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Endogenous Mortgage Current Coupons

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Abstract

We consider the problem of identifying current coupons for agency-backed To-Be-Announced pools of residential mortgage loans. In a doubly stochastic model which allows for prepayment intensities to depend upon current and origination mortgage rates, as well as underlying investment factors, we identify the current coupon with solutions to a degenerate elliptic, non-linear fixed point problem. Using Schaefer's theorem we prove existence of current coupons. We also provide an explicit approximation to the fixed point, valid for compact perturbations off a baseline model where intensities only depend on the underlying factors. Numerical examples are provided which show that the approximation performs well in estimating the current coupon.

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Contents

1 Introduction **2**

1.1 An overview of mortgage-backed securities 2

1.2 Agency MBS and the current coupon 3

1.3 MBS valuation and the problem of prepayments 5

1.4 Review of existing MBS valuation models 7

1.5 Endogenous mortgage rates 13

1.6 Organization of the dissertation 16

2 Model And Problem Formulation **18**

2.1 A basic mortgage contract 18

2.2 Endogenous current coupons 19

2.3 The fixed point problem and the main result 20

3 Perturbation Analysis And Numerical Results **28**

3.1 Perturbation analysis 28

3.2 Numerical approximations 36

4 Proof Of The Main Theorem **40**

4.1 Outline of the proof 40

4.2 *A priori* estimates of Hölder norms 41

4.3 The localized problem 45

4.4 Global existence of a fixed point 62

5 Conclusion And Final Remarks **69**

5.1 Conclusion 69

5.2 Extensions and future work 70

A Appendix **77**

A.1 Auxiliary lemmas 77

A.2 A note on Goncharov's result 78

A.3 On the naive contraction method 78

List of Figures

1.1 Value of MBS issuances in USD billions from 1996 to 2014. (*source:SIFMA statistics, structured finance*) 4

1.2 U.S. bond markets average daily trading volume in USD billions from 2005 to 2015. (*source:SIFMA statistics, structured finance*) 4

1.3 CPR by month for fixed rate agency pools. (*source:www.embs.com)* 7

3.1 Current coupon functions (left plot) and errors (right plot) as a function of the underlying CIR factor. In the left plot, the thick-dash plot is the current coupon function m obtained through naive contraction. The solid line is the approximation $m_0 + m_1$ while the thin dash plot is m_0. Values are given in percentage points. For the right plot, the error is the difference (in basis points) between m and $m_0 + m_1$. Also in the right plot is the invariant pdf for the CIR process r. m_0 is calculated with $\gamma_0(x) = 0$ and m_1 is calculated with $\gamma_1(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*, *Mathematica*. 38

3.2 Current coupon functions (left plot) and errors (right plot) as a function of the underlying CIR factor. In the left plot, the thick-dash plot is the current coupon function m obtained through naive contraction. The solid line is the approximation $m_0 + m_1$ while the thin dash plot is m_0. Values are given in percentage points. For the right plot, the error is the difference (in basis points) between m and $m_0 + m_1$. Also in the right plot is the invariant pdf for the CIR process r. m_0 is calculated with $\gamma_0(x) = \gamma$ and m_1 is calculated with $\gamma_1(x, m, z) = k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*, *Mathematica*. 39

A.1 Current coupon functions as a function of the underlying CIR factor obtained by naive contraction and Monte Carlo method. The dotted line is the initial guess for m. The dash line is the m function after 2 iterations. The dash-dotted line is the m function after 5 iterations. The solid line is the m function after 10 iterations. Values are given in percentage points. The intensity function is $\gamma(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*. 79

A.2 Current coupon functions as a function of the underlying CIR factor obtained by solving the coupled system using naive contraction and finite difference method. The dotted line is the initial guess for m. The dash line is the m function after 2 iterations. The dash-dotted line is the m function after 5 iterations. The solid line is the m function after 10 iterations. Values are given in percentage points. The intensity function is $\gamma(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1, T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*. 83

Chapter 1

Introduction

1.1 An overview of mortgage-backed securities

The U.S. residential mortgage market is enormous with an estimated outstanding value of $11 trillion as of 2015 Q3, according to the mortgage debt outstanding release of the Federal Reserve Board*. The market consists of a *primary* mortgage market, where borrowers who seek to buy a house receive loans from lenders such as banks, credit unions and thrifts; and a *secondary* mortgage market, where mortgage buyers such as the government-sponsored enterprises (GSEs) as well as large financial institutions generate funds by grouping mortgages into securitized pools and selling them in the form of mortgage-backed securities (MBS). The secondary market (commonly referred to as the MBS market), which links homeowners, lenders and fixed income investors, is currently the second largest segment of the U.S. fixed income market (see [29]).

Generally speaking, a mortgage-backed security is a claim to the cash flows generated by a specific pool of mortgages. A mortgage pool is the aggregation of large numbers of mortgage loans with similar (but not identical) characteristics. Such characteristics may include note rate, term to maturity, loan balance, product type and borrower credit quality. Once a pool is created, it may be sold to investors in the form of a pass-through, in which principal and interest are paid to investors based on their *pro rata* share of the pool. Alternatively, the cash flows of the pool may be split to meet the requirements of different types of investors. For example, in collateralized mortgage obligations (CMOs), the underlying pool's cash flows are tranched, or divided, into securities that

*The release is available on FRB's website: *http://www.federalreserve.gov*.

have varying average lives and durations*, different degrees of prepayment† protection or exposure, and different degrees of credit risk. This allows a wide range of investors with different investment objectives and risk tolerances to invest in the MBS market, and at the same time supply the funds that are recycled into new mortgage originations.

Overall, with the creation of MBS, mortgage loans are transformed from a heterogeneous group of disparate assets into sizable and homogeneous securities that can be traded in a liquid market.

1.2 Agency MBS and the current coupon

Mortgage-backed securities can be classified as either *agency* MBS or *non-agency* MBS. Agency MBS are issued by a government-sponsored enterprise such as Fannie Mae or Freddie Mac, or guaranteed by the government agency Ginnie Mae. On the other hand, non-agency MBS are issued by private institutions. Loans that meet the guidelines of the three agencies are assigned an insurance premium, known as the guarantee fee, by the agency and securitized as an agency pool‡. Loans that either do not conform to the agency guidelines, or for which agency pooling execution is not efficient, can be securitized in non-agency, or "private-label" pools.

Agency MBS has been the major component of the MBS market since the financial crisis. In fact, issuance of agency MBS has remained robust since 2007, while mortgage securitization by private financial institutions has declined to very low levels, as shown in Figure 1.1 below. Agency MBS is also actively traded, with an average daily trading volume of about 200 billion USD in 2015, as shown in Figure 1.2 below.

A well-known feature of agency MBS is that each security carries either an explicit government credit guarantee, or is perceived to carry an implicit one. Ginnie Mae, for example, guarantees the timely payment of principal and interest on its mortgage securities, and is backed by the "full faith and credit" of the U.S. government. Thus, holders of Ginnie Mae MBS are assured of receiving payments promptly each month, regardless of whether the underlying homeowners make their payments. They are also guaranteed to receive the full return of remaining principal balance even if the underlying borrowers default on their loans. Fannie Mae and Freddie Mac also generally guarantee timely payments of both principal and interest on their mortgage securities whether or not the payments have been collected from the borrowers. Thus, agency MBS investors are protected from

*Duration is a measure of the sensitivity of the price of a fixed-income investment to a change in interest rates, usually expressed as a number of years.

†Prepayment is the early repayment of principal by the borrower. See Section 1.3 below for more details.

‡More details on agency pooling can be found in [27].

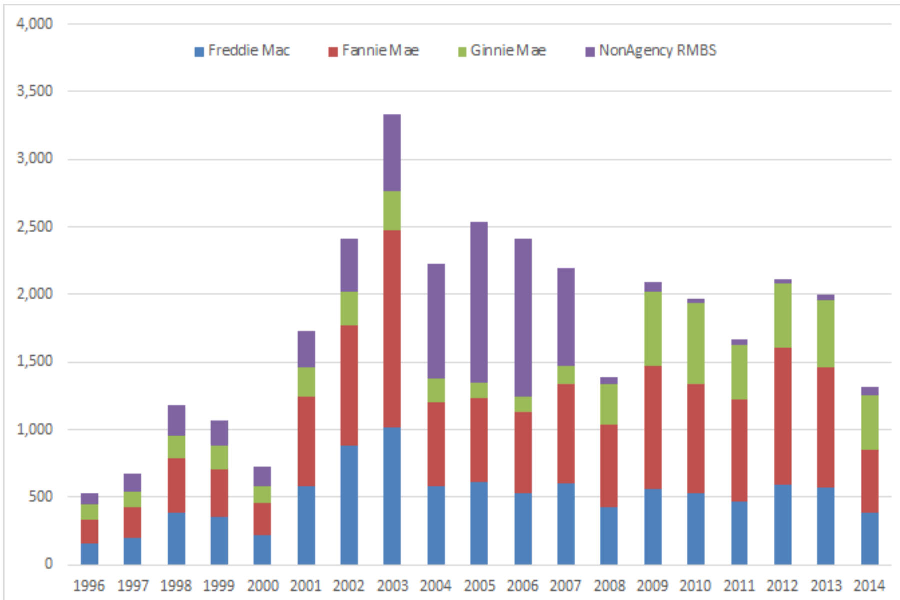
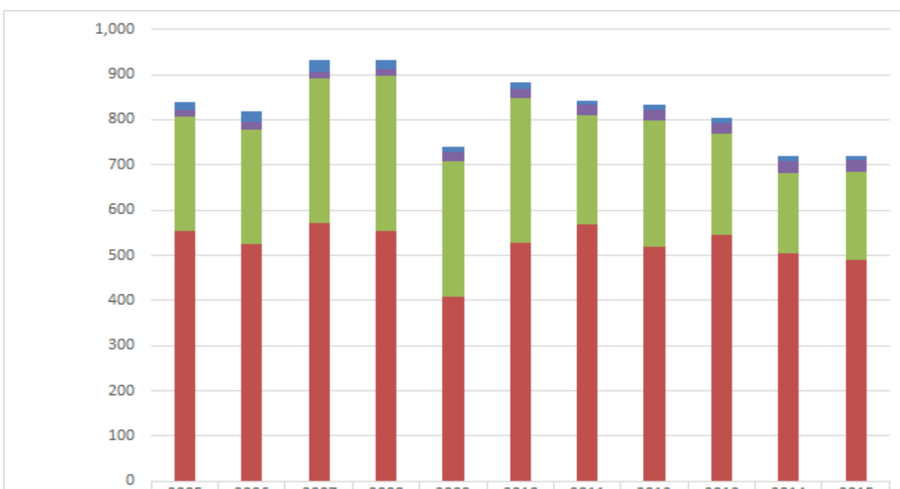


Figure 1.1: Value of MBS issuances in USD billions from 1996 to 2014. (*source:SIFMA statistics, structured finance*)



	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015
Municipal	16.9	23.1	25.1	19.4	12.5	13.3	11.3	11.3	11.2	9.9	8.6
Corporate	16.6	16.9	16.4	14.3	19.9	20.5	20.6	22.6	24.7	26.7	26.6
Agency MBS	251.8	254.6	320.1	344.9	299.9	320.6	243.3	280.4	222.8	178.0	194.4
Treasury	554.5	524.7	570.2	553.1	407.9	528.2	567.8	518.9	545.4	504.2	490.1

Figure 1.2: U.S. bond markets average daily trading volume in USD billions from 2005 to 2015. (*source:SIFMA statistics, structured finance*)

credit losses in case of mortgage borrower defaults, and as such, for valuation purposes, defaults appear to the pool holders nearly identical to prepayments.

Another less well-recognized feature of agency MBS is that more than 90 percent of agency MBS trading volume occurs in a liquid forward market, known as the TBA market (see [38]). The distinguishing feature of a TBA trade is that the actual identity of the securities to be delivered on the settlement date is not specified on the trade date. Instead, the buyer and the seller agree upon general parameters of the securities to be delivered, such as issuer, maturity, coupon, price, par amount and settlement date.

Closely related to TBA mortgage-backed securities is a secondary market rate known as the *current coupon*. The current coupon is a theoretical coupon rate, typically interpolated from the observed TBA prices, that makes the price of a TBA with current delivery month equal to par. As such, the current coupon is an endogenous rate, and it is widely used as a benchmark for MBS pool valuation, playing a key role in the secondary mortgage market. In addition, the current coupon is often used (typically adjusted by a primary-secondary rate spread[*]) to estimate the primary mortgage rates, as in general it is difficult to view the primary rates directly[†]. The goal of this dissertation is to show existence of endogenous current coupons in a MBS valuation model.

1.3 MBS valuation and the problem of prepayments

Valuing mortgage-backed securities is of practical and theoretical interest, and both market practitioners and academic researchers have developed MBS valuation models over the years (more details to follow in Section 1.4 below). The primary difficulty in valuing MBS is the fact that the home buyer has, at any time prior to maturity of the loan, the right to prepay all or part of her mortgage with few, if any, penalties. In particular, the mortgagor may refinance (multiple times) her loan in order to take advantage of current market conditions. Typically, homeowners prepay for the following reasons:

1. The sale of the property (the homeowner is moving, divorced, or dead).

2. The destruction of the property (e.g., by natural disaster).

[*]The primary-second mortgage rate spread is a closely tracked series. It was relatively stable from 1995 to 2000 at about 30 basis points; it subsequently widened to about 50 basis points through early 2008, but then reached more than 100 basis points in early 2009 and during 2012. More details can be found in [2].

[†]Primary mortgage rates are usually published as results of mortgage banking surveys. One popular source is the Freddie Mac Weekly Primary Mortgage Market Survey, or *PMMS*. They are not collected daily, and only few main loan types are covered.

3. The homeowner is refinancing to a lower rate.

4. The homeowner wants to take equity out of her home (known as cash-out refinance).

5. The homeowner is partially paying down principal as a debt reduction strategy (known as *curtailments*).

6. The homeowner is defaulting.

Prepayments due to reasons 1 and 2 above are generally referred to as "turnover". Turnover rates can be influenced by the health of the housing market, for example the levels of real estate appreciation[*] and the volume of existing home sales; and by the so-called seasonality effect, which suggests that turnover rate typically increases during spring and summer months. Non cash-out refinancing (item 3 above), on the other hand, are primarily driven by interest rates movements. Even so, refinancing rates are not entirely predictive because of the well known fact that individual mortgagors vary in their financial sophistication and often do not refinance optimally. For example many mortgagors delay their refinancing decisions even when interest rates decline to a level such that it is financially optimal to do so (see [39]).

Figure 1.3 below shows a 36-month history of aggregated prepayment speeds for fixed-rate agency pools. The prepayment speed is measured in CPR, or conditional prepayment rate, which is an annualized rate that measures prepayments as a percentage of the outstanding loan balance of a pool. For example, a 8% CPR indicates that 8% of the outstanding balance of the mortgage pool is likely to be prepaid over the next year. In reality, since mortgage payments happen monthly, the CPR is calculated based on single monthly mortality (SMM), which is the percentage of the loan balance prepaid in a given month. Figure 1.3 highlights the wide variation in prepayment speeds, which can significantly affect the timing of the MBS cash flows.

Since prepayments generally increase as mortgage rates fall, pass-through MBS generally exhibit *negative convexity*[†] (see [13] for more details). In particular, (non cash-out) refinancing speeds typically rise when interest rates decrease, resulting in early return of principle and leaving investors exposed to reinvestment risk, namely that cash flows come in low rate environments when alternative investments produce lower returns.

[*]Home price appreciation is also a primary driver for cash-out refinances.

[†]Convexity is a measure of the nonlinear relationship between price and duration of a bond to changes in interest rates.

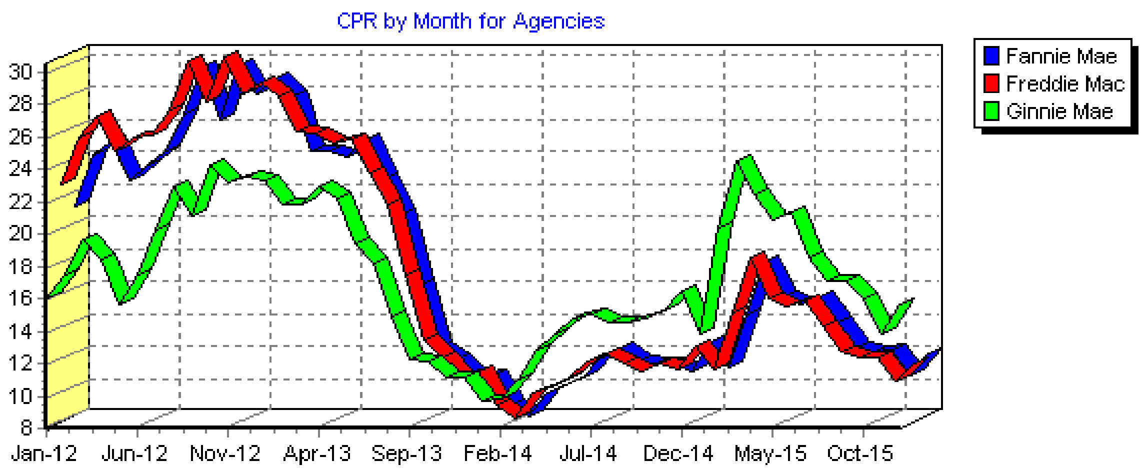


Figure 1.3: CPR by month for fixed rate agency pools. (*source:www.embs.com)*

1.4 Review of existing MBS valuation models

Econometric models

Traditionally, MBS valuations rely on econometric models for prepayment speeds and mortgage rates (including current coupons). For example, prepayments are usually estimated by regression models, where the dependent variable is the prepayment speed (typically measured in SMM), and the independent variables are quantities which are directly linked to the pool itself (e.g. loan size, loan age, loan-to-value ratio*, etc.) or the general economy (e.g. interest rates and home price indices). Once the prepayment model is built, one can price MBS using Monte-Carlo simulation based on the following procedure. First, interest rate paths are simulated according to some model (e.g. CIR). Second, given an interest rate scenario, the prepayment speed is calculated using the prepayment model. Next, the scheduled and prepayment cash flows are calculated using the amortizing schedule of the underlying collateral and the projected prepayment speeds. Lastly, one calculates the theoretical price of the MBS by discounting and averaging the simulated scenarios. In addition, one typically uses Option-Adjusted Spread, or OAS †, to account for the difference between the theoretical (model) price of an MBS and its market price.

*Loan-to-value ratio, or LTV, is the ratio of the amount of money borrowed over the appraised value of the home expressed as a percentage.

†OAS is the yield spread on a benchmark interest rate curve to discount payments of an MBS to match its market price. See [6] for a precise definition.

The Schwartz and Torous ([34]) model is a frequently cited econometric model. Here, maximum likelihood techniques and proportional-hazards models * are used to estimate a prepayment function from recent price data. Let T be a continuous random variable representing the time until prepayment of a mortgage. The prepayment function $p(t)$ specifies the instantaneous rate of prepayment at $T = t$ conditional upon the mortgage not having been prepaid prior to time t, and is formally defined by

$$p(t) = \lim_{\Delta t \to 0^+} \frac{\mathbb{P}\left(t \le T < t + \Delta t \mid T \ge t\right)}{\Delta t}.$$

Adopting a proportional-hazards framework, $p(t)$ in [34] is modeled by

$$p(t; x(t), \theta) = p_0(t, \gamma, \lambda) e^{x_t^{'}\beta}, \tag{1.1}$$

where t is the time from origination of the mortgage, $\theta = (\gamma, \lambda, \beta)$ is the parameter vector which is estimated from historical prepayment data, x_t is the vector of explanatory variables, and $p_0(t, \gamma, \lambda)$ is the baseline hazard function which measures the probability of prepayment under homogeneous conditions $x_t = 0$ and is given by the standard log-logistic hazard function

$$p_0(t, \gamma, \lambda) = \frac{\gamma\lambda(\gamma t)^{\lambda - 1}}{1 + (\gamma t)^{\lambda}}.$$

The log-logistic hazard function admits a variety of relationships between the probability of prepayment and the age of the mortgage. In particular, for $\lambda > 1$, the prepayment probability increases from zero to a maximum at $t^* = (\lambda - 1)^{1/\lambda}/\gamma$, and decreases to zero thereafter, which is consistent with the observation that, all other things being equal, conditional prepayment rates are typically low in the early years of a mortgage, increase as the age of the mortgage increases, and then diminish with further seasoning.

The proportional-hazards framework is also used by Deng *et al.* in [7], who presented a unified econometric model and empirical analysis of the competing risks of mortgage terminations by prepayment and default. The authors consider the two hazards (i.e. prepayment and default) as dependent competing risks and estimate them jointly using maximum likelihood techniques.

All in all, econometric models, despite introduced decades ago, remain the most popular and widely-used MBS valuation model in the financial industry.

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*Proportional-hazards models are a class of survival models in statistics. In a proportional-hazards model, the unique effect of a unit increase in a covariate is multiplicative with respect to the hazard rate.

Academic models

Within the math finance literature, there are two types of models used to value MBS: the "option-theoretic" and "reduced form" models (see [20, 17] for a more thorough introduction and literature review). The option-theoretic method treats the right to prepay as an American style embedded option and MBS valuation is performed using options pricing theory. Early results along this line were obtained in [9, 23, 1, 21]. However, it was quickly recognized that option-theoretic methods suffer due to the non-optimal prepayment behavior of borrowers, and hence the option-theoretic approach has not been widely adopted by mortgage market practitioners.

Alternatively, the reduced form method borrows from the theory of credit derivative valuation and assumes that prepayments are driven by an underlying intensity process which may be estimated from historical data. Here, the non-optimality of prepayment behavior is built into the intensity function. Reduced form methods have been studied in [34, 32, 23, 7, 20, 19, 17, 18, 43] amongst others. In this dissertation, we consider the reduced form method. We pay particular attention to [20], which computes mortgage rates when the intensity is driven by one (or many) economic factors and [19], which considers similar intensities to those we treat. Further connections with [19] are discussed below.

Option-theoretic models

Option-theoretic models treat a mortgage as a portfolio of three assets: 1) a non-callable, amortized loan, 2) an American style call option (the "prepayment option") on the underlying loan with a strike price at par, and, if defaults are considered, 3) an European style put option (the "default option"). In such models, one usually assumes there are two sources of uncertainty: the random fluctuations of house prices and interest rates. We will briefly describe the valuation methodology used in [23], which is often cited for option-theoretic models, and subsequent papers including [1, 36]. To begin with, the house price is modeled as a log-normal process solving the stochastic differential equation (SDE)

$$\frac{dH_t}{H_t} = (r_t - s)dt + \sigma_H dz_t^H,$$

where r is the default-free spot interest rate, z^H is a standard Wiener process, and s and σ_H are positive constants. r itself is assumed to follow a 1-d CIR process, i.e. r solves the SDE

$$dr_t = \gamma(\theta - r_t)dt + \sigma_r \sqrt{r_t} dz_t^r,$$

where γ, θ, σ_r are positive constants such that $\gamma\theta \geq \frac{1}{2}\sigma_r^2$, and z^r is another standard Wiener process. The correlation between z^H and z^r is assumed to be constant and denoted by ρ. Assuming the above dynamics are under some risk-neutral, or pricing measure $\mathbb{Q}$, then the partial differential equation (PDE) for valuation of assets solely dependent on the house price and the interest rate takes the form (here $X_t = X(H_t, r_t, t)$ is the value of the asset at time t)

$$\frac{1}{2}H^2\sigma_H^2\frac{\partial^2 X}{\partial H^2} + \rho H\sqrt{r}\sigma_H\sigma_r\frac{\partial^2 X}{\partial H\partial r} + \frac{1}{2}r\sigma_r^2\frac{\partial^2 X}{\partial r^2} + \gamma(\theta - r)\frac{\partial X}{\partial r} + (r-s)H\frac{\partial X}{\partial H} + \frac{\partial X}{\partial t} - rX = 0. \tag{1.2}$$

Directly adopting the notations used in [23], the value at time t of the mortgage contract V is given by

$$V(H, r, t, i) = A(r, t, i) - D(H, r, t, i) - C(H, r, t, i), \tag{1.3}$$

where i denotes the i-th calendar month since the mortgage origination; $t \in (\tau(i-1), \tau(i)]$* is the time elapsed after the $i-1$-th (and before the i-th) mortgage payment; $A(r, t, i)$ is the value at time t of the promised mortgage payments from i to the term of the loan; $D(H, r, t, i)$ (resp. $C(H, r, t, i)$) is the value at time t of the default (resp. prepayment) option when the next mortgage payment is due at month i.

