{aligned} \mathbb{E} \left[\left(\frac{\widehat{Y}_\tau}{\widehat{Y}_T} \right)^{\frac{1}{p'-1}} \middle| \mathcal{F}_\tau \right] &\leq 1 + C_3^{1-a} \left(\mathbb{E} \left[\frac{\widehat{X}_T Z_T}{\widehat{X}_\tau Z_\tau} \middle| \mathcal{F}_\tau \right] \right)^{1-a} \left(\mathbb{E} \left[\left(\frac{Z_\tau}{Z_T} \right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] \right)^a \\ &\leq 1 + C_3^{1-a} C_1^a. \end{aligned}$$

Hence, $\widehat{Y}$ satisfies $(A_{p'})$. □

II.4 Local martingale property for maximal elements of $\mathcal{Y}$

Even if the dual minimizer $Y(y)$ is uniformly integrable, it may not be a martingale, due to the lack of the local martingale property; see the single-period example for logarithmic utility in [37, Example 5.1′]. Proposition II.4.2

below yields sufficient conditions for every *maximal* element of $\mathcal{Y}$ (in particular, for $Y(y)$) to be a local martingale.

A semimartingale R is called *σ-bounded* if there is a predictable process $h > 0$ such that the stochastic integral $\int hdR$ is bounded. Following [39], we make the following assumption.

Assumption II.4.1. For all X and X' in $\mathcal{X}$ such that $X > 0$, the process X'/X is σ-bounded.

Assumption II.4.1 holds easily if every $X \in \mathcal{X}$ is continuous. Theorem 3 in Appendix of [39] provides a sufficient condition in the presence of jumps. It states that *every* semimartingale R is σ-bounded if there is a finite-dimensional local martingale M such that every bounded purely discontinuous martingale N is a stochastic integral with respect to M. In particular, as Proposition 2 in the Appendix of [39] shows, every semimartingale R is σ-bounded if the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ allows for the existence of a finite-dimensional stock price process S' such that the S'-market is complete.

Proposition II.4.2. *Suppose that Assumption II.4.1 holds. Let $Y \in \mathcal{Y}$ be such that YX' is a local martingale for some $X' \in \mathcal{X}$, $X' > 0$. Then YX is a local martingale for every $X \in \mathcal{X}$. In particular, Y is a local martingale.*

Proof. We assume first that $X' = Y = 1$. Let $X \in \mathcal{X}$. As X is σ-bounded, there is a predictable $h > 0$ such that

$$\left| \int hdX \right| \leq 1.$$

Since the bounded non-negative processes $1 \pm \int hdX$ belong to $\mathcal{X}$, they are supermartingales, which is only possible if $\int hdX$ is a martingale. It follows that X is a non-negative stochastic integral with respect to a martingale:

$$X = X_0 + \int \frac{1}{h} d(\int hdX) \geq 0.$$

Therefore, X is a local martingale, see [3]. Under the condition $X' = Y = 1$, the proof is obtained.

We now consider the general case. Without loss of generality, we can assume that $X'_0 = Y_0 = 1$. By localization, we can also assume that the local martingale YX' is uniformly integrable and then define a probability measure $\mathbb{Q}$ with the density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = X'_T Y_T.$$

Let $X \in \mathcal{X}$. We have that XY is a local martingale under $\mathbb{P}$ if and only if X/X' is a local martingale under $\mathbb{Q}$.

By Assumption II.4.1, the process X/X' is σ-bounded. Elementary computations show that X/X' is a wealth process in the financial market with stock price

$$S' = \left(\frac{1}{X'}, \frac{S}{X'}\right);$$

see [15]. The result now follows by applying the previous argument to the S'-market whose reference probability measure is given by $\mathbb{Q}$. □

II.5 Existence of the optimal martingale measure

Recall that $X(x)$ denotes the optimal wealth process for the primal problem (II.2.2), while $Y(y)$ stands for the minimizer to the dual problem (II.2.3). As usual, the *density process* of a probability measure $\mathbb{R} \ll \mathbb{P}$ is a uniformly integrable martingale (under $\mathbb{P}$) with the terminal value $\frac{d\mathbb{R}}{d\mathbb{P}}$.

