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Radner equilibrium in infinite and finite time-horizon Lévy models

by

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INTRODUCTION

It is commonly known in the US market, and perhaps in the global market, that the historical returns on equity have far exceeded the returns on government bonds. This large magnitude of differential in returns–known as equity premium–cannot be accounted for by standard equilibrium models as presented by Mehra and Prescott (1985) in [24], leading to the so called risk premium puzzle. Since then, despite a large number of proposed resolutions which are generally based on complete Brownian models, no single one has achieved general acceptance. Consequently, this dissertation attempts to overcome such challenge by utilizing incomplete Lévy models. The main feature is a positive impact due to the model's incompleteness on the Sharpe ratio. This leads to a subsequently positive impact on equity premium which captures some disparity between theoretical and empirical equity premiums. More detailed discussion and a literature review are provided in the introduction of Chapter I.

This dissertation is organized as follows. The first chapter investigates Lévy models in the simplest, infinite time-horizon settings. Sufficient conditions on the Lévy measure are stated and the closed-form equilibrium is solved using the Hamilton-Jacobi-Bellman (HJB) equation. A numerical example illustrating the incompleteness impacts on the interest rate and the Sharpe ratio is provided for the compound Poisson model with Gaussian jumps. The work in this chapter was in collaboration with K. Larsen [21] and is forthcoming in Mathematics and Financial Economics.

Chapter II explores finite time-horizon Lévy models with a more relaxed assumption on the Lévy measure. The closed-form equilibrium is derived using Fenchel's conjugate as the main tool for solving the investors' maximization problem. The relaxed assumption on the Lévy measure allows us to extend the equilibrium result to variance gamma models, whose numerical example is provided therein. Finally, we discuss the lack of the minimal martingale measure in these incomplete Lévy models.

CHAPTER I

Radner equilibrium in Lévy models[1]

1.1 Introduction

We construct equilibrium models in which a finite number of heterogeneous exponential investors cannot fully trade their future income streams. We show that the framework of continuous-time Lévy processes produces the Radner equilibrium in closed-form (i.e., optimal strategies, interest rates, drifts, and volatility structures are available in closed-form). In addition to allowing for more model flexibility, we show that by going beyond models based on Brownian motions we can produce the following empirically desirable feature: The class of pure jump Lévy models can simultaneously lower the equilibrium interest rate and increase the equilibrium Sharpe ratio due to investors' income streams being unspanned (i.e., due to model incompleteness).

The first construction of an incomplete continuous-time model which allows for an explicit description of the Radner equilibrium was given in [9]. As an application of this model, [9] showed that model incompleteness can significantly lower the equilibrium interest rate. However, the (instantaneous) Sharpe ratio is unaffected by the model's incompleteness.[2] Beside being of mathematical interest, our motivation

[1] This chapter is the joint work with K. Larsen and is forthcoming in Mathematics and Financial Economics

[2] Theorem 4.1 in [10] shows that no model based on exponential utilities, continuous consumption, and a filtration generated by Brownian motions can ever produce an incompleteness impact on the Sharpe ratio when this ratio is measured instantaneously.

behind extending the Brownian framework in [9] to the more general Lévy framework is to produce simultaneously a negative impact on interest rate and a positive impact on the Sharpe ratio while still maintaining a closed-form equilibrium model. Our desire to construct an incomplete equilibrium model with these features is of course due to Weil's celebrated risk-free rate puzzle (see [33]) as well as Mehra and Prescott's equity premium puzzle (see [24]). These and other asset pricing puzzles are also discussed in detail in the survey [6].

The literature on continuous-time Radner equilibrium theory in models where the investors' income streams are spanned (i.e., complete models) is comprehensive and we refer to the recent references on endogenous completeness [1], [18], [16], and [20] for more information. On the other hand, models with continuous-time trading and unspanned income streams (i.e., incomplete models) are much less developed and only in recent years has progress been made. The papers [35], [34], [8], and [19] consider models with exponential utilities, no dividends (i.e., only financial assets), and discrete-time consumption.[3] These papers differ in how general the underlying state-processes describing the investors' discrete-time income streams can be: [35] considers a Brownian motion and an independent indicator process. [34] and [8] consider multiple Brownian motions ([8] also allow for processes with mean reversion) whereas the recent paper [19] allows for a non-Markovian Brownian setting. The current paper is more related to [9] and [10] who - in Brownian settings - consider both financial and real assets in the case of exponential investors with continuous-time consumption. Indeed, the current paper can be seen as a direct extension of [9] to the setting of discontinuous Lévy processes.

[3]By restricting the investors to only consume at maturity, we obtain a model in which the economy's interest rate is undetermined. Furthermore, by only considering financial assets, we cannot determine the assets' volatility structures. Therefore, the interest rate and the volatility parameters are taken as exogenously specified model input in such models.

The paper is organized as follows: The next section exemplifies the underlying setting in the Gaussian compound Poisson case. Section 1.3 describes the underlying Lévy framework. Section 1.4 provides the solution to the individual investors' problems. Section 1.5 contains our main result which provides the equilibrium price processes in closed-form. Section 1.6 illustrates numerically the equilibrium impacts due to income incompleteness in the specific model from Section 1.3. The last two sections contain all the proofs.

1.2 Example

This section serves to introduce the basic ideas in a specific example. There are $I < \infty$ investors trading and consuming continuously over the infinite time-horizon. Our framework takes as input (i.e., exogenously specified data) the exponential utility investors' time preferences $\delta_i > 0$, risk tolerance coefficients $\tau_i > 0$, income processes $Y_i = (Y_{it})_{t\geq 0}$, and the stock's dividend process $D = (D_t)_{t\geq 0}$. On the other hand, our framework's output (i.e., endogenously determined data) are the economy's equilibrium interest rate process $r = (r_t)_{t\geq 0}$ and stock price process $S = (S_t)_{t\geq 0}$.[4] As a by-product, we also obtain all investors' optimal consumption processes $c_i^* = (c_{it}^*)_{t\geq 0}$, their stock investment strategies $\theta_i^* = (\theta_{it}^*)_{t\geq 0}$, and their money market account investment strategies $\theta_i^{(0)*} = (\theta_{it}^{(0)*})_{t\geq 0}$.

We start by specifying the input processes $(D, Y_1, ..., Y_I)$ in this section's example by means of the widely used compound Poisson process with Gaussian jumps.[5] We let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and we let $(A_n)_{n=1}^{\infty}$ be a sequence of $\mathbb{R}^{I+1}$-valued i.i.d. normals $\mathcal{N}(m, \Sigma)$ which are independent of N. Here m denotes A_n's mean vector and Σ is A_n's variance-covariance matrix (both m and

[4] As usual, the price dynamics of the money account are given by $dS_t^{(0)} = r_t S_t^{(0)} dt$ with $S_0^{(0)} = 1$.

[5] The geometric form of this Lévy process was first used in finance by Merton in his classical paper [25]. This process is also the basis for Bates' asset pricing model developed in [4].

Σ are independent of n). We can then define the dividend and income processes as follows:

$$
\begin{bmatrix} D_t \\ Y_{1t} \\ \vdots \\ Y_{It} \end{bmatrix} = \sum_{n=1}^{N_t} A_n, \quad t \geq 0. \tag{1.1}
$$

The model specification (1.1) has finitely many jumps on all finite intervals. Furthermore, all processes jump at the same time (whenever the Poisson process N jumps) and the various jump sizes are jointly normally distributed. As we shall see in Section 1.6.2, this model produces an incomplete equilibrium model as soon as the jump sizes of the dividend process D are less than perfectly positively correlated with the jump sizes of the incomes Y_i for all $i = 1, 2, ..., I$.

Next, we consider the endogenously determined processes. Throughout the paper we assume that the i'th investor, $i = 1, ..., I$, seeks a consumption process c_i in excess of the income process Y_i and investment strategies $(\theta_i^{(0)}, \theta_i)$ in the money market account and the stock which maximize the expectation[6]

$$
\mathbb{E}\left[\int_0^\infty e^{-\delta_i t} U_i(c_{it} + Y_{it}) dt\right], \quad U_i(c) = -e^{-\frac{c}{\tau_i}}, \quad c \in \mathbb{R}, \quad i = 1, ..., I. \tag{1.2}
$$

Throughout this paper we will consider Radner's notion of equilibrium:[7]

Definition I.1 (Radner)**.** We call $(S^{(0)}, S)$ an equilibrium if these price processes clear the markets in the sense that for all $(t, \omega) \in [0, \infty) \times \Omega$ we have

$$
\sum_{i=1}^I c_{it} = D_t, \quad \sum_{i=1}^I \theta_{it} = 1, \quad \sum_{i=1}^I \theta_{it}^{(0)} = 0, \tag{1.3}
$$

where the processes $(\theta_i, \theta_i^{(0)}, c_i)$ maximize the expectation in (1.2) for $i = 1, 2, ..., I$.

[6] We will need to place various integrability restrictions on investor's possible choices of $(\theta_i, \theta_i^{(0)}, c_i)$; see Definition I.4 below.

[7] Radner's equilibrium notion has a long history in financial economics and is also called security-spot market equilibrium. We refer to Section 10 in the textbook [12] for more details.

The first requirement in (1.3) ensures that the good's market clears, the second requirement ensures that the stock market is in net unit supply (real asset), and the last requirement ensures that the money market account is in net zero supply (financial asset).

Our main result (Theorem I.6 below) provides conditions under which a Radner equilibrium exists. Furthermore, it provides both the equilibrium interest rate r (turns out to be constant) and the equilibrium price process S in closed-form.

In general, it is impossible for a market with only two traded securities (here a money market and a stock) to span all the income risks present in (1.1). Consequently, the investors cannot share their income risks efficiently, which renders the model incomplete. In the next sections we present the equilibrium theory for more general Lévy processes; however, in Section 1.6 we return to this section's compound Poisson process example (1.1). We will illustrate that the model based on (1.1) can simultaneously produce both a lower equilibrium interest rate and a higher equilibrium Sharpe ratio for the stock when compared to a model in which the traded securities span all income risks. In other words, despite the simplicity of both our model's utility functions and the dynamics (1.1), the model is rich enough to simultaneously resolve both the risk-free rate puzzle of [33] and the equity premium puzzle of [24].

1.3 Mathematical setting

1.3.1 Underlying Lévy process

We let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denote the underlying filtered probability space. For some underlying $\mathbb{R}^{I+1}$-dimensional pure jump Lévy process η we denote by $N = N(dt, dz)$ the random counting measure on $[0,\infty) \times \mathbb{R}^{I+1}$ associated with η's jumps. The corresponding compensated random measure is denoted $\tilde{N}(dt, dz) = N(dt, dz) -$

$\nu(dz)dt$ where ν is referred to as the Lévy measure on $\mathbb{R}^{I+1}$ associated with η's jumps, see, e.g., [2] and [30] for more details about these objects. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is the (right-continuous) filtration generated by η.

The following regularity assumption on the Lévy measure ν will be made throughout the paper:

Assumption I.2. *In addition to the usual properties*

$$\nu(\{0\}) = 0, \quad \int_{\mathbb{R}^{I+1}} (||z||^2 \wedge 1)\nu(dz) < \infty, \tag{1.4}$$

the Lévy measure ν satisfies the following three conditions:

$$\int_{||z||<1} |z^{(0)}|\nu(dz) < \infty, \tag{1.5}$$

$$\int_{||z||\geq 1} e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}}\nu(dz) < \infty \textit{ for all } u^{(0)}, u^{(i)} \in \mathbb{R} \textit{ and } i = 1, ..., I, \tag{1.6}$$

$$\nu(z^{(0)} > 0) > 0 \textit{ and } \nu(z^{(0)} < 0) > 0. \tag{1.7}$$

Assumption I.2 requires a few remarks: [9] consider the case of correlated Brownian motions with drift which is why we focus exclusively on the pure jump case. The requirement that (1.6) holds for all $u^{(0)}$ and $u^{(i)}$ in $\mathbb{R}$ can be relaxed to a certain domain at the cost of more cumbersome notation (this can be seen from the proofs in Section 1.8). Condition (1.5) is not implied by (1.4) because it requires that ν can integrate $z^{(0)}$ instead of $(z^{(0)})^2$ on the unit ball and has a number of implications; e.g., (1.5) ensures that the process

$$J_t = \int_0^t \int_{||z||<1} z^{(0)}N(ds, dz), \quad t \geq 0, \tag{1.8}$$

is well-defined and is of finite variation.[8] We note that J can still have infinite activity on finite intervals. The last condition (1.7) can also be relaxed to only requiring that a certain explicit function is onto (see function φ_i of Lemma I.7 in Section 1.7).

[8] The process J will be related to the stock's dividend process D below.

1.3.2 Exogenously specified model input

We consider a pure exchange economy in the sense that the money market account, stock price, income, and dividend processes are all quoted in terms of the model's single consumption good. The i'th investor's income rate process is modeled by

$$dY_{it} = \mu_i dt + \sigma_i \int_{\mathbb{R}^{I+1}} z^{(i)} \tilde{N}(dt, dz), \quad Y_{i0} \in \mathbb{R}, \tag{1.9}$$

where $\mu_i \in \mathbb{R}$ and $\sigma_i > 0$ are constants for $i = 1, ..., I$. The single stock's dividend rate process is modeled by

$$dD_t = \mu_D dt + \sigma_D \int_{\mathbb{R}^{I+1}} z^{(0)} \tilde{N}(dt, dz), \quad D_0 \in \mathbb{R}, \tag{1.10}$$

where $\mu_D \in \mathbb{R}$ and $\sigma_D > 0$ are constants. Because (1.8) is of finite variation, we see that D is of finite variation too (Y_i defined above by (1.9) might not be).

We note that the processes (1.9) and (1.10) are not independent; indeed, the quadratic cross characteristics between D and Y_i are given by[9]

$$d\langle Y_i, D\rangle_t = \sigma_i \sigma_D \int_{\mathbb{R}^{I+1}} z^{(0)} z^{(i)} \nu(dz) dt, \qquad t \geq 0, \quad i = 1, ..., I.$$

We can use (1.4) and (1.6) in Assumption I.2 together with Cauchy-Schwartz's inequality to see that $\langle Y_i, D\rangle_t$ is finitely valued for all $t \geq 0$ and $i = 1, ..., I$.

1.3.3 Endogenously determined price dynamics

We will restrict the financial market to only consist of two traded securities (one financial asset and one real asset). The financial asset is taken to be the zero net supply money market account. Its price process will be shown to have the following equilibrium dynamics

$$dS_t^{(0)} = S_t^{(0)} r dt, \quad S_0^{(0)} = 1, \quad t \geq 0, \tag{1.11}$$

[9]The brackets $\langle \cdot, \cdot \rangle$ are also called the conditional quadratic cross variation; see, e.g., Section III.5 in [27].

where $r > 0$ is a constant. Because the interest rate r is constant, the money market account is equivalent to zero-coupon bonds of all maturities.

The real asset is a stock paying out the dividends at rate D (see (1.10)). This security is in unit net supply and we will show that its equilibrium price dynamics are given by

$$dS_t + D_t dt = \Big(rS_t + \mu\Big)dt + \frac{\sigma_D}{r}\int_{\mathbb{R}^{I+1}} z^{(0)}\tilde{N}(dt, dz), \quad t \geq 0, \tag{1.12}$$

where the excess rate of return $\mu \in \mathbb{R}$ is a constant.

As discussed in the Introduction we are interested in how model incompleteness impacts the interest rate r and the stock's Sharpe ratio. The stock's (instantaneous) Sharpe ratio λ is defined as the constant

$$\lambda = \frac{\mu}{\frac{\sigma_D}{r}\sqrt{\int_{\mathbb{R}^{I+1}}(z^{(0)})^2\nu(dz)}}. \tag{1.13}$$

The Sharpe ratio (1.13) measures the stock's return (cleaned for interest and dividend components) relative to the standard deviation of its noise term. Sharpe ratios have been widely studied and used in the literature and we refer to [11] for an application of the Sharpe ratio (1.13) in a continuous-time jump diffusion setting.