Thus one can effectively view the mortgage contract as a portfolio consisting of three assets: the scheduled payments A, the prepayment option C, and the default option D, and the problem reduces to finding the fair price at origination of A, C, and D respectively. Under the current set up, for any given i, each of $A(r, t, i)$, $D(H, r, t, i)$ and $C(H, r, t, i)$ solves the PDE (1.2) with the appropriate boundary conditions, the details of which can be found in [23, 36]. Here we just point out that the various components of the mortgage contract interact with each other in a complex way: if default occurs in the first place, then prepayment will never occur, and vice versa. We also note that the boundary condition for the prepayment option involves a free boundary, due to the American nature of the option itself. Lastly, we note that the above procedure must be carried out

*Adopting the authors' notations, $\tau(i)$ is the calendar time of the i^{th} month, i.e., $\tau(i) = i/12$.

(backwards) iteratively for all $i = 1, 2, \cdots, n$, which is computationally very demanding.

Apart from the technical difficulties just mentioned, option-theoretic models are often criticized for their limited power to address sub-optimal prepayment behavior. If the prepayment option is exercised optimally*, which is the case in most mathematical models for American-style options, the option value will always be greater than or equal to its intrinsic value. As a result, in the absence of transaction costs the value of the mortgage can never exceed par, which is often violated in market practice. Longstaff ([26]) attempts to address this issue by using a multi-factor term structure to incorporate borrower credit into the analysis. The fact that a borrower's financial situation (such as credit worthiness) affects the rate at which she can refinance is considered: the borrower with a poor credit score will have to refinance at a premium rate, and this is modeled by adding a credit spread to the prepayment cost.

Option-theoretic models are also not well suited to the treatment of pool heterogeneity. Since individual borrowers may vary a lot in their financial sophistication, when interest rates decline and offer refinancing opportunities, many borrowers either delay their refinancing decision or do not act at all until interest rates bounce back. However, the early option-theoretic models typically assume that all borrowers with similar characteristics prepay simultaneously, which does not reflect the reality. Staton ([39]) addresses this issue by assuming the borrowers face heterogeneous transaction costs and make prepayment decisions only at random discrete intervals. The second assumption is needed since even with heterogeneous transaction costs there would still be a critical level for each transaction cost at which all borrowers with the same transaction cost level would prepay immediately.

Kalotay *et al.* in [21] suggested an alternative approach within the class of option-theoretic models that incorporates two different yield curves: one yield curve is used to discount mortgage cash flows and the other one is used to discount MBS cash flows. To model heterogeneity the authors break up the mortgage pool into buckets and assume that each bucket represents different refinancing behavior. Briefly speaking, borrowers are divided into three groups: *financial engineers*, *leapers* and *laggards*. Financial engineers are financially sophisticated enough to be able to exercise their prepayment options optimally. Here, optimality is measured by the so-called "refinancing efficiency", defined by the authors to be the ratio of attainable savings of refinancing to the option value, and financial engineers will choose to refinance when the refinancing efficiency reaches 100%. Leapers, as the name suggests, refer to those borrowers who refinance too early at refinancing efficiency less than 100% of the option value. Laggards, on the other hand, refer to mortgagors

*A detailed study of this optimality assumption can be found in [41].

who continue waiting even after refinancing efficiency has reached 100%. More precisely, each mortgagor is assigned an "imputed coupon", and the mortgagor will refinance whenever a financial engineer would refinance a maturity-matched mortgage with the imputed coupon. For example, a mortgagor who refinances a 7% mortgage when a financial engineer refinances a 6% is referred to as a 1% laggard as she refinances too late. A "laggard spread distribution" is then specified using the factor (current balance as a percentage of the original balance) of the mortgage pool, and used as an input in the valuation process.

Reduced form models

Reduced form models are closely related to the econometric models introduced in the previous section. They offer both mathematical rigor and flexibility with respect to the specification of the intensity process. Inspired by the vast mathematical finance literature on intensity-based credit risk modeling (see e.g. [8, 4, 10, 24]), reduced form models treat prepayments as a "default" on the mortgage contract with "recovery" paid at the time of default and equal to the outstanding principal balance. The first continuous-time, intensity based mortgage valuation model was introduced by Goncharov in [17], where the author derived the following formula for the value of a T-year maturity mortgage at time t:

$$M_t = \mathbb{1}_{\tau > t} \mathbb{E}\left[\int_t^T (c_s + Z_s \gamma_s)\, e^{-\int_t^s (r_\theta + \gamma_\theta) d\theta} ds \,\middle|\, \mathcal{F}_t \right].$$

Above, the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ is assumed to model the flow of observations available to the mortgage lender prior to the mortgage prepayment time. The expectation is understood to be taken under a martingale measure $\mathbb{Q}$. τ is a stopping time representing the time of prepayment. c_s is the (continuous) coupon payment rate that the borrower pays up to time $\tau \wedge T$, and Z_τ is a lump sum payment that the borrower pays at time τ, if $\tau \leq T$. r_t is the short-term interest rate, and finally, γ_t is the intensity of τ under $\mathbb{Q}$. A detailed discussion on the specification of the intensity process γ via the so-called "refinancing incentive" is also presented.

Gorovoy and Linetsky ([20]) adopted a similar intensity-based framework, where the prepayment time τ is modeled as the first time when the hazard process $\int_0^t \gamma_u du$ is greater or equal to the random level $e \sim Exp(1)$, and the intensity process γ is assumed to take the form

$$\gamma_t = h_0(t) + \gamma(k - r_t)^+,$$

where $h_0(t)$ is a deterministic function of time, r_t is the short rate, and γ and k are scalar parameters. The first term models exogenous prepayments independent of interest rates (e.g. turnover). The second term models refinancing due to declining interest rates and is proportional to the positive part of the distance between a constant threshold level and the current short rate. When the short rate follows a CIR diffusion, the authors are able to find expressions for the present value of the mortgage.

Ti Zhou ([43]) incorporated intensity-based, utility indifference pricing methodologies to value individual mortgages and mortgage-backed securities. The totally inaccessible prepayment time is modeled by its intensity that can be either deterministic or stochastic. In the deterministic intensity case, the author derives explicit formulas for the MBS investor's indifference price (using exponential utility) of a mortgage contract. While in the case of stochastic hazard rates, the indifference price is expressed in terms of the solution to a quasilinear PDE.

1.5 Endogenous mortgage rates

Recall that in Section 1.2, we mention that current coupons are obtained by linearly interpolating TBA prices and they are in nature endogenous rates. Furthermore, it is clear that the refinancing incentive is heavily dependent on the mortgage rates. Thus, by contrast with the previous sections, when modeling current coupons, one needs to take into account the current coupon within the prepayment intensity function. In fact, aside from the amortizing nature of a mortgage loan, the key difference between MBS and credit derivative valuation is the dependence of the mortgage pool value on the mortgage rate. Indeed, one has the heuristic relationship

$$\text{Mortgage Rate: } m_0 \quad \Longrightarrow \text{ Prepayment Time: } \tau(m_0) \quad \Longrightarrow \text{ Pool Value: } M(m_0).$$

Thus, there is a natural and delicate fixed point problem in finding m_0 so that $M(m_0)$ is par valued. In reduced form models, this circular dependence is captured in the intensity function. This is in contrast to credit valuation, where one typically expresses the default intensity γ as a function of the underlying economic factors X (e.g. interest rates, house prices, etc.). Indeed, whereas an intensity specification $\gamma_t = \gamma(X_t)$ may be appropriate for credit derivatives, for MBS valuation as mentioned above, it is desirable to allow γ to additionally depend upon both the mortgage origination rate m_0 and the current mortgage rate m_t available for refinancing, i.e. $\gamma_t = \gamma(X_t, m_0, m_t)$. Thus, in a time-homogeneous Markovian setting such as in the above models where the interest rate and housing price process follow a diffusion, one hypothesizes that $m_t = m(X_t)$ is a function of the underlying economic factors and hence

$$\gamma_t = \gamma(X_t, m(X_0), m(X_t)), \tag{1.4}$$

for an exogenously specified function $\gamma = \gamma(x, m, z)$.

Our main goal in this dissertation is to prove existence of endogenous mortgage rates in a reduced form model where the intensity admits the form (1.4). Above, the mortgage rates m_0 and m_t have been understood as the primary mortgage rates. However, as mentioned in Section 1.2, it is difficult to view the primary rates directly, and it has become a common modeling convention to use the continuously available secondary market MBS rate instead, and adjust it by the primary-secondary rate spread as necessary (see [6] for more details). In the sequel, unless otherwise noted, we will interpret $m_t = m(X_t)$ as the pass-through MBS rate. In other words, we assume there are no guarantee fees or servicing fees for servicing the mortgage and passing through payments to investors. With this specification, the goal is then to find a *current coupon function* m so that the pool value $M(m(X_0)) = P_0$* for all values X_0.

The idea of endogenous mortgage rates has been widely discussed by market practitioners and in financial economics literatures. The Mortgage Option-Adjusted Term Structure (MOATS) model developed by Citigroup is one of the first industrial models based on an endogenously defined mortgage rate process. A detailed explanation of the MOATS model can be found in [3].

Campbell and Cuoco study household decisions for endogenously determined mortgage rates in [5]. The authors use a theoretical model of a rational utility-maximizing household who finances the purchase of a house with a mortgage, and who must in each period decide whether to exercise the options to default or to prepay the loan. They first model the cash flows of the loans, including a loss on the value of the house in the event the household defaults. Then risk-adjusted discount rates and a zero-profit condition are used to determine the mortgage premia that in equilibrium should be applied to each contract. Finally, the authors solve several iterations of the model for each mortgage contract to find a fixed point (however, the existence of such fixed points are not proved).

Longstaff in [26] studies the valuation of mortgage-backed securities when borrowers may have to refinance at premium rates because of their credit. The author develops a model that solves for the optimal refinancing strategy of a borrower whose objective is to minimize his lifetime mortgage costs. Of particular note is the fact that the model involves solving for the "par mortgage rate" that

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*Here and in the sequel, P_0 denotes the initial balance of the pool.

by definition is the mortgage rate such that the present value of a newly issued mortgage is equal to par.

There is a relative lack of research on endogenous mortgage rates from the math finance community. Plisaka ([32]) and Goncharov ([17, 19]) first incorporated the endogenous mortgage rate into an intensity-based framework, taking into account the dependence of γ on m. In particular, [19] presented a proof of the existence of a current coupon in a diffusion model similar to the model we will consider. However, we wish to point out three key differences between [19] and the present work. First and foremost, there is an error in [19] (Proposition 4.1 therein is evidently incorrect for the discontinuous intensities considered. A detailed discussion can be found in Section A.2). Second, the existence proof, based on a so-called "Lebesgue set method", is highly non-standard, whereas our proof of existence uses standard topological fixed point theorems. Third, our method of proof has the added benefit that we are able to show regularity in the current coupon function, whereas in [19] only measurable solutions are obtained.

Equally important as identifying existence of current coupons is actually computing the current coupon (see [20, 18, 6]). Indeed, a naive application of the contraction principle where one fixes an initial function m_0 and then sets $m_n(X_0) = \mathcal{A}(m_{n-1}(X_0)), n = 1, 2, ...$ for some operator $\mathcal{A}$ with the idea that m_n converges, while not only theoretically unjustified, is also prohibitively slow. To overcome this problem, [20] writes the intensity as solely a function of the underlying factors with the idea that this captures the bulk of prepayments. Then, for CIR interest rates, the endogenous rate is computed using eigen-function expansions. In [18], a non-iterative method is proposed borrowing ideas from PDE theory. In the current dissertation we take an alternate approach, approximating the current coupon via perturbation analysis. This uses the well known fact (see [19]) that unique current coupon functions in the sense that $M(m(X_0)) = P_0$ exist when $\gamma_t = \gamma(X_t)$ only depends upon the factors. Next, we note that for $\gamma = \gamma(x, m, z)$ from (1.4) one may always write $\gamma(x, m, z) = \gamma_0(x) + \gamma_1(x, m, z)$ by taking $\gamma_0(x) \equiv 0$, but also in the case where the full intensity is assumed to be a constant $\gamma > 0$ plus an additional component, as the intensity specification in [20]. We then embed this decomposition via

$$\gamma^\varepsilon(x, m, z) = \gamma_0(x) + \varepsilon\gamma_1(x, m, z); \qquad \varepsilon > 0.$$

For $\varepsilon = 0$, there is a unique current coupon function $m_0(x)$. For small, positive ε we will obtain a unique, explicit, closed form expression for $m_1(x)$ so that $m^\varepsilon(x) = m_0(x) + \varepsilon m_1(x) + o(\varepsilon)$. With this decomposition, valid for any continuous fixed point m^ε we naturally consider the numerical approximation (at $\varepsilon = 1$) of $m(x) \approx m_0(x) + m_1(x)$. It turns out this approximation does very well in practice: differing by ≤ 10 basis points (on absolute rate levels of $4\% - 12\%$) from the theoretical fixed point determined by naive contraction, see Section 3.2.

1.6 Organization of the dissertation

In Chapter 2 we set up the model framework. Section 2.1 introduces a continuous-time model for a fully amortized fixed rate mortgage contract. Section 2.2 contains a heuristic derivation of the fixed point problem for the endogenous current coupon. Section 2.3 specializes the fixed point problem to a Markovian framework where X is a non-explosive locally elliptic diffusion on a general domain in $\mathbb{R}^d$, making precise assumptions on the model coefficients, as well as the intensity function. In particular, as the mortgage market is typically incomplete, a rigorous construction of the particular risk neutral measures used here for pricing is given. Aside from being done for the sake of mathematical rigor, this shows that when pricing the mortgage pool, one may assume the intensity processes coincide between the physical and risk neutral measures and hence can be estimated using observed prepayment data. Chapter 2 culminates with Theorem 2.1 which proves existence of a current coupon function, under the assumption that $\gamma(x, m, z)$ is approximately constant in m for large values of m (see Remark 2.2 for more discussion on our main assumption).

In Chapter 3, we perform a perturbation analysis where the intensity γ is perturbed off of a baseline intensity γ_0 which only depends upon the factor process X. The goal of this analysis is to uniquely identify m up to leading orders of the perturbation. We provide a numerical approximation to the fixed point and examine its performance in Section 3.2.

In Chapter 4 we present the proof of Theorem 2.1. Due to the considerable complexity of the proof, we first describe the technical difficulties and provide an outline of the major steps of the proof in Section 4.1. In Section 4.2 we prove several auxiliary lemmas that establish *a priori* estimates for the parabolic Hölder norms of some conditional expectation expressions which will be essential in the proof of the main result. In Section 4.3 we localize the original fixed point problem, prove local existence by applying Schaefer's fixed point theorem (Theorem 4.1) and establish several key regularity properties of the localized functional operator. Finally, in Section 4.4 we unwind the localization and prove the global existence of a fixed point, using the Arzelà-Ascoli theorem and a standard diagonal-subsequence argument.

Finally, Chapter 5 contains the conclusion and a summary of possible extensions and future studies. The Appendix contains auxiliary lemmas and additional comments on Goncharov's result

([19]), as well as the naive contraction method used in our numerical examples.

Chapter 2

Model And Problem Formulation

2.1 A basic mortgage contract

We begin by considering a level-payment, fully amortized, T-year fixed rate mortgage which is originated at time $t = 0$. The borrower takes a loan of P_0 dollars at origination and pays a continuous coupon stream at a constant rate of $c > 0$ dollars per annum during the lifetime of the mortgage $[0, T]$. The interest is compounded at a constant mortgage rate m which is fixed at origination. In the absence of prepayments, the scheduled outstanding principal of the mortgage, denoted by $p(t, m)$ for $0 \leq t \leq T$ and $m > 0$, satisfies the following ordinary differential equation (ODE):

$$p_t(t, m) = mp(t, m) - c; \quad p(0, m) = P_0, \ p(T, m) = 0. \tag{2.1}$$

(2.1) has solution

$$p(t, m) = P_0 \frac{1 - e^{-m(T-t)}}{1 - e^{-mT}}. \tag{2.2}$$

As can be seen, the outstanding principal for a fully amortized fixed rate level payment mortgage is a smooth and bounded positive function which decreases from the initial loan amount P_0 to zero at the maturity of the mortgage contract. In the sequel, unless otherwise noted, we will assume $P_0 = 1$ to simplify the presentation.

We can also express the coupon stream payment c in terms of m and T as well:

$$c = c(m) = \frac{m}{1 - e^{-mT}}. \tag{2.3}$$

2.2 Endogenous current coupons

Consider an agency pool of loans secured by a government-sponsored enterprise. We assume there are no defaults or curtailments. We also make the following *homogeneous pool* assumption: A) the mortgage pool is comprised of $I \in \mathbb{N}$ mortgages with the same time-to-maturity T, mortgage rate m, and B) conditional on the common market factors (e.g. interest rates, house prices), the borrowers of the underlying mortgages have the same prepayment intensity. We will explain item B) in detail in the next section, and we will describe in Section 5.2 how item B) can be generalized.

As previously stated, the current coupon is defined as the theoretical coupon rate that makes the TBA price equal to par. In the current work, we assume there are no guarantee fees or servicing costs so that the TBA coupon rate is equal to the initial fixed mortgage rate m. We now informally derive a fixed point equation for the current coupon m. This argument will be made rigorous in Section 2.3 below. In the absence of prepayments, the mortgage balance $p(t,m)$ evolves according to (2.2). Consider now when there is a random prepayment time τ under a pricing measure $\mathbb{Q}$ on the underlying measure space $(\Omega, \mathcal{G})$. The borrower prepays the remaining balance $p(\tau, m)$ at time τ provided $\tau \leq T$. Assuming an interest rate process $r = \{r_t\}_{t \leq T}$, the initial value of the mortgage is

$$M(m) = \mathbb{E}^{\mathbb{Q}}\Bigg[\underbrace{\int_0^{\tau \wedge T} c(m) e^{-\int_0^t r_u du} dt}_{\text{coupon payments}} + \underbrace{\mathbb{1}_{\tau \leq T} p(\tau, m) e^{-\int_0^\tau r_u du}}_{\text{prepayment}}\Bigg]. \tag{2.4}$$

Next, assume that the interest rate process is adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \leq T}$, where $\mathcal{F} = \vee_{t \leq T} \mathcal{F}_t \subset \mathcal{G}$ and that τ has an intensity $\gamma = \{\gamma_t\}_{t \leq T}$ with respect to $(\mathbb{Q}, \mathbb{F})$:

$$\mathbb{Q}\left[\tau > t \mid \mathcal{F}\right]^{*} = \mathbb{Q}\left[\tau > t \mid \mathcal{F}_t\right] = e^{-\int_0^t \gamma_u du}; \quad t \geq 0, \tag{2.5}$$

where γ is non-negative, adapted and integrable. The initial value of the mortgage is now obtained using integration by parts (see [20, 19]) as

$$M(m) = 1 + \mathbb{E}^{\mathbb{Q}}\left[\int_0^T p(t,m)(m - r_t) e^{-\int_0^t (r_u + \gamma_u) du} dt\right]. \tag{2.6}$$

The mortgage rate (current coupon) m is said to be *endogenous* if $M(m) = 1$. In other words we

*This equality requires an additional hypothesis on how τ is constructed and will be shown to hold in the current setup.

seek m so that

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T p(t,m)(m-r_t)e^{-\int_0^t (r_u+\gamma_u)du}dt\right]. \tag{2.7}$$

2.3 The fixed point problem and the main result

Model set-up

The analysis in Section 2.2 is now specified to a doubly stochastic, intensity based model for the mortgage prepayment time τ. To begin with, we fix a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. The measure $\mathbb{Q}$ is interpreted as a pricing, or risk neutral measure and we write $\mathbb{E}[\cdot]$ for $\mathbb{E}^{\mathbb{Q}}[\cdot]$ throughout. Later in this section we provide two rigorous constructions of $\mathbb{Q}$. In particular we show that when estimating the prepayment intensity function γ, one may use observed prepayment data under "physical" measure, rather than estimating prepayments under the risk neutral measure $\mathbb{Q}$.

Let W be a standard d-dimensional Brownian motion under $\mathbb{Q}$ and denote by $\mathbb{F}^W$ the $\mathbb{Q}$-augmented version of the natural filtration so that $\mathbb{F}^W$ satisfies the usual conditions. The underlying economic factors which affect prepayments are governed by a process X satisfying the following SDE

$$dX_t = b(X_t)dt + a(X_t)dW_t; \quad X_0 = x \in D. \tag{2.8}$$

The state space of X is an open, connected region $D \subset \mathbb{R}^d$ which satisfies

Assumption 2.1. $D = \bigcup_{n\geq 1} D_n$ where for each $n \in \mathbb{N}$, D_n is open and bounded with smooth boundary. Furthermore, $\bar{D}_n \subset D_{n+1}$.

We assume that $b : D \mapsto \mathbb{R}^d$ and we take $a = \sqrt{A}$, the unique positive definite symmetric square root of $A : D \mapsto \mathbb{S}^d_{++}$, where $\mathbb{S}^d_{++}$ is the space of symmetric positive definite $d \times d$ matrices. We assume b, A satisfy the following regularity and local-ellipticity conditions:

Assumption 2.2.

1. A is locally elliptic: for each n there exists $K_1(n) > 0$ so that for all $\xi \in \mathbb{R}^d \setminus \{0\}$ and $x \in D_n$ we have

$$\xi' A(x)\xi \geq K_1(n)\xi'\xi.$$

2. b and A are locally Lipschitz with Lipschitz constant $K_2(n)$ on D_n.

The above assumptions imply existence of a local solution to the SDE in (2.8). To ensure the existence of a global solution, we further assume X does not explode to the boundary of D, i.e.,

Assumption 2.3. For all $x \in D$ and $T > 0$, $\mathbb{Q}^x \left[X_t \in D, \forall t \leq T\right]^* = 1$.

Under Assumptions 2.2 and 2.3 it follows that X has a unique strong solution. Since the short term interest rate r plays a key role in mortgage modeling, we assume the first coordinate of X is the short term interest rate and that the state space of $X^{(1)}$ is $(0, \infty)$, i.e.

Assumption 2.4. $X_t^{(1)} = r_t > 0$.

Next we precisely define the intensity γ in equation (2.5). Let $m : D \mapsto [0, \infty)$ be a candidate endogenous current coupon function in that, for given $X_0 = x \in D$, $m(x)$ is the endogenous current coupon. We hypothesize γ is a function of

1. The underlying factor process X.
2. The initial contract mortgage rate $m(x)$.
3. The mortgage rate available via refinancing $m(X)$.

Remark 2.1. Due to the time-homogeneity of the coefficients for X and fixed maturity T, it suffices to consider $m(X) = m(X_t)$.

Thus, given the function m, the intensity at time t is given by $\gamma_t = \gamma(X_t, m(X_0), m(X_t))$ where $\gamma = \gamma(x, m, z)$ is an exogenously specified function, with further assumptions on γ given below.

To facilitate our main assumption on γ, we next define the auxiliary function

$$\Xi(x) = \inf_{0<\beta<1} \frac{\beta e^{-\beta x}}{(1-\beta)(1-e^{-\beta x})}; \quad x > 0. \tag{2.9}$$

It is straightforward to show that $\Xi(x)$ is decreasing with respect to x and

$$\Xi(x) = \frac{1}{x} \text{ for } x \leq 2; \quad \lim_{x\uparrow\infty} \frac{\Xi(x)}{xe^{-(x-1)}} = 1. \tag{2.10}$$

We are now ready to present the main assumptions on γ. For ease of presentation we define $E := D \times (0, \infty) \times (0, \infty)$ and $E_n := D_n \times (0, n) \times (0, n)$, $n \in \mathbb{N}$.

Assumption 2.5. $\gamma : E \mapsto [0, \infty)$ satisfies:

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*$\mathbb{Q}^x$ denotes the measure conditional on $X_0 = x$.

1. $\gamma \in C^2(E)$, and for each n, the derivatives of order ≤ 2 can be continuously extended to $\bar{E_n}$, and are Lipschitz continuous on $\bar{E_n}$ with Lipschitz constant $L_\gamma(n)$.*

2. $\gamma(x,m,z)$ and $\gamma_m(x,0,z)$ are locally bounded in x, uniformly in m and z respectively. In other words for each n there is $B_\gamma(n) > 0$ such that

$$\sup_{x\in D_n, m,z\geq 0} \gamma(x,m,z) \leq B_\gamma(n); \qquad \sup_{x\in D_n, z\geq 0} \gamma_m(x,0,z) \leq B_\gamma(n). \tag{2.11}$$

3. γ_m admits the following lower and upper bounds:

$$0 \leq \gamma_m(x,m,z) \leq \Xi(mT); \quad x \in D, m, z \geq 0. \tag{2.12}$$

Remark 2.2. Regarding condition 3 in Assumption 2.5, $\gamma_m \geq 0$ is expected as prepayments rise with the current coupon. We note that under the given regularity assumptions (which themselves are not overly restrictive since no global bounds on the derivatives' sizes are placed) we have

$$\gamma_m(x,m,z) \leq B_\gamma(n) + L_\gamma(n)m; \quad x \in D_n; m, z \in [0,n]. \tag{2.13}$$

Since $\Xi(mT) = \frac{1}{mT}$ for small m, we see that (2.12) is not restrictive for small m. For large m it does however imply that γ is approximately constant in m.