The following is the main result of the paper.

Theorem II.5.1. *Let Assumption II.4.1 hold. Suppose that there are constants $0 < a < 1$, $b \geq a$ and $C > 0$ such that*

$$\frac{1}{C}\left(\frac{y}{x}\right)^a \leq \frac{U'(x)}{U'(y)} \leq C\left(\frac{y}{x}\right)^b, \quad x \leq y, \tag{II.5.1}$$

and there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ whose density process Z satisfies (A_p) with

$$p = \frac{1}{1-a}. \tag{II.5.2}$$

Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process $Y(y)/y$ satisfies $(A_{p'})$ with

$$p' = 1 + \frac{b}{1-a}.$$

Proof. From Theorem II.3.4 we obtain that the dual minimizer $Y(y)$ exists and satisfies $(A_{p'})$ and then from Lemma II.3.2 that it is of class $(\mathbf{D})$. The local martingale property of $Y(y)$ follows from Proposition II.4.2, if we account for Assumption II.4.1 and the martingale property of $X(-v'(y))Y(y)$. Thus, $Y(y)$ is a uniformly integrable martingale and hence, $Y(y)/y$ is the density process of the optimal martingale measure $\mathbb{Q}(y)$. □

We refer the reader to Remark II.3.1 for a discussion of the conditions of Theorem II.5.1.

Example II.5.2. In a typical situation, the role of the "testing" martingale measure $\mathbb{Q}$ is played by the *minimal* martingale measure, that is, by the optimal martingale measure for logarithmic utility. For a model of stock prices driven by a Brownian motion, its density process Z has the form:

$$Z_t = \mathcal{E}\left(-\lambda \cdot B\right)_t := \exp\left(-\int_0^t \lambda dB - \frac{1}{2}\int_0^t \left|\lambda_s\right|^2 ds\right), \quad t \in [0,T],$$

where B is an N-dimensional Brownian motion and λ is a predictable N-dimensional process of the *market price of risk*. We readily deduce that Z satisfies (A_p) for all $p > 1$ if both λ and the maturity T are bounded. This fact implies the assertions of Theorem II.5.1, provided that inequalities (II.5.1) hold for some $a \in (0,1)$, $b \geq a$ and $C > 0$ or, in particular, if the relative risk-aversion of U is bounded away from 0 and ∞.

The following result shows that the key bound (II.5.2) is the best possible.

Theorem II.5.3. *Let constants a and p be such that*

$$0 < a < 1 \quad \textit{and} \quad p > \frac{1}{1-a}.$$

Then there exists a financial market with a continuous stock price S such that

1. *There is a $\mathbb{Q} \in \mathcal{Q}$ whose density process Z satisfies (A_p).*

2. *In the optimal investment problem with the power utility function*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0,$$

   the dual minimizers $Y(y) = y\widehat{Y}$, $y > 0$, are well-defined, but are not uniformly integrable martingales. In particular, the optimal martingale measure $\widehat{\mathbb{Q}} = \mathbb{Q}(y)$ does not exist.

The proof of Theorem II.5.3 follows from Proposition II.6.1 below, which contains an exact counterexample.

We conclude the section with a couple of corollaries of Theorem II.5.1 which exploit connections between the (A_p) condition and BMO martingales. Hereafter, we shall refer to [34] and therefore, restrict ourselves to the continuous case.

Assumption II.5.4. All local martingales on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ are continuous.

From Assumption II.5.4 we deduce that for every $\mathbb{Q} \in \mathcal{Q}$ the density process Z is a continuous uniformly integrable martingale and that the dual minimizer $Y(y)$ is a continuous local martingale. We also obtain that for every $X \in \mathcal{X}$ the local martingale ZX is continuous. In particular, the wealth process X is continuous and therefore, Assumption II.4.1 holds.