We conclude this section with a lemma which we need in the proof section. Because our model is necessarily incomplete there are infinitely many martingale densities (state-price densities).[10] For our purpose the following particularly simple martingale density suffices. We define $Z = (Z_t)_{t \geq 0}$ as the solution to the linear equation

$$dZ_t = Z_{t-}\int_{\mathbb{R}^{I+1}} \psi(z)\tilde{N}(dt, dz), \quad t \geq 0, \quad Z_0 = 1. \tag{1.14}$$

[10]The technical difficulties related to the existence of equivalent martingale measures on infinite time-horizons already arise in Black-Scholes' model; see, e.g., the textbook discussion in Section 6.N in [12]. The same technical issues are also present in our jump setting which is why we prefer to use martingale densities.

We refer to Section II.8 in [27] for the unique explicit solution of (1.14). In (1.14) the deterministic and time independent integrand ψ is defined by

$$\psi(z) = e^{f\left(-\frac{\mu r}{\sigma_D}+\int_{\mathbb{R}^{I+1}} \tilde{z}^{(0)}\nu(d\tilde{z})\right)z^{(0)}} - 1, \quad z \in \mathbb{R}^{I+1}, \tag{1.15}$$

where f is the inverse of the mapping

$$\mathbb{R} \ni u^{(0)} \to \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)}z^{(0)}} \nu(dz). \tag{1.16}$$

Lemma I.3. *Under Assumption I.2 there exists a unique solution $Z = (Z_t)_{t\geq 0}$ of (1.14). This solution is a positive martingale. Furthermore, let $(S^{(0)}, S)$ be given by (1.11) and (1.12). Then the process $S_tZ_t/S_t^{(0)} + \int_0^t D_uZ_u/S_u^{(0)}du$, $t \geq 0$, is a sigma-martingale.*

1.4 Individual investors' optimization problems

Throughout this section the price dynamics (1.11) and (1.12) are taken as input. Given these price dynamics, the i'th investor is assumed to maximize exponential utility of running consumption over the infinite time-horizon:

$$\sup_{(\theta_i,c_i)\in\mathcal{A}_i} \mathbb{E}\left[\int_0^\infty e^{-\delta_i t}U_i(c_{it} + Y_{it})dt\right]. \tag{1.17}$$

Here the exponential utility function U_i is defined by (1.2) and $\mathcal{A}_i$ is the i'th investor's admissible set (see Definition I.4 below). Because $S_0^{(0)} = 1$, the investor's initial wealth is given by $X_{i0} = \theta_{i0-}^{(0)} + \theta_{i0-}S_0$ where investor i'ths initial endowments are $\theta_{i0-}^{(0)}$ units of the money market account and θ_{i0-} units of the stock. In (1.17) the process c_i denotes the consumption rate in excess of the income rate Y_i, i.e., investor i'ths cumulative consumption at time $t \in [0,\infty)$ is given by $\int_0^t (c_{iu} + Y_{iu})du$. We refer to [31] and [32] for more information about the optimal investment problem (1.17) for exponential investors receiving partly unspanned income over the infinite time-horizon in various Brownian settings.

We next specify the admissible set of controls $\mathcal{A}_i$. The i'th investor chooses predictable processes $\theta_i = (\theta_{it})_{t\geq 0}$ and $c_i = (c_{it})_{t\geq 0}$ to generate the self-financing gain dynamics

(1.18)
$$dX_{it} = \Big(rX_{it} - c_{it} + \theta_{it}\mu\Big)dt + \theta_{it}\frac{\sigma_D}{r}\int_{\mathbb{R}^{I+1}} z^{(0)}\tilde{N}(dt,dz), \quad t\geq 0, \quad X_{i0}\in\mathbb{R},$$

provided that the various integrals exist. As usual, the investor's investment $\theta_i^{(0)}$ in the money market account is implicitly specified by (1.18); see, e.g., Section 6.L in [12]. To state the additional properties (θ_i, c_i) will be required to satisfy in order to be deemed admissible, we first note that Lemma I.7 in Section 1.7 ensures that the function

$$\mathbb{R}\ni u^{(0)} \to \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}}\nu(dz), \quad u^{(i)}\in\mathbb{R}, \tag{1.19}$$

has a well-defined continuous inverse $f^i_{u^{(i)}}(\cdot)$ with domain $\mathbb{R}$. We can then define the constants $\theta_i^*\in\mathbb{R}$ by (here we use (1.5))

$$\theta_i^* = -\frac{\tau_i}{\sigma_D} f^i_{-\frac{1}{\tau_i}\sigma_i}\left(-\frac{\mu r}{\sigma_D} + \int_{\mathbb{R}^{I+1}} z^{(0)}\nu(dz)\right). \tag{1.20}$$

This allows us to define the function

$$V_i(x,y) = -\frac{1}{r}e^{-\frac{r}{\tau_i}x-\frac{1}{\tau_i}y-\frac{1}{r}g_i}, \quad x,y\in\mathbb{R}, \tag{1.21}$$

where we have defined the constants g_i by

$$\begin{aligned} g_i &= \delta_i - r + \frac{1}{\tau_i}\theta_i^* r\mu + \frac{1}{\tau_i}\mu_i \\ &\quad - \int_{\mathbb{R}^{I+1}}\Big(e^{-\frac{1}{\tau_i}\theta_i^*\sigma_D z^{(0)} - \frac{1}{\tau_i}\sigma_i z^{(i)}} - 1 + \frac{1}{\tau_i}\theta_i^*\sigma_D z^{(0)} + \frac{1}{\tau_i}\sigma_i z^{(i)}\Big)\nu(dz). \end{aligned} \tag{1.22}$$

As usual, infinite time-horizon problems require a transversality condition to hold (see, e.g., Section 9.D in the textbook [12]). In our case, this condition takes the form

$$\lim_{t\to\infty} e^{-\delta_i t}\mathbb{E}[V_i(X_{it}, Y_{it})] = 0. \tag{1.23}$$

We can then state the precise constraints on the investor's choice of controls in (1.17). The last requirement placed on (θ_i, c_i) in the next definition is purely technical and is needed in the verification part of the proof of Theorem I.5 given in Section 1.8.[11]

Definition I.4. A pair of predictable processes (θ_i, c_i) is deemed admissible if (i) the wealth dynamics (1.18) are well-defined, (ii) the expectation in (1.17) is finite, (iii) the transversality condition (1.23) holds, and (iv) the stochastic integral

$$\int_0^t \int_{\mathbb{R}^{I+1}} \Big(V_i(X_{is-} + \theta_{is}\frac{\sigma_D}{r}z^{(0)}, Y_{is-} + \sigma_i z^{(i)}) - V_i(X_{is-}, Y_{is-})\Big)\tilde{N}(ds, dz), \quad t \geq 0, \tag{1.24}$$

is well-defined and is a martingale. When (i)-(iv) hold we write $(\theta_i, c_i) \in \mathcal{A}_i$.

$\diamondsuit$

In terms of these objects, the following result provides the explicit solution to (1.17); see Section 1.8 for the proof.

Theorem I.5. *Let Assumption I.2 hold and let the price dynamics* (1.11) *and* (1.12) *with $r > 0$ be given. Then the processes $(\theta_i^*, c_i^*) \in \mathcal{A}_i$ attain the supremum in* (1.17) *where θ_i^* is defined by* (1.20) *and*

$$c_{it}^* = rX_{it}^* + \frac{\tau_i g_i}{r}, \quad t \geq 0. \tag{1.25}$$

In (1.25) *the process X_i^* denotes the gain process* (1.18) *produced by (θ_i^*, c_i^*). Furthermore, the resulting optimal expected utility is given by $V_i(X_{i0}, Y_{i0})$ where V_i is defined by* (1.21).

[11] For continuous-time optimal control problems, martingale conditions are always needed to verify optimality (see, e.g., Section V.15 in the textbook [29]).

From this theorem we note that the marginal utilities

$$e^{-\delta_i t}U_i'(c_{it}^* + Y_{it}), \quad i = 1, ..., I,$$

are not in general proportional across investors. This is due to the fact that the market $(S^{(0)}, S)$ is insufficient for the investors to share all risks imbedded in their various income streams $(Y_i)_{i=1}^I$. In other words, because there are non-traded income claims, the investors cannot share all income risks efficiently and this feature renders the model incomplete.

1.5 Radner equilibrium

This section contains our main result which provides the Radner equilibrium in closed-form. Our main existence result is Theorem I.6 below (the proof is in Section 1.8) which is stated in terms of the constants (r, μ) as well as the martingale $Z = (Z_t)_{t\geq 0}$ from Lemma I.3. To define the constants, we start by defining the Sharpe ratio $\lambda \in \mathbb{R}$ (constant) through the requirement

$$\sigma_D + \sum_{i=1}^I \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i}\left(-\lambda\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2\nu(dz)} + \int_{\mathbb{R}^{I+1}} z^{(0)}\nu(dz)\right) = 0, \tag{1.26}$$

where the inverse functions $(f^i)_{i=1}^I$ are defined by (1.19). Lemma I.7 in Section 1.7 ensures that (1.26) uniquely determines λ. Then we can define the constants

$$\theta_i^* = -\frac{\tau_i}{\sigma_D} f^i_{-\frac{1}{\tau_i}\sigma_i}\left(-\lambda\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2\nu(dz)} + \int_{\mathbb{R}^{I+1}} z^{(0)}\nu(dz)\right), \quad i = 1, ..., I. \tag{1.27}$$

In turn, this allows us to define the constant

$$\begin{aligned} r = \frac{1}{\tau_\Sigma}\Big\{&\mu_D + \sum_{i=1}^I \tau_i\delta_i + \sum_{i=1}^I \mu_i \\ &- \int_{\mathbb{R}^{I+1}} \Big(\sum_{i=1}^I \tau_i e^{-\frac{1}{\tau_i}\theta_i^*\sigma_D z^{(0)} - \frac{1}{\tau_i}\sigma_i z^{(i)}} - \tau_\Sigma + \sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\Big)\nu(dz)\Big\}, \end{aligned} \tag{1.28}$$

where $\tau_\Sigma = \sum_{i=1}^I \tau_i$. We can then finally define the constant

$$\mu = \frac{\lambda\sigma_D}{r}\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)}. \tag{1.29}$$

The next theorem is our main existence result.

Theorem I.6. *Let Assumption I.2 hold, let* $\sum_{i=1}^I \theta_{i0-}^{(0)} = 0$, $\sum_{i=1}^I \theta_{i0-} = 1$, *define* (r,μ) *by* (1.28)-(1.29), *and assume that* $r > 0$. *Then* $(S^{(0)}, S)$ *with* $S^{(0)}$ *defined by* (1.11) *and* S *defined by*

$$S_t = \frac{1}{Z_t}\int_t^\infty e^{-r(s-t)}\mathbb{E}[Z_s D_s | \mathcal{F}_t] ds, \quad t \geq 0, \tag{1.30}$$

where the martingale density Z *is defined by* (1.14), *constitute a Radner equilibrium in the sense of Definition I.1.*

We would like to mention that we can replace the exogenous process parameters $\mu_D, \sigma_D, \mu_i, \sigma_i$, $i = 1, ..., I$, by deterministic functions of time and still perform the equilibrium analysis provided that we replace the constant coefficients in the dynamics (1.11) and (1.12) with time dependent functions.

1.6 Application

In this section we will compare the incomplete equilibrium of Theorem I.6 with the corresponding complete equilibrium based on the representative agent. In the second part of this section we specify the Lévy measure ν to be the compound Poisson process with Gaussian jumps from Section 1.2. We illustrate numerically the impacts on the resulting equilibrium parameters due to model incompleteness.

1.6.1 Representative agent's equilibrium

It is well-known that when all investors have exponential utilities, then so does the sup-convolution describing the representative agent's preferences with risk tolerance coefficient $\tau_\Sigma = \sum_{i=1}^I \tau_i$ and time preference parameter $\delta_\Sigma = \frac{1}{\tau_\Sigma}\sum_{i=1}^I \tau_i\delta_i$; see, e.g., Section 5.26 in [17]. We therefore define the representative agent's utility function by

$$U_{\text{rep}}(c) = -e^{-c/\tau_\Sigma}, \quad c \in \mathbb{R}.$$

The consumption-based capital asset pricing model developed in [5] (and extended in [15] to certain incomplete models) is based on constructing price processes by applying the first-order condition for optimality in the representative agent's problem through the proportionality requirement

$$e^{-\delta_\Sigma t}U'_{\text{rep}}\Big(D_t + \sum_{i=1}^I Y_{it}\Big) \propto e^{-r^{\text{rep}}t}Z_t^{\text{rep}}, \quad t \geq 0. \tag{1.31}$$

Here r^{rep} is the interest rate and Z^{rep} is the model's (unique) martingale density. This model (i.e., r^{rep} and Z^{rep}) will serve as the basis for our comparison. Itô's lemma produces the following dynamics of the left-hand-side of (1.31)

$$\begin{aligned}
&\frac{de^{-\delta_\Sigma t}U'_{\text{rep}}(D_t + \sum_{i=1}^I Y_{it})}{e^{-\delta_\Sigma t}U'_{\text{rep}}(D_t + \sum_{i=1}^I Y_{it})}\\
&= -\delta_\Sigma dt + \int_{\mathbb{R}^{I+1}}\Big(e^{-\frac{1}{\tau_\Sigma}\left(\sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\right)} - 1\Big)\tilde{N}(dt,dz) - \frac{1}{\tau_\Sigma}\Big(\mu_D + \sum_{i=1}^I \mu_i\Big)dt\\
&+ \int_{\mathbb{R}^{I+1}}\Big(e^{-\frac{1}{\tau_\Sigma}\left(\sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\right)} - 1 + \frac{1}{\tau_\Sigma}\Big(\sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\Big)\Big)\nu(dz)dt.
\end{aligned}$$

By matching coefficients with the right-hand-side of (1.31) we find

$$dZ_t^{\text{rep}} = Z_{t-}^{\text{rep}}\int_{\mathbb{R}^{I+1}}\psi^{\text{rep}}(z)\tilde{N}(dt,dz), \quad \psi^{\text{rep}}(z) = e^{-\frac{1}{\tau_\Sigma}\left(\sum_{i=1}^I \sigma_i z^{(i)} + \sigma_D z^{(0)}\right)} - 1, \tag{1.32}$$

$$\begin{aligned}
r^{\text{rep}} &= \delta_\Sigma + \frac{1}{\tau_\Sigma}\Big(\mu_D + \sum_{i=1}^I \mu_i\Big)\\
&\quad - \int_{\mathbb{R}^{I+1}}\Big(e^{-\frac{1}{\tau_\Sigma}\left(\sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\right)} - 1 + \frac{1}{\tau_\Sigma}\Big(\sigma_D z^{(0)} + \sum_{i=1}^I \sigma_i z^{(i)}\Big)\Big)\nu(dz).
\end{aligned} \tag{1.33}$$

We find the parameters describing the stock dynamics by computing the dynamics of (similar to (1.30))

$$S_t^{\mathrm{rep}} = \frac{1}{Z_t^{\mathrm{rep}}} \int_t^\infty e^{-r^{\mathrm{rep}}(s-t)} \mathbb{E}\big[Z_s^{\mathrm{rep}} D_s \big| \mathcal{F}_t\big] ds, \quad t \geq 0. \tag{1.34}$$

As in the proof of Theorem I.6 in Section 1.8, we find that the dynamics of (1.34) are of the form (1.12) but with $r = r^{\mathrm{rep}}$ and $\mu = \mu^{\mathrm{rep}}$, where

$$\mu^{\mathrm{rep}} = -\frac{\sigma_D}{r^{\mathrm{rep}}} \int_{\mathbb{R}^{I+1}} \psi^{\mathrm{rep}}(z) z^{(0)} \nu(dz),$$

with ψ^{rep} defined by (1.32). Finally, the Sharpe ratio based on the representative agent is defined as:

$$\lambda^{\mathrm{rep}} = \frac{\mu^{\mathrm{rep}}}{\frac{\sigma_D}{r^{\mathrm{rep}}} \sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)}}, \tag{1.35}$$

which is the analogue of (1.13).