With all the assumptions in place, our formal definition of the current coupon function is the following

Definition 2.1. $m : D \mapsto [0,\infty)$ is a *current coupon function* if, for all $x \in D$,

$$0 = \mathbb{E}^x\left[\int_0^T p(t,m(x))(m(x)-r_t)e^{-\int_0^t (r_u+\gamma(X_u,m(x),m(X_u)))du}dt\right]. \tag{2.14}$$

As $m(x)$ is deterministic, we can rewrite (2.14) as

$$m(x) = \mathcal{A}[m](x) := \frac{\mathbb{E}^x\left[\int_0^T p(t,m(x))r_t e^{-\int_0^t (r_u+\gamma(X_u,m(x),m(X_u)))du}dt\right]}{\mathbb{E}^x\left[\int_0^T p(t,m(x))e^{-\int_0^t (r_u+\gamma(X_u,m(x),m(X_u))du}dt\right]}. \tag{2.15}$$

As such, current coupon functions are fixed points of the operator $\mathcal{A}$. However, $\mathcal{A}$ is highly

*We will assume γ and its derivatives of order ≤ 2 are defined on $D \times [0,\infty) \times [0,\infty)$. The values at zero are defined via the continuous extensions.

nonlinear in m, and depends jointly on $(m(x), m(X_t))$, which make it difficult to verify if $\mathcal{A}$ is a contraction. Thus, we will have to consider a topological fixed point theorem to show existence of solutions. Unfortunately, the presence of $m(x)$ inside the expectation means there is no "smoothing" effect of the operator $\mathcal{A}$. In other words, we should not expect that $\mathcal{A}[m]$ possesses a higher order of regularity than the input function m; or that $\mathcal{A}$ possesses the compactness properties needed in classical topological fixed point theorems. Despite this, we will show that a fixed point does exist under the current assumptions. The proof is done through a delicate localization argument and is presented in Chapter 4.

On the construction of the risk neutral measure $\mathbb{Q}$

Before proving existence of current coupon functions, in this section we first offer two rigorous ways to construct the risk neutral measure $\mathbb{Q}$. The first construction is valid for "large" mortgage pools in which the number of borrowers tends to infinity. The second construction is valid for a single loan pool. Worth mentioning is the fact that, in both constructions, one may use the prepayment intensity under the physical probability measure.

To begin with, let $\tilde{b} : D \mapsto \mathbb{R}^d$ and $A : D \mapsto \mathbb{S}^d$ satisfy Assumption 2.2, where D is described in Assumption 2.1. $\tilde{b}$, A will describe the dynamics of X under the physical measure $\mathbb{P}$. In order that X not explode under $\mathbb{P}$ we assume that there exists a unique solution to the Martingale problem (see [40]) for the second order linear operator $\tilde{L}$ associated to $(\tilde{b}, A)$ on D, given by

$$\tilde{L} = \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} A_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{d} \tilde{b}_i(x)\frac{\partial}{\partial x_i}.^* \tag{2.16}$$

Denote by $\widetilde{W}$ a d-dimensional Brownian motion on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Set $\mathbb{F}^{\widetilde{W}}$ to be the $\mathbb{P}$-augmentation of the natural filtration for $\widetilde{W}$, so that $\mathbb{F}^{\widetilde{W}}$ satisfies the usual conditions. Since the Martingale problem for $\tilde{L}$ is well posed, there exists a unique strong solution to the SDE (see [33])

$$dX_t = \tilde{b}(X_t)dt + a(X_t)d\widetilde{W}_t; \quad X_0 = x \in D, \tag{2.17}$$

where $a = \sqrt{A}$. Next, let $\mu : D \mapsto \mathbb{R}^d$ and $\Sigma : D \mapsto \mathbb{S}^d$ also satisfy Assumption 2.2 and let $\sigma = \sqrt{\Sigma}$. The financial market is formed via trading instruments (S, S^0), where $S = \left(S^1, \cdots, S^d\right)$

*Going forward we will omit the sums so that $\tilde{L} = \frac{1}{2}A_{ik}\frac{\partial^2}{\partial x_i \partial x_k} + \tilde{b}_i\frac{\partial}{\partial x_i}$.

have dynamics

$$\frac{dS_t^i}{S_t^i} = \mu^i(X_t)dt + \sum_{j=1}^{d} \sigma^{ij}(X_t)d\widetilde{W}_t^j; \quad i = 1, \cdots, d, \tag{2.18}$$

and $S_t^0 := e^{\int_0^t r_u du}$ is the money market account.

Now, define $b : D \mapsto \mathbb{D}^d$ by

$$b(x) = \tilde{b}(x) - a(x)\sigma(x)^{-1}\left(\mu(x) - r\mathbb{1}\right), \tag{2.19}$$

where $\mathbb{1} = (1, \cdots, 1) \in \mathbb{R}^d$ is the vector of ones. Note that b also satisfies Assumption 2.2.

Lastly, assume the Martingale problem for $L = \frac{1}{2}A_{ik}(x)\frac{\partial^2}{\partial x_i \partial x_k} + b_i(x)\frac{\partial}{\partial x_i}$ associated to (b, A) is also well posed on D. Under the above hypotheses the above market with $\mathbb{F}^{\widetilde{W}}$ adapted, S-integrable trading strategies is complete. Furthermore, the unique risk neutral measure $\mathbb{Q}$ on $\mathcal{F}_T^{\widetilde{W}}$ has Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T^{\widetilde{W}}} = Z_T; \quad Z_t := \mathcal{E}\left(-\int_0^{\cdot} \left(\mu(X_t) - r_t\mathbb{1}\right)' \sigma^{-1}(X_t)d\widetilde{W}_t\right)_t, \; t \leq T. \tag{2.20}$$

With $\mathbb{Q}$ well-defined on $\mathcal{F}_T^{\widetilde{W}}$, we recall that (see [22]) if $\mathcal{C} = \{\mathcal{C}_t\}_{t\leq T}$ represents a cumulative cash-flow stream adapted to $\mathbb{F}^{\widetilde{W}}$ with rate $C(t) := \dot{\mathcal{C}}(t)$, and which satisfies the requisite integrability conditions, then the unique price for the stream is given by $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T C(t)e^{-\int_0^t r_u du}dt\right]$.

Large pool

Now we assume that the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ also supports a $\mathbb{P}$-independent, identically distributed (i.i.d.) sequence of $U(0,1)$ random variables, denoted by $U_1, U_2, \cdots$, which are also $\mathbb{P}$ independent of $\widetilde{W}$. Let $\{\gamma_i\}_{i=1,2,\cdots}$ be a sequence of non-negative, integrable, $\mathbb{F}^{\widetilde{W}}$ adapted process. The random times $\{\tau_i\}_{i=1,\cdots}$ are constructed via

$$\tau_i := \inf\{t \geq 0 \mid U_i = e^{-\int_0^t \gamma_u^i du}\}; \quad i = 1, 2, \cdots. \tag{2.21}$$

In this section, we will further assume that $\gamma^i = \gamma$ is identical, for all $i \in \mathbb{N}$. This implies $\{\tau_i\}_{i\in\mathbb{N}}$ are $\mathbb{P}$ conditionally i.i.d. given $\mathcal{F}_T^{\widetilde{W}}$, each with common $\mathbb{P}$-intensity γ.

Consider a large pool consisting of infinitely many mortgages with uniformly infinitesimally small initial balances. More precisely, we first fix $N \in \mathbb{N}$ and for $i = 1, \cdots, N$ set τ_i as the prepayment time of the i[th] borrower in an N-loan pool, where each loan is of size $1/N$. We assume that the pool has common contract rate m so that the respective principal balances and coupon payments are given by

$$p^i(t,m) = (1/N)p(t,m) = \frac{1}{N}\frac{1-e^{-m(T-t)}}{1-e^{-mT}}, \tag{2.22}$$

and

$$c^i(t,m) = (1/N)c(m) = \frac{1}{N}\frac{m}{1-e^{-mT}}. \tag{2.23}$$

The cumulative cash-flows of the mortgage pool is

$$\mathcal{C}_N(t) = \frac{1}{N}\sum_{i=1}^{N} c(m)(t\wedge\tau_i) + \frac{1}{N}\sum_{i=1}^{N} p^i(\tau_i,m)\mathbb{1}_{\tau_i\le t}. \tag{2.24}$$

We now invoke the conditional version of Kolmogorov's strong law of large numbers (see, for example Theorem 4.2. in [28]). For each $t \le T$ it follows that $\mathcal{C}_N(t) \to \mathcal{C}(t)$ almost surely where for τ a generic copy of τ_i:

$$\begin{aligned}\mathcal{C}(t) &= c(m)\mathbb{E}\left[t\wedge\tau \mid \mathcal{F}_T^{\widetilde{W}}\right] + \mathbb{E}\left[p(\tau,m)\mathbb{1}_{\tau\le t} \mid \mathcal{F}_T^{\widetilde{W}}\right] \\ &= c(m)te^{-\int_0^t \gamma_u du} + c(m)\int_0^t u\gamma_u e^{-\int_0^u \gamma_v dv}du + \int_0^t p(u,m)\gamma_u e^{-\int_0^u \gamma_v dv}du.\end{aligned} \tag{2.25}$$

Proposition 2.1. $\mathbb{P}\big(\lim_{N\to\infty}\mathcal{C}_N(t) = \mathcal{C}(t); \quad \forall t\le T\big)= 1.$

Proof. First, it follows by the above argument that $\mathcal{C}_N(t) \to \mathcal{C}(t)$ almost surely for all rational $t \in [0,T]$.

Next, using the fact that $\mathcal{C}_N$ is non-decreasing in t and $\mathcal{C}$ is continuous in t, we get the desired result. □

The cash flow rate for $\mathcal{C}$ is given by $C(t) := c(m)e^{-\int_0^t \gamma_u du} + p(t,m)\gamma_t e^{-\int_0^t \gamma_u du}$. As such, it follows that the price of the large pool is

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (c(m)+p(t,m)\gamma_t)e^{-\int_0^t (r_u+\gamma_u)du}dt\right].$$

Using (2.1) and integration by parts, we can rewrite the above as

$$1+\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (m-r_t)p(t,m)e^{-\int_0^t (r_u+\gamma_u)du}dt\right].$$

This yields (2.6) and the fixed point equation (2.7). Note that γ here is the $\mathbb{P}$ prepayment intensity. As such, one does not need to obtain the $\mathbb{Q}$ prepayment intensity in order to solve for the endogenous current coupon.

Single loan pool

Now we turn to the "single loan pool". We assume that in addition to $\widetilde{W}$, $(\Omega, \mathcal{G}, \mathbb{P})$ also supports a $U(0,1)$ random variable U that is $\mathbb{P}$-independent of $\widetilde{W}$. We create the random time τ as in (2.21), where γ is a non-negative, integrable, $\mathbb{F}^{\widetilde{W}}$ adapted process. Let $H = \{H_t\}_{t\geq 0}$ be the indicator process associated to τ, with $H_t = \mathbb{1}_{\tau > t}$, and let $\mathbb{F}^H = \{\mathcal{H}_t\}_{t\geq 0}$ be the filtration generated by H via $\mathcal{H}_t = \sigma(H_s; s \leq t)$. Note that τ is an $\mathbb{F}^H$-stopping time. The *enlarged filtration* $\mathbb{G}$ is generated by both $\mathbb{F}^{\widetilde{W}}$ and the $\mathbb{P}$-augmented version of $\mathbb{F}^H$. Note that $\mathbb{G}$ is right continuous ([42, Theorem 1]). We now enlarge the market described above by allowing for $\mathbb{G}$ adapted trading strategies. Obviously this market is incomplete. However it is well known (see [12, 35] for details) that the "minimal martingale measure" $\mathbb{Q}$ satisfies (where Z is from (2.20))

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{G}_T} = Z_T; \quad T \geq 0.$$

Now, let $A \in \mathcal{F}^{\widetilde{W}}$ and $t \geq 0$. Clearly we have that $\mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{\tau>t}\mathbb{1}_A\right] = \mathbb{E}^{\mathbb{P}}\left[e^{-\int_0^t \gamma_u du}\mathbb{1}_A\right]$ and hence

$$\mathbb{P}\left[\tau > t \,\middle|\, \mathcal{F}^{\widetilde{W}}\right] = \mathbb{P}\left[\tau > t \,\middle|\, \mathcal{F}_t^{\widetilde{W}}\right] = e^{-\int_0^t \gamma_u du},$$

and it follows that γ is the $(\mathbb{P}, \mathbb{F}^{\widetilde{W}})$ intensity of τ. Furthermore:

Proposition 2.2. γ is the $(\mathbb{Q}, \mathbb{F}^{\widetilde{W}})$ intensity of τ.

Proof. First we note that

$$\mathbb{Q}\left[U \leq u\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{U\leq u} Z_T\right] = \mathbb{P}\left[U \leq u\right] = u,$$

and hence $U \sim U(0,1)$ under $\mathbb{Q}$. Next, for all $A \in \mathcal{F}_T^{\widetilde{W}}$ and $T \geq 0$ we have

$$\mathbb{Q}\left[U \leq u, A\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{U\leq u}\mathbb{1}_A Z_T\right] = \mathbb{P}\left[U \leq u\right]\mathbb{Q}\left[A\right] = \mathbb{Q}\left[U \leq u\right]\mathbb{Q}\left[A\right].$$

Hence U is independent of $\mathbb{F}^{\widetilde{W}}$ under $\mathbb{Q}$. Lastly, for all $A \in \mathcal{F}^{\widetilde{W}}$ and $t \geq 0$:

$$\mathbb{Q}\left[\tau > t, A\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\tau > t} \mid \mathcal{F}^{\widetilde{W}}\right]\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_A e^{-\int_0^t \gamma_u du}\right],$$

which shows that γ is the $(\mathbb{Q}, \mathbb{F}^{\widetilde{W}})$ intensity of τ. □

By Lemma 2.2, we have $\mathbb{Q}\left[\tau > t \,\middle|\, \mathcal{F}^{\widetilde{W}}\right] = e^{-\int_0^t \gamma_u du}$ for all $t > 0$. Starting with (2.4) where $\mathbb{Q}$ is now the minimal measure in the enlarged market, we again have that both (2.6) and (2.7) hold.

Statement of the main theorem

Having rigorously constructed the pricing measure, we now state our main result, Theorem 2.1, which proves existence of current coupon functions as defined in Definition 2.1. To this end, it suffices to show the operator $\mathcal{A}$ from equation (2.15) has a fixed point. As already pointed out, the contraction mapping theorem is not applicable. However, fixed points with nice regularity conditions do exist under the current setup:

Theorem 2.1. Under Assumptions 2.1-2.5, there exists a strictly positive current coupon function m such that (2.14) holds. The function m is locally α-Hölder continuous for all $\alpha \in (0, 1)$.

Chapter 3

Perturbation Analysis And Numerical Results

Theorem 2.1 asserts the existence of current coupon function. However, since our method of proof (see Chapter 4) does not involve the contraction mapping principle, we do not know if solutions are unique, nor do we have a method to compute them. One may certainly try an iterative procedure to solve (2.15), starting with an arbitrary function m_0 on D and, defining $m_n = \mathcal{A}[m_{n-1}], n = 1, 2, \ldots$, but in the absence of a contraction mapping, it is not clear if this procedure converges. In this chapter, we will perform a perturbation analysis where the intensity γ is perturbed off of a baseline intensity γ_0 which only depends upon the factors process X. The goal of this analysis is to uniquely identify m up to leading orders of the perturbation. With this identification, we will provide a numerical approximation to the fixed point and examine its performance.

3.1 Perturbation analysis

As a starting point, we present a proposition similar to [19, Lemma 2.1], which shows that there is a unique current coupon function in the special case where $\gamma = \gamma^0(X)$ only depends on the factor process X.

Proposition 3.1. Let Assumptions 2.1-2.4 hold. Assume $\gamma(x, m, z) = \gamma(x)$, where γ satisfies 1)-2) in Assumption 2.5. Then there exists a unique current coupon function $m = m_0$ that solves (2.14), which in this case reduces to

$$0 = \mathbb{E}^x \left[\int_0^T p(t, m(x))(m(x) - r_t) e^{-\int_0^t (r_u + \gamma(X_u))du} dt \right]. \tag{3.1}$$

The function m_0 is locally α-Hölder continuous in D for any $\alpha \in (0, 1)$.

Proof. Fix $x \in D$. For $t \leq T$ define

$$\begin{aligned} f(t) &:= \mathbb{E}^x \left[e^{-\int_0^t (r_u + \gamma(X_u))du} \right]; \qquad F(t) := \int_0^t f(u)du, \\ g(t) &:= \mathbb{E}^x \left[r_t e^{-\int_0^t (r_u + \gamma(X_u))du} \right]; \qquad G(t) := \int_0^t g(u)du. \end{aligned} \tag{3.2}$$

Next, define

$$h(T, m) := e^{mT} \int_0^T \left(1 - e^{-m(T-t)}\right) (mf(t) - g(t))\, dt; \qquad T > 0, m > 0.$$

Note that we will have a solution to (3.1) if for each $x \in D, T > 0$ we can find a number $m = m(x) > 0$ such that $h(T, m) = 0$. Indeed, this follows by plugging in $p(t, m)$ from (2.2) and noting that $e^{mT}, 1 - e^{-m(T-t)}$ are strictly positive. To find such an m, note that $h(0, m) = 0$ and

$$\frac{\partial}{\partial T} h(T, m) = m e^{mT} \int_0^T (mf(t) - g(t))\, dt = m e^{mT} (mF(T) - G(T)),$$

so that $h(T, m) = \int_0^T m e^{mt} (mF(t) - G(t))\, dt$. Now, for G from (3.2) we have

$$\begin{aligned} G(t) &= \mathbb{E}^x \left[\int_0^t (r_u \pm \gamma(X_u))\, e^{-\int_0^u (r_v + \gamma(X_v))dv} du \right] \\ &= 1 - \mathbb{E}^x \left[\int_0^t \gamma(X_u) e^{-\int_0^u (r_v + \gamma(X_v))dv} du \right] - \mathbb{E}^x \left[e^{-\int_0^t (r_v + \gamma(X_v))dv} \right] \\ &= H(t) - \dot{F}(t), \end{aligned}$$

where we have set $H(t) := 1 - \mathbb{E}^x \left[\int_0^t \gamma(X_u) e^{-\int_0^u (r_v + \gamma(X_v))dv} du \right]$. Since $r > 0$:

$$H(t) > 1 - \mathbb{E}^x \left[\int_0^t (r_u + \gamma(X_u))\, e^{-\int_0^u (r_v + \gamma(X_v))dv} du \right] = \dot{F}(t) > 0. \tag{3.3}$$

Coming back to h we have

$$h(T,m) = \int_0^T me^{mt}\left(mF(t) + \dot{F}(t) - H(t)\right)\,dt = m\left(e^{mT}F(T) - \int_0^T e^{mt}H(t)\,dt\right).$$

Hence, $h(T,m) = 0$ is equivalent to $F(T) - \int_0^T e^{-m(T-t)}H(t)dt = 0$. Using (3.3) it is clear that, as a function of m, the left hand side is strictly increasing, takes the value $F(T) - \int_0^T H(t)dt < 0$ at 0, and converges to $F(T) > 0$ as $m \uparrow \infty$. Thus, there is a unique m so that $h(T,m) = 0$. The statement regarding the regularity of m follows from Theorem 2.1, since fixed points are unique in this case. $\square$

Now that the rate m_0 is uniquely determined. We would like to investigate the uniqueness of the general endogenous current coupon function up to leading orders of expansion around $\varepsilon \approx 0$. To this end, one may consider linear perturbations using smooth functions with compact support. I.e. one may consider the following

Assumption 3.1. Let $\gamma(x,m,z) = \gamma_0(x) + \varepsilon\gamma_1(x,m,z)$ where γ_0 satisfies parts $1), 2)$ of Assumption 2.5 and $\gamma_1 \in C^2(E)$ is compactly supported with derivatives which are continuously extendable to $D \times \{0\} \times \{0\}$.

Under this assumption, as well as Assumptions 2.1-2.4, it follows from Theorem 2.1 that for $\varepsilon > 0$ small enough, there exists a continuous current coupon function m^ε. In fact, as the following proposition shows, m^ε is unique up to leading orders of ε and is explicitly identifiable.

Proposition 3.2. Let Assumptions 2.1–2.4 and 3.1 hold. For $\varepsilon > 0$ small enough, let m^ε be *any* current coupon function, continuous on D. Then we have

$$m^\varepsilon(x) = m_0(x) + \varepsilon m_1(x) + \mathrm{o}(\varepsilon). \tag{3.4}$$

Above, the convergence is locally uniform for $x \in D$. The function m_0 is the unique fixed point from Proposition 3.1 and for all $x \in D$,

$$m_1(x) = \frac{\mathbb{E}^x\left[\int_0^T (m_0(x) - r_t)\,p(t,m_0(x))\left(\int_0^t \gamma_1(X_u, m_0(x), m_0(X_u))du\right)e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]}{\mathbb{E}^x\left[\int_0^T \left((m_0(x) - r_t)p_m(t,m_0(x)) + p(t,m_0(x))\right)e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]}. \tag{3.5}$$

The point of Proposition 3.2 is that, although the formula for m_1 is complicated, it is explicitly identifiable given m_0, the unique fixed point in the baseline case. Additionally, as will be shown in the following section, the formula for m_1 makes perfect sense as long as the relevant random variables and expectations are well defined. In particular, γ_1 needs not be compactly supported, and γ_0, γ_1 need not be C^2 in order for the above formula to make sense.

Proof of Proposition 3.2. For $\varepsilon > 0$ small enough, let $m^\varepsilon(x)$ be *any* continuous solution of (2.14) with $\gamma = \gamma_0 + \varepsilon\gamma_1$. From Theorem 2.1 we know at least one such function exists. Since $p(t,m) \leq 1, \gamma \geq 0, r \geq 0$, the numerator in (2.15) is bounded above by

$$\mathbb{E}^x\left[\int_0^T r_t e^{-\int_0^t r_u du} dt\right] \leq 1. \tag{3.6}$$

Now using that γ_1 is compactly supported (and hence bounded above by some C_{γ_1}) and Lemma A.1 below, it follows for any $\varepsilon_0 > 0$ small enough, for $\varepsilon < \varepsilon_0$ the denominator in (2.15) is bounded below by $(1/2)\exp(-\varepsilon_0 C_{\gamma_1} T)\mathbb{E}^x\left[\int_0^{\frac{T}{2}} e^{-\int_0^t r_u + \gamma_0(X_u)du} dt\right]$, which is continuous and strictly positive in D as a function of x, where this latter fact follows from the elliptic Harnack inequality (see [31, Chapter 4]). Thus, m^ε is locally bounded on D, uniformly in $0 < \varepsilon < \varepsilon_0$.