We recall that a continuous local martingale M with $M_0 = 0$ belongs to BMO if there is a constant $C > 0$ such that

$$\mathbb{E}\left[\langle M\rangle_T - \langle M\rangle_\tau \middle| \mathcal{F}_\tau\right] \leq C \quad \text{for every stopping time } \tau, \tag{II.5.3}$$

where $\langle M\rangle$ is the quadratic variation process for M. It is known that BMO is a Banach space with the norm

$$\|M\|_{\mathrm{BMO}} \triangleq \inf\left\{\sqrt{C} > 0 : \text{ (II.5.3) holds for } C > 0\right\}.$$

We also recall that for a continuous local martingale M with $M_0 = 0$,

1. The stochastic exponential $\mathcal{E}(M) \triangleq e^{M-\langle M\rangle/2}$ satisfies (A_p) for *some* $p > 1$ if and only if $M \in \mathrm{BMO}$; see Theorem 2.4 in [34].

2. The stochastic exponentials $\mathcal{E}(M)$ and $\mathcal{E}(-M)$ satisfy (A_p) for *all* $p > 1$ if and only the martingale

$$q(M)_t \triangleq \mathbb{E}\left[\langle M\rangle_T \middle| \mathcal{F}_t\right] - \mathbb{E}\left[\langle M\rangle_T\right], \quad t \in [0, T], \tag{II.5.4}$$

is well-defined and belongs to the closure in $\|\cdot\|_{\mathrm{BMO}}$ of the space of bounded martingales; see Theorem 3.12 in [34].

Corollary II.5.5. *Let Assumption II.5.4 hold. Suppose that there are constants $b \geq 1$ and $C > 0$ such that*

$$\frac{1}{C}\left(\frac{y}{x}\right) \leq \frac{U'(x)}{U'(y)} \leq C\left(\frac{y}{x}\right)^b, \quad x \leq y, \tag{II.5.5}$$

and there is a martingale measure $\mathbb{Q} \in \mathcal{Q}$ with density process $Z = \mathcal{E}(M)$ with $M \in \mathrm{BMO}$. Then for every $y > 0$ the optimal martingale measure $\mathbb{Q}(y)$ exists and its density process is given by $Y(y)/y = \mathcal{E}(M(y))$ with $M(y) \in \mathrm{BMO}$.

Proof. From 1 we deduce that Z satisfies (A_p) for some $p > 1$. Clearly, (II.5.5) implies (II.5.1) for every $a \in (0, 1)$ and in particularly for a satisfying (II.5.2). Theorem II.5.1 then implies that $Y(y)/y$ satisfies $(A_{p'})$ for some $p' > 1$ and another application of 1 yields the result. □

We notice that by 1 and Theorem II.5.3 the power 1 in the first inequality of (II.5.5) cannot be replaced with any $a \in (0, 1)$, in order to guarantee that the optimal martingale measure $\mathbb{Q}(y)$ exists.

Corollary II.5.6. *Let Assumption II.5.4 hold and let inequality* (II.5.1) *be satisfied for some constants* $0 < a < 1$*,* $b \geq a$ *and* $C > 0$*. Suppose also that there is a martingale measure* $\mathbb{Q} \in \mathcal{Q}$ *whose density process* $Z = \mathcal{E}(M)$ *is such that the martingale* $q(M)$ *in* (II.5.4) *is well-defined and belongs to the closure in* $\|\cdot\|_{\mathrm{BMO}}$ *of the space of bounded martingales. Then for every* $y > 0$ *the optimal martingale measure* $\mathbb{Q}(y)$ *exists and its density process is given by* $Y(y)/y = \mathcal{E}(M(y))$ *with* $M(y) \in \mathrm{BMO}$*.*

Proof. The result follows directly from 2 and Theorem II.5.1. □

II.6 Counterexample

In this section we construct an example of financial market satisfying the conditions of Theorem II.5.3. For a semimartingale R, we denote by $\mathcal{E}(R)$ its stochastic exponential, that is, the solution of the linear equation:

$$d\mathcal{E}(R) = \mathcal{E}(R)_- dR, \quad \mathcal{E}(R)_0 = 1.$$

We start with an auxiliary filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, which supports a Brownian motion $B = (B_t)$ and a counting process $N = (N_t)$ with the stochastic intensity $\lambda = (\lambda_t)$ given in (II.6.3) below; $B_0 = N_0 = 0$. We define the process

$$S_t \triangleq \mathcal{E}(B)_t = e^{B_t - t/2}, \quad t \geq 0,$$

and the stopping times

$$\begin{aligned} T_1 &\triangleq \inf\{t \geq 0 : \ S_t = 2\}, \\ T_2 &\triangleq \inf\{t \geq 0 : \ N_t = 1\}, \\ T &\triangleq T_1 \wedge T_2 = \min(T_1, T_2). \end{aligned}$$