1.6.2 Incompleteness impacts in a numerical example

In this section we consider the Lévy measure corresponding to a compound Poisson process with Gaussian jumps (i.i.d. zero-mean normals with covariance matrix Σ) and a unit constant Poisson intensity. In other words, for a symmetric positive definite matrix Σ with unit diagonal elements, we consider the Lévy measure

$$\nu(dz) = \frac{1}{\sqrt{(2\pi)^{I+1} \det(\Sigma)}} e^{-\frac{1}{2} z' \Sigma^{-1} z} dz, \quad z \in \mathbb{R}^{I+1}.$$

This measure satisfies Assumption I.2. Furthermore, the functions f^i and f (the inverse functions of 1.19 and 1.16) can be expressed via the Gaussian moment generating function $e^{\frac{1}{2} h' \Sigma h}$, $h \in \mathbb{R}^{I+1}$, and its derivatives.

Based on (1.28) and (1.33), we see that the incompleteness impact on the equilibrium interest rate is given by

$$r^{\mathrm{rep}} - r = \int_{\mathbb{R}^{I+1}} \Big(\sum_{i=1}^{I} \frac{\tau_i}{\tau_\Sigma} e^{-\frac{1}{\tau_i} \theta_i^* \sigma_D z^{(0)} - \frac{1}{\tau_i} \sigma_i z^{(i)}} - e^{-\frac{1}{\tau_\Sigma} \left(\sigma_D z^{(0)} + \sum_{i=1}^{I} \sigma_i z^{(i)} \right)} \Big) \nu(dz).$$

Jensen's inequality and the clearing property $\sum_{i=1}^{I}\theta_i^* = 1$ (coming from (1.26) and (1.27)) can be used to see that this difference is always non-negative (a similar observation is made in [9] and [10]). On the other hand, the impact on the instantaneous Sharpe ratio due to model incompleteness, i.e., $\lambda - \lambda^{\text{rep}}$, can be either positive or negative. Here λ^{rep} is defined by (1.35) and the (instantaneous) Sharpe ratio λ in the incomplete equilibrium is defined by (1.13). The constant $\lambda \in \mathbb{R}$ is found implicitly by solving

$$\sigma_D + \sum_{i=1}^{I} \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i}(-\lambda) = 0.$$

This follows from (1.26) and the zero-mean and unit variance properties of the Lévy measure ν.

To proceed with the numerics, we will use a flat correlation matrix in the sense that $\Sigma_{ij} = \rho$ for $i \neq j$ and $\Sigma_{ii} = 1$ for $i, j = 0, 1, ..., I$ where $\rho \in [0, 1)$. The remaining parameters used to generate Figure 1.1 below are[12]

$$\sigma_D = .2I, \quad \sigma_i = .1, \quad \tau_i = \tau, \quad i = 1, ..., I. \tag{1.36}$$

From Figure 1.1 we see that our model can simultaneously produce a positive impact on the equilibrium (instantaneous) Sharpe ratio and a negative impact on the equilibrium interest rate. As discussed in the Introduction, these effects are empirically desirable because they are linked to the asset pricing puzzles in [33] and [24]. We note that as $\rho \uparrow 1$ the resulting model approaches the representative agent's complete model and both incompleteness impacts vanish.

Because all the investors in the model behind Figure 1.1 are homogeneous they hold the same number of stocks and there are no differences across investors. We would like to see how investor heterogeneity affects the equilibrium. The next table

[12]The dividend's parameters are scaled up to the size of the economy, while the investors' income parameters are independent to the size as they represent individual characteristic of investors.

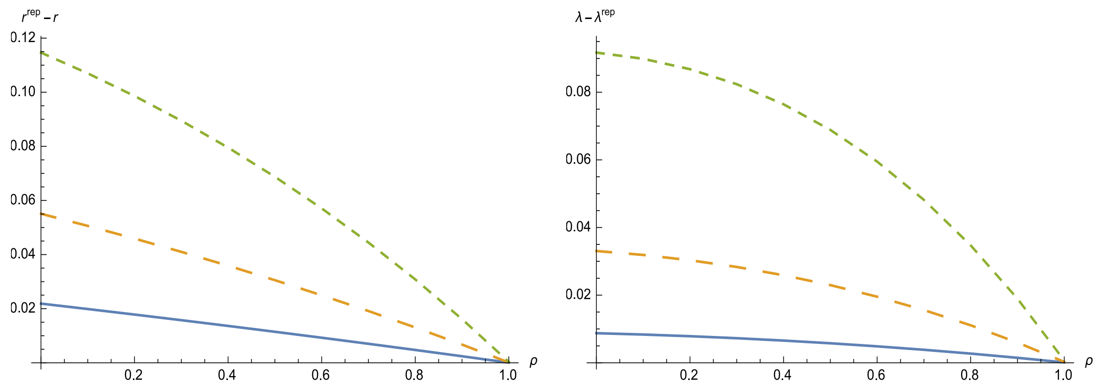

Figure 1.1: Plot of impacts due to model incompleteness on $r^{\text{rep}} - r$ (left) and $\lambda - \lambda^{\text{rep}}$ (right) seen as a function of the correlation coefficient ρ. We consider the limiting economy ($I \to \infty$) whereas the remaining parameters are given by (1.36) for the various risk tolerance coefficients τ: $\tau = \frac{1}{2}$ (———), $\tau = \frac{1}{3}$ (– – –), and $\tau = \frac{1}{4}$ (- - - -).

illustrates investor heterogeneity effects on the equilibrium for the independent case where $\Sigma_{ij} = 0$ for $i \neq j$ and $\Sigma_{ii} = 1$ for $i, j = 0, 1, ..., I$. We split the population into two homogeneous groups (A, B) and we let $w \in [0, 1]$ denote the relative weight of group A. The investors in group A all have the same high income volatility coefficient $\sigma_A = .2$ whereas the investors in group in B all have the coefficient $\sigma_B = .1$. In Table 1.1 below we also report how the investors' strategies vary between the two groups (A, B) which is done as follows: For $I \in \mathbb{N}$ we have (see 1.3)

$$1 = \sum_{i=1}^{I} \theta_i^* = Iw\theta_A^* + I(1-w)\theta_B^*.$$

Table 1.1 reports $\lim_{I\to\infty} Iw\theta_A^*$ which measures how big a fraction of the equity group A collectively holds in equilibrium (in the limiting model $I \to \infty$).

w	(τ_A, τ_B)			
	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{3})$
1.00	1 [.090], (.036)	1, [.090], (.036)	1, [.236], (.142)	1, [.236], (.142)
0.75	.740, [.073], (.029)	.814, [.084], (.037)	.639, [.159], (.082)	.731, [.190], (.113)
0.50	.487, [.056], (.022)	.593, [.077], (.037)	.369, [.101], (.046)	.475, [.145], (.085)
0.25	.240, [.039], (.015)	.327, [.067], (.036)	.162, [.057], (.024)	.232, [.100], (.058)
0.00	0, [.022], (.009)	0, [.055], (.033)	0, [.022], (.009)	0, [.055], (.033)

Table 1.1: Values of $\lim_{I\to\infty} wI\theta_A^*, [r^{\text{rep}} - r], (\lambda - \lambda^{\text{rep}})$ in the limiting economy ($I \to \infty$) for various weights w and various risk tolerance parameters (τ_A, τ_B). The values are based on the parameters $\sigma_A = .2$, $\sigma_B = .1$, $\sigma_D = .2I$, and $\Sigma_{ij} = 0$ for $i \neq j$ and $\Sigma_{ii} = 1$ for $i, j = 0, 1, ..., I$.

By comparing Columns 2 and 3 in Table 1.1 we see that the incompleteness effects on the interest rate and the Sharpe ratio are largest when the most risk averse investors have the highest income uncertainty. A similar observation regarding the interest rate is made in Table 1 in [9]. Compared to all Brownian models (see Theorem 4.1 in [10]), our framework's main new feature is that discontinuous Lévy models can produce a significant impact on the Sharpe ratio while still being as tractable as the arithmetic Brownian model developed in [9].

Finally, we note from Table 1.1 how the equilibrium equity distribution between the two groups depends on the groups' risk aversion coefficients. When the two groups are equally sized ($w = 0.5$) and have the same risk aversion coefficients (Columns 1 and 4 in Table 1.1), group B, who has the lowest income volatility, holds slightly more equity than group A. However, by changing the risk aversion coefficients we dramatically change the equilibrium equity distribution. Indeed, if group B's risk aversion is increased from 2 to 3, group A will hold close to 60% of the economy's equity even though group A has the highest income volatility (Column 2 in Table 1.1).

1.7 An auxiliary lemma

Lemma I.7. *Suppose Assumptions I.2 holds. Then the partial derivative*

$$\varphi_i(u^{(0)}, u^{(i)}) = \frac{\partial}{\partial u^{(0)}} \int_{\mathbb{R}^{I+1}} \Big(e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}} - 1\Big)\nu(dz), \quad u^{(0)}, u^{(i)} \in \mathbb{R},$$

is a well-defined function and satisfies the following properties:

1. *The function φ_i has the representation*

$$\varphi_i(u^{(0)}, u^{(i)}) = \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}} \nu(dz), \quad u^{(0)}, u^{(i)} \in \mathbb{R}. \tag{1.37}$$

2. *The function φ_i is jointly continuous.*

3. *For fixed $u^{(i)} \in \mathbb{R}$, the function $u^{(0)} \to \varphi_i(u^{(0)}, u^{(i)})$ is strictly increasing and onto $\mathbb{R}$. Consequently, the inverse function $f^i_{u^{(i)}}(\cdot)$ exists and is continuous on $\mathbb{R}$.*

Proof. For the first claim, we can use the bound

$$\frac{|e^{hz} - 1|}{|h|} \leq |z|e^{|z|}, \quad z^{(0)} \in \mathbb{R}, \quad |h| \leq 1. \tag{1.38}$$

This bound is integrable by (1.5) and (1.6) of Assumption I.2. Therefore, the dominated convergence theorem can be used to produce the representation (1.37). The second claim follows similarly. The strict monotonicity property in the last claim follows directly from (1.37). Finally, (1.7) ensures that the map $\varphi_i(\cdot, u^{(i)})$ is onto $\mathbb{R}$. ♢

1.8 Proofs

Proof of Lemma I.3. The integrability property

$$\int_{\mathbb{R}^{I+1}} \big(\psi(z)\big)^2 \nu(dz) < \infty,$$

follows from the definition of ψ (see (1.15)), the bound (1.38), and the integrability requirements in Assumption I.2. Furthermore, because ψ defined by (1.15) satisfies $\psi+1>0$ we see from Theorem 37 in Section II.8 in [27] that there is a unique strictly positive solution Z of (1.14). The martingale property of Z follows from Novikov's condition for Lévy processes (see Theorem 9 in [28]). The claimed sigma-martingale property follows from Itô's product rule applied to $Z\frac{S}{S^{(0)}}$ combined with the no drift property

$$\frac{\mu r}{\sigma_D}+\int_{\mathbb{R}^{I+1}}\psi(z)z^{(0)}\nu(dz)=0. \tag{1.39}$$

The latter condition (1.39) follows from the definition of ψ.

$\diamondsuit$

Proof of Theorem I.5. Throughout this proof we let V_i be defined by (1.21) and we let (θ_i^*, c_i^*) be defined by (1.20) and (1.25). We split the proof into two steps:

Step 1: (Admissibility of θ_i^*, c_i^*). By inserting (θ_i^*, c_i^*) into (1.18) we produce the gain dynamics

$$dX_{it}^*=\Big(\theta_i^*\mu-\frac{\tau_i g_i}{r}\Big)dt+\theta_i^*\frac{\sigma_D}{r}\int_{\mathbb{R}^{I+1}}z^{(0)}\tilde{N}(dt,dz),\quad t\geq 0.$$

To see that X_{it}^* has all exponential moments we let $a\in\mathbb{R}$ be arbitrary. The integrability conditions (1.4) and (1.6) ensure that

$$\int_{\mathbb{R}^{I+1}}\Big(e^{az^{(0)}}-1-az^{(0)}\Big)\nu(dz)<\infty. \tag{1.40}$$

Consequently, Itô's lemma ensures that

$$e^{a\int_0^t\int_{\mathbb{R}^{I+1}}z^{(0)}\tilde{N}(du,dz)-t\int_{\mathbb{R}^{I+1}}\left(e^{az^{(0)}}-1-az^{(0)}\right)\nu(dz)}$$

is a nonnegative sigma-martingale. Furthermore, Ansel-Stricker's Theorem ensures that it is a supermartingale which combined with the deterministic property of (1.40) produces

$$\mathbb{E}\left[e^{a\int_0^t\int_{\mathbb{R}^{I+1}} z^{(0)}\tilde{N}(du,dz)}\right] \leq e^{t\int_{\mathbb{R}^{I+1}}\left(e^{az^{(0)}}-1-az^{(0)}\right)\nu(dz)} < \infty. \tag{1.41}$$

In a similar fashion we can see that Y_{it} has all exponential moments. This shows that the expected utility of (θ_i^*, c_i^*) is finite.

To see that the stochastic integral (1.24) is a martingale, it suffices to show the square integrability property

$$\mathbb{E}\left[\int_0^t V_i(X^*_{iu-}, Y_{iu-})^2 \int_{\mathbb{R}^{I+1}} \left(e^{-\frac{\theta_i^*\sigma_D}{\tau_i}z^{(0)} - \frac{\sigma_i}{\tau_i}z^{(i)}} - 1\right)^2 \nu(dz)du\right] < \infty,$$

for all $t \geq 0$. The integrand in the ν-integral does not depend on $\omega \in \Omega$. The bound (1.38) combined with (1.4) and (1.6) ensures that this ν-integral is finite. Furthermore, the first inequality in (1.41) shows that the functions

$$[0,t] \ni u \to \mathbb{E}\left[e^{bX^*_{iu}}\right] \quad \text{and} \quad [0,t] \ni u \to \mathbb{E}\left[e^{b'Y_{iu}}\right],$$

are bounded by exponential functions for all $b, b' \in \mathbb{R}$. Because such functions on finite intervals are bounded, the square integrability property follows from Cauchy-Schwartz's inequality.