Now, recall (2.14), specified to the current setup (without loss of generality we take $p(t,m) = 1 - e^{-m(T-t)}$, since we can always multiply both sides of (2.14) by $1 - e^{-m(x)T}$) :

$$0 = \mathbb{E}^x\left[\int_0^T \left(m^\varepsilon(x) - r_t\right) p(t, m^\varepsilon(x)) e^{-\int_0^t (r_u + \gamma_0(X_u) + \varepsilon\gamma_1(X_u, m^\varepsilon(x), m^\varepsilon(X_u)))du} dt\right]. \tag{3.7}$$

We first claim that for each $x \in D$, $\lim_{\varepsilon\downarrow 0} m^\varepsilon(x) = m_0(x)$. Indeed, since m^ε is locally bounded in D, uniformly in $0 < \varepsilon < \varepsilon_0$, it follows for each $x \in D$ that $\{m^\varepsilon(x)\}_{\varepsilon < \varepsilon_0}$ is uniformly bounded. Let $\varepsilon_n \to 0$ and assume $m^{\varepsilon_n}(x) \to \tilde{m}(x)$ for some $\tilde{m}(x)$. Since γ_1 is continuous and compactly supported, the dominated convergence theorem yields

$$0 = \mathbb{E}^x\left[\int_0^T (\tilde{m}(x) - r_t) p(t, \tilde{m}(x)) e^{-\int_0^t (r_u + \gamma_0(X_u))du} dt\right],$$

and so by the uniqueness of m_0 from Proposition 3.1 we know that $\tilde{m}(x) = m_0(x)$. Since this works for all subsequences $\varepsilon_n \to 0$ the convergence result holds. In fact, since m^ε is continuous, the above convergence is uniform on compact subsets of D. Next, define $\overline{m}$ through

$$m^\varepsilon(x) = m_0(x) + \varepsilon\overline{m}(x,\varepsilon); \qquad x \in D, \varepsilon < \varepsilon_0. \tag{3.8}$$

We first show that the term $\overline{m}(x,\varepsilon)$ is upper bounded uniformly in ε for small ε. Using Taylor's theorem we write

$$\begin{aligned} p(t,m^{\varepsilon}(x)) &= p(t,m_0(x)) + \varepsilon\overline{m}(x,\varepsilon)p_m(t,\xi(x,\varepsilon)); \\ e^{-\varepsilon\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du} &= 1-\varepsilon\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du \\ &\quad + \frac{1}{2}\varepsilon^2\left(\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du\right)^2\hat{\xi}(x,\varepsilon,t), \end{aligned} \tag{3.9}$$

where

$$|\xi(x,\varepsilon)-m_0(x)| \le \varepsilon|\overline{m}(x,\varepsilon)|; \qquad 0\le \hat{\xi}(x,\varepsilon,t) \le e^{\varepsilon\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du}.$$

Plugging these expansions back into (3.7), using that

$$\mathbb{E}^x\left[\int_0^T (m_0(x)-r_t)p(t,m_0(x))e^{-\int_0^t(r_u+\gamma_0(X_u))du}dt\right]=0,$$

where the equality follows from Proposition 3.1, and rearranging terms, we can express the term $\overline{m}(x,\varepsilon)$ by the following:

$$\overline{m}(x,\varepsilon) = \frac{\begin{aligned}\mathbb{E}^x\Bigg[&\textstyle\int_0^T (m_0(x)-r_t)p(t,m_0(x))\Big[\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u)) \\ &\quad + \tfrac{1}{2}\varepsilon\big(\int_0^t \gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du\big)^2\hat{\xi}(x,\varepsilon,t)\Big]e^{-\int_0^t r_u+\gamma_0(X_u)du}dt\Bigg]\end{aligned}}{\mathbb{E}^x\left[\int_0^T \big((m_0(x)-r_t)p_m(t,\xi(x,\varepsilon))+p(t,m^{\varepsilon}(x))\big)e^{-\int_0^t r_u+\gamma_0(X_u)+\varepsilon\gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du}dt\right]}. \tag{3.10}$$

Under the given regularity, local boundedness and compactly supported assumptions, the numerator in equation (3.10) is bounded above for $\varepsilon<\varepsilon_0$ and $\varepsilon_0>0$. It remains to show the denominator is bounded below uniformly in ε when ε is small. For this we write

$$\begin{aligned} &\mathbb{E}^x\left[\int_0^T \big((m_0(x)-r_t)p_m(t,\xi(x,\varepsilon))+p(t,m^{\varepsilon}(x))\big)e^{-\int_0^t r_u+\gamma_0(X_u)+\varepsilon\gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du}dt\right] \\ &\qquad = L(x,\varepsilon)+R(x,\varepsilon), \end{aligned}$$

where for ease of presentation we have set

$$L(x,\varepsilon) := \mathbb{E}^x\left[\int_0^T \big((m_0(x)-r_t)p_m(t,m_0(x))+p(t,m_0(x))\big)e^{-\int_0^t r_u+\gamma_0(X_u)+\varepsilon\gamma_1(X_u,m^{\varepsilon}(x),m^{\varepsilon}(X_u))du}dt\right].$$

Borrowing the notations in the proof of Proposition 3.1, we know that

$$
\begin{aligned}
&\mathbb{E}^x\left[\int_0^T \big((m_0(x)-r_t)p_m(t,m_0(x))+p(t,m_0(x))\big)e^{-\int_0^t r_u+\gamma_0(X_u)du}dt\right]\\
&\quad=\frac{\partial}{\partial m}\Big|_{m=m_0(x)}\mathbb{E}^x\left[\int_0^T (m-r_t)p(t,m)e^{-\int_0^t r_u+\gamma_0(X_u)du}dt\right]\\
&\quad=\frac{\partial}{\partial m}\Big|_{m=m_0(x)}\left(mF(T)-m\int_0^T H(t)e^{-m(T-t)}dt\right)\\
&\quad\geq m_0(x)e^{-m_0(x)T}\int_0^T H_t(T-t)dt\\
&\quad\geq \frac{m_0(x)T^2e^{-m_0(x)T}}{2}\mathbb{E}^x\left[e^{-\int_0^T r_u+\gamma_0(X_u)du}\right].
\end{aligned}
$$

Therefore the term $L(x,\varepsilon)$ is bounded away from 0 uniformly for $\varepsilon<\varepsilon_0$. As for the term $R(x,\varepsilon)$, using that $\lim_{\varepsilon\downarrow 0} m^\varepsilon(x)=m_0(x)$ it is evident that $R(x,\varepsilon)\to 0$ as $\varepsilon\downarrow 0$, locally uniformly in x.

We next turn to the proof of (3.5). Plugging the second order Taylor expansion

$$
p(t,m^\varepsilon(x))=p(t,m_0(x))+\varepsilon\overline{m}(x,\varepsilon)p_m(t,m_0(x))+\frac{1}{2}\varepsilon^2\overline{m}(x,\varepsilon)^2p_{mm}(t,\xi(x,\varepsilon)),
$$

as well as (3.9) back into (3.7), we then collect terms by explicit powers of ε. The first order (in ε) terms inside expectation and time integral are

$$
\begin{aligned}
&\overline{m}(x,\varepsilon)p(t,m_0(x))+\overline{m}(x,\varepsilon)(m_0(x)-r_t)p_m(t,m_0(x))\\
&\qquad-(m_0(x)-r_t)p(t,m_0(x))\int_0^t\gamma_1(X_u,m^\varepsilon(x),m^\varepsilon(X_u))du.
\end{aligned}
$$

Using the given regularity, local boundedness and compactly supported assumptions, and that $\overline{m}(x,\varepsilon)$ is bounded above uniformly in ε for small ε, we see that all higher order terms together are $O(\varepsilon^2)$, uniformly on compact subsets of D. Since the zeroth order term vanishes, we may divide (3.7) by $\varepsilon>0$ to obtain

$$
\begin{aligned}
0=\;&\overline{m}(x,\varepsilon)\mathbb{E}^x\left[\int_0^T\left(p(t,m_0(x))+(m_0(x)-r_t)p_m(t,m_0(x))\right)e^{-\int_0^t(r_u+\gamma_0(X_u))du}dt\right]\\
&+\mathbb{E}^x\left[\int_0^T(m_0(x)-r_t)p(t,m_0(x))\int_0^t\gamma_1(X_u,m^\varepsilon(x),m^\varepsilon(X_u))du\,e^{-\int_0^t(r_u+\gamma_0(X_u))du}dt\right]+\frac{O(\varepsilon^2)}{\varepsilon},
\end{aligned}
$$

which can be re-written as

$$\overline{m}(x,\varepsilon) = \frac{\mathbb{E}^x\left[\int_0^T (m_0(x)-r_t)p(t,m_0(x))\int_0^t \gamma_1(X_u,m^\varepsilon(x),m^\varepsilon(X_u))du\, e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]+\frac{O(\varepsilon^2)}{\varepsilon}}{\mathbb{E}^x\left[\int_0^T \left(p(t,m_0(x))+(m_0(x)-r_t)p_m(t,m_0(x))\right)e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]}$$
$$= m_1(x) + \frac{\mathbb{E}^x\left[\int_0^T (m_0(x)-r_t)p(t,m_0(x))R(t;x,\varepsilon)e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]+\frac{O(\varepsilon^2)}{\varepsilon}}{\mathbb{E}^x\left[\int_0^T \left(p(t,m_0(x))+(m_0(x)-r_t)p_m(t,m_0(x))\right)e^{-\int_0^t (r_u+\gamma_0(X_u))du}dt\right]},$$

where

$$R(t;x,\varepsilon) := \int_0^t \left(\gamma_1(X_u,m^\varepsilon(x),m^\varepsilon(X_u)) - \gamma_1(X_u,m_0(x),m_0(X_u))\right)du.$$

It thus follows by the dominated convergence theorem that $\lim_{\varepsilon\downarrow 0}\overline{m}(x,\varepsilon) - m_1(x) = 0$ with uniform convergence on compact subsets of D, finishing the result. □

As will be shown in the next section, linear perturbations in the form of Assumption 3.1 and the associated closed-form formula for m^ε in Proposition 3.2 perform well in practice. Nonetheless, we can consider a more general setting: a family of intensity functions $\gamma(x,m,z;\varepsilon)$ defined on $D\times[0,\infty)\times[0,\infty)$ and indexed by $\varepsilon\in[0,\varepsilon_0]$ for some fixed $\varepsilon_0>0$. In fact, we will treat the intensity function $\gamma(x,m,z;\varepsilon)$ as a function of four variables: x, m, z and ε, defined on $D\times[0,\infty)\times[0,\infty)\times[0,\varepsilon_0]$. As before, to ease presentation, we define $E := D\times(0,\infty)\times(0,\infty)$ and $E_n := D_n\times(0,n)\times(0,n), n\in\mathbb{N}$.

Assumption 3.2. The family of functions $\{\gamma(\,\cdot\,;\varepsilon)\}_{0\le\varepsilon\le\varepsilon_0}$ satisfies the following regularity properties:

1. For all $\varepsilon\in[0,\varepsilon_0]$, the function $\gamma(x,m,z;\varepsilon): E\mapsto[0,\infty)$ satisfies 1)-3) of Assumption 2.5. Furthermore, the bounding constants L_γ, B_γ are uniform for all ε.

2. $\gamma(x,m,z;\varepsilon)$ is C^2 in ε for $\varepsilon\in(0,\varepsilon_0)$, and the first derivative $\gamma_\varepsilon(x,m,z;\varepsilon)$ can be continuously extended to $\varepsilon=0$. $\gamma_\varepsilon(x,m,z;0)$ can be continuously extend to $\bar{E_n}$ and is Lipschitz continuous on E_n for each $n\in\mathbb{N}$. Furthermore, both $\gamma_\varepsilon(x,m,z;0)$ and $\gamma_{\varepsilon\varepsilon}(x,m,z;\varepsilon)$ are uniformly upper bounded.

3. There exists a compact subset of D, denoted by K, such that $\gamma_\varepsilon(x,m,z;0)=\hat{\gamma}(x)$ for $x\in D\setminus K$ and for some function $\hat{\gamma}$ defined on $D\setminus K$*.

*Essentially we want $\gamma_\varepsilon(x,m,z;0)$ to be “flat” in variables (m,z) when x is outside of the compact set K.

The above setup is a strict generalization of smooth, compactly supported perturbations that we have seen earlier. For $\varepsilon \in [0, \varepsilon_0]$, by Theorem 2.1, we let $m^\varepsilon(x)$ be a solution of the fixed point equation

$$m^\varepsilon(x) = \frac{\mathbb{E}^x\left[\int_0^T r_t p(t, m^\varepsilon(x)) e^{-\int_0^t r_\theta + \gamma(X_\theta, m^\varepsilon(x), m^\varepsilon(X_\theta);\varepsilon) d\theta} dt\right]}{\mathbb{E}^x\left[\int_0^T p(t, m^\varepsilon(x)) e^{-\int_0^t r_\theta + \gamma(X_\theta, m^\varepsilon(x), m^\varepsilon(X_\theta);\varepsilon) d\theta} dt\right]},$$

or equivalently

$$\mathbb{E}^x\left[\int_0^T (m^\epsilon(x) - r_t) p(t, m^\epsilon(x)) e^{-\int_0^t X_\theta^{(1)} + \gamma(X_\theta, m^\varepsilon(x), m^\varepsilon(X_\theta);\varepsilon) d\theta} dt\right] = 0. \tag{3.11}$$

The "baseline-intensity" case is specified by the following

Assumption 3.3. When $\varepsilon = 0$, the intensity function reduces to a function of the factor process X:

$$\gamma(x, m, z; 0) = \gamma^0(x),$$

for some function γ^0 that satisfies 1)-2) of Assumption 2.5. In particular, when $\epsilon = 0$, the baseline endogenous current coupon function $m_0(x)$ solves the equation

$$\mathbb{E}^x\left[\int_0^T (m_0(x) - r_t) p(t, m_0(x)) e^{-\int_0^t r_\theta + \gamma^0(X_\theta) d\theta} dt\right] = 0. \tag{3.12}$$

We close this section by the following

Proposition 3.3. *Any* fixed point m^ϵ that solves (3.11) satisfies

$$m^\epsilon(x) = m_0(x) + \epsilon m_1(x) + o(\epsilon), \tag{3.13}$$

uniformly on compact subsets of D, where m_1 is given by

$$\frac{\mathbb{E}^x\left[\int_0^T \big(m_0(x) - r_t\big) p(t, m_0(x)) e^{-\int_0^t r_\theta + \gamma^0(X_\theta) d\theta} \big(\int_0^t \gamma_\varepsilon(X_\theta, m_0(x), m_0(X_\theta); 0) d\theta\big) dt\right]}{\mathbb{E}^x\left[\int_0^T \big(p(t, m_0(x)) + (m_0(x) - r_t) p_m(t, m_0(x))\big) e^{-\int_0^t r_\theta + \gamma^0(X_\theta) d\theta} dt\right]}.$$

Proof. The proof is much like that of Proposition 3.2 and will be omitted. Here we only point out that, Item 3 of Assumption 3.2 implies

$$\lim_{\epsilon\to 0}\mathbb{E}^x\left[\int_0^T (m_0(x)-r_t)p(t,m_0(x))\left(e^{-\int_0^t r_\theta+\gamma^0(X_\theta)d\theta}\int_0^t \gamma_\varepsilon(X_\theta,m^\epsilon(x),m^\epsilon(X_\theta);0)d\theta\right)dt+O(\epsilon)\right]$$
$$=\mathbb{E}^x\left[\int_0^T (m_0(x)-r_t)p(t,m_0(x))e^{-\int_0^t r_\theta+\gamma^0(X_\theta)d\theta}\left(\int_0^t \gamma_\varepsilon(X_\theta,m_0(x),m_0(X_\theta);0)d\theta\right)dt\right],$$

which is used in the proof of the uniform convergence on compact subsets. □

3.2 Numerical approximations

Proposition 3.2 offers a natural numerical approximation for computing current coupon functions. Namely, for a given intensity function γ we first identify if there is a decomposition

$$\gamma(x,m,z)=\gamma_0(x)+\gamma_1(x,m,z), \tag{3.14}$$

and then we compute m_0 from γ_0, define m_1 as in (3.5) and output the approximation from Proposition 3.2 at $\varepsilon=1$: i.e.

$$m(x)\approx m_0(x)+m_1(x). \tag{3.15}$$

Note that this approximation is obtainable as long as m_0, m_1 are well defined, and does not necessarily require γ_0,γ_1 to satisfy the regularity and growth conditions in Assumption 2.5. The computational advantage of this approximation over naive contraction method is clear: for each $x\in D$ along a given mesh, there is only one Monte Carlo simulation needed to compute m_1. Note also that a decomposition of the form (3.14) is always obtainable as one may take $\gamma_0=0$. In this instance, $m_0(x)$ from Proposition 3.1 solves

$$\frac{1-e^{-m_0(x)T}}{m_0(x)T}=\frac{1}{T}\int_0^T \mathbb{E}^{\mathbb{Q}^x}\left[e^{-\int_0^t r_u du}\right]dt; \qquad x\in D. \tag{3.16}$$

For many models of interest (e.g. see [37, Example 6.5.2] for when r follows a CIR process), the expectation on the right hand side is explicitly computable and m_0 is easily obtained by inverting the strictly decreasing function $y\mapsto(1-e^{-y})/y$. Alternatively, one can take $\gamma_0(x)=\gamma$ and $\gamma_1(x,m,z)=\gamma(x,m,z)-\gamma$ if there is some $\gamma>0$ such that $\gamma(x,m,z)\geq\gamma$. In this case, for constant $\gamma_0 = \gamma$, a simple calculation shows that m_0 satisfies

$$\frac{1 - e^{-m_0(x)T}}{m_0(x)} = \int_0^T e^{-\gamma t}\mathbb{E}^{\mathbb{Q}^x}\left[e^{-\int_0^t r_u du}\right]\left(1 + \gamma\frac{1 - e^{-m_0(x)(T-t)}}{m_0(x)}\right)dt, \tag{3.17}$$

which is easy to compute numerically given an explicit formula for $\mathbb{E}^x\left[e^{-\int_0^t r_u}\right]$ exists. Once m_0 is known, one may then compute m_1 using Monte Carlo simulation.

We now take an example similar to that in [20, Section 6] and assume X is a one dimensional CIR process (i.e. $d = 1$, $D = (0, \infty)$ and $X^{(1)} = r$ is a CIR process) and γ takes the form

$$\gamma(x, m, z) = \gamma + k(m - z)^+. \tag{3.18}$$

In other words, there is a constant baseline prepayment intensity γ, and the full intensity is adjusted upwards by the difference between the contract rate m and refinancing rate z, when this value is positive. This adjustment is then scaled by a factor $k > 0$. As in [20], we will assume $k = 5$ so this is not necessarily a small perturbation off the baseline case.

We will perform two approximations. The first sets $\gamma_0(x) = 0$, $\gamma_1(x) = \gamma + k(m - z)^+$, computes m_0 from (3.16), and then computes m_1 from (3.5). The second approximation takes $\gamma_0(x) = \gamma, \gamma_1(x, m, z) = k(m - z)^+$, computes m_0 from (3.17) and then m_1 from (3.5). For each approximation we compare $m_0 + m_1$ to the "theoretical fixed point" m obtained by naive contraction, which will be described in section A.3 below, which in this instance converges rapidly (e.g. after approximately five iterations) to a fixed function for a random initial guess $m^{(0)}$. The model parameters are the same as [20]. If we write $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$, then $\kappa = 0.25, \theta = 0.06$, $\sigma = 0.1$. Additionally, $\gamma = 0.045$ and $k = 5$.

Figure 3.1 below compares $m_0 + m_1$ to m when $\gamma_0(x) = 0$. As shown in the right plot, the approximation does very well, differing by less than 20 basis points (for an absolute level of $4\% - 12\%$) within the $(2.5\%, 97.5\%)$ percentiles of the CIR invariant distribution. In the "middle" of the invariant distribution, the approximation is virtually identical to the naive fixed point, with errors consistently between $0 - 5$ basis points.

Figure 3.2 below makes a similar comparison, using $\gamma_0(x) = \gamma$. Here, the performance is significantly improved with the $(2.5\%, 90\%)$ percentiles in that the approximation m_0+m_1 is nearly identical to the function m obtained through naive contraction. Indeed, the difference between $m_0 + m_1$ and m is less than 3 basis points. However, for large values of r the error is a bit larger than in the previous method, approaching approximately 7 basis points.

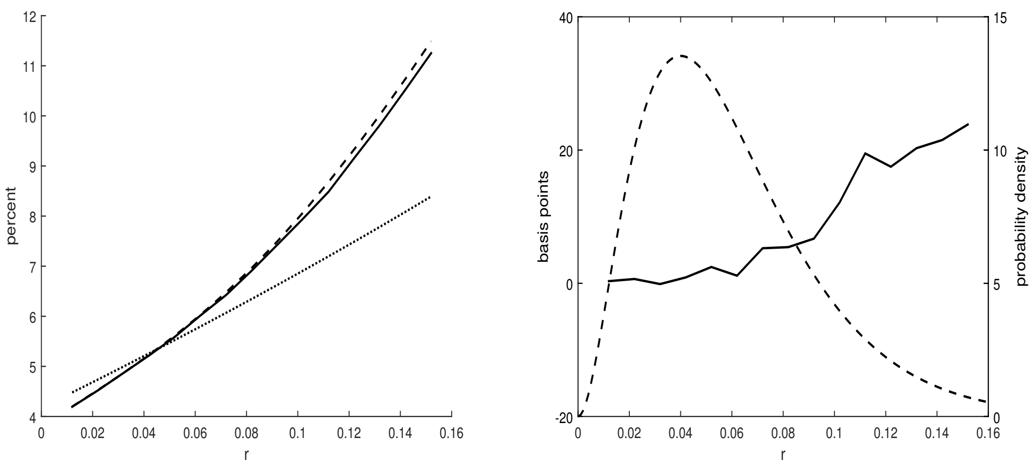

Figure 3.1: Current coupon functions (left plot) and errors (right plot) as a function of the underlying CIR factor. In the left plot, the thick-dash plot is the current coupon function m obtained through naive contraction. The solid line is the approximation $m_0 + m_1$ while the thin dash plot is m_0. Values are given in percentage points. For the right plot, the error is the difference (in basis points) between m and $m_0 + m_1$. Also in the right plot is the invariant pdf for the CIR process r. m_0 is calculated with $\gamma_0(x) = 0$ and m_1 is calculated with $\gamma_1(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*, *Mathematica*.

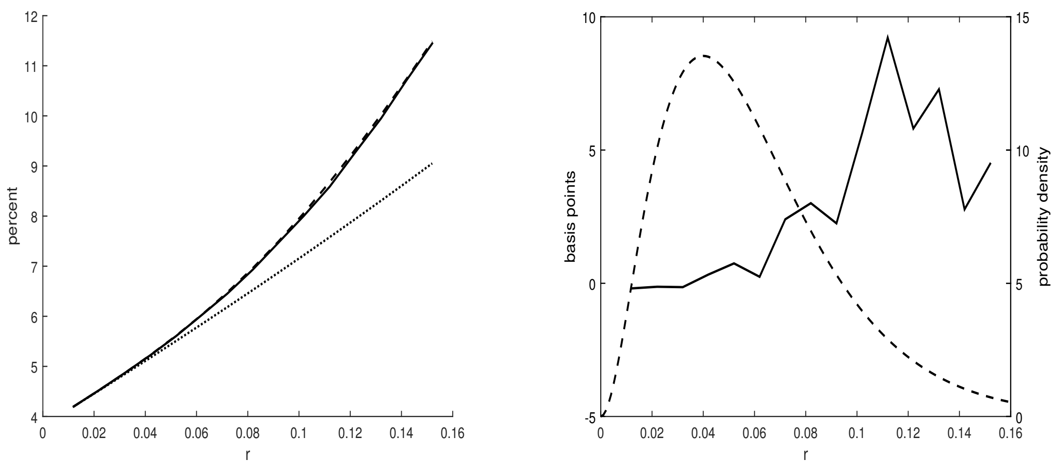

Figure 3.2: Current coupon functions (left plot) and errors (right plot) as a function of the underlying CIR factor. In the left plot, the thick-dash plot is the current coupon function m obtained through naive contraction. The solid line is the approximation $m_0 + m_1$ while the thin dash plot is m_0. Values are given in percentage points. For the right plot, the error is the difference (in basis points) between m and $m_0 + m_1$. Also in the right plot is the invariant pdf for the CIR process r. m_0 is calculated with $\gamma_0(x) = \gamma$ and m_1 is calculated with $\gamma_1(x, m, z) = k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*, *Mathematica*.

Chapter 4

Proof Of The Main Theorem

4.1 Outline of the proof

The goal is to show the existence of a function $m : D \mapsto (0, \infty)$ that satisfies $m(x) = \mathcal{A}[m](x)$ for all $x \in D$, where $\mathcal{A}$ is defined in (2.15), and which possesses the required regularity conditions. To do this, we will appeal to Schaefer's fixed point theorem (see [11]), which we now state

Theorem 4.1. Let K be a closed convex subset of a Banach space X with $0 \in K$. Assume $\mathcal{A} : K \mapsto K$ is continuous, compact and such that the set $\{u \in K \mid u = \lambda\mathcal{A}[u], 0 \leq \lambda \leq 1\}$ is bounded. Then $\mathcal{A}$ has a fixed point in K.

Remark 4.1. In [11], Schaefer's theorem is stated for an operator $\mathcal{A}$ which maps a Banach space to itself. However, we are applying the results when $\mathcal{A}$ maps a closed convex subset of a Banach space (with 0 in the set) to itself. The proof in [11] carries over easily to our setting. In fact, choose $M > 0$ such that $\|u\| < M$ if $u = \lambda\mathcal{A}[u]$ for $0 \leq \lambda \leq 1$ and define operator $\tilde{\mathcal{A}}$ by

$$\tilde{\mathcal{A}}[u] = \begin{cases} \mathcal{A}[u]; & \|\mathcal{A}[u]\| \leq M, \\ \frac{M\mathcal{A}[u]}{\|\mathcal{A}[u]\|}; & \|\mathcal{A}[u]\| > M. \end{cases}$$

Clearly $\tilde{\mathcal{A}}$ maps $B(0, M) \cap K$ to $B(0, M) \cap K$. Now set C to be the closed convex hull of $\tilde{\mathcal{A}}(B(0, M) \cap K)$. Since $\mathcal{A}$ and thus $\tilde{\mathcal{A}}$ are compact mappings, $C \subset K$ is a compact, convex subset of X. Furthermore $\tilde{\mathcal{A}}$ maps C to itself. By Schauder's fixed point theorem, there exists $u \in C$ such that $\tilde{\mathcal{A}}[u] = u$. We claim that u is also a fixed point of $\mathcal{A}$. Suppose this is not true, then we would have $\|\mathcal{A}[u]\| > M$ and $u = \lambda\mathcal{A}[u]$ for $\lambda := \frac{M}{\|\mathcal{A}[u]\|} < 1$. But this implies

$M > \|u\| = \left\|\tilde{\mathcal{A}}[u]\right\| = M$, a contradiction.