We fix constants a and p such that

$$0 < a < 1 \quad \text{and} \quad p > \frac{1}{1-a} \tag{II.6.1}$$

and choose a constant b such that

$$a < b < \frac{1}{q} \quad \text{and} \quad \gamma \leq \frac{1}{2}\delta(1-\delta), \tag{II.6.2}$$

where

$$\begin{aligned} q &\triangleq \frac{p}{p-1} < \frac{1}{a}, \\ \delta &\triangleq b - a > 0, \\ \gamma &\triangleq \frac{b}{2}(1 - qb) > 0. \end{aligned}$$

With this notation, we define the stochastic intensity $\lambda = (\lambda_t)$ as

$$\lambda_t \triangleq \frac{\gamma}{1 - (S_t/2)^\delta} 1_{\{t < T_1\}} + \gamma 1_{\{t \geq T_1\}}, \quad t \geq 0. \tag{II.6.3}$$

Recall that $N - \int \lambda dt$ is a local martingale under $\mathbb{Q}$.

Finally, we introduce a probability measure $\mathbb{P} \ll \mathbb{Q}$ with the density

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \frac{1}{\mathbb{E}^{\mathbb{Q}}\left[S_T^b\right]} S_T^b.$$

Notice that

$$\left\{\frac{d\mathbb{P}}{d\mathbb{Q}} = 0\right\} = \{S_T = 0\} = \{\mathcal{E}(B)_T = 0\} = \{T = \infty\} \tag{II.6.4}$$

and therefore, the stopping time T is finite under $\mathbb{P}$:

$$\mathbb{P}(T < \infty) = 1.$$

Proposition II.6.1. *Assume* (II.6.1) *and* (II.6.2) *and consider the financial market with the price process S and the maturity T defined on the filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Then*

1. *The probability measure $\mathbb{Q}$ belongs to $\mathcal{Q}$ and the density process Z of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies (A_p).*
2. *In the optimal investment problem with the power utility function*

$$U(x) = \frac{x^{1-a}}{1-a}, \quad x > 0, \tag{II.6.5}$$

the dual minimizers $Y(y) = y\widehat{Y}$, $y > 0$, are well-defined but are not uniformly integrable martingales. In particular, the optimal martingale measure $\widehat{\mathbb{Q}} = \mathbb{Q}(y)$ does not exist.

The proof is divided into a series of lemmas.

Lemma II.6.2. *The stopping time T is finite under $\mathbb{Q}$ and the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.*

Proof. In view of (II.6.4), we only have to show that

$$\mathbb{Q}(T < \infty) = 1.$$

Indeed, by (II.6.3), the intensity λ is bounded below by $\gamma > 0$ and hence,

$$\mathbb{Q}(T > t) \leq \mathbb{Q}(T_2 > t) \leq e^{-\gamma t} \to 0, \quad t \to \infty.$$

□

From the construction of the model and Lemma II.6.2 we deduce that $\mathbb{Q} \in \mathcal{Q}$. To show that the density process Z of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies (A_p) we need the following estimate.

Lemma II.6.3. *Let $0 < \epsilon < 1$ be a constant and τ be a stopping time. Then*

$$\mathbb{E}^{\mathbb{Q}}\left[S_T^{\epsilon} \middle| \mathcal{F}_\tau \right] \leq S_\tau^{\epsilon} \leq \left(1 + \frac{\epsilon(1-\epsilon)}{2\gamma}\right) \mathbb{E}^{\mathbb{Q}}\left[S_T^{\epsilon} \middle| \mathcal{F}_\tau \right].$$

Proof. We denote

$$\theta = \frac{1}{2}\epsilon(1-\epsilon)$$

and deduce that

$$S_t^{\epsilon} = \mathcal{E}(B)_t^{\epsilon} = \mathcal{E}(\epsilon B)_t e^{-\theta t}, \quad t \in [0, T].$$

In particular, S^ϵ is a $\mathbb{Q}$-supermartingale, and the first inequality in the statement of the lemma follows.