To verify the transversality condition (1.23) we use Itô's lemma to compute the dynamics

$$\begin{aligned}
dV_i(X^*_{it},Y_{it}) &= \int_{\mathbb{R}^{I+1}} \Big(V_i(X^*_{it-}+\theta^*_i\frac{\sigma_D}{r}z^{(0)},Y_{it-}+\sigma_i z^{(i)}) - V_i(X^*_{it-},Y_{it-})\Big)\tilde{N}(dt,dz)\\
&+ V_i(X^*_{it-},Y_{it-})\int_{\mathbb{R}^{I+1}}\Big(\frac{1}{\tau_i}\big(\theta^*_i\sigma_D z^{(0)}+\sigma_i z^{(i)}\big)+e^{-\frac{\theta^*_i\sigma_D}{\tau_i}z^{(0)}-\frac{1}{\tau_i}\sigma_i z^{(i)}}-1\Big)\nu(dz)dt\\
&- V_i(X^*_{it-},Y_{it-})\Big(\frac{r}{\tau_i}\mu\theta^*_i - g_i+\frac{\mu_i}{\tau_i}\Big)dt\\
&= \int_{\mathbb{R}^{I+1}} \Big(V_i(X^*_{it-}+\theta^*_i\frac{\sigma_D}{r}z^{(0)},Y_{it-}+\sigma_i z^{(i)}) - V_i(X^*_{it-},Y_{it-})\Big)\tilde{N}(dt,dz)\\
&+ V_i(X^*_{it-},Y_{it-})\big(\delta_i - r\big)dt.
\end{aligned}$$

The just proven martingale property of the stochastic integral (1.24) produces

$$\mathbb{E}[V_i(X^*_{it},Y_{it})] = V_i(X_{i0},Y_{i0}) + \int_0^t \mathbb{E}[V_i(X^*_{iu},Y_{iu})](\delta_i - r)du.$$

Therefore, because $r>0$, we have

$$\lim_{t\to\infty} e^{-\delta_i t}\mathbb{E}[V_i(X^*_{it},Y_{it})] = \lim_{t\to\infty} e^{-\delta_i t}V_i(X_{i0},Y_{i0})e^{(\delta_i-r)t} = 0.$$

Step 2 (Verification): Itô's lemma produces the following dynamics for arbitrary controls $(\theta_i,c_i)\in\mathcal{A}_i$

$$\begin{aligned}
& d\Big(\int_0^t e^{-\delta_i u}U_i(c_{iu}+Y_{iu})du + e^{-\delta_i t}V_i(X_{it},Y_{it})\Big)\\
&= e^{-\delta_i t}\Big\{U_i(c_{it}+Y_{it})dt - \delta_i V_i(X_{it},Y_{it})dt\\
&+ \int_{\mathbb{R}^{I+1}}\big(V_i(X_{it-}+\theta_{it}\frac{\sigma_D}{r}z^{(0)},Y_{it-}+\sigma_i z^{(i)}) - V_i(X_{it-},Y_{it-})\big)\tilde{N}(dt,dz)\\
&+ V_i(X_{it-},Y_{it-})\int_{\mathbb{R}^{I+1}}\Big(\frac{1}{\tau_i}\big(\theta_{it}\sigma_D z^{(0)}+\sigma_i z^{(i)}\big)+e^{-\frac{\theta_{it}\sigma_D}{\tau_i}z^{(0)}-\frac{1}{\tau_i}\sigma_i z^{(i)}}-1\Big)\nu(dz)dt\\
&- V_i(X_{it-},Y_{it-})\frac{r}{\tau_i}(rX_{it-}-c_{it}+\mu\theta_{it})dt - V_i(X_{it-},Y_{it-})\frac{\mu_i}{\tau_i}dt\Big\}.
\end{aligned}$$

A direct calculation shows that the drift is maximized pointwise by the processes (θ^*_i,c^*_i) given by (1.20) and (1.25). Furthermore, the definition of g_i ensures that this maximal value is zero. Therefore, the martingale property of the integrals in (1.24) produces

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-\delta_i u} U_i(c_{iu}+Y_{iu})du + e^{-\delta_i t} V_i(X_{it}, Y_{it})\right] \\
&\leq V(X_{i0}, Y_{i0}) \\
&= \mathbb{E}\left[\int_0^t e^{-\delta_i u} U_i(c^*_{iu}+Y_{iu})du + e^{-\delta_i t} V_i(X^*_{it}, Y_{it})\right].
\end{aligned}
$$

We can then use the monotone convergence theorem to pass $t \to \infty$ as well as the transversality condition (1.23) to see

$$
\mathbb{E}\left[\int_0^\infty e^{-\delta_i u} U_i(c_{iu}+Y_{iu})du\right] \leq V(X_{i0}, Y_{i0}) = \mathbb{E}\left[\int_0^\infty e^{-\delta_i u} U_i(c^*_{iu}+Y_{iu})du\right].
$$

$\diamondsuit$

Proof of Theorem I.6. Fix the time points $0 \leq t \leq s < \infty$. Based on Lemma I.3 we can define the $\mathbb{P}$-equivalent measure $\mathbb{Q}_s$ on $\mathcal{F}_s$ by the Radon-Nikodym derivative $\frac{d\mathbb{Q}_s}{d\mathbb{P}} := Z_s$. Girsanov's theorem and the definition of ψ (see 1.15) produce the $\mathbb{Q}_s$-dynamics

$$
dD_u = \Big(\mu_D - \mu r\Big)du + \sigma_D \int_{\mathbb{R}^{I+1}} z^{(0)} \Big(N(du,dz) - \big(1+\psi(z)\big)\nu(dz)du\Big), \quad u \in [0,s].
$$

To ensure that the stochastic integral is a $\mathbb{Q}_s$-martingale it suffices to show

$$
\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \big(1+\psi(z)\big)\nu(dz) < \infty.
$$

This follows from ψ's definition (1.15) and the integrability requirements in Assumption I.2. Bayes' rule then produces

$$
\frac{\mathbb{E}[Z_s D_s | \mathcal{F}_t]}{Z_t} = \mathbb{E}^{\mathbb{Q}_s}[D_s|\mathcal{F}_t] = D_t + (\mu_D - \mu r)(s-t).
$$

This gives us the representation

$$
S_t = \int_t^\infty e^{-r(s-t)} \frac{\mathbb{E}[Z_s D_s|\mathcal{F}_t]}{Z_t} ds = \frac{D_t}{r} + \frac{\mu_D - \mu r}{r^2}. \tag{1.42}
$$

This representation and (1.10) produce the dynamics (1.12).

To see that the clearing conditions (1.3) hold, we first note that inserting μ defined by (1.29) into (1.27) gives us (1.20); hence, θ_i^* is optimal by Theorem I.5. Therefore, we have

$$\sum_{i=1}^{I} \theta_i^* = -\frac{1}{\sigma_D} \sum_{i=1}^{I} \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i} \left(-\frac{\mu r}{\sigma_D} + \int_{\mathbb{R}^{I+1}} z^{(0)} \nu(dz) \right) = 1,$$

where the last equality follows from inserting μ defined by (1.29) into (1.26). Clearing for the money market account market is equivalent to $S_t = \sum_{i=1}^{I} X_{it}^*$. For $t = 0$ this holds. By inserting the optimal consumption processes (1.25) into the gain dynamics (1.18) and using the already established property $\sum_{i=1}^{I} \theta_i^* = 1$ we find

$$\sum_{i=1}^{I} dX_{it}^* = \left(\mu - \sum_{i=1}^{I} \frac{\tau_i g_i}{r} \right) dt + \frac{\sigma_D}{r} \int_{\mathbb{R}^{I+1}} z^{(0)} \tilde{N}(dt, dz).$$

On the other hand, the representation (1.42) produces

$$dS_t = \frac{\mu_D}{r} dt + \frac{\sigma_D}{r} \int_{\mathbb{R}^{I+1}} z^{(0)} \tilde{N}(dt, dz).$$

The claim therefore follows as soon as we establish

$$\sum_{i=1}^{I} \tau_i g_i = \mu r - \mu_D. \tag{1.43}$$

To see that this relationship holds we can insert the definition of g_i (see (1.22)) and use the definition of r (see (1.28)). This argument also produces clearing in the good's market because

$$\sum_{i=1}^{I} c_{it}^* = \sum_{i=1}^{I} \left(r X_{it}^* + \frac{\tau_i g_i}{r} \right) = r S_t + \sum_{i=1}^{I} \frac{\tau_i g_i}{r} = D_t + \frac{\mu_D - \mu r}{r} + \sum_{i=1}^{I} \frac{\tau_i g_i}{r} = D_t.$$

Here the third equality follows from the representation (1.42) and the last equality comes from (1.43).

$\diamondsuit$

CHAPTER II

Model extensions

In this chapter, we consider the finite time horizon version of the Lévy model introduced in Chapter I. In these finite time horizon models, we relax the assumption on the Lévy measure in order to extend our equilibrium result to more general models, including the variance gamma model. The following sections are organized as follows: first we re-state the new finite time horizon problem including the new assumption, then we state and prove the new equilibrium results, and finally we consider a variance gamma model and its numerical results. In addition, the discussion about the lack of the minimal martingale measure is included at the end of this chapter.

2.1 Finite time-horizon setting

2.1.1 Underlying Lévy process

Recall that η is the underlying $\mathbb{R}^{I+1}$-dimensional pure jump Lévy process which generates the (right-continuous) filtration $(\mathcal{F}_t)_{t\geq 0}$. $N = N(dt, dz)$ is the random counting measure on $[0,\infty) \times \mathbb{R}^{I+1}$ and ν is the Lévy measure on $\mathbb{R}^{I+1}$ associated with η's jumps. We define the sets of exponentially integrable domain $\mathcal{D}_i$ by

$$\mathcal{D}_i = \Big\{(u^{(0)}, u^{(i)}) \in \mathbb{R}^2 : \int_{||z||\geq 1} e^{u^{(0)}z^{(0)} + u^{(i)}z^{(i)}} \nu(dz) < \infty\Big\}, \quad \forall i = 1, \dots, I.$$

Below we state our relaxed assumption on the Lévy measure ν.

Assumption II.1. *In addition to the usual properties* (1.4) *the Lévy measure* ν *satisfies the following conditions:*

$$\int_{||z||<1} |z^{(0)}|\nu(dz) < \infty, \tag{2.1}$$

$$\mathcal{D}_i \textit{ is open}, \quad \forall i = 1, \ldots, I, \tag{2.2}$$

$$\nu(z^{(0)} > 0) > 0 \textit{ and } \nu(z^{(0)} < 0) > 0. \tag{2.3}$$

This new assumption requires a few remarks and lemmas.

Remark II.2. While Conditions (1.5)-(2.1) and (1.7)-(2.3) are identical, Condition (1.6) implies (2.2) with $\mathcal{D}_i = \mathbb{R}^2, \quad \forall i = 1, \ldots, I$.

Lemma II.3. *Suppose Condition* (2.2) *in Assumption II.1 holds. Then for every* $(u^{(0)}, u^{(i)}) \in \mathcal{D}_i$, *there exist a* $\nu|_{||z||\geq 1}$*-integrable function* h *and an open neighborhood* U *of* $(u^{(0)}, u^{(i)})$ *such that*

$$e^{xz^{(0)}+yz^{(i)}} \leq h(z), \quad \forall z \in \{||z|| \geq 1\}, \forall (x, y) \in U. \tag{2.4}$$

Proof. Let $(u^{(0)}, u^{(i)}) \in \mathcal{D}_i$. Since $\mathcal{D}_i$ is open, we can find $\epsilon > 0$ such that

$$[(u^{(0)} - \epsilon, u^{(0)} + \epsilon] \times [u^{(i)} - \epsilon, u^{(i)} + \epsilon] \subset \mathcal{D}_i.$$

We define the dominating function h by

$$\begin{aligned} h(z) = \quad & e^{(u^{(0)}+\epsilon)z^{(0)}+(u^{(i)}+\epsilon)z^{(i)}} + e^{(u^{(0)}+\epsilon)z^{(0)}+(u^{(i)}-\epsilon)z^{(i)}} \\ & + e^{(u^{(0)}-\epsilon)z^{(0)}+(u^{(i)}+\epsilon)z^{(i)}} + e^{(u^{(0)}-\epsilon)z^{(0)}+(u^{(i)}-\epsilon)z^{(i)}}. \end{aligned}$$

Then h satisfies (2.4) with $U = (u^{(0)} - \epsilon, u^{(0)} + \epsilon) \times (u^{(i)} - \epsilon, u^{(i)} + \epsilon)$ and

$$\int_{||z||\geq 1} h(z)\nu(dz) < \infty.$$

$\diamondsuit$

Corollary II.4. *Suppose Condition* (2.2) *in Assumption II.1 holds, then for every compact subset K of $\mathcal{D}_i$, there exists a $\nu|_{||z||\geq 1}$-integrable function h such that*

$$e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}} \leq h(z), \quad \forall z \in \{||z|| \geq 1\}, \quad \forall (u^{(0)}, u^{(i)}) \in K.$$

The following lemma is the analogue of Lemma I.7 under the relaxed Assumption II.1.

Lemma II.5. *Suppose Assumption II.1 holds. Then the partial derivative*

$$\varphi_i(u^{(0)}, u^{(i)}) = \frac{\partial}{\partial x} \int_{\mathbb{R}^{I+1}} \left(e^{xz^{(0)}+u^{(i)}z^{(i)}} - 1\right)\nu(dz)\bigg|_{x=u^{(0)}}, \quad (u^{(0)}, u^{(i)}) \in \mathcal{D}_i,$$

is well-defined and satisfies the following properties:

1. *The function φ_i has the representation*

$$\varphi_i(u^{(0)}, u^{(i)}) = \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}} \nu(dz), \quad (u^{(0)}, u^{(i)}) \in \mathcal{D}_i.$$

2. *The function φ_i is jointly continuous on $\mathcal{D}_i$.*

3. *For fixed $u^{(i)}$, the function $u^{(0)} \to \varphi_i(u^{(0)}, u^{(i)})$ is strictly increasing and onto $\mathbb{R}$; hence, admits the continuous inverse function $f^i_{u^{(i)}}(\cdot)$ whose domain is the entire space $\mathbb{R}$.*

4. *The map $(t, u^{(i)}) \to f^i_{u^{(i)}}(t)$ is jointly continuous.*

Proof. The first property is a direct consequence of Lemma II.3. For the second property, we use the bound

$$\frac{|e^{hz} - 1|}{|h|} \leq |z|(e^{hz} + e^{-hz}), \quad z^{(0)} \in \mathbb{R}, \quad |h| \leq 1. \tag{2.5}$$

The joint continuity follows from the bound (2.5) and the dominated convergence theorem.

Next, we proceed to prove the third property that the map $u^{(0)} \to \varphi_i(u^{(0)}, u^{(i)})$ is onto $\mathbb{R}$. Since φ_i is continuous in $\mathcal{D}_i$ it suffices to show that inside $\mathcal{D}_i$ the function $\varphi_i(\cdot, u^{(i)})$ is unbounded from above and below. To prove this claim, we fix $(u^{(0)}, u^{(i)}) \in \mathcal{D}_i$ and let

$$\left\{x \in \mathbb{R} : \int_{||z|| \geq 1} e^{xz^{(0)} + u^{(i)} z^{(i)}} \nu(dz) < \infty\right\} = (\alpha, \beta),$$

for some $-\infty \leq \alpha < \beta \leq \infty$. Here we know the above set is an open interval because $\mathcal{D}_i$ is convex and open. We then prove the lack of an upper bound in two following separate cases.