We would like to choose X as the space of α-Hölder continuous functions on D, and K as the subset of non-negative functions. However, we should not expect the operator $\mathcal{A}$ defined in (2.15) to possess the requisite continuity and compactness properties, due to the following reasons:

- The domain D is not necessarily bounded.
- The coefficient matrix a is not necessarily uniformly elliptic on D.
- $\mathcal{A}[m]$ possesses at most the same order of regularity as the input function m.

Therefore we must first localize the problem, and obtain a fixed point at the localized level using Schaefer's theorem. We will then unwind the localization to get a global fixed point. Hence, the outline of our proof is:

1. Define an operator $\mathcal{A}^n$, and show that $\mathcal{A}^n$ has a fixed point $m^n > 0$ which is defined on the set D_n introduced in Assumption 2.1, and is α-Hölder continuous for $\alpha \in (0, 1)$.
2. For each $\tilde{n} \geq 1$, obtain uniform (in n) Hölder norm estimates on $D_{\tilde{n}}$ for m^n, $n \geq \tilde{n} + 1$.
3. Show that m^n has a convergent subsequence with limit m that solves the full fixed point problem.

To begin with, we need to obtain several *a-priori* Hölder norm estimates of solutions to certain partial differential equations, which are defined through expectations.

4.2 *A priori* estimates of Hölder norms

We first recall the standard definitions of elliptic and parabolic Hölder spaces. A more thorough introduction to these spaces can be found in [16] for the elliptic case and [14, 11, 25] for the parabolic case.

Fix $n \in \mathbb{N}$. As D_n is bounded with smooth boundary, for $k \in N$ we denote by $C^k(D_n)$ the collection of functions u on D_n such that all partial derivatives of order $\leq k$ are continuous, and by $C^k(\overline{D_n})$ the subspace of functions with partial derivatives of order $\leq k$ that can be continuously extended to ∂D_n.

For a given function u on D_n and $\alpha \in (0, 1]$, set

$$|u|_{D_n} := \sup_{x \in D_n} |u(x)|,$$
$$[u]_{\alpha, D_n} := \sup_{\substack{x,y \in D_n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

The elliptic Hölder space $C^{k,\alpha}(\overline{D_n})$ is defined as the subset of $C^k(\overline{D_n})$ consisting of functions f whose partial derivatives of order $\leq k$ have finite $|\cdot|_{D_n}$ norm and whose partial derivatives of order k have finite $[\cdot]_{\alpha, D_n}$ seminorm. For $f \in C^{k,\alpha}(\overline{D_n})$ define the norm

$$\|f\|_{k,\alpha,\overline{D_n}} = |f|_{D_n} + \sum_{j=1}^{k} \sup_{|\beta| = j} \left| D^\beta f \right|_{D_n} + \sum_{|\beta| = k} \left[D^\beta f \right]_{\alpha, D_n},$$

where β is a multi-index: $\beta = (\beta_1, \cdots, \beta_d)$ such that $|\beta| = \sum_{i=1}^d \beta_i$, $D^\beta u = \partial^{|\beta|}_{\beta_1, \cdots, \beta_d}$. We write $C^\alpha(\overline{D_n})$ for $C^{0,\alpha}(\overline{D_n})$ and $\|\cdot\|_{\alpha,\overline{D_n}}$ for $\|\cdot\|_{0,\alpha,\overline{D_n}}$. We also note that $C^{k,\alpha}(\overline{D_n})$ with norm $\|\cdot\|_{k,\alpha,\overline{D_n}}$ is a Banach space.

For the parabolic Hölder space, we first define the parabolic domain $Q_n := (0, T) \times D_n$. For $P_1 = (t, x)$, $P_2 = (\bar{t}, \bar{x}) \in Q_n$, the parabolic distance between P_1, P_2 is defined by

$$d(P_1, P_2) = (|x - \bar{x}|^2 + |t - \bar{t}|)^{\frac{1}{2}}. \tag{4.1}$$

For $\alpha \in (0, 1]$, we next recall the Hölder norms defined on Q_n:

$$|u|_{0,n} := \sup_{P \in Q_n} |u(P)|;$$
$$[u]_{\alpha,n} := \sup_{\substack{P_1, P_2 \in Q_n \\ P_1 \neq P_2}} \frac{|u(P_1) - u(P_2)|}{d(P_1, P_2)^\alpha};$$
$$|u|_{\alpha,n} := |u|_{0,n} + [u]_{\alpha,n};$$
$$|u|_{2+\alpha,n} := |u|_{0,n} + \sum_{i=1}^d |D_i u|_{0,n} + \sum_{i,j=1}^d \left| D^2_{ij} u \right|_{\alpha,n} + |D_t u|_{\alpha,n},$$

where $D_i = D^1_{0,\cdots,1,\cdots,0}$ and $D^2_{ij} = D^2_{0,\cdots,1,\cdots,1,\cdots,0}$ are the corresponding partial differential operators with the 1-s at i and i, j, respectively.

We now prove three lemmas which establish *a priori* estimates (both local and global) for the

$\|\cdot\|_{\alpha,\overline{D}_n}$ norm and $\|\cdot\|_{2,\alpha,\overline{D}_n}$ norm of some conditional expectation expressions, which will be essential in the proof of Theorem 2.1. For each n, denote by τ_n the first exit time of the process X from D_n. Each of the lemmas below concern the function $u : D_n \mapsto \mathbb{R}$ defined by

$$u(x) := \mathbb{E}^x \left[\int_0^{T\wedge\tau_n} g(t, X_t) e^{-\int_0^t h(u,X_u)du} dt \right]; \qquad x \in D_n, \tag{4.2}$$

where $g(t, x)$ and $h(t, x)$ are functions defined on Q_n. To ease presentation, the bounding constants below may change from line to line, and the n in the constants is assumed to absorb $K_1(n)$, $K_2(n)$, $B_\gamma(n)$, $L_\gamma(n)$ of Assumptions 2.1-2.5, as well as the dimension d, parabolic domain Q_n, and time horizon T. However we will make the dependence upon α (the Hölder exponent) explicit.

Lemma 4.1 (Global $C^{2,\alpha}$ estimate)**.** Let $u(x) : D_n \to \mathbb{R}$ be defined as in (4.2), and assume for some $\alpha \in (0, 1]$ and $K_3(n) > 0$, that g and h satisfy

$$|g|_{\alpha,n} < \infty; \quad |h|_{\alpha,n} \le K_3(n);$$
$$\lim_{y\to x, t\to T} g(t, y) = 0, \ x \in \partial D_n.$$

Then

$$\|u\|_{2,\alpha,\overline{D_n}} \le C(n, K_3(n), \alpha) \cdot |g|_{\alpha,n}.$$

Proof. We first write $u(x) = U(0, x)$, where

$$U(t, x) := \mathbb{E}^x \left[\int_t^{T\wedge\tau_n} g(s, X_s) e^{-\int_t^s h(\theta,X_\theta)d\theta} \, dt \right]; \quad t \le T, \ x \in D_n. \tag{4.3}$$

Under the given regularity and ellipticity assumptions, [14, Theorem 3.7] implies that $U(t, x)$ is the unique solution to the Cauchy-Dirichlet problem

$$\begin{cases} U_t + \mathcal{L}U - h(t, x)U = -g(t, x), \quad (t, x) \in Q_n, \\ U(T, x) = 0, \quad x \in D_n, \\ U(t, x) = 0, \quad (t, x) \in [0, T] \times \partial D_n. \end{cases} \tag{4.4}$$

Above, $\mathcal{L}$ denotes the infinitesimal generator of X, given by $\mathcal{L}U = \frac{1}{2}A_{ik}(x)\frac{\partial^2 U}{\partial x_i \partial x_k} + b_i(x)\frac{\partial U}{\partial x_i}$. The boundary Schauder estimate ([14, Theorems 3.6, 3.7]) for parabolic equations (note that g satisfies the compatibility condition of [14, Theorems 3.7]) yields

$$\|u\|_{2,\alpha,\overline{D_n}} \leq |U|_{2+\alpha,n} \leq C(n, K_3(n), \alpha) \cdot |g|_{\alpha,n} .$$

□

Lemma 4.2 (Global C^α estimate)**.** Let $u(x) : D_n \to \mathbb{R}$ be defined as in (4.2), and assume for some $\alpha_0 \in (0, 1]$ and $K_4(n) > 0$, that g, h satisfy

$$|g|_{\alpha_0,n} < \infty; \quad |h|_{\alpha_0,n} < \infty; \quad |h|_{0,n} \leq K_4(n).$$

Then for all $\alpha \in (0, 1)$:

$$\|u\|_{\alpha,\overline{D}_n} \leq C(n, K_4(n), \alpha) \cdot |g|_{0,n}.$$

Proof. Since g, h are α_0-Hölder continuous, we can invoke the stochastic representation of solutions to parabolic PDEs (see [15, Theorem 5.2]) to write $u(x) = U(0, x)$, where $U(t, x)$ solves the parabolic PDE in (4.4). Now we invoke the boundary $W_p^{2,1}$ estimate for parabolic equations ([25, Theorem 7.32]) to get, for all $p > 1$,

$$\|U\|_{L^p(Q_n)} + \|DU\|_{L^p(Q_n)} + \|U_t\|_{L^p(Q_n)} \leq C(n, K_4(n), p) \cdot |g|_{0,n} .$$

Now let $\alpha \in (0, 1)$. Since Q_n is a Lipschitz domain, the Sobolev embedding (see [11, Theorem 5.5]) yields

$$\|u\|_{\alpha,\overline{D}_n} \leq |U|_{\alpha,n} \leq C(n, \alpha)\|U\|_{W^{1,p}(Q_n)} \leq C(n, K_4(n), \alpha) \cdot |g|_{0,n},$$

where $p > 1$ is sufficiently large depending upon α. □

Lemma 4.3 (Interior C^α estimate)**.** Let $u : D_n \mapsto \mathbb{R}$ be defined in (4.2) and assume for some $\alpha_0 \in (0, 1]$ and $K_4(n) > 0$, that g, h satisfy

$$|g|_{\alpha_0,n} < \infty; \quad |h|_{\alpha_0,n} < \infty; \quad |h|_{0,n} \leq K_4(n).$$

Then for all $\alpha \in (0, 1)$ and for all $m < n$:

$$\|u\|_{\alpha,\overline{D}_m} \leq C(m, K_4(m + 1), \alpha) \cdot (|g|_{0,m+1} + |U|_{0,m+1}) ,$$

where U satisfies the parabolic PDE (4.4).

Proof. Again we write $u(x) = U(0, x)$, where $U(t, x)$ satisfies (4.4). Set

$$Q'_m := \left(0, \frac{T}{2}\right) \times D_m.$$

For $p \geq 2$, the interior $W_p^{2,1}$ estimate for parabolic equations ([25, Theorem 7.22]) yields

$$\|U\|_{L^p(Q'_m)} + \|DU\|_{L^p(Q'_m)} + \|U_t\|_{L^p(Q'_m)} \leq C(m, K_4(m+1), p)\left(|g|_{0,m+1} + |U|_{0,m+1}\right).$$

Since Q'_m is a Lipschitz domain, Sobolev embedding yields for any $\alpha \in (0,1)$ by taking p large enough that

$$\begin{aligned}\|u\|_{\alpha,\overline{D}_m} &\leq \|U\|_{\alpha,Q'_m} \leq C(m,\alpha)\|U\|_{W^{1,p}(Q'_m)} \\ &\leq C(m, K_4(m+1), \alpha)\left(|g|_{0,m+1} + |U|_{0,m+1}\right),\end{aligned}$$

where above we have set $\|\cdot\|_{\alpha,Q'_m}$ as the α-Hölder norm on the parabolic domain Q'_m. $\square$

4.3 The localized problem

Throughout this section we enforce Assumptions 2.1-2.5. To localize the original fixed point problem (2.15), we first seek functions $m = m^n$ on D_n that satisfies for each $x \in D_n$:

$$\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} (m(x) - r_t)p(t, m(x))e^{-\int_0^t (r_u + \gamma(X_u, m(x), m(X_u)))du} dt\right] + \frac{m(x)^2}{n(1 - e^{-m(x)T})} = 0. \tag{4.5}$$

Note that the second term in (4.5) is a correction term that vanishes as $n \uparrow \infty$. This term is in place to establish local regularity of solutions m^n.

To establish existence of solutions to (4.5), we let $\alpha \in (0,1)$ and fix a function $\eta \in \mathbb{K}_n$ where

$$\mathbb{K}_n := \left\{\eta \in C^\alpha(\overline{D}_n) : \eta \geq 0\right\}, \tag{4.6}$$

and look for functions $m = m^{n,\eta}$ that solve, for $x \in D_n$:

$$\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} (m(x) - r_t)p(t, m(x))e^{-\int_0^t (r_u + \gamma(X_u, m(x), \eta(X_u)))du} dt\right] + \frac{m(x)^2}{n(1 - e^{-m(x)T})} = 0. \tag{4.7}$$

Note that we have substituted $\eta(X_t)$ for $m(X_t)$ in γ. Since $\lim_{m\downarrow 0} m^2/(1-e^{-mT}) = 0$, we define the second term above to be 0 when $m(x) = 0$. To facilitate the proof, for $\eta \in K_n$ we define the function $k^n(m, x; \eta)$ for $x \in D_n, m > 0$ by

$$k^n(m,x;\eta) := \frac{1}{m}\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} (m - r_t)\left(1 - e^{-m(T-t)}\right) e^{-\int_0^t (r_u + \gamma(X_u, m, \eta(X_u)))du} dt\right] + \frac{m}{n}. \tag{4.8}$$

Note from (2.2) that (4.7) holds if for each $x \in D_n$ we can find $m = m(x) = m^{n,\eta}(x) > 0$ so that $k^n(m, x; \eta) = 0$. We now state the following existence result:

Proposition 4.1. For $\alpha \in (0, 1)$ and $\eta \in \mathbb{K}_n$, there exists a unique function $m = m^{n,\eta}$ that is strictly positive, and solves (4.7) in D_n. $m^{n,\eta}$ is continuously differentiable in D_n with gradient

$$\nabla_x m^{n,\eta}(x) = -\frac{\nabla_x k^n(m,x;\eta)}{\partial_m k^n(m,x;\eta)}\bigg|_{m=m^{n,\eta}(x)}. \tag{4.9}$$

Furthermore, $\forall\beta \in (\alpha, 1)$, m satisfies the following *a priori* estimate of the β-Hölder norm:

$$\|m^{n,\eta}\|_{\beta,\overline{D}_n} \le C(n, \beta), \tag{4.10}$$

where $C(n, \beta)$ *does not* depend upon η.

Remark 4.2. Proposition 4.1 above establishes existence and uniqueness of functions $m^{n,\eta}$ for each given $\eta \in \mathbb{K}_n$, hence defines the mapping $\mathcal{A}^n : \eta \mapsto m^{n,\eta}$. Using the *a-prioi* estimates established in the previous section we then verify this mapping satisfies the hypotheses of Schaefer's theorem. We thus conclude that there is a fixed point m^n for the operator $\mathcal{A}^n$: $m^n = \mathcal{A}^n[m^n]$. But this is equivalent to m^n solving (4.5).

Before proving Proposition 4.1 we state and prove two technical lemmas. For the ease of presentation we define

$$C_n^{(1)} := \sup\left\{x^{(1)} : x \in D_n\right\}, \tag{4.11}$$

and

$$C_n := \sup\{|x| : x \in D_n\}, \tag{4.12}$$

and note that any solution of (4.5) must satisfy $0 \le m^n(x) < C_n^{(1)}$.

The first technical lemma establishes regularity of k^n in (x, m) for a fixed η. As in the previous section, the bounding constants below may change from line to line and their dependence on n is understood to absorb the dependence upon the constants $K_1(n), K_2(n), L_\gamma(n), B_\gamma(n)$ of Assumptions 2.2, 2.5, as well as the region D_n, dimension d and time-to-maturity T.

Lemma 4.4. Let $\alpha \in (0,1)$ and $\eta \in \mathbb{K}_n$ and define k^n as in (4.8). Then

1. For a fixed $x \in D_n$, $k^n(\cdot, x; \eta)$ is continuously differentiable on $(0,\infty)$. Furthermore, there exists a constant $A(n)$ such that for all $\eta \in \mathbb{K}_n$, $m > 0$ and $x \in D_n$:

$$\frac{1}{n} \le \partial_m k^n(m, x; \eta) \le A(n). \tag{4.13}$$

2. For a fixed $m > 0$, $k^n(m, \cdot; \eta) \in C^{2,\alpha}(\overline{D}_n)$ and there exists a constant $\Lambda(n, \|\eta\|_{\alpha,\overline{D}_n})$ such that for all $0 < m \le C_n^{(1)}$:

$$\|k^n(m, \cdot; \eta)\|_{2,\alpha,\overline{D}_n} \le \Lambda(n, \|\eta\|_{\alpha,\overline{D}_n}). \tag{4.14}$$

   For $R > 0$, $\Lambda(n, \|\eta\|_{\alpha,\overline{D}_n})$ can be made uniform (i.e. depending only upon n, R) for $\|\eta\|_{\alpha,\overline{D}_n} \le R$.

Proof. Note that $r_t, \gamma(X_t, m, \eta(X_t))$ are non-negative and uniformly bounded above by $C_n^{(1)} + B_\gamma(n)$ for $t \le \tau_n$. Additionally, from (2.11) and (2.12) we have that for all $x \in D_n, m, z \ge 0$ that

$$\gamma_m(x, m, z) \le \min\left\{B_\gamma(n) + L_\gamma(n)m, \Xi(mT)\right\} \le \begin{cases} B_\gamma(n) + L_\gamma(n); & m \le 1 \\ \Xi(T); & m > 1 \end{cases} := \overline{M}(n), \tag{4.15}$$

so that $\gamma_m(X_t, m, \eta(X_t))$ is almost surely bounded above on $t \le \tau_n$ by a constant depending only upon n. By the bounded convergence theorem we can switch the order of the differential operator and the integral in (4.8) to get

$$\partial_m k^n(m, x; \eta) = \mathbb{E}^x\left[\int_0^{T\wedge\tau_n} \partial_m \left(\left(1 - \frac{r_t}{m}\right)\left(1 - e^{-m(T-t)}\right) e^{-\int_0^t (r_u + \gamma(X_u, m, \eta(X_u)))du}\right) dt\right] + \frac{1}{n}. \tag{4.16}$$

Similarly we can verify the following calculation by differentiating and collecting terms:

$$
\begin{aligned}
&e^{\int_0^t (r_u+\gamma(X_u,m,\eta(X_u)))du} \times \partial_m \left(\left(1-\frac{r_t}{m}\right)\left(1-e^{-m(T-t)}\right) e^{-\int_0^t (r_n+\gamma(X_u,m,\eta(X_u)))du} \right) \\
&\quad = r_t \left(\frac{1-e^{-m(T-t)}}{m^2} - \frac{(T-t)e^{-m(T-t)}}{m} + \frac{1-e^{-m(T-t)}}{m} \int_0^t \gamma_m(X_u,m,\eta(X_u))du \right) \\
&\qquad + (T-t)e^{-m(T-t)} - (1-e^{-m(T-t)}) \int_0^t \gamma_m(X_u,m,\eta(X_u))du.
\end{aligned}
\tag{4.17}
$$

For all $m > 0, t \le T$ calculation shows

$$
0 \le \frac{1-e^{-m(T-t)}}{m^2} - \frac{(T-t)e^{-m(T-t)}}{m} \le \frac{1}{2}(T-t)^2; \qquad 0 \le \frac{1-e^{-m(T-t)}}{m} \le (T-t). \tag{4.18}
$$

Since $0 \le \gamma_m(x,m,z) \le \overline{M}(n)$ and $0 \le r_t \le C_n^{(1)}$ almost surely in D_n it follows that the right hand side of (4.17) is bounded below by

$$
(T-t)e^{-m(T-t)} - (1-e^{-m(T-t)}) \int_0^t \gamma_m(X_u,m,\eta(X_u))du, \tag{4.19}
$$

and from above by

$$
C_n^{(1)} \left(\frac{1}{2}(T-t)^2 + (T-t)t\overline{M}(n) \right) + (T-t).
$$

The upper bound in (4.13) readily follows.

For the lower bound in (4.13), we first note that (writing $\beta = 1-t/T$ and multiplying numerator and denominator by T):

$$
\Xi(mT) = \inf_{\beta\in(0,1)} \frac{\beta e^{-\beta m T}}{(1-\beta)(1-e^{-\beta m T})} = \inf_{t\in(0,T)} \frac{(T-t)e^{-m(T-t)}}{t(1-e^{-m(T-t)})}.
$$

From (2.12) we then have

$$
\begin{aligned}
(T-t)e^{-m(T-t)} &- (1-e^{-m(T-t)}) \int_0^t \gamma_m(X_u,m,\eta(X_u))du \\
&\ge (T-t)e^{-m(T-t)} - \Xi(mT)t(1-e^{-m(T-t)}) \\
&\ge 0.
\end{aligned}
\tag{4.20}
$$

It follows from (4.17) that almost surely for all $m > 0$ and $t \leq T \wedge \tau_n$:

$$\partial_m \left(\left(1 - \frac{r_t}{m}\right) \left(1 - e^{-m(T-t)}\right) e^{-\int_0^t (r_n + \gamma(X_u, m, \eta(X_u)))du} \right) \geq 0,$$

which yields the lower bound in (4.13).

Lastly, it is evident from (4.17) that the map

$$m \mapsto \partial_m \left(\left(1 - \frac{r_t}{m}\right) \left(1 - e^{-m(T-t)}\right) e^{-\int_0^t (r_n + \gamma(X_u, m, \eta(X_u)))du} \right)$$

is almost surely continuous in m and non-negative with upper bound $C_n^{(1)}\left(\frac{1}{2}T^2 + T^2\overline{M}(n)\right) + T$, and hence by the bounded convergence theorem the map $m \mapsto \partial_m k^n(m, x; \eta)$ is continuous for all $m > 0$. Now, regarding the regularity of $k^n(m, \cdot; \eta)$ for fixed m and η, write $k^n(m, \cdot; \eta) = u^{m,\eta} + m/n$ where

$$u^{m,\eta}(x) := \mathbb{E}^x \left[\int_0^{T \wedge \tau_n} (m - r_t) \frac{1 - e^{-m(T-t)}}{m} e^{-\int_0^t (r_u + \gamma(X_u, m, \eta(X_u)))du} dt \right]; \qquad x \in D_n. \tag{4.21}$$

Note that $u^{m,\eta}$ is of the form (4.2) with

$$\begin{aligned} g^m(t, x) &:= (m - x^{(1)}) \frac{1 - e^{-m(T-t)}}{m}; \\ h^{m,\eta}(t, x) = h^{m,\eta}(x) &:= x^{(1)} + \gamma(x, m, \eta(x)). \end{aligned} \tag{4.22}$$

Calculation shows for $0 < m \leq C_n^{(1)}$ that

$$\begin{aligned} &\lim_{t \uparrow T, y \to x} g^m(t, y) = 0; \quad x \in \partial D_n; \\ &|g^m|_{0,n} \leq C_n^{(1)} T; \\ &[g^m]_{\alpha,n} \leq 2C_n^{(1)} T^{1-\alpha/2} + T(C_n^{(1)})^{1-\alpha}; \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} &|h^{m,\eta}|_{0,n} \leq C_n^{(1)} + B_\gamma(n); \\ &[h^{m,\eta}]_{\alpha,n} \leq (C_n^{(1)})^{1-\alpha} + L_\gamma(n \vee C_n^{(1)} \vee \|\eta\|_{\alpha,\overline{D}_n}) \left((2C_n)^{1-\alpha} + \|\eta\|_{\alpha,\overline{D}_n} \right). \end{aligned} \tag{4.24}$$

Lemma 4.1 thus yields the upper bound in (4.14). Note that the upper bounds in (4.23) and (4.24) can be made uniform for all $\|\eta\|_{\alpha,\overline{D}_n} \leq R$ for any $R > 0$.

$\square$

The second lemma establishes regularity of k^n with respect to changes in both m and η.

Lemma 4.5. For $\eta_1, \eta_2 \in \mathbb{K}_n$ and $0 < m_1, m_2 \leq C_n^{(1)}$ there exists a constant $\Lambda'(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})$ so that

$$\begin{aligned} &\|k^n(m_1,\cdot;\eta_1) - k^n(m_2,\cdot;\eta_2)\|_{2,\alpha,\overline{D}_n} \\ &\quad \leq \Lambda'(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n}) \left(\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n} + |m_1-m_2| + \|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|m_1-m_2|\right), \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} &\sup_{x\in D_n} |\partial_m k^n(m_1,x;\eta_1) - \partial_m k^n(m_2,x;\eta_2)| \\ &\quad \leq \Lambda'(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n}) \left(|m_1-m_2| + \|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\right). \end{aligned} \tag{4.26}$$

The constant Λ' can be made uniform for all $\|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n} \leq R$ for any $R > 0$.