To verify the second inequality, we define local martingales L and M under $\mathbb{Q}$ as

$$\begin{aligned} L_t &= \int_0^t \frac{\theta}{\lambda_r}(dN_r - \lambda_r dr), \\ M_t &= \mathcal{E}(\epsilon B)_t \mathcal{E}(L)_t, \end{aligned}$$

and observe that

$$\begin{aligned} M_t &= S_t^{\epsilon}, \quad t \leq T,\ t < T_2, \\ M_T &= \left(1 + \frac{\theta}{\lambda_T}\right) S_T^{\epsilon}, \quad T = T_2. \end{aligned}$$

Since $\lambda \geq \gamma$, we obtain that

$$S_t^{\epsilon} \leq M_t \leq \left(1 + \frac{\theta}{\gamma}\right) S_t^{\epsilon}, \quad t \in [0, T].$$

As $S \leq 2$, we deduce that M is a bounded $\mathbb{Q}$-martingale and the result readily follows. $\square$

Lemma II.6.4. *The density process Z of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies (A_p).*

Proof. Fix a stopping time τ. As $\mathbb{Q} \sim \mathbb{P}$, we have

$$\begin{aligned} \mathbb{E}\left[\left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] &= \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{Z_\tau}{Z_T}\right)^{1+\frac{1}{p-1}} \middle| \mathcal{F}_\tau \right] = \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{Z_\tau}{Z_T}\right)^{q} \middle| \mathcal{F}_\tau \right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\widetilde{Z}_T}{\widetilde{Z}_\tau}\right)^{q} \middle| \mathcal{F}_\tau \right], \end{aligned}$$

where $\widetilde{Z} = 1/Z$ is the density process of $\mathbb{P}$ with respect to $\mathbb{Q}$.

Recall that

$$\widetilde{Z}_T = C S_T^b,$$

for some constant $C > 0$. Since $0 < b < bq < 1$, Lemma II.6.3 yields that

$$\begin{aligned}
\widetilde{Z}_\tau = \mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T \middle| \mathcal{F}_\tau\right] &= C\mathbb{E}^{\mathbb{Q}}\left[S_T^b \middle| \mathcal{F}_\tau\right] \geq C\left(1 + \frac{b(1-b)}{2\gamma}\right)^{-1} S_\tau^b, \\
\mathbb{E}^{\mathbb{Q}}\left[\widetilde{Z}_T^q \middle| \mathcal{F}_\tau\right] &= C^q\mathbb{E}^{\mathbb{Q}}\left[S_T^{qb} \middle| \mathcal{F}_\tau\right] \leq C^q S_\tau^{qb},
\end{aligned}$$

which implies the result. □

We now turn our attention to the second item of Proposition II.6.1. Of course, our financial market has been specially constructed in such a way that the solutions $X(x)$ and $Y(y)$ to the primal and dual problems are quite explicit.

Lemma II.6.5. *In the optimal investment problem with the utility function U from* (II.6.5)*, it is optimal to buy and hold stocks:*

$$X(x) = xS, \quad x > 0.$$

The dual minimizers have the form $Y(y) = y\widehat{Y}$, $y > 0$, with

$$\widehat{Y} = \mathcal{E}(L)Z, \tag{II.6.6}$$

where Z is the density process of $\mathbb{Q}$ with respect to $\mathbb{P}$ and

$$L_t = \int_0^t \frac{\gamma}{\lambda_r}(\lambda_r dr - dN_r), \quad t \in [0, T]. \tag{II.6.7}$$

Proof. We verify the conditions of Lemma II.2.1. For the stochastic exponential $\mathcal{E}(L)$ we obtain that

$$\mathcal{E}(L)_t = e^{\gamma t}, \quad t < T,$$

and, as $S_{T_1} = 2$, that

$$\begin{aligned}
\mathcal{E}(L)_T &= e^{\gamma T}\left(1_{\{T=T_1\}} + \left(1 - \frac{\gamma}{\lambda_T}\right)1_{\{T=T_2\}}\right) \\
&= e^{\gamma T}\left(1_{\{T=T_1\}} + \left(\frac{S_T}{2}\right)^\delta 1_{\{T=T_2\}}\right) \\
&= e^{\gamma T}\left(\frac{S_T}{2}\right)^\delta.
\end{aligned}$$

Hence for $\widehat{Y}$ defined by (II.6.6) we have

$$\widehat{Y}_T = \mathcal{E}(L)_T Z_T = C S_T^{-a} = C U'(S_T),$$

for some constant $C > 0$.