Case 1: $\beta = \infty$. We know from the first property that the function φ_i is finite inside the set $\mathcal{D}_i$, i.e.,

$$\int_{\mathbb{R}^{I+1}} z^{(0)} e^{xz^{(0)} + u^{(i)} z^{(i)}} \nu(dz) < \infty, \quad \forall x \in (\alpha, \infty).$$

Since $u^{(0)} \to \varphi_i(u^{(0)}, u^{(i)})$ is strictly increasing, we can use condition (2.3) to conclude that

$$\int_{\mathbb{R}^{I+1}} z^{(0)} e^{xz^{(0)} + u^{(i)} z^{(i)}} \nu(dz) \to \infty, \qquad \text{as } x \uparrow \infty.$$

Case 2: $\beta < \infty$. We take an increasing sequence $(\beta_n)_{n \in \mathbb{N}}$ such that $\beta_n \uparrow \beta$ and $(\beta_n, u^{(i)}) \in \mathcal{D}_i$. We have the inequality

$$\int_{\{||z|| \geq 1\} \cap \{z^{(0)} \leq 1\}} e^{\beta z^{(0)} + u^{(i)} z^{(i)}} \nu(dz) \leq e^{\beta - \beta_n} \int_{\{||z|| \geq 1\} \cap \{z^{(0)} \leq 1\}} e^{\beta_n z^{(0)} + u^{(i)} z^{(i)}} \nu(dz) < \infty.$$

Because $\mathcal{D}_i$ is open and $(\beta, u^{(i)})$ is at the boundary of $\mathcal{D}_i$, we have $(\beta, u^{(i)}) \notin \mathcal{D}_i$. This, together with the above inequality, implies

$$\int_{\{||z|| \geq 1\} \cap \{z^{(0)} > 1\}} e^{\beta z^{(0)} + u^{(i)} z^{(i)}} \nu(dz) = \infty.$$

By the monotone convergence theorem, we deduce that

$$\begin{aligned}\lim_{n\uparrow\infty}\int_{\{||z||\geq 1\}\cap\{z^{(0)}>1\}} z^{(0)}e^{\beta_n z^{(0)}+u^{(i)}z^{(i)}}\nu(dz) &\geq \lim_{n\uparrow\infty}\int_{\{||z||\geq 1\}\cap\{z^{(0)}>1\}} e^{\beta_n z^{(0)}+u^{(i)}z^{(i)}}\nu(dz)\\ &= \int_{\{||z||\geq 1\}\cap\{z^{(0)}>1\}} e^{\beta z^{(0)}+u^{(i)}z^{(i)}}\nu(dz)\\ &= \infty.\end{aligned}$$

A similar argument shows that $\varphi_i(\cdot, u^{(i)})$ is unbounded from below in $\mathcal{D}_i$; therefore the claim is proved.

Finally, we want to show the joint continuity of the map $(t, u^{(i)}) \to f^i_{u^{(i)}}(t)$ in the last property. We let $(t_n, u^{(i)}_n) \to (t, u^{(i)})$ as $n \uparrow \infty$ and write $u^{(0)}_n = f^i_{u^{(i)}_n}(t_n)$ and $u^{(0)} = f^i_{u^{(i)}}(t)$. We assume for the moment that $\big\{(u^{(0)}_n, u^i_n), (u^{(0)}_n, u^i)\big\}_{n\in\mathbb{N}}$ is contained in a compact subset of $\mathcal{D}_i$. We write

$$t_n - t = \Big(\varphi_i(u^{(0)}_n, u^{(i)}_n) - \varphi_i(u^{(0)}_n, u^{(i)})\Big) + \Big(\varphi_i(u^{(0)}_n, u^{(i)}) - \varphi_i(u^{(0)}, u^{(i)})\Big). \tag{2.6}$$

By Corollary II.4 and the dominated convergence theorem, we have

$$\big|\varphi_i(u^{(0)}_n, u^{(i)}_n) - \varphi_i(u^{(0)}_n, u^{(i)})\big| \to 0, \qquad \text{as } n \uparrow \infty.$$

The left hand side of (2.6) converges to zero by initial assumption. Consequently, we have

$$\varphi_i(u^{(0)}_n, u^{(i)}) \to \varphi^i(u^{(0)}, u^{(i)}), \qquad \text{as } n \uparrow \infty.$$

We deduce that $u^{(0)}_n \to u^{(0)}$ using the continuity of $f^i_{u^{(i)}}(\cdot)$ (for a fixed $u^{(i)}$) from the third property.

To see that $\big\{(u^{(0)}_n, u^{(i)}_n), (u^{(0)}_n, u^{(i)})\big\}_{n\in\mathbb{N}}$ is compactly contained in $\mathcal{D}_i$, we rely on the strict monotonicity of $f^i_{u^{(i)}}$. We let $\delta > 0$ and denote $u^{(0)}_\delta = f^i_{u^{(i)}}(t+2\delta)$. By joint continuity of φ_i, we can find $\epsilon > 0$ such that

$$\big|\varphi_i(x, y) - (t+2\delta)\big| < \delta, \qquad \text{for } (x, y) \in (u^{(0)}_\delta, u^{(i)}) + (-\epsilon, \epsilon) \times (-\epsilon, \epsilon).$$

In particular,

$$t+\delta<\varphi_i(u_\delta^{(0)},y)<t+3\delta,\qquad \text{for } y\in(u^{(i)}-\epsilon,u^{(i)}+\epsilon).$$

As $u_n^{(i)}\to u^{(i)}$ and $t_n\to t$, we have, for some $M\in\mathbb{N}$ and any $n\geq M$,

$$u^{(i)}-\epsilon<u_n^{(i)}<u^{(i)}+\epsilon, \tag{2.7}$$

$$t_n<t+\delta. \tag{2.8}$$

Applying strict monotonicity of $f^i_{u_n^{(i)}}$ to (2.8), we have

$$u_n^{(0)}<u_\delta^{(0)},\qquad \text{for } n\geq M.$$

Similarly, we can find a lower bound $u_{-\delta}^{(0)}\in\mathbb{R}$ such that

$$u_{-\delta}^{(0)}<u_n^{(0)}<u_\delta^{(0)},\qquad \text{for } n\geq M. \tag{2.9}$$

The compact containment follows immediately from (2.7) and (2.9). $\diamondsuit$

2.1.2 Economy

We assume that the economic activities happen continuously on a finite time horizon $[0,T]$, $T\in(0,\infty)$. As usual the economy consists of $I<\infty$ investor-consumers all having exponential utility of consumption, a single non-durable consumption good, and two traded assets. The i^{th} investor equipped with exponential utility function

$$U_i(c)=e^{-\frac{c}{\tau_i}},\quad \tau_i>0,\quad c\in\mathbb{R},$$

receives random endowment over the time period $[0,T]$. The endowment is specified by the rate process $(Y_{it})_{t\in[0,T]}$ which evolves as

$$dY_{it}=\mu_i(t)dt+\int_{\mathbb{R}^{I+1}}\sigma_i(t)z^{(i)}\tilde{N}(dt,dz),\quad Y_{i0}\in\mathbb{R}.$$

Here the functions $\mu_i(t)$ and $\sigma_i(t)$ are deterministic and exogenously specified. The two traded assets are the money market account and a risky asset. The money market is in zero net supply and its equilibrium price is conjectured to have the dynamics

$$dS_t^{(0)} = r(t)S_t^{(0)}dt, \quad \forall t \in [0,T], \quad S_0^{(0)} = 1, \tag{2.10}$$

for some deterministic rate of return $r(t)$. The risky asset is taken to be a unit net supply stock which pays out continuous dividends at rate D. The dividend rate D is exogenously specified by the dynamics

$$dD_t = \mu_D(t)dt + \int_{\mathbb{R}^{I+1}} \sigma_D(t)z^{(0)}\tilde{N}(dt,dz), \quad \forall t \in [0,T], \quad D_0 \in \mathbb{R},$$

which contain the deterministic (input) functions μ_D and σ_D. We will show that the equilibrium price dynamics of the stock take the form

$$dS_t + D_t dt = r(t)S_t dt + \mu(t)dt + \int_{\mathbb{R}^{I+1}} \sigma(t)z^{(0)}\tilde{N}(dt,dz), \quad \forall t \in [0,T), \tag{2.11}$$

$$S_{T-} = S_T = 0, \tag{2.12}$$

for some deterministic functions μ and σ. The insertion of terminal condition (2.12) guarantees no stock price jumps at the terminal time. Consequently, the gain process is left-continuous at the terminal time; a result which is crucial in our equilibrium verification. To sum up, $(\mu_D, \sigma_D, \mu_i, \sigma_i)$ are the model's exogenously specified input, whereas the functions (r, μ, σ) are endogenously determined.

2.1.3 Assumptions on the market structure

In order to keep our model as simple as possible, we place several assumptions on the prices of the traded assets, their dividends, and the investors' endowments. The first assumption is placed on the choice of input parameters in the dividend process and the investors' endowment processes. These parameters are completely exogenously specified.

Assumption II.6. *The deterministic functions* $(\mu_D, \sigma_D, \mu_i, \sigma_i), \quad i = 1, \dots, I,$ *are continuous and finitely valued on the interval* $[0, T]$*. Furthermore,* $\sigma_D \geq 0$ *on the interval* $[0, T]$*.*

The second assumption is placed on the prices of the traded assets which, on the other hand, are endogenously determined. We need to verify that our candidate equilibrium price processes satisfy this assumption. Before we state the assumption we first define the price of zero coupon bonds, $\beta(t, s)$, and the price of the annuity, $A(t)$, by

$$\beta(t, s) = e^{-\int_t^s r(u)du}, \quad 0 \leq t \leq s \leq T,$$

$$A(t) = \int_t^T \beta(t, u)du, \quad 0 \leq t \leq T.$$

Assumption II.7. *The functions* (r, μ, σ) *appearing in the price of the traded assets are deterministic, continuous, and finitely valued on the interval* $[0, T]$*.*

We remark that if Assumption II.7 is satisfied then there exists a constant $\bar{r}$ such that $r(t) \geq \bar{r} > -\infty$ for $t \in [0, T]$. As a consequence, the price of annuity satisfies the condition

$$\lim_{s \uparrow T} \int_0^s A(t)^{-1} dt = +\infty. \tag{2.13}$$

There is a third and final assumption on the choice of functions (σ_D, σ_i). The purpose of this assumption is to ensure sufficient integrability of our optimal investment strategies. Since the assumption is technical and requires the definition of admissible strategies, we postpone the statement of this assumption to Section 2.2 (see Assumption II.16).

2.1.4 Individual investor's problem

Throughout this section the price dynamics (2.10) and (2.11) are given as input. Each investor has to choose his investment and consumption rates for every time $t \in [0,T]$. The choice of investment of the i^{th} investor is represented by the self-financing predictable pair $(\theta_{it}^{(0)}, \theta_{it})_{t\in[0,T]}$, where $\theta_{it}^{(0)}$ and θ_t denote the number of units invested in the money market account and the stock at time t respectively. Initially, the i^{th} investor starts with his endowments $(\theta_{i0-}^{(0)}, \theta_{i0-})$. The initial aggregate holdings of money market account and stock satisfy

$$\sum_{i=1}^{I} \theta_{i0-}^{(0)} = 0, \quad \sum_{i=1}^{I} \theta_{i0-} = 1.$$

As before, we denote $c_i = (c_{it})_{t\in[0,T]}$ the rate of consumption in excess of the endowment. Due to self-financing, the investor's choice of investment-consumption strategy can be identified with just the pair (θ_i, c_i). The investor's objective is to maximize the expected utility of running consumption over $[0,T]$:

$$\sup_{(\theta_i,c_i)\in\mathcal{A}_i} \mathbb{E}\left[\int_0^T e^{-\delta_i t} U_i(c_{it}+Y_{it})dt\right]. \tag{2.14}$$

We will specify the admissible set of controls $\mathcal{A}_i$ later in this section.

We next define the gain process and the state-price densities. The i^{th} investor's gain process is defined by $X_{it}^{(c_i,\theta_i)} = \theta_{it}^{(0)} S_t^{(0)} + \theta_{it} S_t$ with initial value $X_{i0}^{(c_i,\theta_i)} = \theta_{i0-}^{(0)} + \theta_{i0-} S_0$. By imposing the self-financing condition, the dynamics of the gain process can be written as

$$\begin{aligned}
&dX_{it}^{(c_i,\theta_i)} = \big(r(t)X_{it}^{(c_i,\theta_i)} - c_{it} + \theta_{it}\mu(t)\big)dt + \int_{\mathbb{R}^{I+1}} \theta_{it}\sigma(t)z^{(0)}\tilde{N}(dt,dz), \quad t\in[0,T),\\
&X_{iT}^{(c_i,\theta_i)} = X_{iT-}^{(c_i,\theta_i)},
\end{aligned}$$

where the terminal gain equals its left limit due to no terminal price jumps. We also need some integrability requirements for the above process to be well-defined. These requirements are

(2.15)
$$\int_0^T |c_{it}|dt < \infty, \quad \int_0^T \int_{\mathbb{R}^{I+1}} (\theta_{it}\sigma(t)z^{(0)})^2 \nu(dz)dt < \infty, \quad \int_0^T |\theta_{it}\mu(t)|dt < \infty,$$

$\mathbb{P}$-a.s. We then define the state-price densities below.

Definition II.8. A cadlag, adapted process $\xi = (\xi_t)_{t\in[0,T]}$ is said to be a state-price density if it satisfies the following three properties:

(i) $\xi_0 = 1$ and ξ_t is strictly positive for $t \in [0,T]$.

(ii) The process $Z_t = \frac{\xi_t}{S_t^{(0)}}$ is a martingale.

(iii) The process $\left(\xi_t X_{it}^{(c_i,\theta_i)} + \int_0^t \xi_s c_{is} ds\right)_{t\in[0,T]}$ is a σ-martingale for any (c_i,θ_i) satisfying (2.15) and for $i = 1, 2, \ldots, I$.

Note that for a state-price density ξ_t we call $Z_t = \frac{\xi_t}{S_t^{(0)}}$ martingale density.

To derive the representation of the state-price densities we consider the processes ξ^ϑ whose dynamics are of the form

$$d\xi_t^\vartheta = \xi_{t-}^\vartheta \Big\{ - r(t)dt + \int_{\mathbb{R}^{I+1}} \vartheta_t(z)\tilde{N}(dt,dz)\Big\}, \quad \xi_0^\vartheta = 1, \quad t \in [0,T], \tag{2.16}$$

for some predictable integrand $\vartheta_t(z) = \vartheta_t(z,\omega)$. However, not every predictable integrand ϑ gives rise to a state-price density. The three conditions on ϑ below are sufficient for ξ^ϑ to be a state-price density:

$$\vartheta_t(z) > -1, \quad \forall t \in [0,T], \tag{2.17}$$

$$\mathbb{E}\Big[e^{\int_0^T \int_{\mathbb{R}^{I+1}} \vartheta_t(z)^2 \nu(dz)dt}\Big] < \infty, \tag{2.18}$$

$$\frac{\mu(t)}{\sigma(t)} + \int_{\mathbb{R}^{I+1}} \vartheta_t(z) z^{(0)} \nu(dz) = 0. \tag{2.19}$$

In particular if ϑ satisfies the above conditions, then the state-price density has the explicit exponential form stated in (2.22). The above conditions deserve a few remarks. Condition (2.17) guarantees the exponential form in (2.22) is well-defined. Condition (2.18) is a Novikov-type condition that guarantees the martingality of Z^{ϑ}; see Thm 9 in [28]. Lastly, condition (2.19) guarantees the σ-martingality of the processes in (iii) of Definition II.8.