Proof. We have $k^n(m_1,\cdot;\eta_1) - k^n(m_2,\cdot;\eta_2) = u^{m_1,\eta_1} - u^{m_2,\eta_2} + (m_1-m_2)/n$ where $u^{m,\eta}$ is from (4.21). For $0 < m_1, m_2 \leq C_n^{(1)}$, it follows from Lemma 4.1 and (4.23), (4.24) that $u^{m_i,\eta_i} = U^{m_i,\eta_i}(0,\cdot)$, where U^{m_i,η_i} solves the linear parabolic PDE given in (4.4). Furthermore, $\overline{|U^{m_i,\eta_i}|}_{2,\alpha,\overline{D}_n} \leq C(n, \|\eta_i\|_{\alpha,\overline{D}_n})$ where the bounded constant can be made uniform for $\|\eta_i\|_{\alpha,\overline{D}_n} \leq R$.

Set

$$V := U^{m_1,\eta_1} - U^{m_2,\eta_2},$$

and

$$\tilde{g}(t,x) := g^{m_1}(t,x) - g^{m_2}(t,x) + U^{m_2,\eta_2}(t,x)(h^{m_2,\eta_2} - h^{m_1,\eta_1})(x). \tag{4.27}$$

It is straightforward to check that V solves the linear parabolic PDE

$$\begin{cases} V_t + \mathcal{L}V - h^{m_1,\eta_1}V = -\tilde{g}, \quad (t,x) \in Q_n, \\ V(T,x) = 0, \quad x \in D_n, \\ V(t,x) = 0, \quad (t,x) \in [0,T] \times \partial D_n. \end{cases} \tag{4.28}$$

From (4.24) we have that $\overline{|h^{m_1,\eta_1}|}_{\alpha,n}$ is bounded from above by a constant which only depends upon $n, \|\eta_1\|_{\alpha,\overline{D}_n}$ and can be made uniform if $\|\eta_1\|_{\alpha,\overline{D}_n} \leq R$.

On the other hand, a lengthy yet straightforward calculation yields

$$|g^{m_1} - g^{m_2}|_{0,n} \leq \left(T + \frac{1}{2}C_n^{(1)}T^2\right)|m_2 - m_1|;$$
$$|h^{m_1,\eta_1} - h^{m_2,\eta_2}|_{0,n} \leq L_\gamma(n \vee C_n^{(1)} \vee \|\eta_1\|_{\alpha,\overline{D}_n} \vee \|\eta_2\|_{\alpha,\overline{D}_n})\left(\|\eta_2 - \eta_1\|_{\alpha,\overline{D}_n} + |m_2 - m_1|\right).$$

Note that the bounding constants in the above can also be made uniform for $\|\eta_i\|_{\alpha,\overline{D}_n} \leq R$. Lemma 4.7 below shows that there is a constant $\tilde{\Lambda}(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})$, which is uniform for $\|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n} \leq R$, so that

$$\begin{aligned} [g^{m_1} - g^{m_2}]_{\alpha,n} &\leq \left((1 + 2TC_n^{(1)})T^{1-\alpha/2} + \frac{1}{2}T^2(C_n^{(1)})^{1-\alpha}\right)|m_2 - m_1|; \\ [h^{m_1,\eta_1} - h^{m_2,\eta_2}]_{\alpha,n} &\leq \tilde{\Lambda}\left(|m_1 - m_2| + \|\eta_2 - \eta_1\|_{\alpha,\overline{D}_n} + |m_1 - m_2|\|\eta_2 - \eta_1\|_{\alpha,\overline{D}_n}\right). \end{aligned} \tag{4.29}$$

From (4.27) and that $\overline{|U^{m_2,\eta_2}|}_{2,\alpha,\overline{D}_n} \leq C(n, \|\eta_2\|_{\alpha,\overline{D}_n})$ we have

$$|\tilde{g}|_{\alpha,n} \leq \tilde{\Lambda}(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1 - m_2| + \|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n} + |m_1 - m_2|\|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n}\right).$$

(4.25) now follows from Lemma 4.1 since both g^m and U^{m_2,η_2} satisfy the compatibility condition.

We then turn to (4.26). As follows from (4.16) and (4.17) we have

$$\begin{aligned} &\partial_m k^n(m_1, x; \eta_1) - \partial_m k^n(m_2, x; \eta_2) \\ &= \mathbb{E}^x\left[\int_0^{T\wedge\tau_n} (A_1(t)\,(B(t)C_1(t) + D_1(t)) - A_2(t)\,(B(t)C_2(t) + D_2(t)))\,dt\right], \end{aligned} \tag{4.30}$$

where for $i = 1, 2$:

$$
\begin{aligned}
A_i(t) &= e^{-\int_0^t (r_u + \gamma(X_u, m_i, \eta_i(X_u)))du};\\
B(t) &= r_t;\\
C_i(t) &= \frac{1}{m_i^2}\left(1 - e^{-m_i(T-t)} - m_i(T-t)e^{-m_i(T-t)}\right) + \frac{1 - e^{-m_i(T-t)}}{m_i}\int_0^t \gamma_m(X_u, m_i, \eta_i(X_u))du;\\
D_i(t) &= (T-t)e^{-m_i(T-t)} - (1 - e^{-m_i(T-t)})\int_0^t \gamma_m(X_u, m_i, \eta_i(X_u))du.
\end{aligned}
$$

Using that $\gamma \geq 0$, $0 \leq r_t \leq C_n^{(1)}$ on $t \leq \tau_n$, (4.18), and (4.15), we have the following almost sure inequalities:

$$
\begin{aligned}
|A_1(t)| &\leq 1;\\
|B(t)| &\leq C_n^{(1)};\\
|C_2(t)| &\leq T^2\left(\frac{1}{2} + \overline{M}(n)\right);\\
|D_2(t)| &\leq T(1 + \overline{M}(n)).
\end{aligned}
$$

To estimate (4.30) we will use the elementary estimate

$$
|A_1(BC_1 + D_1) - A_2(BC_2 + D_2)| \leq |A_1||B||C_1 - C_2| + (|B||C_2| + |D_2|)|A_1 - A_2| + |A_1||D_1 - D_2|,
$$

and it remains to find suitable upper bounds for the terms $|C_1 - C_2|$, $|A_1 - A_2|$ and $|D_1 - D_2|$, respectively.

First we have

$$
\begin{aligned}
&|C_1(t) - C_2(t)|\\
&\quad\leq \left|\frac{1 - e^{-m_1(T-t)} - m_1(T-t)e^{-m_1(T-t)}}{m_1^2} - \frac{1 - e^{-m_2(T-t)} - m_2(T-t)e^{-m_2(T-t)}}{m_2^2}\right|\\
&\quad+ \frac{1 - e^{-m_1(T-t)}}{m_1}\int_0^T |\gamma_m(X_u, m_1, \eta_1(X_u)) - \gamma_m(X_u, m_2, \eta_2(X_u))|\, du\\
&\quad+ \int_0^T \gamma_m(X_u, m_2, \eta_2(X_u))du\left|\frac{1 - e^{-m_1(T-t)}}{m_1} - \frac{1 - e^{-m_2(T-t)}}{m_2}\right|\\
&\quad=: I_1 + I_2 + I_3.
\end{aligned}
$$

Since the partial derivative

$$\frac{\partial}{\partial m}\left[(1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)})/m^2\right]$$
$$=-(2/m^3)(1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}-(1/2)m^2(T-t)^2e^{-m(T-t)})$$

is non-positive and bounded in absolute value by $(T-t)^3/3 \leq T^3/3$, we have, for term I_1 that

$$\left|\frac{1-e^{-m_1(T-t)}-m_1(T-t)e^{-m_1(T-t)}}{m_1^2}-\frac{1-e^{-m_2(T-t)}-m_2(T-t)e^{-m_2(T-t)}}{m_2^2}\right| \leq \frac{T^3}{3}|m_1-m_2|.$$

For the term I_2 we have

$$\frac{1-e^{-m_1(T-t)}}{m_1}\int_0^T |\gamma_m(X_u,m_1,\eta_1(X_u))-\gamma_m(X_u,m_2,\eta_2(X_u))|\,du$$
$$\leq T^2L_\gamma(n\vee C_n^{(1)}\vee\|\eta_1\|_{\alpha,\overline{D}_n}\vee\|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1-m_2|+\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\right).$$

Using that $\left|\frac{\partial}{\partial m}(1-e^{-m(T-t)})/m\right| \leq (T-t)^2/2$ we have, for term I_3 that

$$\int_0^T \gamma_m(X_u,m_2,\eta_2(X_u))du\left|\frac{1-e^{-m_1(T-t)}}{m_1}-\frac{1-e^{-m_2(T-t)}}{m_2}\right| \leq \frac{1}{2}T^3\overline{M}(n)|m_1-m_2|.$$

Therefore there exists some positive constant $C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})$ so that almost surely for $t \leq T$:

$$|C_1(t)-C_2(t)| \leq C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1-m_2|+\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\right).$$

Next, by the non-negativity of r,γ and the elementary inequality $|e^{-a}-e^{-b}| \leq |a-b|$ for $a,b \geq 0$ we have almost surely for $t \leq T\wedge\tau_n$ that

$$|A_1(t)-A_2(t)| \leq \int_0^T |\gamma(X_u,m_1,\eta_1(X_u))-\gamma(X_u,m_2,\eta_2(X_u))|\,du$$
$$\leq TL_\gamma(n\vee C_n^{(1)}\vee\|\eta_1\|_{\alpha,\overline{D}_n}\vee\|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1-m_2|+\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\right)$$
$$=C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1-m_2|+\|\eta_1-\eta_2\|_{\alpha,\overline{D}_m}\right).$$

Lastly, we have

$$
\begin{aligned}
&|D_1(t) - D_2(t)| \\
&\quad \le T\left|e^{-m_1(T-t)} - e^{-m_2(T-t)}\right| + (1 - e^{-m_2(T-t)})\int_0^T |\gamma_m(X_u, m_1, \eta_1(X_u)) - \gamma_m(X_u, m_2, \eta_2(X_u))|\, du \\
&\qquad + \int_0^T \gamma_m(X_u, m_2, \eta_2(X_u))du \left|e^{-m_2(T-t)} - e^{-m_1(T-t)}\right| \\
&\quad \le T^2|m_1 - m_2| + TL_\gamma(n \vee C_n^{(1)} \vee \|\eta_1\|_{\alpha,\overline{D}_n} \vee \|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1 - m_2| + \|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n}\right) \\
&\qquad + \overline{M}(n)T^2|m_1 - m_2| \\
&\quad \le C(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1 - m_2| + \|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n}\right).
\end{aligned}
$$

Putting the above pieces together in (4.30) yields

$$
|\partial_m k^n(m_1, x; \eta_1) - \partial_m k^n(m_2, x; \eta_2)| \le C(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1 - m_2| + \|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n}\right),
$$

for all $x \in D_n$, which is the desired result. □

Now that we have established regularity of k^n, we can now prove Proposition 4.1.

Proof of Proposition 4.1. It suffices to find $m = m(x) = m^{n,\eta}(x)$ for each $x \in D_n$ so that $k^n(m, x; \eta) = 0$. From Lemma 4.4 we know that k^n is strictly increasing in m. Additionally, by the dominated convergence theorem, the non negativity of γ, and that $r_t \le C_n^{(1)}$, $t \le \tau_n$ we have

$$
\begin{aligned}
&\lim_{m \downarrow 0} k^n(m, x; \eta) = -\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} r_t(T-t)e^{-\int_0^t (r_u + \gamma(X_u, 0, \eta(X_u)))du} dt\right] < 0; \\
&\lim_{m \uparrow \infty} k^n(m, x; \eta) = \infty.
\end{aligned}
$$

So for any $x \in D_n$ there exists an unique $m(x) > 0$ such that $k^n(m(x), x; \eta) = 0$. This defines the map $m = m^{n,\eta} : D_n \mapsto (0, \infty)$. We next show the *a priori* estimate for the Hölder norm of m in (4.10). By definition, for all $x, y \in D_n$,

$$
k^n(m(x), x; \eta) = k^n(m(y), y; \eta) = 0, \tag{4.31}
$$

which implies

$$
k^n(m(y), y; \eta) - k^n(m(x), y; \eta) = k^n(m(x), x; \eta) - k^n(m(x), y; \eta). \tag{4.32}
$$

Since y is fixed, the mean value theorem applied to $m \mapsto k^n(m, y; \eta)$ (which is C^1 in m from Lemma 4.4) asserts the existence of ξ between $m(x)$ and $m(y)$ such that

$$\partial_m k^n(\xi, y; \eta) \cdot (m(y) - m(x)) = k^n(m(x), x; \eta) - k^n(m(x), y; \eta). \tag{4.33}$$

By Lemma 4.4 we have

$$|m(x) - m(y)| \leq n|k^n(m(x), x; \eta) - k^n(m(x), y; \eta)|. \tag{4.34}$$

Now, fix x and note that $k^n(m(x), \cdot; \eta) = u^{m(x),\eta} + m(x)/n$ where $u^{m,\eta}$ is defined in (4.21). Here we may think of x as a parameter. By (4.22), (4.23), (4.24), as well as that $m(x) \leq C_n^{(1)}$ and that $0 \leq y^{(1)} + \gamma(y, m(x), \eta(y) \leq C_n^{(1)} + B_\gamma(n)$ on D_n we may apply Lemma 4.2 to obtain that for all $\beta \in (\alpha, 1)$:

$$\|u^{m(x),\eta}\|_{\beta,\overline{D}_n} \leq C(n, K_4(n), \beta) \sup_{(t,y)\in Q_n} \left| m(x) - y^{(1)} \right| \frac{1 - e^{-m(x)(T-t)}}{m(x)} \leq C(n, K_4(n), \beta),$$

where the constant $K_4(n)$ does not depend upon η. From (4.34) we obtain

$$|m(x) - m(y)| \leq n|k^n(x, m(x); \eta) - k^n(y, m(x); \eta)| \leq C(n, K_4(n), \beta)|x - y|^\beta.$$

Since it is clear from (4.7) that $m^{n,\eta} < C_n^{(1)}$, the upper bound in (4.10) clearly holds.

Lastly, (4.9) follows immediately from the implicit function theorem since Lemmas 4.4, 4.5 imply that $k^n(m, x; \eta)$ is C^1 in $(0, C_n^{(1)}) \times D_n$ for fixed $\eta \in \mathbb{K}_n$. □

In light of Proposition 4.1 we define the map $\mathcal{A}^n : \mathbb{K}_n \mapsto \mathbb{K}_n$ by

$$\mathcal{A}^n[\eta] = m^{n,\eta}; \qquad \eta \in \mathbb{K}_n. \tag{4.35}$$

The following lemma will be needed in the proof of the continuity of the operator $\mathcal{A}^n$.

Lemma 4.6. Let $\alpha \in (0, 1)$ and $\eta_1, \eta_2(x) \in \mathbb{K}_n$. Let $m_1 = \mathcal{A}^n[\eta_1]$, $m_2 = \mathcal{A}^n[\eta_2]$. There is a constant $\tilde{\Lambda}(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})$ which can be made uniform for $\|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n} \leq R$ such that

$$\sup_{x\in D_n}|m_1(x)-m_2(x)|\leq\tilde{\Lambda}(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n};$$

$$\sup_{x\in D_n}|\nabla_x k^n(x,m_1(x);\eta_1)-\nabla_x k^n(x,m_2(x);\eta_2)|\leq\tilde{\Lambda}(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n};$$

$$\sup_{x\in D_n}|\partial_m k^n\left(x,m_1(x);\eta_1\right)-\partial_m k^n\left(x,m_2(x);\eta_2\right)|\leq\tilde{\Lambda}(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}.$$

Proof. By definition of m_1, m_2 we have that $0 = k^n(m_1(x), x; \eta_1) = k^n(m_2(x), x; \eta_2)$ for all $x \in D_n$, and hence

$$k^n(m_2(x),x;\eta_2)-k^n(m_1(x),x;\eta_2)=k^n(m_1(x),x;\eta_1)-k^n(m_1(x),x;\eta_2).$$

By the mean value theorem applied to the map $m \mapsto k^n(m, x; \eta_2)$ (which is C^1 from Lemma 4.4) there is some ξ between $m_1(x), m_2(x)$ so that

$$\partial_m k^n(\xi,x;\eta_2)(m_2(x)-m_1(x))=k^n(m_2(x),x;\eta_2)-k^n(m_1(x),x;\eta_2).$$

It thus follows that

$$\begin{aligned}|m_2(x)-m_1(x)|&=\frac{|k^n(m_1(x),x;\eta_2)-k^n(m_1(x),x;\eta_1)|}{|\partial_m k^n(\xi,x;\eta_2)|}\\&\leq n\Lambda'\left(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n}\right)\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\\&=\tilde{\Lambda}(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}.\end{aligned}$$

The above inequality follows from (4.25) in Lemma 4.5 since $0 < m_i(x) < C_n^{(1)}$ on D_n $(i = 1, 2)$. The second inequality follows immediately from the first by (4.25). Similarly, the third inequality follows from the first by (4.26). □

The following Proposition establishes a fixed point in $\mathbb{K}_n$:

Proposition 4.2. Let $\alpha \in (0, 1)$. There exists $m^n \in \mathbb{K}_n$ that is strictly positive for $x \in D_n$ and solves the fixed point equation $m^n = \mathcal{A}^n[m^n]$ in D_n. Equivalently, m^n satisfies (4.5). Furthermore, $\forall\beta \in (\alpha, 1)$, m^n satisfies the following *a priori* estimate of the β-Hölder norm on D_n:

$$\|m^n\|_{\beta,\overline{D}_n}\leq C(n,\beta).$$

Proof. The existence of a fixed point m^n follows from Theorem 4.1 by verifying the steps below. Here, the Banach space is $X = C^\alpha(\overline{D}_n)$, the closed convex subset containing 0 is $\mathbb{K}_n$ and the operator $\mathcal{A}$ is $\mathcal{A}^n$ from (4.35).

1. *The mapping $\mathcal{A}^n : \mathbb{K}_n \mapsto \mathbb{K}_n$ is continuous.* For any $\eta_1, \eta_2 \in \mathbb{K}_n$, let $m_1 = \mathcal{A}^n[\eta_1]$ and $m_2 = \mathcal{A}^n[\eta_2]$. In light of the first inequality in Lemma 4.6, we only need to consider the Hölder semi-norm $[m_1 - m_2]_{\alpha,n}$, and it suffices to show that $\sup_{x\in D_n} |\nabla_x(m_1(x) - m_2(x))| \leq C(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n}$.
   Note that from Proposition 4.1 we have for $i = 1, ..., d$ and $x \in D_n$:
$$\begin{aligned}\partial_{x_i}\left(m_1(x) - m_2(x)\right) &= -\left(\frac{\partial_{x_i}k^n\left(m_1(x), x; \eta_1\right)}{\partial_m k^n(m_1(x), x; \eta_1)} - \frac{\partial_{x_i}k^n\left(m_2(x), x; \eta_2\right)}{\partial_m k^n(m_2(x), x; \eta_2)}\right)\\ &= -\frac{\partial_{x_i}k^n\left(m_1(x), x; \eta_1\right) - \partial_{x_i}k^n\left(m_2(x), x; \eta_2\right)}{\partial_m k^n(m_1(x), x; \eta_1)}\\ &\quad + \frac{\partial_{x_i}k^n(m_2(x), x; \eta_2) \times \left(\partial_m k^n\left(m_1(x), x; \eta_1\right) - \partial_m k^n\left(m_2(x), x; \eta_2\right)\right)}{\partial_m k^n(m_1(x), x; \eta_1)\partial_m k^n(m_2(x), x; \eta_2)},\end{aligned}$$
   and so from Lemmas 4.4, 4.6 we have
$$\begin{aligned}\left|\partial_{x_i}\left(m_1(x) - m_2(x)\right)\right| &\leq n\left|\partial_{x_i}k^n\left(m_1(x), x; \eta_1\right) - \partial_{x_i}k^n\left(m_2(x), x; \eta_2\right)\right|\\ &\qquad + n^2\Lambda(n, \|\eta_2\|_{\alpha,\overline{D}_n})\left|\partial_m k^n\left(m_1(x), x; \eta_1\right) - \partial_m k^n\left(m_2(x), x; \eta_2\right)\right|\\ &\leq \tilde{\Lambda}(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})\left(n + n^2\Lambda(n, \|\eta_2\|_{\alpha,\overline{D}_n})\right)\|\eta_1 - \eta_2\|_{\alpha,\overline{D}_n},\end{aligned}$$
   which proves continuity.

2. *The mapping $\mathcal{A}^n : \mathbb{K}_n \to \mathbb{K}_n$ is compact.* Let us fix some $\beta \in (\alpha, 1)$. Given any bounded sequence $\{\eta_i\}_{i\in\mathbb{N}}$ in $\mathbb{K}_n$, Proposition 4.1 yields, $\forall i \in \mathbb{N}$,
$$\|\mathcal{A}^n[\eta_i]\|_{C^\beta(\overline{D_n})} \leq C(n, \beta).$$
   By the standard compact embeddings of Hölder spaces (see [16, Lemma 6.33]), there exists a subsequence $\{\mathcal{A}^n[\eta_{i_k}]\}_{k\in\mathbb{N}}$ of $\{\mathcal{A}^n[\eta_i]\}_{i\in\mathbb{N}}$ such that $\{\mathcal{A}^n[\eta_{i_k}]\}_{k\in\mathbb{N}}$ converges in $\|\cdot\|_{C^\alpha(\overline{D}_n)}$ norm to some limit in $\mathbb{K}_n$.

3. *The set $\{m \in \mathbb{K}_n \ : \ m = \lambda\mathcal{A}^n[m]$ for some $0 \leq \lambda \leq 1\}$ is bounded.* Suppose $m \in \mathbb{K}_n$ satisfies $m = \lambda\mathcal{A}^n[m]$ for some $0 \leq \lambda \leq 1$. We have from Proposition 4.1
$$\|m\|_{C^\alpha(\overline{D_n})} = \lambda\|\mathcal{A}^n[m]\|_{C^\alpha(\overline{D_n})} \leq C(n, \alpha).$$

Schaefer's Theorem thus asserts that the operator $\mathcal{A}^n$ has a fixed point m^n in $\mathbb{K}_n$. By Proposition 4.1, m^n is strictly positive. Moreover, m^n satisfies the following *a priori* estimate of the β-Hölder norm on D_n:

$$\|m\|_{C^\beta(\overline{D_n})} \leq C(n,\beta),\ \forall \beta \in (\alpha, 1).$$

□

Lemma 4.7. For $0 < m_1, m_2 \leq C_n^{(1)}$, $\eta_1, \eta_2 \in \mathbb{K}_n$ and g^m, h^m as in (4.22) the inequalities in (4.29) hold.

Proof. The proof is a lengthy calculation based on Taylor's formula. We will use the fact that γ is C^2, has derivatives of order ≤ 2 that can be continuously extended to $D \times \{0\} \times \{0\}$, as well as that all derivatives of order ≤ 2 are Lipschitz continuous in $\bar{D}_n \times [0,n] \times [0,n]$ with Lipschitz constant $L_\gamma(n)$. In particular, for any partial derivative v of γ with order ≤ 2, any n and constants $m_n, z_n > 0$:

$$\sup_{\substack{x \in D_n,\\ m \leq m_n, z \leq z_n}} |v(x,m,z)| < \infty,$$

$$\sup_{\substack{x,x' \in D_n,\\ m,m' \leq m_n,\\ z,z' \leq z_n}} |v(x,m,z) - v(x',m',z')| \leq L_\gamma(n \vee m_n \vee z_n)\left(|x - x'| + |m - m'| + |z - z'|\right).$$

The above inequalities are repeatedly used in the proof.

Also, $C(n, \|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n})$ is a constant which may change from line to line and can always be made uniform in η_1, η_2 for $\|\eta_1\|_{\alpha,\overline{D}_n}, \|\eta_2\|_{\alpha,\overline{D}_n} \leq R$.