Let $X \in \mathcal{X}$. Under $\mathbb{Q}$, the product $X\mathcal{E}(L)$ is a local martingale, because X is a stochastic integral with respect to the Brownian motion B and $\mathcal{E}(L)$ is a purely discontinuous local martingale. It follows that $X\widehat{Y} = X\mathcal{E}(L)Z$ is a non-negative local martingale (hence, a supermartingale) under $\mathbb{P}$. Thus,

$$\widehat{Y} \in \mathcal{Y}.$$

Observe that the convex conjugate to U is given by

$$V(y) = \frac{a}{1-a} y^{-\frac{1-a}{a}}, \quad y > 0.$$

It follows that

$$V(y\widehat{Y}_T) = V(y)\widehat{Y}_T \widehat{Y}_T^{-1/a} = V(y) C^{-1/a} \widehat{Y}_T S_T$$

and therefore,

$$\mathbb{E}\left[V(y\widehat{Y}_T)\right] \le V(y) C^{-1/a} < \infty, \quad y > 0.$$

To conclude the proof we only have to show that the local martingale $S\widehat{Y} = S\mathcal{E}(L)Z$ under $\mathbb{P}$ is of class (**D**) or, equivalently, that the local martingale $S\mathcal{E}(L)$ under $\mathbb{Q}$ is of class (**D**). Actually, we have a stronger property:

$$\{S_\tau \mathcal{E}(L)_\tau : \ \tau \text{ is a stopping time}\} \text{ is bounded in } \mathbf{L}^q(\mathbb{Q}).$$

Indeed,

$$S_t \mathcal{E}(L)_t \le S_t e^{\gamma t} \le 2^{1-b} S_t^b e^{\gamma t}, \quad t \in [0, T],$$

and then for a stopping time τ,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[(S_\tau \mathcal{E}(L)_\tau)^q\right] &\le 2^{q(1-b)} \mathbb{E}^{\mathbb{Q}}\left[\left(S_\tau^b e^{\gamma\tau}\right)^q\right] = 2^{q(1-b)} \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(B)_\tau^{qb} e^{q\gamma\tau}\right] \\ &= 2^{q(1-b)} \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(qbB)_\tau\right] \le 2^{q(1-b)}. \end{aligned}$$

□

The following lemma completes the proof of the proposition.

Lemma II.6.6. *For the dual minimizer $\widehat{Y}$ constructed in Lemma II.6.5 we have*

$$\mathbb{E}\left[\widehat{Y}_T\right] < 1.$$

Thus, $\widehat{Y}$ is not a uniformly integrable martingale.

Proof. Recall from the proof of Lemma II.6.5 that for the local martingale L defined in (II.6.7),

$$\mathcal{E}(L)_T = e^{\gamma T} \left(\frac{S_T}{2} \right)^{\delta}.$$

Using (II.6.2), we deduce that

$$\mathcal{E}(L)_T = \frac{1}{2^\delta} e^{\gamma T} \left(\mathcal{E}(B)_T \right)^{\delta} = \frac{1}{2^\delta} e^{\gamma T} \mathcal{E}(\delta B)_T e^{-\frac{1}{2}\delta(1-\delta)T} \leq \frac{1}{2^\delta} \mathcal{E}(\delta B)_T.$$

It follows that

$$\mathbb{E}\left[\widehat{Y}_T \right] = \mathbb{E}\left[\mathcal{E}(L)_T Z_T \right] = \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(L)_T \right] \leq \frac{1}{2^\delta} \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}(\delta B)_T \right] \leq \frac{1}{2^\delta}.$$

$\square$

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