We would like to set up a non-trivial admissible set of investment-consumption strategies that rules out the doubling-type strategies and makes it possible to perform verification. In the following setup, two additional restrictions on the possible strategies are made. To state these restriction conditions we define the deterministic function θ_i^* by

$$\theta_i^*(t) = -\frac{\tau_i A(t)}{\sigma(t)} f^i_{-\frac{1}{\tau_i}\sigma_i(t)}\Big(-\frac{\mu(t)}{\sigma(t)} + \int_{\mathbb{R}^{I+1}} z^{(0)} \nu(dz)\Big), \tag{2.20}$$

where function $f^i_{u^{(i)}}$ is the inverse of the map

$$u^{(0)} \to \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)} z^{(0)} + u^{(i)} z^{(i)}} \nu(dz)$$

as defined in 3. of Lemma II.5. Equivalently, there is an implicit form of θ_i^* obtained from the unique solution to the equation

$$\frac{\mu(t)}{\sigma(t)} + \int_{\mathbb{R}^{I+1}} z^{(0)} \Big(e^{-\frac{1}{\tau_i} A(t)^{-1} \theta_i^*(t) \sigma(t) z^{(0)} - \frac{1}{\tau_i}\sigma_i(t) z^{(i)}} - 1\Big) \nu(dz) = 0. \tag{2.21}$$

Lemma II.5 shows that θ_i^* defined by (2.20) is continuous on $[0, T]$, hence, there is no difference between writing $\theta_i^*(t-)$ and $\theta_i^*(t)$. We let

$$\vartheta_{it}^*(z) = e^{-\frac{1}{\tau_i} A(t)^{-1} \theta_i^*(t) \sigma(t) z^{(0)} - \frac{1}{\tau_i}\sigma_i(t) z^{(i)}} - 1,$$

and we define the state-price density specific to the i^{th} investor, ξ_i^*, by

$$d\xi_{it}^* = \xi_{it-}^* \Big\{ -r(t)dt + \int_{\mathbb{R}^{I+1}} \vartheta_{it}^*(z) \tilde{N}(dt, dz) \Big\}, \quad \xi_{i0}^* = 1.$$

Lemma II.13 in Section 2.1.5 shows that the state-price density specific to i^{th} investor has the explicit form

(2.22)

$$\begin{aligned}\xi_{it}^* =\exp\Big\{ & -\int_0^t r(s)ds + \int_0^t \int_{\mathbb{R}^{I+1}} \big(-\frac{1}{\tau_i} A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{1}{\tau_i}\sigma_i(s)z^{(i)}\big)\tilde{N}(ds,dz) \\ & + \int_0^t \int_{\mathbb{R}^{I+1}} \big(- e^{-\frac{1}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{1}{\tau_i}\sigma_i(s)z^{(i)}} + 1\big)\nu(dz)ds \\ & + \int_0^t \int_{\mathbb{R}^{I+1}} \big(-\frac{1}{\tau_i} A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{1}{\tau_i}\sigma_i(s)z^{(i)}\big)\nu(dz)ds \Big\}.\end{aligned}$$

We are now ready to state our choice of admissible sets.

Definition II.9. An investment-consumption strategy (c_i, θ_i) said to be admissible for the i^{th} investor, and we write $(c_i, \theta_i) \in \mathcal{A}_i$, if it satisfies the following three conditions:

1. The integrability requirements in (2.15) are satisfied.
2. $\mathbb{P}\Big(X_{iT}^{(c_i,\theta_i)} \geq 0 \Big) = 1$.
3. The σ-martingale $\Big(\xi_{it}^* X_{it}^{(c_i,\theta_i)} + \int_0^t \xi_{is}^* c_{is} ds \Big)_{t\in[0,T]}$ is a $\mathbb{P}$-supermartingale.

In what follows, we state another assumption on both the endogenous and exogenous functions (μ, σ, σ_i). This assumption requires that θ_i^* implicitly obtained from (μ, σ, σ_i) is sufficiently smooth and sufficiently integrable.

Assumption II.10. $\forall i = 1, \ldots, I$, *the map* $t \mapsto A^{-1}(t)\theta_i^*(t)\sigma(t)$ *is continuous, finitely valued on* $[0,T]$, *and*

$$\Big(-\frac{2}{\tau_i} A^{-1}(t)\theta_i^*(t)\sigma(t), -\frac{2}{\tau_i}\sigma_i(t) \Big) \in \mathcal{D}_i, \quad \forall t \in [0,T].$$

Once we find an equilibrium candidate (μ, σ), we will return to this assumption and restate it in terms of the exogenous input (σ_D, σ_i) (see Assumption II.16). We now conclude the section with the main optimization result.

Theorem II.11. *Under Assumptions II.1, II.6, II.7, and II.10, the optimal consumption-investment strategy* $(\hat{c}_i, \hat{\theta}_i)$ *is deterministic and is given by*

$$\hat{\theta}_{it} = \theta_i^*(t), \tag{2.23}$$

$$\hat{c}_{it} = A(t)^{-1} X_t^{(\hat{c}_i, \hat{\theta}_i)} + \tau_i A(t)^{-1} \int_t^T \beta(t,s) \int_t^s g_i(u) du ds, \tag{2.24}$$

where g_i *is the continuous deterministic function defined by*

(2.25)

$$g_i(t) = \delta_i - r(t) + \frac{1}{\tau_i}\theta_i^*(t) A(t)^{-1}\mu(t) + \frac{1}{\tau_i}\mu_i(t)$$
$$- \int_{\mathbb{R}^{I+1}} \Big(e^{-\frac{1}{\tau_i}\theta_i^*(t)A(t)^{-1}\sigma(t)z^{(0)} - \frac{1}{\tau_i}\sigma_i(t)z^{(i)}} - 1 + \frac{1}{\tau_i}\theta_i^*(t)A(t)^{-1}\sigma(t)z^{(0)} + \frac{1}{\tau_i}\sigma_i(t)z^{(i)} \Big)\nu(dz).$$

2.1.5 Auxiliary lemmas and proofs

Lemma II.12. *Suppose that Assumption II.7 holds. Suppose* m *and* c *are two continuous deterministic functions which are finitely-valued on [0,T], then the solution to SDE*

$$dX_t = \Big((r(t) - A(t)^{-1}) X_t + m(t) \Big) dt + \int_{\mathbb{R}^{I+1}} A(t) c(t) z^{(0)} \tilde{N}(dt, dz), \qquad X_0 \in \mathbb{R},$$

exists and is unique on $[0, T)$. *Furthermore,* $X_t \to 0$ *a.s. as* $t \uparrow T$.

Proof. Let $b(t) = r(t) - A(t)^{-1}$. Ito's product rule for Lévy processes gives us that

$$X_t = e^{\int_0^t b(u)du} \Big(X_0 + \int_0^t e^{-\int_0^s b(u)du} m(s) ds + \int_0^t \int_{\mathbb{R}^{I+1}} e^{-\int_0^s b(u)du} A(s)c(s)z^{(0)} \tilde{N}(ds, dz) \Big)$$

is the unique solution to the given SDE. To see the convergence as $t \uparrow T$, we first note that $b(t) \to -\infty$ as $t \uparrow T$. We would like to justify the convergence to zero in each term in the above equation. Condition (2.13) in Assumption II.7 provides the convergence of the first term:

$$e^{\int_0^t b(u)du} X_0 \to 0.$$

The convergence of the second term follows from L'Hopital's rule:

$$\lim_{t\uparrow T} \frac{\int_0^t e^{-\int_0^s b(u)du} m(s)ds}{e^{-\int_0^t b(u)du}} = \lim_{t\uparrow T} \frac{m(t)}{-b(t)} = 0.$$

For the third term, we claim that the stochastic integral defined by

$$I_t = \int_0^t \int_{\mathbb{R}^{I+1}} e^{-\int_0^s b(u)du} A(s)c(s)z^{(0)} \tilde{N}(ds, dz)$$

is bounded in $\mathbb{L}^2$. We start by giving a bound to its quadratic variation:

$$\begin{aligned} \mathbb{E}\Big(\big[I, I\big]_t\Big) &= \int_0^t \int_{\mathbb{R}^{I+1}} e^{-2\int_0^s b(u)du} A(s)^2 c(s)^2 (z^{(0)})^2 \nu(dz)ds \\ &\leq t\Big(\sup_{s\in[0,T]} e^{-2\int_0^s b(u)du} A(s)^2 c(s)^2\Big) \int_{\mathbb{R}^{I+1}} ||z||^2 \nu(dz). \end{aligned}$$

If we can show that

$$\sup_{s\in[0,T]} e^{-2\int_0^s b(u)du} A(s)^2 c(s)^2 < \infty,$$

then I becomes a martingale bounded in $\mathbb{L}^2$. Due to continuity of the functions A and c, it suffices to show

$$\lim_{t\uparrow T} e^{-2\int_0^t b(u)du} A(t)^2 c(t)^2 < \infty.$$

Since $\frac{\partial}{\partial t} A(t)^{-2} = -2A(t)^{-2} b(t)$, we obtain the representation

$$A(t)^{-2} = A(0)^{-2} e^{-2\int_0^t b(u)du}.$$

Inserting this expression in the limit, we have

$$\lim_{t\uparrow T} e^{-2\int_0^t b(u)du} A(t)^2 c(t)^2 = \lim_{t\uparrow T} A(0)^2 c(t)^2 = A(0)^2 c(T)^2 < \infty,$$

and the claim is proved. Now that $\{I_t\}_{t<T}$ is a martingale bounded in $\mathbb{L}^2$, we know it converges almost surely to a finitely-valued random variable I_T. Consequently,

$$e^{\int_0^t b(u)du} I_t \to 0, \quad \text{as } t \uparrow T,$$

and the lemma is proved. $\diamondsuit$

Lemma II.13. *Suppose that Assumptions II.6, II.7, and II.10 hold. The state-price density specific to the i^{th} investor, ξ_i^*, has the explicit solution as in* (2.22)*. Moreover, there exists $p > 2$ such that ξ_i^* is bounded in $\mathbb{L}^p$ uniformly on $[0,T]$.*

Proof. The explicit form of ξ_{it}^* follows immediately from Ito's formula. To prove the latter part, we choose $p > 2$ such that $\Big(-\frac{p}{\tau_i} A(t)^{-1}\theta_i^*(t)\sigma(t), -\frac{p}{\tau_i}\sigma_i(t)\Big) \in \mathcal{D}_i$ for $t \in [0,T]$. We define $M = (M_t)_{t\in[0,T]}$ by

$$M_t = \int_0^t \int_{\mathbb{R}^{I+1}} \Big(e^{-\frac{p}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{p}{\tau_i}\sigma_i(s)z^{(i)}} - 1\Big)\tilde{N}(ds,dz),$$

and define $\mathcal{E}(M)$ to be the Doleans exponential of M given by

$$\begin{aligned}\mathcal{E}(M)_t = &\exp\Big\{ \int_0^t \int_{\mathbb{R}^{I+1}} \Big(-\frac{p}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{p}{\tau_i}\sigma_i(s)z^{(i)}\Big)\tilde{N}(ds,dz) \\ &- \int_0^t \int_{\mathbb{R}^{I+1}} \Big(e^{-\frac{p}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{p}{\tau_i}\sigma_i(s)z^{(i)}} - 1\Big)\nu(dz)ds\Big\}.\end{aligned}$$

Since $\Delta M > -1$, $\mathcal{E}(M)$ is a positive supermartingale and hence

$$\mathbb{E}[\mathcal{E}(M)_t] \le 1, \quad \forall t \in [0,T]. \tag{2.26}$$

We then write

(2.27)

$$\begin{aligned}\big(\xi_{it}^*\big)^p = \mathcal{E}(M)_t \exp\Big\{ &-p\int_0^t r(s)ds + \int_0^t \int_{\mathbb{R}^{I+1}} \big(e^{-\frac{p}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{p}{\tau_i}\sigma_i(s)z^{(i)}} - 1\big)\nu(dz)ds \\ &+ p\int_0^t \int_{\mathbb{R}^{I+1}} \big(-e^{-\frac{1}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{1}{\tau_i}\sigma_i(s)z^{(i)}} + 1\big)\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}^{I+1}} \big(-\frac{p}{\tau_i}A(s)^{-1}\theta_i^*(s)\sigma(s)z^{(0)} - \frac{p}{\tau_i}\sigma_i(s)z^{(i)}\big)\nu(dz)ds\Big\}.\end{aligned}$$

It follows that ξ_i^* is bounded uniformly in $\mathbb{L}^p$ because of (2.26) and that the remaining terms in (2.27) are deterministic and bounded. $\diamondsuit$

Lemma II.14. *Suppose that Assumptions II.1, II.6, and II.10 hold. Let $\hat{X}_i$ denote the gain process corresponding to the optimal strategy $(\hat{c}_i, \hat{\theta}_i)$ defined in Theorem II.11, then it is bounded in $\mathbb{L}^p$ uniformly for any $p > 0$.*

Proof. We begin by writing the dynamics of the optimal gain process as

$$d\hat{X}_{it} = \Big\{(r(t) - A(t)^{-1})\hat{X}_{it} + \theta_i^*(t)\mu(t) - \tau_i K_i(t)\Big\}dt + \int_{\mathbb{R}^{I+1}} \theta_i^*(t)\sigma(t)z^{(0)}\tilde{N}(dt, dz),$$

where the deterministic function K_i is defined by

$$K_i(t) = A(t)^{-1}\int_t^T \beta(t,s)\int_t^s g_i(u)duds.$$

Using $\frac{\partial}{\partial t}A(t)^{-1} = A(t)^{-2} - r(t)A(t)^{-1}$, we can write the dynamics of $A^{-1}\hat{X}$ as

$$dA(t)^{-1}\hat{X}_{it} = \big(A(t)^{-1}\theta_i^*(t)\mu(t) - \tau_i A(t)^{-1}K_i(t)\big)dt + \int_{\mathbb{R}^{I+1}} A(t)^{-1}\theta_i^*(t)\sigma(t)z^{(0)}\tilde{N}(dt, dz).$$

Under Assumption II.10 and that $\mathcal{D}_i$ is open at zero, we can find $\epsilon > 0$ such that

$$\Big(\epsilon A(t)^{-1}\theta_i^*(t)\sigma(t), 0\Big), \Big(-\epsilon A(t)^{-1}\theta_i^*(t)\sigma(t), 0\Big) \in \mathcal{D}_i, \quad \forall t \in [0, T].$$

An argument similar to that in Lemma II.13 shows

$$\sup_{t\in[0,T)} \mathbb{E}\Big[e^{\epsilon A(t)^{-1}\hat{X}_{it}}\Big] < \infty, \qquad \sup_{t\in[0,T)} \mathbb{E}\Big[e^{-\epsilon A(t)^{-1}\hat{X}_{it}}\Big] < \infty.$$

We then use the inequality $|x|^p \leq C_p(e^{\epsilon x} + e^{-\epsilon x})$, $\forall x \in \mathbb{R}$ for positive constant C_p to see that

$$\sup_{t\in[0,T)} \mathbb{E}\Big[\big|A(t)^{-1}\hat{X}_{it}\big|^p\Big] < \infty.$$

Finally, due to continuity of the function A on $[0, T]$ and Fatou's lemma, we have

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big|\hat{X}_{it}\big|^p\Big] < \infty.$$

$\diamondsuit$

Corollary II.15. *Under Assumptions II.6, II.7, and II.10, we have*

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big(\xi_{it}^*\hat{X}_{it}\big)^2\Big] < \infty.$$

Proof. The bound follows immediately from Lemmas II.13, II.14 and Holder's inequality. ♢

Proof of Theorem II.11. We divide the proof into two steps.

Step 1: Admissibility conditions.

In this step we will verify three key ingredients needed for the next step. For simplicity, we write $\hat{X}_t = X_t^{(\hat{c}_i, \hat{\theta}_i)}$ for $(\hat{c}_i, \hat{\theta}_i)$ defined in (2.23)-(2.24). The three key ingredients are

(i) $e^{-\delta_i t} U_i'(\hat{c}_{it} + Y_{it}) = \alpha \xi_{it}^*$, $\forall t \in [0,T]$, for some positive constant $\alpha > 0$.

(ii) The stochastic process $\left(\xi_{it}^* \hat{X}_{it} + \int_0^t \xi_{is}^* \hat{c}_{is} ds\right)_{t \in [0,T]}$ is a martingale.

(iii) $\hat{X}_{iT} = 0$ a.s. $\mathbb{P}$.