Now, for $s, t < T, x, y \in D_n$ we have

$$\begin{aligned}
&g^{m_1}(t,x) - g^{m_2}(t,x) - (g^{m_1}(s,y) - g^{m_2}(s,y)) \\
&= (m_1 - x^{(1)})\frac{1 - e^{-m_1(T-t)}}{m_1} - (m_2 - x^{(1)})\frac{1 - e^{-m_2(T-t)}}{m_2} \\
&\quad - \left((m_1 - y^{(1)})\frac{1 - e^{-m_1(T-s)}}{m_1} - (m_2 - y^{(1)})\frac{1 - e^{-m_2(T-s)}}{m_2}\right) \\
&= \int_{m_2}^{m_1} \left((T-t)e^{-m(T-t)} + \frac{x^{(1)}}{m^2}\left(1 - e^{-m(T-t)} - m(T-t)e^{-m(T-t)}\right)\right) dm \\
&\quad - \int_{m_2}^{m_1} \left((T-s)e^{-m(T-s)} + \frac{y^{(1)}}{m^2}\left(1 - e^{-m(T-s)} - m(T-s)e^{-m(T-s)}\right)\right) dm.
\end{aligned}$$

First,

$$\left|\int_{m_2}^{m_1}\left((T-t)e^{-m(T-t)}-(T-s)e^{-m(T-s)}\right)dm\right| = \left|\int_{m_2}^{m_1}\int_s^t e^{-m(T-\tau)}(m(T-\tau)-1)d\tau dm\right|$$
$$\leq (1+C_n^{(1)}T)|t-s||m_1-m_2|.$$

Next,

$$\left|\frac{x^{(1)}}{m^2}\left(1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}\right)-\frac{y^{(1)}}{m^2}\left(1-e^{-m(T-s)}-m(T-s)e^{-m(T-s)}\right)\right|$$
$$\leq x^{(1)}\left|\frac{1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}}{m^2}-\frac{1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}}{m^2}\right|$$
$$+|x^{(1)}-y^{(1)}|\frac{1-e^{-m(T-s)}-m(T-s)e^{-m(T-s)}}{m^2}.$$

For any $k \geq 0$ the function $m \mapsto m^{-2}\left(1-e^{-km}-kme^{-km}\right)$ is non-negative and deceasing in $m>0$ with limit as $m\to 0$ of $(1/2)k^2$. Using this we have

$$|x^{(1)}-y^{(1)}|\frac{1-e^{-m(T-s)}-m(T-s)e^{-m(T-s)}}{m^2} \leq \frac{1}{2}(T-s)^2|x^{(1)}-y^{(1)}| \leq \frac{T^2}{2}|x^{(1)}-y^{(1)}|.$$

For any $m>0$, the partial derivative

$$\frac{\partial}{\partial m}\left[m^{-2}\left(1-e^{-m(T-\tau)}-m(T-\tau)e^{-m(T-\tau)}\right)\right] = -(T-\tau)e^{-m(T-\tau)}$$

is bounded above in absolute value by T on $\tau \leq T$. This implies

$$x^{(1)}\left|\frac{1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}}{m^2}-\frac{1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}}{m^2}\right| \leq C_n^{(1)}T|t-s|.$$

Putting these two terms together gives

$$\left|\int_{m_2}^{m_1}\left(\frac{x^{(1)}}{m^2}\left(1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}\right)-\frac{y^{(1)}}{m^2}\left(1-e^{-m(T-s)}-m(T-s)e^{-m(T-s)}\right)\right)dm\right|$$
$$\leq\left(\frac{T^2}{2}|x^{(1)}-y^{(1)}|+C_n^{(1)}T|t-s|\right)|m_1-m_2|.$$

Therefore

$$\begin{aligned}&|g^{m_1}(t,x)-g^{m_2}(t,x)-(g^{m_1}(s,y)-g^{m_2}(s,y))|\\&\qquad\leq|m_1-m_2|\left((1+2C_n^{(1)}T)|t-s|+\frac{T^2}{2}|x^{(1)}-y^{(1)}|\right),\end{aligned}$$

and hence

$$[g^{m_1}-g^{m_2}]_{\alpha,n}\leq|m_1-m_2|\left((1+2C_n^{(1)}T)T^{1-\alpha/2}+\frac{T^2}{2}(C_n^{(1)})^{1-\alpha}\right),$$

which is (4.29) for g.

We then turn to estimates for h. Write $\mathbf{a}_i(x):=(x,m_i,\eta_i(x))$ for $i=1,2$ and $x\in D_n$ and set

$$M_n\;:=\;n\vee C_n^{(1)}\vee\|\eta_1\|_{\alpha,\overline{D}_n}\vee\|\eta_2\|_{\alpha,\overline{D}_n}, \tag{4.36}$$

and note that

$$\mathbf{a}_i(x)\in\bar{E}_{M_n}=\bar{D}_{M_n}\times[0,M_n]\times[0,M_n]\,;\qquad x\in D_n. \tag{4.37}$$

The second order Taylor formula yields

$$\begin{aligned}&h^{m_1,\eta_1}(x)-h^{m_2,\eta_2}(x)-(h^{m_1,\eta_1}(y)-h^{m_1,\eta_1}(y))\\&=\gamma(\mathbf{a}_1(x))-\gamma(\mathbf{a}_2(x))-(\gamma(\mathbf{a}_1(y))-\gamma(\mathbf{a}_2(y)))\\&=(m_1-m_2)\left(\gamma_m(\mathbf{a}_2(x))-\gamma_m(\mathbf{a}_2(y))\right)\\&\qquad+\gamma_z(\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x))-\gamma_z(\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))\\&\qquad+(m_1-m_2)^2\left(R_{mm}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))-R_{mm}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))\right)\\&\qquad+R_{zz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x))^2-R_{zz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))^2\\&\qquad+2(m_1-m_2)\left(R_{mz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x))-R_{mz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))\right)\\&\qquad=:I_1+I_2+I_3+I_4+I_5.\end{aligned} \tag{4.38}$$

Note we have set above for $\mathbf{a}_1(x),\mathbf{a}_2(x),x\in D_n$ that

$$\begin{aligned}R_{mm}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))&=\int_0^1(1-u)\gamma_{mm}\left(\mathbf{a}_1(x)+u(\mathbf{a}_2(x)-\mathbf{a}_1(x))\right)du\\&=\int_0^1(1-u)\gamma_{mm}\left(x,m_2+u(m_1-m_2),\eta_2(x)+u(\eta_1(x)-\eta_2(x))\right)du,\end{aligned}$$

with analogous formulas for R_{zz} and R_{mz}.

Since $m_2+u(m_1-m_2)$ is in between m_1 and m_2, and $\eta_2(x)+u(\eta_1(x)-\eta_2(x))$ is in between $\eta_1(x)$ and $\eta_2(x)$ the above formula implies (recall (4.37))

$$\left|R_{mm}(\mathbf{a}_1(x)\middle|\mathbf{a}_2(x))\right| \leq \frac{1}{2}\sup_{(x,m,z)\in E_n}|\gamma_{mm}(x,m,z)| = C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n}), \tag{4.39}$$

with analogous formulas for R_{mz}, R_{zz}, as well as

$$\begin{aligned}
&\left|R_{mm}(\mathbf{a}_1(x)\middle|\mathbf{a}_2(x)) - R_{mm}(\mathbf{a}_1(y)\middle|\mathbf{a}_2(y))\right| \\
&\quad\leq L_\gamma(M_n)\int_0^1 (1-u)\left(|x-y| + |(1-u)(\eta_2(x)-\eta_2(y)) + u(\eta_1(x)-\eta_1(y))|\right)du \\
&\quad\leq \frac{1}{2}L_\gamma(M_n)\left(|x-y| + \|\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha + \|\eta_1\|_{\alpha,\overline{D}_n}|x-y|^\alpha\right) \\
&\quad= C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|x-y|^\alpha,
\end{aligned} \tag{4.40}$$

again with analogous formulas for R_{zz}, R_{mz} too.

We now use (4.39), (4.40) to bound the terms I_1-I_5 in (4.38) separately. First,

$$\begin{aligned}
&|(m_1-m_2)\left(\gamma_m(\mathbf{a}_2(x)) - \gamma_m(\mathbf{a}_2(y))\right)| \\
&\quad\leq |m_1-m_2|L_\gamma(M_n)\left(|x-y| + \|\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha\right) \\
&\quad\leq C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|m_1-m_2||x-y|^\alpha.
\end{aligned}$$

Second,

$$\begin{aligned}
&|\gamma_z(\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x)) - \gamma_z(\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))| \\
&\quad\leq |\gamma_z(\mathbf{a}_2(x))|\,|\eta_1(x)-\eta_2(x)-(\eta_1(y)-\eta_2(y))| + |\eta_1(y)-\eta_2(y)|\,|\gamma_z(\mathbf{a}_2(x))-\gamma_z(\mathbf{a}_2(y))| \\
&\quad\leq \sup_{(x,m,z)\in\bar{E}_{M_n}}|\gamma_z(x,m,z)|\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha \\
&\qquad + \|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}L_\gamma(M_n)\left(|x-y| + \|\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha\right) \\
&\quad= C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha.
\end{aligned}$$

Third, (recall (4.40)),

$$\begin{aligned}
&(m_1-m_2)^2\left(R_{mm}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))-R_{mm}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))\right)\\
&\qquad\le C_n^{(1)}C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|m_1-m_2||x-y|^\alpha\\
&\qquad= C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|m_1-m_2||x-y|^\alpha.
\end{aligned}$$

Fourth (recall (4.39) and (4.40),

$$\begin{aligned}
&\left|R_{zz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x))^2-R_{zz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))^2\right|\\
&\quad\le\left|R_{zz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))\right|\left|(\eta_1(x)-\eta_2(x))^2-(\eta_1(y)-\eta_2(y))^2\right|\\
&\qquad+(\eta_1(y)-\eta_2(y))^2\left|R_{zz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))-R_{zz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))\right|\\
&\quad\le 2C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha\\
&\qquad+\|\eta_1-\eta_2\|^2_{\alpha,\overline{D}_n}C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|x-y|^\alpha\\
&\quad= C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha.
\end{aligned}$$

Lastly,

$$\begin{aligned}
&\left|2(m_1-m_2)\left(R_{mz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))(\eta_1(x)-\eta_2(x))-R_{mz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))(\eta_1(y)-\eta_2(y))\right)\right|\\
&\quad\le 2|m_1-m_2||R_{mz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))|\,|\eta_1(x)-\eta_2(x)-(\eta_1(y)-\eta_2(y))|\\
&\qquad+2|m_2-m_2|\,|\eta_1(y)-\eta_2(y)|\left|R_{mz}(\mathbf{a}_1(x)\big|\mathbf{a}_2(x))-R_{mz}(\mathbf{a}_1(y)\big|\mathbf{a}_2(y))\right|\\
&\quad\le 2|m_1-m_2|\left(C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha+C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|x-y|^\alpha\right)\\
&\quad= C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})|m_1-m_2|\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}|x-y|^\alpha.
\end{aligned}$$

Putting together the five estimates above in equation (4.38) we obtain

$$\begin{aligned}
&|h^{m_1,\eta_1}(x)-h^{m_2,\eta_2}(x)-(h^{m_1,\eta_1}(y)-h^{m_1,\eta_1}(y))|\\
&\quad\le C(n,\|\eta_1\|_{\alpha,\overline{D}_n},\|\eta_2\|_{\alpha,\overline{D}_n})\left(|m_1-m_2|+\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}+|m_1-m_2|\|\eta_1-\eta_2\|_{\alpha,\overline{D}_n}\right)|x-y|^\alpha,
\end{aligned}$$

which yields the result in (4.29). □

4.4 Global existence of a fixed point

For an arbitrary $\alpha\in(0,1)$ and $n\in\mathbb{N}$ we now choose $m^n\in\mathbb{K}_n$ such that m^n is a fixed point of the operator $\mathcal{A}^n$ in $\mathbb{K}_n$, where $\mathcal{A}^n$ is defined in (4.35). Let us now fix an arbitrary $\tilde{n}\in\mathbb{N}$. The following lemma establishes *a priori* estimates for the α-Hölder norms of $\{m^n(x)\}_{n>\tilde{n}}$ in $D_{\tilde{n}}$. We adopt the notation $\Lambda(\tilde{n})$ to denote some positive constant that changes from line to line and may depend on the dimension d, the model coefficients $K_1(\tilde{n}+1), K_2(\tilde{n}+1)$ from Assumption 2.2, the local Lipschitz constant $L_\gamma(\tilde{n}+1)$ and local boundedness constant $B_\gamma(\tilde{n}+1)$ from Assumption 2.5, the time horizon T, and domains $D_{\tilde{n}}, D_{\tilde{n}+1}$. If the constant depends additionally upon the Hölder exponent β we will write $\Lambda(\tilde{n},\beta)$ to highlight the dependence. As such, in the sequel when we write $\Lambda(\tilde{n})$ the constant *does not* depend on β.

Lemma 4.8. Let $\beta \in (0,1)$. For any $\tilde{n} \in \mathbb{N}$ there exists a positive constant $\Lambda(\tilde{n},\beta)$ such that for all $n > \tilde{n}$, $\|m^n\|_{C^\beta(\overline{D_{\tilde{n}}})} \leq \Lambda(\tilde{n},\beta)$.

Proof of Lemma 4.8. By rearranging terms in (4.7) we have that, for all $n \geq \tilde{n}+1$ and $x \in D_{\tilde{n}}$:

$$\begin{aligned} m^n(x) &= \frac{\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} r_t p(t,m^n(x)) e^{-\int_0^t (r_u+\gamma(X_u,m^n(x),m^n(X_u)))du} dt\right]}{\mathbb{E}^x\left[\int_0^{T\wedge\tau_n} p(t,m^n(x)) e^{-\int_0^t (r_u+\gamma(X_u,m^n(x),m^n(X_u)))du} dt\right] + \frac{m^n(x)}{n\left(1-e^{-m^n(x)T}\right)}} \\ &\leq \frac{2}{\inf\limits_{x\in D_{\tilde{n}}} \mathbb{E}^x\left[\int_0^{\frac{T}{2}\wedge\tau_{\tilde{n}+1}} e^{-\int_0^t (r_u du + C_\gamma(\tilde{n}+1))du} dt\right]} \\ &\leq \Lambda(\tilde{n}). \end{aligned} \tag{4.41}$$

Note that the second inequality above has used equation (3.6) and Lemma A.1 below.

We next turn to the β-Hölder semi-norm. From (4.33), for all $x,y \in D_{\tilde{n}}$ we have

$$|m^n(x) - m^n(y)| = \left|\frac{k^n(m^n(x),x;m^n) - k^n(m^n(x),y;m^n)}{\partial_m k^n(\xi,y;m^n)}\right|, \tag{4.42}$$

where ξ is some number between $m^n(x)$ and $m^n(y)$. From (4.16), (4.17) and (4.20), we obtain

$$
\begin{aligned}
&\frac{\partial k^n}{\partial m}(\xi, y; m^n) \\
&\geq \mathbb{E}^{\mathbb{Q}^y}\left[\int_0^{T\wedge\tau_n} r_t e^{-\int_0^t (r_u+\gamma(X_u, m^n(x), m^n(X_u)))du} \frac{1-e^{-\xi(T-t)}-\xi(T-t)e^{-\xi(T-t)}}{\xi^2} dt\right] \\
&\geq \mathbb{E}^{\mathbb{Q}^y}\left[\int_0^{\frac{T}{2}\wedge\tau_{\tilde{n}+1}} r_t e^{-\int_0^t (r_u+\gamma(X_u, m^n(x), m^n(X_u)))du} \frac{1-e^{-m(T-t)}-m(T-t)e^{-m(T-t)}}{m^2}\Big|_{m=m^n(x)\vee m^n(y)} dt\right] \\
&\geq \frac{1-e^{-mT/2}-m(T/2)e^{-m(T/2)}}{m^2}\Big|_{m=m^n(x)\vee m^n(y)} \mathbb{E}^{\mathbb{Q}^y}\left[\int_0^{\frac{T}{2}\wedge\tau_{\tilde{n}+1}} r_t e^{-\int_0^t (r_u+\gamma(X_u, m^n(x), m^n(X_u)))du} dt\right] \\
&\geq \Lambda(\tilde{n})\mathbb{E}^{\mathbb{Q}^y}\left[\int_0^{\frac{T}{2}\wedge\tau_{\tilde{n}+1}} r_t e^{-\int_0^t r_u du} dt\right] \\
&\geq \Lambda(\tilde{n}).
\end{aligned}
$$

The second and third inequalities above follow since the mapping $m \mapsto m^{-2}(1 - e^{-m(T-u)} - m(T-u)e^{-m(T-u)})$ is strictly positive and decreasing in m. The fourth inequality has used equation (4.41) and the fact that $\gamma(X_u, m^n(x), m^n(X_u)) \leq B_\gamma(\tilde{n}+1)$ almost surely for $t \leq \frac{T}{2} \wedge \tau_{\tilde{n}+1}$. The last inequality follows by taking the infimum of $\mathbb{E}^{\mathbb{Q}^y}\left[\int_0^{\frac{T}{2}\wedge\tau_{\tilde{n}+1}} r_t e^{-\int_0^t r_u du} dt\right]$ over $y \in D_{\tilde{n}}$ and noting that this value is strictly positive given that $D_{\tilde{n}}$ is strictly contained in $D_{\tilde{n}+1}$.

For the numerator in (4.42) we have

$$k^n(m^n(x), x; m^n) - k^n(m^n(x), y; m^n) = u^{m^n(x), m^n}(x) - u^{m^n(x), m^n}(y),$$

where $u^{m,\eta}$ is as defined in (4.21). Note that $u^{m^n(x), m^n}$ is of the form (4.2) with $g = g^{m^n(x)}$ and $h = h^{m^n(x), m^n}$ from (4.22). Precisely, we have:

$$
\begin{aligned}
g^{m^n(x)}(t, y) &= (m^n(x) - y^{(1)})\frac{1-e^{-m^n(x)(T-t)}}{m^n(x)}; \\
h^{m^n(x), m^n}(y) &= y^{(1)} + \gamma(y, m^n(x), m^n(y)).
\end{aligned}
$$

Since $0 < m^n(x) < C_n^{(1)}$ we have from (4.23) and (4.24) that the assumptions of Lemma 4.3 are satisfied with $\alpha_0 = \alpha$. Hence, for all $\beta \in (0,1)$ we have

$$\begin{aligned}\|u^{m^n(x),m^n}\|_{C^\beta(\overline{D}_{\tilde{n}})} &\leq \Lambda(\tilde{n},\beta)\left(|g^{m^n(x)}|_{0,\tilde{n}+1} + \left|U^{m^n(x),m^n}\right|_{0,\tilde{n}+1}\right)\\ &\leq \Lambda(\tilde{n},\beta)\left(\Lambda(\tilde{n}) + \left|U^{m^n(x),m^n}\right|_{0,\tilde{n}+1}\right),\end{aligned}$$

where $U^{m^n(x),m^n}$ is of the form (4.3) with $g = g^{m^n(x)}$ and $h = h^{m^n(x),m^n}$. Now, for $t \in [0,T]$ and $y \in D_{\tilde{n}+1}$:

$$\begin{aligned}\left|U^{m^n(x),m^n}(t,y)\right| &\leq \mathbb{E}^y\left[\int_t^{T\wedge\tau_n}(m^n(x)+r_s)\frac{1-e^{-m^n(x)(T-s)}}{m^n(x)}e^{-\int_t^s r_\theta+\gamma(X_\theta,m^n(x),m^n(X_\theta))d\theta}ds\right]\\ &\leq T(1+\mathbb{E}^y\left[\int_t^T r_s e^{-\int_t^s r_\theta d\theta}ds\right])\\ &\leq 2T.\end{aligned}$$

Hence we conclude that $\left\|u^{m^n(x),m^n}\right\|_{\beta,\overline{D}_{\tilde{n}+1}} \leq \Lambda(\tilde{n},\beta)$ and thus

$$|k^n(m^n(x),x;m^n) - k^n(m^n(x),y;m^n)| \leq \Lambda(\tilde{n},\beta)|x-y|^\beta.$$

Putting these two estimates together in equation (4.42) gives

$$|m^n(x) - m^n(y)| \leq \Lambda(\tilde{n},\beta)|x-y|^\beta,\ \forall x,\ y \in D_{\tilde{n}}.$$

This finishes the proof (recall (4.41)). □

Having established the local fixed points and the necessary *a priori* estimates, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. We note that (2.14) is equivalent to

$$m(x) = \frac{\mathbb{E}^x\left[\int_0^T r_t p(t,m(x))e^{-\int_0^t(r_u+\gamma(X_u,m(x),m(X_u)))du}dt\right]}{\mathbb{E}^x\left[\int_0^T p(t,m(x))e^{-\int_0^t(r_u+\gamma(X_u,m(x),m(X_u)))du}dt\right]}; \qquad x \in D.$$

Let $\alpha \in (0,1)$. From Lemma 4.8, there exists a positive constant $\Lambda(1,\alpha)$ such that $\forall n > 1$, $\|m^n\|_{\alpha,\overline{D}_1} \leq \Lambda(1,\alpha)$. The Arzelà-Ascoli theorem asserts the existence of a subsequence of $\{m^n(x)\}_{n>1}$, which we denote by $\left\{m^{n_k^{(1)}}(x)\right\}_{k\in\mathbb{N}}$, and some $m^{(1)} \in \mathbb{K}_1$ such that for each $n_k^{(1)}$, $m^{n_k^{(1)}}$ satisfies the equality in (4.41) for $x \in D_1$ and such that $m^{n_k^{(1)}}(x)$ converge to $m^{(1)}(x)$ uniformly in D_1 as $k \to \infty$, with $\|m^{(1)}\|_{\alpha,\overline{D}_1} \leq \Lambda(1,\alpha)$.

Applying Lemma 4.8 again, we have that there exists a positive constant $\Lambda(2,\alpha)$ such that $\forall n_k^{(1)} > 2$, we have $\|m^{n_k^{(1)}}\|_{\alpha,\overline{D}_2} \leq \Lambda(2,\alpha)$. The Arzelà-Ascoli theorem again assures the existence of a subsequence of $\left\{m^{n_k^{(2)}}(x)\right\}_{k\in\mathbb{N}}$ and some $m^{(2)} \in \mathbb{K}_2$ such that $m^{n_k^{(2)}}$ converge to $m^{(2)}$ uniformly in D_2 as $k \to \infty$, with $\|m^{(2)}\|_{\alpha,\overline{D}_2} \leq \Lambda(2,\alpha)$. Furthermore, by construction we have $m^{(2)}(x) = m^{(1)}(x)$ for $x \in D_1$.

The above procedure can be carried out iteratively and we conclude that for all $l \in \mathbb{N}$, there exists a subsequence of $\{m^{n_k^{(l)}}\}_{k>1}$, denoted by $\{m^{n_k^{(l+1)}}\}_{k\in\mathbb{N}}$, and function $m^{(l+1)} \in \mathbb{K}_{l+1}$, such that $m^{n_k^{(l+1)}}$ converge to $m^{(l+1)}$ uniformly in D_{l+1} as $k \to \infty$, and $\|m^{(l+1)}\|_{\alpha,\overline{D}_{l+1}} \leq \Lambda(l+1,\alpha)$. Moreover, by construction, $m^{(l+1)}(x) = m^{(l)}(x)$ for $x \in D_l$.

Now, for fixed, arbitrary $x \in D$, there is some $l \in \mathbb{N}$ such that $x \in D_k$, $\forall k \geq l$. We define $m : D \to [0,\infty)$ by

$$m(x) := m^{(l)}(x). \tag{4.43}$$

Note that by construction, m is well defined and $m(x) \in C^{\alpha}_{loc}(D)$, $\forall \alpha \in (0,1)$. We claim that m is the desired fixed point.

Indeed, for fixed l we note that for $x \in D_l$ we have $m(x) = \lim_{k\to\infty} m^{n_k^{(l')}}(x)$ for any $l' \geq l$. Therefore, by using (4.41), we can write

$$\begin{aligned} m(x) &= \frac{\lim\limits_{k\to\infty} \mathbb{E}^x\left[\int_0^{T\wedge\tau_{n_k^{(l')}}} r_t p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u))\right) du} dt\right]}{\lim\limits_{k\to\infty} \mathbb{E}^x\left[\int_0^{T\wedge\tau_{n_k^{(l')}}} p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u))\right) du} dt\right] + \frac{m^{n_k^{(l')}}(x)}{n_k^{(l')}\left(1 - e^{-m^{n_k^{(l')}}(x)T}\right)}} \\ &=: \frac{\mathbf{A}(l')}{\mathbf{B}(l')}, \end{aligned} \tag{4.44}$$

for any $l' \geq l$, where (recall $x \in D_l$ and l is fixed):

$$\begin{aligned}\mathbf{A}(l') &= \lim_{k\to\infty} \mathbb{E}^x \left[\int_0^{T\wedge\tau_{l'}} r_t p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u)) \right) du} \right] \\ &\quad + \lim_{k\to\infty} \mathbb{E}^x \left[\int_{T\wedge\tau_{l'}}^{T\wedge\tau_{n_k^{(l')}}} r_t p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u)) \right) du} dt \right] \\ &= \mathbb{E}^x \left[\int_0^{T\wedge\tau_{l'}} r_t p(t, m(x)) e^{-\int_0^t (r_u + \gamma(X_u, m(x), m(X_u))) du} dt \right] \\ &\quad + \lim_{k\to\infty} \mathbb{E}^x \left[\int_{T\wedge\tau_{l'}}^{T\wedge\tau_{n_k^{(l')}}} r_t p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u)) \right) du} dt \right].\end{aligned}$$

The second equality above follows from the bounded convergence theorem, if we recall that $0 \leq p \leq 1$, $0 \leq r_t \leq C_{l'}^{(1)}$, $\gamma \geq 0$, that $m^{n_k^{(l')}}(X_u) \to m(X_u)$ almost surely for $u \leq \tau_{l'}$, and that $m^{n_k^{(l')}}(x) \to m(x)$ since for $l' \geq l$, $x \in D_l \subset D_{l'}$.

To take care of the second term above, we note that

$$\begin{aligned}0 &\leq \mathbb{E}^x \left[\int_{T\wedge\tau_{l'}}^{T\wedge\tau_{n_k^{(l')}}} r_t p(t, m^{n_k^{(l')}}(x)) e^{-\int_0^t \left(r_u + \gamma(X_u, m^{n_k^{(l')}}(x), m^{n_k^{(l')}}(X_u)) \right) du} dt \right] \\ &\leq \mathbb{E}^x \left[\int_{T\wedge\tau_{l'}}^{T} r_t e^{-\int_0^t r_u du} dt \right].\end{aligned}$$

So by taking $l' \uparrow \infty$ and using the non-explosivity of X along with the monotone convergence theorem we get

$$\lim_{l'\uparrow\infty} \mathbf{A}(l') = \mathbb{E}^x \left[\int_0^T r_t p(t, m(x)) e^{-\int_0^t (r_u + \gamma(X_u, m(x), m(X_u))) du} dt \right].$$

The same calculation can be applied to $\mathbf{B}(l')$ as well. Here the only differences are

1. The absence of r_t, which is bounded for $t \leq \tau_{l'}$.

2. The fraction $m^{n_k^{(l')}}(x) / (n_k^{(l')} (1 - e^{-m^{n_k^{(l')}}(x) T}))$, which vanishes as $k \uparrow \infty$.

It thus follows by a similar argument as above that

$$\lim_{l'\uparrow\infty} \mathbf{B}(l') = \mathbb{E}^x \left[\int_0^T p(t, m(x)) e^{-\int_0^t (r_u + \gamma(X_u, m(x), m(X_u)))du} dt \right]; \qquad x \in D_l.$$

Thus, since $m(x)$ on the left hand side of (4.44) did not depend upon l', we obtain the desired result.

$\square$

Chapter 5

Conclusion And Final Remarks

5.1 Conclusion

The current coupon is a critical input to virtually all MBS valuation models. Defined as the theoretical coupon rate such that a TBA pass-through is valued at par, the current coupon is by nature an endogenous rate. As we have seen, finding the current coupon involves modeling the mortgage borrower's prepayment decision, which in turn is dependent on the current coupon itself. This circular relationship naturally yields a fixed point problem. By identifying the current coupon with solutions to a degenerate elliptic, non-linear fixed point problem in a doubly stochastic factor based model which allows for prepayment intensities to depend upon current and origination mortgage rates, we have shown the existence of a current coupon function with nice regularity properties. The major difficulty in the proof has been how to deal with the non-compact and non-smoothing nature of the original fixed point operator $\mathcal{A}$. Our approach involves first localizing the original fixed point problem and establishing local existence and regularity results using results from PDE theory and *a priori* Hölder norm estimates. We then unwind the localization using a delicate diagonal subsequence argument. Equally important as identifying existence of current coupons is to efficiently compute them. Unfortunately, any iterative procedure that naively applies the contraction mapping principle is not only theoretically unjustified, but also prohibitively slow. To overcome this problem, we have approximated the current coupon via a perturbation analysis when the prepayment intensity takes the form $\gamma(x,m,z) = \gamma_0(x) + \varepsilon\gamma_1(x,m,z)$, using the well known fact that unique current coupon functions exist when the prepayment intensity $\gamma_t = \gamma(X_t)$ only depends upon the factors, and have obtained a unique, explicit, closed form expression $m_1(x)$ such that the current coupon function admits the decomposition $m^\varepsilon(x) = m_0(x) + \varepsilon m_1(x) + o(\varepsilon)$. The point is that m^ε

can be *any* fixed point, not necessarily unique, however it is uniquely identified up to leading orders of expansion around $\varepsilon \approx 0$. Moreover, the closed form of m_1 is explicitly identifiable given m_0 (which itself is unique and easy to compute for many models), and can be easily calculated using standard Monte-Carlo method. We have shown the power of this approximation in a simple one dimensional CIR specification, where the approximation differs by ≤ 10 basis points (on absolute rate levels of $4\% - 12\%$) from the theoretical fixed point determined by naive contraction.

5.2 Extensions and future work

Throughout this dissertation, we have been focused on the modeling of the current coupon for agency TBA pools of residential mortgage loans. Let us recall two important assumptions we have made on the underlying collateral:

- The mortgage pool is "homogeneous" in the sense that all borrowers in the pool share the same prepayment intensity function (refer to the construction of the risk neutral measure $\mathbb{Q}$ in Section 2.3 for details).
- The cash flow of the pool is subject to prepayment risk, but not default risk (typical assumption for agency pools).

Although the above assumptions are not overly restrictive for modeling agency pools, we do point out that our model framework has the potential to be generalized to adapt to the following topics for future studies:

- Existence of endogenous current coupon function for heterogeneous mortgage pools (i.e. pools consisting of borrowers with different prepayment intensity specifications).
- Existence of endogenous mortgage rates in the presence of default options (e.g. for non-agency mortgage pools).
- Higher order approximations in the perturbation analysis.

We now briefly describe how to generalize the existing model to allow for the inclusion of the default option. Set $\tau_1 = \inf\{t: \ e^{-\int_0^t \gamma_1(u)du} \leq U_1\} \wedge T$ and $\tau_2 = \inf\{t: \ e^{-\int_0^t \gamma_2(u)du} \leq U_2\} \wedge T$, where U_1, U_2 are random variables in the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, and γ_1, γ_2 are nonnegative, $\mathbb{F}^W$-adapted, integrable processes. Let $D_t^{(i)} := \mathbb{1}_{\{\tau_i \leq t\}}$ $(i = 1, 2)$, and denote by $\mathbb{D}^{(i)}$ the filtration generated by the right-continuous process $D_t^{(i)}$, i.e. $\mathcal{D}_t^{(i)} := \sigma(D_u^{(i)} : \ u \leq t)$. Finally, we set $\mathcal{G}_t = \mathcal{D}_t^{(1)} \vee \mathcal{D}_t^{(2)} \vee \mathcal{F}_t$.

Assumption 5.1. For the random variables U_1 and U_2:

1. For $i = 1, 2$ the random variable U_i is uniformly distributed on $[0, 1]$ under $\mathbb{Q}$, and is independent from $\mathcal{F}^W$.

2. The 2-dimensional vector $\mathbf{U} = (U_1, U_2)$ is distributed according to the 2-dimensional copula $C(\mathbf{u})$, which is continuously differentiable. Furthermore, $\mathbf{U}$ is independent from $\mathcal{F}^W$.

Under the above assumption, the joint conditional survival function of τ_1, τ_2 is given by

$$\mathbb{Q}\left[\tau_1 > t, \tau_2 > s | \mathcal{F}_\infty\right] = C\left(\Gamma_1(t), \Gamma_2(s)\right),$$

where for $i = 1, 2$, $\Gamma_i(t) = \exp(-\int_0^t \gamma_i(u)\, du)$.

The prepayment time τ_P and default time τ_D are defined by

$$\tau_P := \tau_1 \mathbb{1}_{\{\tau_1 < \tau_2\}} + T\mathbb{1}_{\tau_1 \geq \tau_2};$$
$$\tau_D := \tau_2 \mathbb{1}_{\{\tau_2 < \tau_1\}} + T\mathbb{1}_{\tau_2 \geq \tau_1}.$$

Note that $\tau_P = \tau_D$ if and only if $\tau_P = \tau_D = T$, namely neither prepayment nor default occurs during the lifetime of the mortgage. If $\tau_P < T$(resp. $\tau_D < T$) then $\tau_D = T$(resp. $\tau_P = T$) and we say that prepayment (resp. default) occurs in this case.

To derive the endogenous mortgage rate function, we begin with the cash flow analysis of the mortgage contract with both prepayment and default option. The cash flow received by the mortgage issuer (or by the MBS investor, assuming no guarantee fee or servicing fee) is comprised of three parts:

1. A continuous coupon stream of c dollars per unit time, during the lifetime of the mortgage till the prepayment time (if prepayment occurs) or the default time (if default occurs).

2. A lump-sum payment of the remaining principal of the mortgage $P(\tau_P, m)$, if prepayment occurs.

3. A lump-sum recovery payment Z_{τ_D}, if default occurs. We assume that the nonnegative recovery process Z_t is adapted to the market filtration $\{\mathcal{F}_t\}_t$ and is bounded above by the housing price process H_t (which is a factor).

The initial value of the mortgage contract is now given by the following risk-neutral pricing formula:

$$
\begin{aligned}
M_0 &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^{\tau_1\wedge\tau_2} ce^{-\int_0^u r_\theta\, d\theta}du\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_1<\tau_2\}}P(\tau_1,m)e^{\int_0^{\tau_1} r_\theta d\theta}\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_2<\tau_1\}}Z_{\tau_2}e^{\int_0^{\tau_2} r_\theta d\theta}\right] \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Calculation shows:

$$
\begin{aligned}
I_1 &= P_0 + \mathbb{E}^{\mathbb{Q}}\Bigg[\int_0^T e^{-\int_0^u r_\theta d\theta}\Big((m-r_u)p(u,m)C(\Gamma_1(u),\Gamma_2(u)) \\
&\quad - \gamma_1(u)p(u,m)C_1(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u \gamma_1(\theta)d\theta} - \gamma_2(u)p(u,m)C_2(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u \gamma_2(\theta)d\theta}\Big)\, du\Bigg]; \\
I_2 &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T p(u,m)\gamma_1(u)C_1(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u r_\theta+\gamma_1(\theta)d\theta}\, du\right]; \\
I_3 &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^T Z_u\gamma_2(u)C_2(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u r_\theta+\gamma_2(\theta)d\theta}\, du\right].
\end{aligned}
$$

Setting $M_0 = P_0$, we obtain

$$
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^u r\theta d\theta}\big((m-r_u)p(u,m)C(\Gamma_1(u),\Gamma_2(u)) - \gamma_2(u)(p(u,m)-Z_u)C_2(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u \gamma_2(\theta)d\theta}\big)\, du\right] = 0.
$$

The fixed point problem for the endogenous mortgage rate is thus given by

$$
m(x) = \frac{\mathbb{E}^x\left[\int_0^T e^{-\int_0^u r_\theta d\theta}\big(r_u p(u,m)C(\Gamma_1(u),\Gamma_2(u)) + \gamma_2(u)(p(u,m)-Z_u)C_2(\Gamma_1(u),\Gamma_2(u))e^{-\int_0^u \gamma_2(\theta)d\theta}\big)\, du\right]}{\mathbb{E}^x\left[\int_0^T e^{-\int_0^u r_\theta d\theta}p(u,m)C(\Gamma_1(u),\Gamma_2(u))\, du\right]}.
$$

We note that under additional assumptions on the recovery process Z and the copula function C, our proof method may be used to obtain fixed points in this generalized model as well.

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Appendix A

Appendix

A.1 Auxiliary lemmas

Recall the remaining balance function $p(t,m)$ defined in (2.2). Direct calculation yields

$$p_t(t,m) = \frac{\partial}{\partial t}p(t,m) = -\frac{me^{mt}}{e^{mT}-1}, \tag{A.1}$$

and

$$p_m(t,m) = \frac{\partial}{\partial m}p(t,m) = \frac{e^{-m(T-t)}(T-t)}{1-e^{-mT}} - \frac{e^{-mT}\left(1-e^{-m(T-t)}\right)T}{(1-e^{-mT})^2}. \tag{A.2}$$

It is straightforward to verify that both $p_t(t,m)$ and $p_m(t,m)$ are continuous functions on $[0,T]\times[0,\infty)$ and that for each $T>0$ there exists some constant $L(T)$ depending only on T such that $|p_t(t,m)| \le L(T)$, $|p_m(t,m)| \le L(T)$, $\forall (t,m) \in [0,T]\times[0,\infty)$.

The following elementary property of $p(t,m)$ is used in the proofs.

Lemma A.1. $\forall m \ge 0$, $\inf\left\{t \in [0,T]:\ p(t,m) \le \frac{1}{2}\right\} \ge \frac{T}{2}$.

Proof. Assume for $m>0$ and $t\in[0,T]$ the pair (t,m) satisfies

$$\frac{1-e^{-m(T-t)}}{1-e^{-mT}} = \frac{1}{2},$$

then

$$t = T + \frac{\log\left[\frac{1}{2}\left(1+e^{-mT}\right)\right]}{m}.$$

It is clear

$$t > \frac{T}{2} \iff \frac{\log\left[\frac{1}{2}(1+e^{-mT})\right]}{m} > -\frac{T}{2} \iff \frac{1}{2}\left(1+e^{-mT}\right) > e^{-\frac{mT}{2}}.$$

The last inequality holds for all $m > 0$ and $T > 0$. $\square$

A.2 A note on Goncharov's result

The proof of [19, Theorem 5.1] relies heavily on [19, Proposition 4.2], which is an application of the intermediate value theorem for continuous functions. It relies on the conclusion of [19, Proposition 4.1] which establishes continuity of the operator $\mathcal{L}[f(\,\cdot\,)](\mu, x)$ defined therein. Unfortunately, [19, Proposition 4.1] fails in general. In fact we can come up with the following counter-example: adopting the notations in Goncharov [19], let $f(x) := 1$, $\gamma_t := \mathbb{1}_{\{m^0 - m^t > \delta\}}$, then

$$\begin{aligned}\mathcal{L}[f(\,\cdot\,)](\mu, x) &= \mathbb{E}_x\left[\int_0^T (\mu - r_t)p(t,\mu)e^{-\int_0^t \mathbb{1}_{\{\mu-1>\delta\}}+r_\theta\, d\theta}\, dt\right] \\ &= \mathbb{E}_x\left[\int_0^T (\mu - r_t)p(t,\mu)e^{-\int_0^t r_\theta\, d\theta}e^{-\mathbb{1}_{\{\mu>1+\delta\}}t}\, dt\right],\end{aligned}$$

which is discontinuous at $\mu = 1 + \delta$.

A.3 On the naive contraction method

For the sake of completeness, in the rest of this section we describe the naive contraction method. The method involves, starting with some initial guess for $m(x)$ on a grid of x-values, solving equation (2.15) iteratively and updating the value of m on the grid points in each iteration. Standard Monte Carlo methods can be applied to evaluate the expectations in (2.15). Alternatively, we may first transfer equation (2.15) into a coupled system of parabolic partial differential equations via Feynmann-Kac representations, and then solve the resulting system using numerical PDE methods. The hope here is that (although completely unjustified) this process will converge to a stable numerical solution after a reasonable number of iterations, regardless of the choice of the initial guess m.

Monte Carlo method

Recall that in the current setup, the state process X is a CIR diffusion representing the interest rate. We choose a set of grid points $x_i = r_i$ that cover the $(2.5\%, 97.5\%)$ percentiles of the CIR invariant distribution, with increment $dr = 0.01$. Figure A.1 shows the initial random guess for m, and the numerical solution m obtained via Monte Carlo method after 2, 5 and 10 iterations, respectively. We note that this procedure converges rapidly after only a few iterations, despite the fact that the initial guess for m is deliberately chosen at random.

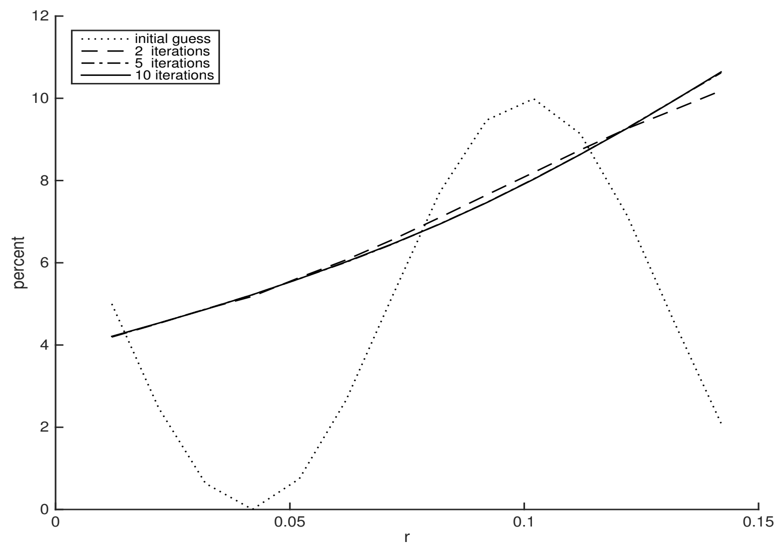

Figure A.1: Current coupon functions as a function of the underlying CIR factor obtained by naive contraction and Monte Carlo method. The dotted line is the initial guess for m. The dash line is the m function after 2 iterations. The dash-dotted line is the m function after 5 iterations. The solid line is the m function after 10 iterations. Values are given in percentage points. The intensity function is $\gamma(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*.

Numerical PDE method

First of all we note that equation (2.15) can be transformed to the following coupled system of parabolic PDEs:

$$\begin{cases} U_t + \frac{1}{2}\sigma^2 x U_{xx} + \kappa(\theta - x)U_x - (x + \gamma + k(m(y) - m(x))^+)U = -xp(t, m(y)),\ (x, y) \in D \times D,\ t \in [0, T); \\ U(T, x, y) = 0,\ (x, y) \in \overline{D} \times \overline{D}; \\ V_t + \frac{1}{2}\sigma^2 x V_{xx} + \kappa(\theta - x)V_x - (x + \gamma + k(m(y) - m(x))^+)V = -p(t, m(y)),\ (x, y) \in D \times D,\ t \in [0, T); \\ V(T, x, y) = 0,\ (x, y) \in \overline{D} \times \overline{D}; \\ m(x) = U(0, x, x)/V(0, x, x),\ x \in D. \end{cases} \tag{A.3}$$

Note that in the above coupled system the only spatial derivative appearing in the equations is the partial derivative with respect to x. However we cannot simply treat the spatial variable y as a parameter, since in each iteration we need to update the m function, which calls for the values of U and V along the diagonal of the spatial domain: $\{(x, y) \in D \times D;\ x = y\}$. Thus in each iteration we need to solve the coupled system for all possible values of y.

Finite Difference Schemes

In the current setting, the factor process X is a 1-d CIR process, so the state space D is simply the positive half line: $D = (0, \infty)$, and the domain for the coupled system is $D \times D = (0, \infty) \times (0, \infty)$. First we need to truncate the domain by choosing a far boundary location $R > 0$, and define the numerical method on the truncated domain $D_R := [0, R] \times [0, R]$. We will determine the value of R via numerical experiments. In order for the coupled system and the associated numerical method to be well defined on D_R, we also need to specify suitable boundary conditions for U and V on the boundary of the truncated domain D_R. Thus for each fixed value of $y \in [0, R]$, we will specify suitable boundary conditions for x on the left, or near, boundary $x = 0$ and the right, or far, boundary $x = R$. First, since the left boundary $x = 0$ is a degenerate boundary, we first verify that Fichera's condition (see [30]) holds and then employ the following internal boundary conditions:

$$U_t + \kappa\theta U_x - (\gamma + k(m(y) - m(0))^+)U(0, y) = 0,\ (t, y) \in [0, T] \times [0, M];$$
$$V_t + \kappa\theta V_x - (\gamma + k(m(y) - m(0))^+)V(0, y) + 1 = 0,\ (t, y) \in [0, T] \times [0, M].$$

As for the right boundary, we propose the following artificial boundary conditions:

$$U(t,R,y) = \mathbb{E}^R\left[\int_t^T X_u e^{-\int_t^u X_\theta + \gamma d\theta}\, du\right];$$
$$V(t,R,y) = \mathbb{E}^R\left[\int_t^T e^{-\int_t^u X_\theta + \gamma d\theta}\, du\right].$$

In the case where X is a 1-d CIR process, the above boundary conditions can be simplified:

$$U(t,R,y) = 1 - \gamma\int_t^T e^{-RC(0,u-t)-A(0,u-t)-\gamma(u-t)}\, du - e^{-RC(0,T-t)-A(0,T-t)-\gamma(T-t)};$$
$$V(t,R,y) = \int_t^T e^{-RC(0,u-t)-A(0,u-t)-\gamma(u-t)}\, du,$$

where

$$C(t,T) := \frac{\sinh(\alpha(T-t))}{\alpha\cosh(\alpha(T-t)) + \frac{1}{2}\kappa\sinh(\alpha(T-t))};$$
$$A(t,T) := -\frac{2\kappa\theta}{\sigma^2}\log\left[\frac{\alpha e^{\frac{1}{2}\kappa(T-t)}}{\alpha\cosh(\alpha(T-t)) + \frac{1}{2}\kappa\sinh(\alpha(T-t))}\right];$$
$$\alpha := \frac{1}{2}\sqrt{\kappa^2 + 2\sigma^2}.$$

Now let $N > 0$, $M > 0$ be the number of mesh points on the x and t direction, respectively. Let $\Delta t = \frac{T}{M}$, $\Delta x = \Delta y = \frac{R}{N}$. We set up the uniform mesh points by $t_l = l\Delta t$, $x_i = i\Delta x$, $y_j = j\Delta y$, and set $U_{ij}^l = U(t_l, x_i, y_j)$, $V_{ij}^l = V(t_l, x_i, y_j)$ for $l = 0,\ 1,\ \cdots,\ M$ and $i, j = 0,\ 1,\ \cdots,\ N$. The following simple iterative algorithm can be used to solve the coupled system A.3:

Algorithm 1 A simple iterative finite difference algorithm

```
m ← m_0                          ▷ initialize m along the mesh points using function m_0
for k in 1:K do
    for j in 0:N do              ▷ for each y_j, solve the coupled system by finite difference
        for l in M:-1:1 do
            U_ij^{l-1} ← finite difference method
            V_ij^{l-1} ← finite difference method
        end for
        m_j ← U_jj^0 / V_jj^0    ▷ update the value of m at the j-th diagonal mesh point
    end for
end for
```

Any standard finite difference method can be used to solve for U and V in each iteration. For example, in the Crank-Nicolson method, the finite difference approximations to the partial derivatives are given by

$$\begin{aligned}
U_t(t_{l+\frac{1}{2}}, x_i, y_j) &= \frac{U_{ij}^l - U_{ij}^{l-1}}{\Delta t}; \\
U_x(t_{l+\frac{1}{2}}, x_i, y_j) &= \frac{1}{2}\left(\frac{U_{i+1,j}^{l+1} - U_{i-1,j}^{l+1}}{2\Delta x} + \frac{U_{i+1,j}^{l} - U_{i-1,j}^{l}}{2\Delta x}\right); \\
U_{xx}(t_{l+\frac{1}{2}}, x_i, y_j) &= \frac{1}{2}\left(\frac{U_{i-1,j}^{l+1} - 2U_{ij}^{l+1} + U_{i+1,j}^{l+1}}{\Delta x^2} + \frac{U_{i-1,j}^{l} - 2U_{ij}^{l} + U_{i+1,j}^{l}}{\Delta x^2}\right).
\end{aligned}$$

Figure A.2 below shows the initial random guess for m (the same as in Figure A.1), and the numerical solution m obtained via numerical PDE method after 2, 5 and 10 iterations, respectively. Again this procedure converges rapidly after around 5 iterations. Moreover, the resulting plot for m after 10 iterations overlays almost exactly as the corresponding plot in Figure A.1. The benefit of the PDE approach is that it is much faster compared to the Monte Carlo approach when the dimension of the state process is low.

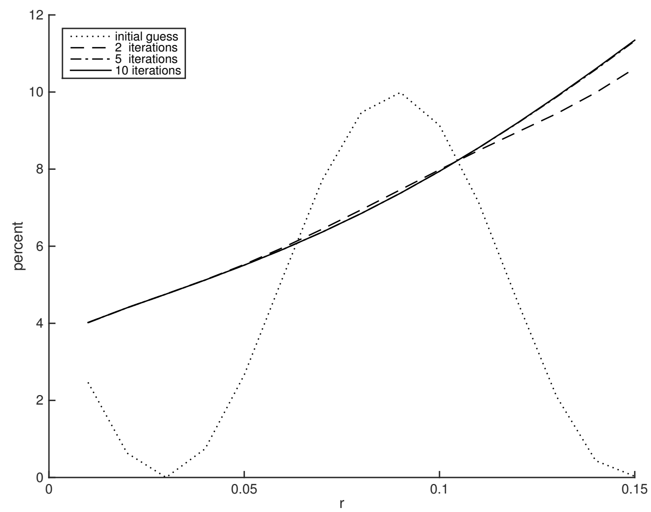

Figure A.2: Current coupon functions as a function of the underlying CIR factor obtained by solving the coupled system using naive contraction and finite difference method. The dotted line is the initial guess for m. The dash line is the m function after 2 iterations. The dash-dotted line is the m function after 5 iterations. The solid line is the m function after 10 iterations. Values are given in percentage points. The intensity function is $\gamma(x, m, z) = \gamma + k(m - z)^+$. Parameters are $\kappa = 0.25, \theta = 0.06, \sigma = 0.1$, $T = 30$, $k = 5$ and $\gamma = 0.045$. Computations were performed using *Matlab*.