To prove claim (i), it suffices to show that the process $E_{it} = e^{-\frac{\hat{c}_{it} + Y_{it}}{\tau_i}}$ satisfies the same SDE as $e^{\delta_i t} \xi_{it}^*$. Inserting the representation of $\hat{c}_{it}$ from (2.24) and applying Ito's formula to E_{it}, we have

$$
\begin{aligned}
dE_{it} = E_{it-} \Big\{ \Big(& - \frac{1}{\tau_i} A(t)^{-1} \theta_i^*(t) \mu(t) - \frac{1}{\tau_i} \mu_i(t) + g_i(t) \\
& + \int_{\mathbb{R}^{I+1}} (e^{-\frac{1}{\tau_i} A(t)^{-1} \theta_i^*(t) \sigma(t) z^{(0)} - \frac{1}{\tau_i} \sigma_i(t) z^{(i)}} - 1 + \frac{1}{\tau_i} A(t)^{-1} \theta_i^*(t) \sigma(t) z^{(0)} + \frac{1}{\tau_i} \sigma_i(t) z^{(i)}) \nu(dz) \Big) dt \\
& + \int_{\mathbb{R}^{I+1}} (e^{-\frac{1}{\tau_i} A(t)^{-1} \theta_t^* \sigma(t) z^{(0)} - \frac{1}{\tau_i} \sigma_i(t) z^{(i)}} - 1) \tilde{N}(dt, dz) \Big\} \\
= E_{it-} \Big\{ & \big(\delta_i - r(t)\big) dt + \int_{\mathbb{R}^{I+1}} \vartheta_{it}^*(z) \tilde{N}(dt, dz) \Big\}.
\end{aligned}
$$

Hence, the claim (i) is proved.

For claim (ii), applying Ito's product rule to $\xi^*_{it}\hat{X}_{it}$, we have

$$\begin{aligned} d(\xi^*_{it}\hat{X}_{it}) + \xi^*_{it}\hat{c}_{it}dt &= \xi^*_{it-}\Big\{\big(\mu(t)\theta^*_i(t) + \int_{\mathbb{R}^{I+1}} \theta^*_i(t)\sigma(t)\vartheta^*_{it}(z)z^{(0)}\nu(dz)\big)dt \\ &\qquad + \int_{\mathbb{R}^{I+1}} \Big(\hat{X}_{it-}\vartheta^*_{it}(z) + \theta^*_i(t)\sigma(t)\big(\vartheta^*_{it}(z)+1\big)z^{(0)}\Big)\tilde{N}(dt,dz)\Big\} \\ &= \xi^*_{it-}\int_{\mathbb{R}^{I+1}} \Big(\hat{X}_{it-}\vartheta^*_{it}(z) + \theta^*_i(t)\sigma(t)\big(\vartheta^*_{it}(z)+1\big)z^{(0)}\Big)\tilde{N}(dt,dz), \end{aligned}$$

where we have used the state-price density condition (2.21) for ξ^*_i to cancel the drift term. We need to show that the remaining stochastic integral is a true martingale. It suffices to show the following two conditions:

$$\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^{I+1}} \big(\xi^*_{it-}\hat{X}_{it-}\vartheta^*_{it}(z)\big)^2\nu(dz)dt\Big] < \infty,$$

$$\mathbb{E}\Big[\int_0^T \int_{R^{I+1}} \Big(\xi^*_{it-}\theta^*_i(t)\sigma(t)z^{(0)}\big(\vartheta^*_{it}(z)+1\big)\Big)^2\nu(dz)dt\Big] < \infty.$$

The first integrability condition is obtained from Corollary II.15 and the bound

$$\begin{aligned} &\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}^{I+1}} \big(\xi^*_{it-}\hat{X}_{it-}\vartheta^*_{it}(z)\big)^2\nu(dz)dt\Big] \\ &\quad = \int_0^T \mathbb{E}\big(\xi^{*2}_{it-}\hat{X}^2_{it-}\big)\int_{\mathbb{R}^{I+1}} \big(e^{-\frac{1}{\tau_i}A(t)^{-1}\theta^*_i(t)\sigma(t)z^{(0)} - \frac{1}{\tau_i}\sigma_i(t)z^{(i)}} - 1\big)^2\nu(dz)dt \\ &\quad < \infty. \end{aligned}$$

The second integrability condition follows from Lemma II.13 in a similar fashion.

Claim (iii) follows directly from Lemma II.12 and the assumption that $t \to A(t)^{-1}\theta^*_i(t)\sigma(t)$ is continuous and finitely valued on the interval $[0,T]$.

Step 2: Verification step.

We denote by V_i the Fenchel's conjugate of U_i. We have

$$
\begin{aligned}
\mathbb{E}\Big[\int_0^T e^{-\delta_i t} U_i(c_{it}+Y_{it})dt\Big] &\leq \mathbb{E}\Big[\int_0^T e^{-\delta_i t} V_i(\alpha e^{\delta_i t}\xi^*_{it})dt + \int_0^T \alpha \xi^*_{it}(c_{it}+Y_{it})dt\Big] \\
&\leq \mathbb{E}\Big[\int_0^T e^{-\delta_i t} V_i(\alpha e^{\delta_i t}\xi^*_{it})dt + \int_0^T \alpha \xi^*_{it}(c_{it}+Y_{it})dt + \alpha\xi^*_{iT}X_{iT}\Big] \\
&\leq \mathbb{E}\Big[\int_0^T e^{-\delta_i t} V_i(\alpha e^{\delta_i t}\xi^*_{it})dt + \int_0^T \alpha \xi^*_{it}Y_{it}dt + \alpha X_{i0}\Big] \\
&= \mathbb{E}\Big[\int_0^T e^{-\delta_i t} V_i(\alpha e^{\delta_i t}\xi^*_{it})dt + \int_0^T \alpha \xi^*_{it}(\hat{c}_{it}+Y_{it})dt + \alpha\xi^*_{iT}\hat{X}_{iT}\Big] \\
&= \mathbb{E}\Big[\int_0^T e^{-\delta_i t} V_i(\alpha e^{\delta_i t}\xi^*_{it})dt + \int_0^T \alpha \xi^*_{it}(\hat{c}_{it}+Y_{it})dt\Big] \\
&= \mathbb{E}\Big[\int_0^T e^{-\delta_i t} U_i(\hat{c}_{it}+Y_{it})dt\Big].
\end{aligned}
$$

We note that the first inequality follows from Fenchel's inequality, whereas the supermartingality of an admissible strategy gives the third inequality. Meanwhile, the first equality holds because of (iii), the martingality of the optimal strategy is used in the second, and the last equality follows from (i). $\diamondsuit$

2.2 Finite time-horizon equilibrium

In order for the economy to be in equilibrium, all markets must clear: The money market account must be in zero net demand, while the stock must be in net unit demand. Furthermore, since the consumption good is non-storable, the net excess consumption must equal the dividends from the risky asset. We recall the full definition of Radner equilibrium price processes $(S^{(0)}, S)$ in Definition I.1 where now the investor's objective is (2.14) and the admissible sets are $\mathcal{A}_i$ defined in Definition II.9. Our main goal is find deterministic functions (r, μ, σ) that induce equilibrium prices $(S^{(0)}, S)$. We will assume throughout this section that Assumptions II.1 and II.6 hold.

Our main equilibrium result is developed from solving the following fixed point problem:

$$\sum_{i=1}^{I} \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i(t)}\Big(-\lambda(t)\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)} + \int_{\mathbb{R}^{I+1}} z^{(0)} \nu(dz)\Big) = -\sigma_D(t). \tag{2.28}$$

Before deriving the equilibrium, let us first motivate why the above fixed point problem comes up naturally. Suppose that we find (r, μ, σ), a candidate for equilibrium processes satisfying $\sigma(t) = \sigma_D(t)A(t)$ for $t \in [0, T]$. We recall from the previous section that the optimal investment strategy for the i^{th} investor can be written in terms of (r, μ, σ) as in (2.20). We let the Sharpe ratio λ (deterministic function) be given by

$$\lambda(t) = \frac{\mu(t)}{\sigma(t)\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)}}. \tag{2.29}$$

In terms of λ, the i^{th} investor's optimal investment strategy can be written as

(2.30)

$$\theta_i^*(t) = -\frac{\tau_i}{\sigma_D(t)} f^i_{-\frac{1}{\tau_i}\sigma_i(t)}\Big(-\lambda(t)\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)} + \int_{\mathbb{R}^{I+1}} z^{(0)} \nu(dz)\Big), \quad i = 1, \ldots, I.$$

We see from (2.30) that the fixed point problem (2.28) is equivalent to the clearing condition in the stock market.

We start over and redefine (λ, θ_i^*) according to their chronological order. Initially, we are given $(\mu_D, \sigma_D, \mu_i, \sigma_i)$ as input. We first define $\lambda(t)$ to be the (unique) solution to the fixed point problem (2.28) for $t \in [0, T]$ (See Assumption II.16 and Lemma II.17 for sufficient conditions). Then, we define $\theta_i^*(t)$ in terms of $\lambda(t)$ from equation (2.30). We let $\tau_\Sigma = \sum_{i=1}^{I} \tau_i$. Next, we state our last assumption on choice of input parameters (σ_D, σ_i) that had been postponed in Section 2.1.4. The main purpose of this assumption is to provide first a sufficient condition for the existence of solution to (2.28), and second a sufficient integrability condition for verification.

Assumption II.16. *We assume that the following two conditions hold for* (σ_D, σ_i)*:*

$$\Big(-\frac{1}{\tau_\Sigma}\sigma_D(t), -\frac{1}{\tau_i}\sigma_i(t)\Big) \in \mathcal{D}_i, \quad \forall t \in [0,T], \quad \forall i = 1, \dots, I, \tag{2.31}$$

$$\Big(-\frac{2}{\tau_i}\theta_i^*(t)\sigma_D(t), -\frac{2}{\tau_i}\sigma_i(t)\Big) \in \mathcal{D}_i, \quad \forall t \in [0,T], \quad \forall i = 1, \dots, I. \tag{2.32}$$

Lemma II.17. *Suppose* (2.31) *of Assumption II.16 holds. Then there exists a unique solution* $\lambda(t)$ *to equation* (2.28) *for* $t \in [0,T]$*.*

Proof. By Assumption II.16, we have

$$\tau_i f^i_{-\frac{1}{\tau_i}\sigma_i(t)}(x) > -\frac{\tau_i}{\tau_\Sigma}\sigma_D(t), \quad \text{as } x \uparrow \infty, \quad \forall i = 1, \dots, I.$$

Summing over $i \le I$, we obtain

$$\sum_{i=1}^{I} \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i(t)}(x) > -\sigma_D, \quad \text{as } x \uparrow \infty.$$

We can show similarly that

$$\sum_{i=1}^{I} \tau_i f^i_{-\frac{1}{\tau_i}\sigma_i(t)}(x) < -\sigma_D, \quad \text{as } x \downarrow -\infty.$$

Because $f^i_{u^{(i)}}$ is continuous and strictly monotone, we therefore conclude that equation (2.28) has a unique solution for $\lambda(t), \quad \forall t \in [0,T]$. ♢

Similar to the infinite time-horizon model, this model is incomplete and has infinitely many martingale measures. In addition, the minimal martingale measure does not exist in this model (more detail is discussed in Section 2.4). In order to properly state the equilibrium result, we will define a simple martingale measure which is independent of individual investor's parameters. We first define the deterministic functions θ° and ϑ° by

$$\theta^\circ(t) = f\Big(-\lambda(t)\sqrt{\int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz)} + \int_{\mathbb{R}^{I+1}} z^{(0)} \nu(dz)\Big),$$

$$\vartheta^\circ_t(z) = e^{\theta^\circ(t) z^{(0)}} - 1,$$

where f is the inverse of the map $u^{(0)} \to \int_{\mathbb{R}^{I+1}} z^{(0)} e^{u^{(0)} z^{(0)}} \nu(dz)$. We define $Z^{\vartheta^\circ} = (Z^{\vartheta^\circ})_{t\in[0,T]}$ as the solution to the linear equation

$$dZ_t^{\vartheta^\circ} = Z_{t-}^{\vartheta^\circ}\Big(\int_{\mathbb{R}^{I+1}} \vartheta_t^\circ(z)\tilde{N}(dt, dz)\Big), \quad t \in [0,T], \quad Z_0^\circ = 1.$$

We will construct μ such that (2.29) holds. Then Z^{ϑ° is a martingale density because ϑ° satisfies conditions (2.17)-(2.19). We let $\mathbb{Q}^\circ$ be the martingale measure generated by the Radon-Nikodym derivative $\frac{d\mathbb{Q}^\circ}{d\mathbb{P}} = Z_T^{\vartheta^\circ}$ and write $\mathbb{E}^\circ$ for the expectation under $\mathbb{Q}^\circ$.

We can now state our equilibrium result, which is an analogue of Theorem I.6.

Theorem II.18. *In addition to Assumptions II.1 and II.6, we suppose that Assumption II.16 holds. Assume that the initial aggregate holdings of money market account and stock satisfy*

$$\sum_{i=1}^{I} \theta_{i0-}^{(0)} = 0, \quad \sum_{i=1}^{I} \theta_{i0-} = 1.$$

We define λ implicitly through (2.28) and θ_i^* by (2.30). Then the functions (r, μ, σ) defined by

$$\begin{aligned} r(s) = \frac{1}{\tau_\Sigma}\Big\{&\mu_D(s) + \sum_{i=1}^{I} \tau_i\delta_i + \sum_{i=1}^{I} \mu_i(s) \\ &- \int_{\mathbb{R}^{I+1}} \Big(\sum_{i=1}^{I} \tau_i e^{-\frac{1}{\tau_i}\theta_i^*(s)\sigma_D(s)z^{(0)} - \frac{1}{\tau_i}\sigma_i(s)z^{(i)}} - \tau_\Sigma\Big)\nu(dz) \\ &+ \int_{\mathbb{R}^{I+1}} \Big(\sigma_D(s)z^{(0)} + \sum_{i=1}^{I} \sigma_i(s)z^{(i)}\Big)\nu(dz)\Big\}, \end{aligned} \tag{2.33}$$

$$\sigma(s) = A(s)\sigma_D(s), \tag{2.34}$$

$$\mu(s) = \lambda(s)\sigma(s)\sqrt{\int_{\mathbb{R}^{I+1}} z^{(0)}\nu(dz)}, \tag{2.35}$$

constitute equilibrium price processes $(S^{(0)}, S)$ defined in (2.10)-(2.11). Furthermore, *the stock price process satisfies*

$$S_t = \mathbb{E}^\circ\Big[\int_t^T \beta(t,u) D_u du \Big| \mathcal{F}_t\Big], \quad t \in [0,T]. \tag{2.36}$$

Proof. To prove the representation of S in the theorem, we instead define S as in (2.36) and will show that its dynamics satisfies (2.11). We begin by computing the dynamics of $Z^{\vartheta^\circ} D$:

$$\begin{aligned} dZ_t^{\vartheta^\circ} D_t = Z_{t-}^{\vartheta^\circ}\Bigg\{&\Big(\mu_D(t) + \int_{\mathbb{R}^{I+1}} \vartheta_t^\circ(z)\sigma_D(t) z^{(0)}\nu(dz)\Big)dt \\ &+ \int_{\mathbb{R}^{I+1}} \Big\{\big(1+\vartheta_t^\circ(z)\big)\sigma_D(t) z^{(0)} + D_{t-}\vartheta_t^\circ(z)\Big\}\tilde{N}(dt,dz)\Bigg\} \\ = Z_{t-}^{\vartheta^\circ}\Bigg\{&\big(\mu_D(t) - A(t)^{-1}\mu(t)\big)dt + \int_{\mathbb{R}^{I+1}} \Big\{\big(1+\vartheta_t^\circ(z)\big)\sigma_D(t) z^{(0)} + D_{t-}\vartheta_t^\circ(z)\Big\}\tilde{N}(dt,dz)\Bigg\}, \end{aligned}$$

where have used the fact that ϑ° satisfies (2.21) in the second equality. Rewriting the above dynamics as an integral and taking conditional expectation, we obtain

$$\begin{aligned} \mathbb{E}^\circ\big[D_u \big| \mathcal{F}_t\big] &= \frac{\mathbb{E}\big[Z_u^{\vartheta^\circ} D_u \big| \mathcal{F}_t\big]}{Z_t^{\vartheta^\circ}} \\ &= \frac{Z_t^{\vartheta^\circ} D_t + \mathbb{E}\Big[\int_t^u Z_{s-}^{\vartheta^\circ}\big(\mu_D(s) - A(s)^{-1}\mu(s)\big)ds \Big| \mathcal{F}_t\Big]}{Z_t^{\vartheta^\circ}}, \end{aligned} \tag{2.37}$$

for $u > t$. Since Z^{ϑ° is a martingale, we have, for $s > t$,

$$\begin{aligned} \mathbb{E}\big[Z_{s-}^{\vartheta^\circ} \big| \mathcal{F}_t\big] &= \mathbb{E}\big[Z_s^{\vartheta^\circ} - \Delta Z_s \big| \mathcal{F}_t\big] \\ &= Z_t^{\vartheta^\circ} - \mathbb{E}\Big[\mathbb{E}\big[\Delta Z_s^{\vartheta^\circ} \big| \mathcal{F}_{s-}\big] \Big| \mathcal{F}_t\Big] \\ &= Z_t^{\vartheta^\circ}. \end{aligned}$$

Using the fact that $\mu_D(s) - A(s)^{-1}\sigma_D(s)$ is deterministic, equation (2.37) can be further simplified to

$$\mathbb{E}^\circ\big[D_u \big| \mathcal{F}_t\big] = D_t + \int_t^u \Big(\mu_D(s) - A(s)^{-1}\mu(s)\Big)ds.$$

It then follows from Fubini's theorem that

$$\begin{aligned} S_t &= \int_t^T \beta(t,u)\mathbb{E}^\circ\big[D_u\big|\mathcal{F}_t\big]du \\ &= A(t)D_t + \int_t^T \beta(t,u)\int_t^u \Big(\mu_D(s) - A(s)^{-1}\mu(s)\Big)dsdu. \end{aligned} \tag{2.38}$$

Computing the dynamics in (2.38), we see that the dynamics dS_t match with (2.11) with the terminal condition (2.12).

Next, we need to check the market clearing conditions. The stock market automatically clears due to the construction of λ and θ_i^*. Provided the stock market clears, the money market account clears if and only if

$$S_t = \sum_{i=1}^I \hat{X}_{it}, \quad \forall t \in [0,T]. \tag{2.39}$$

Initially, equation (2.39) holds at $t=0$ because

$$\sum_{i=1}^I \hat{X}_{i0} = \sum_{i=1}^I (\theta_{i0-}^{(0)} S_0^{(0)} + \theta_{i0-}S_0) = S_0.$$

It also holds at terminal time as $S_T = 0$ and $\sum_{i=1}^I \hat{X}_{it} \to 0$ as $t \uparrow T$ by Lemma II.12. We claim that, prematurely, both S_t and $\sum_{i=1}^I \hat{X}_{it}$ are the unique solution to the SDE

$$\begin{aligned} dR_t =& \Big\{(r(t) - A(t)^{-1})R_t + \mu(t) - A(t)^{-1}\int_t^T \beta(t,u)\int_t^u \sum_{i=1}^I \tau_i g_i(s)dsdu\Big\}dt \\ &+ \int_{\mathbb{R}^{I+1}} \sigma(t)z^{(0)}\tilde{N}(dt,dz), \qquad t \in [0,T), \quad R_0 = S_0. \end{aligned} \tag{2.40}$$

It is clear from the optimal choice of consumption $\hat{c}_{it}$ in (2.24) and Lemma II.12 that $\sum_{i=1}^I \hat{X}_{it}$ satisfies the SDE (2.40). To see that the same holds for S, we write dynamics of S as

$$dS_t = \Big\{(r(t) - A(t)^{-1})S_t + \mu(t) + A(t)^{-1}S_t - D_t\Big\}dt + \int_{\mathbb{R}^{I+1}} \sigma(t)z^{(0)}\tilde{N}(dt,dz).$$

Comparing the above dynamics with (2.40), we wish to have

$$A(t)^{-1}S_t - D_t = -A(t)^{-1}\int_t^T \beta(t,u)\int_t^u \sum_{i=1}^N \tau_i g_i(s)dsdu. \tag{2.41}$$

By definition of g_i and r in (2.25) and (2.33), we have

$$-\sum_{i=1}^I \tau_i g_i(s) = \mu_D(s) - A(s)^{-1}\mu(s). \tag{2.42}$$

Equation (2.41) follows immediately from (2.42) and (2.38), therefore the claim is proved and the money market account clears. It remains to show that the goods market clears. To see this we write

$$\begin{aligned}
\sum_{i=1}^I \hat{c}_{it} &= A(t)^{-1}\sum_{i=1}^I \hat{X}_{it} + A(t)^{-1}\int_t^T \beta(t,s)\int_t^s \tau_i g_i(u)duds \\
&= A(t)^{-1}S_t + A(t)^{-1}\int_t^T \beta(t,s)\int_t^s \tau_i g_i(u)duds \\
&= D_t,
\end{aligned}$$

where the third equality comes from (2.41). ♢

2.3 Variance gamma model

As an example we consider the Variance Gamma process as our underlying Lévy process η; see [23]. The VG process suits our model because it is a well-known pure jump process. Below we follow [3] for the construction of the multivariate VG process through a linear transformation of independent one-dimensional VG processes.

We let $Z_i \sim CGMY(C_i, G_i, M_i, 0)$ be independent VG processes characterized as a special case of the CGMY process as the last parameter $Y_i = 0$ [7]. In other words, the Lévy measure ν_i of Z_i is defined by

$$\frac{\nu_i(dz)}{dz} = \begin{cases} \frac{C_i \exp(G_i z)}{-z}, & z < 0 \\ \frac{C_i \exp(-M_i z)}{z}, & z > 0. \end{cases}$$

Let $\alpha \in \mathbb{R}^{I+1}$ with $\alpha_0 = 1$. We define the multivariate VG process η as

$$\eta = (\eta^{(0)}, \eta^{(1)}, \ldots, \eta^{(I)}) = (Z_0 + \alpha_0 Z_{I+1}, \ldots, Z_I + \alpha_I Z_{I+1}).$$

According to Proposition 11.10 in [30], the Lévy measure ν of the multivariate process η satisfies

$$\nu(E) = \sum_{i=0}^{I} \nu_i(E_i) + \nu_{I+1}(E_\Delta), \qquad \forall E \in \mathcal{B}(\mathbb{R}^{I+1} \setminus \{0\}),$$

where

$$E_i = \{x_i \in \mathbb{R} : (0, 0, \ldots, x_i, \ldots, 0) \in E\},$$

$$E_\Delta = \{s \in \mathbb{R} : s\alpha \in E\}.$$

In particular, this Lévy measure ν have positive measure only on the $z^{(i)}$-axes and on the line $\{z \in \mathbb{R}^{I+1} : z = s\alpha\}$. We observe that ν fails to satisfy Condition (1.6) in Assumption I.2. We will instead derive the exponentially integrable domains $\mathcal{D}_i$ and show that the conditions in Assumption II.1 are satisfied. Since

$$\int_{|z^{(i)}|\geq 1} e^{u^{(i)}z^{(i)}} \nu_i(dz^{(i)}) = C_i \int_1^\infty \frac{e^{(u^{(i)}-M_i)z^{(i)}}}{z^{(i)}} + \frac{e^{-(u^{(i)}+G_i)z^{(i)}}}{z^{(i)}} dz^{(i)},$$

we have

$$\int_{|z^{(i)}|\geq 1} e^{u^{(i)}z^{(i)}} \nu_i(dz^{(i)}) < \infty \iff u^{(i)} \in (-G_i, M_i). \tag{2.43}$$

It follows similarly as in (2.43) that $\int_{||z||\geq 1} e^{u^{(0)}z^{(0)}+u^{(i)}z^{(i)}} \nu(dz) < \infty$ if and only if

$$(u^{(0)}, u^{(i)}) \in (-G_0, M_0) \times (-G_i, M_i) \quad \text{and} \quad \alpha_0 u^{(0)} + \alpha_i u^{(i)} \in (-G_{I+1}, M_{I+1}).$$

Hence the domain $\mathcal{D}_i$ is open and is given by

$$\mathcal{D}_i = \Big\{(u^{(0)}, u^{(i)}) \in (-G_0, M_0) \times (-G_i, M_i) : u^{(0)} + \alpha_i u^{(i)} \in (-G_{I+1}, M_{I+1})\Big\}.$$

Next we proceed to derive the explicit form of function φ_i. We have

$$\begin{aligned}\varphi_i(u^{(0)},u^{(i)}) &= \int_{\mathbb{R}} z^{(0)}e^{u^{(0)}z^{(0)}}\nu_0(dz^{(0)}) + \int_{\mathbb{R}} se^{(u^{(0)}+\alpha_i u^{(i)})s}\nu_{I+1}(ds)\\ &= C_0\left(\frac{1}{M_0-u^{(0)}}-\frac{1}{G_0+u^{(0)}}\right)\\ &\quad + C_{I+1}\left(\frac{1}{M_{I+1}-(u^{(0)}+\alpha_i u^{(i)})}-\frac{1}{G_{I+1}+(u^{(0)}+\alpha_i u^{(i)})}\right).\end{aligned}$$

Unlike φ_i, the function $f^i_{u^{(i)}}$ can only be obtained implicitly through the inverse of the $\varphi_i(\cdot,u^{(i)})$. As a result, we can only verify numerically the conditions in Assumption II.16.

The remainder of this section is dedicated to numerical results from a VG model. We consider the economy with a large number of identical exponential investors with $\tau_i=\tau$ for $i=1,\dots,I$. We set the dividend volatility parameter $\sigma_D=.1I$. The Lévy measure is constructed by iid one-dimensional VG processes

$$Z_i \sim CGMY(1,2,2,0), \quad \alpha_0=1, \quad \alpha_i=\rho, \qquad \text{for } i=2,\dots,I+1. \tag{2.44}$$

This particular parameter setting reflects an economy whose investors face the same income uncertainty with ρ indicating degree of the shared income risk. Figure 2.1 below shows the impacts due to the incompleteness in this model as ρ changes between $[-1,1]$.

Next we show the effects of investor heterogeneity on the equilibrium for the case where $\rho=1$. As usual, we split the population into two homogeneous groups (A,B), each containing investors with identical characteristic. Investors in group A bear higher income risk through the increase in jump sizes as $\sigma_A=.2$, whereas in group B, $\sigma_B=.1$. We let $w\in[0,1]$ denote the weight of group A against the whole population. Table 2.1 below reports the impacts on fraction of equity, equilibrium interest rate, and Sharpe ratio as the weight changes.

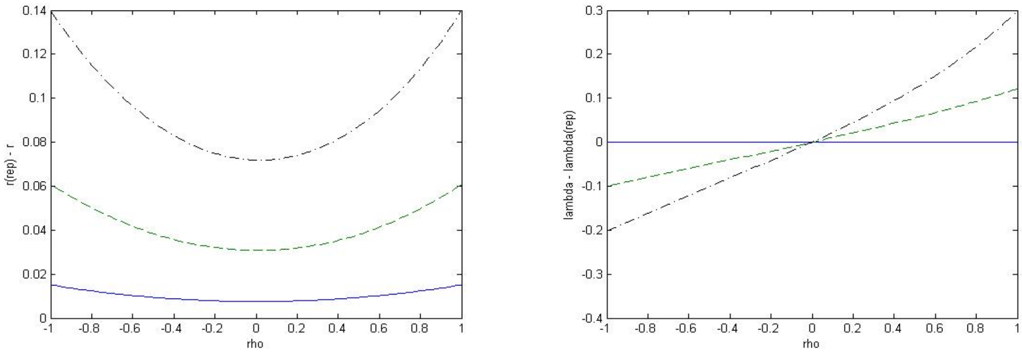

Figure 2.1: Plot of impacts due to model incompleteness on $r^{\text{rep}} - r$ (left) and $\lambda - \lambda^{\text{rep}}$ (right) seen as a function of the parameter ρ. We consider the limiting economy ($I \to \infty$) whereas the remaining parameters are given by $\sigma_D = .1I$ and (2.44) for the various risk tolerance coefficients τ: $\tau = 1$ (———), $\tau = \frac{1}{2}$ (− − −), and $\tau = \frac{1}{3}$ (− · − ·).

w	(τ_A, τ_B)			
	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{3}, \frac{1}{3})$
0.75	.648, [.052], (.104)	.780, [.053], (.122)	.474, [.120], (.200)	.638, [.121], (.248)
0.50	.365, [.041], (.087)	.544, [.045], (.122)	.175, [.086], (.136)	.352, [.094], (.202)
0.25	.149, [.027], (.071)	.287, [.034], (.122)	.031, [.049], (.090)	.141, [.061], (.160)

Table 2.1: Values of $\lim_{I\to\infty} wI\theta^*_A, [r^{\text{rep}} - r], (\lambda - \lambda^{\text{rep}})$ in the limiting economy ($I \to \infty$) for various weights w and various risk tolerance parameters (τ_A, τ_B). The values are based on the parameters $\sigma_A = .2$, $\sigma_B = .1$, $\sigma_D = .1I$, $\rho = 1$, and the remaining parameters as in (2.44).

2.4 Lack of the minimal martingale measure

This section is dedicated to the discussion of the (lack of) minimal martingale measure in our jump model. The existence of the minimal martingale measure for models with continuous stock prices is well studied by [13]. They established that the minimal martingale measure exists under a square-integrable condition of its martingale density. In our pure jump Lévy model, the stock price is not continuous. Therefore, the result from [13] does not apply in our case. In fact, we will show that the minimal martingale measure does not exist. We will proceed by proving that the unique candidate minimal measure cannot be a martingale measure.

We first introduce necessary notation to define the minimal martingale measure. We let $\mathbb{M}^2$ be the set of all $\mathbb{L}^2$-martingales M such that $\sup_{t\geq 0}\mathbb{E}\{M_t^2\} < \infty$, and $M_0 = 0$ a.s. The space $\mathbb{M}^2$ endowed with the norm

$$||M|| = \mathbb{E}\{M_\infty^2\}^{\frac{1}{2}}$$

is a Hilbert space. Two martingales $N, M \in \mathbb{M}^2$ are said to be strongly orthogonal if their product $L = NM$ is a (uniformly integrable) martingale. Furthermore, we can extend this definition of orthogonality to a pair of locally square-integrable martingales $M, N \in \mathbb{M}^2_{loc}$ in the natural way by stopping. We write $\mathcal{A}^\times$ to denote the set of all elements of $\mathbb{M}^2$ strongly orthogonal to each element of $\mathcal{A}$.

We now state the necessary assumption and define the minimal martingale measure. We let $\tilde{S} = \frac{S}{S^{(0)}}$ be the discounted stock price, and assume that it admits a semimartingale decomposition $\tilde{S} = \tilde{S}_0 + M + A$, for a local $\mathbb{P}$-martingale M and an adapted process A of finite variation. The following definition of the minimal martingale measure is taken from [14].

Definition II.19 (Structure condition)**.** $\tilde{S}$ satisfies the structure condition (SC) if M is locally $\mathbb{P}$-square-integrable and A has the form $A = \int \lambda d\langle M\rangle$ for a predictable process λ such that the increasing process $\int \lambda'\lambda d\langle M\rangle$ is finitely valued.

Definition II.20 (Minimal martingale measure)**.** Suppose $\tilde{S}$ satisfies (SC). A martingale measure $\hat{\mathbb{P}}$ with $\mathbb{P}$-square-integrable density $d\hat{\mathbb{P}}/d\mathbb{P}$ is minimal if $\hat{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_0$ and if every local $\mathbb{P}$-martingale L which is locally $\mathbb{P}$-square-integrable and strongly $\mathbb{P}$-orthogonal to M is also a local $\hat{\mathbb{P}}$-martingale.

In our Lévy model, the discounted risky asset satisfies

$$d\tilde{S}_t = \frac{\mu(t) - D_t}{S_t^{(0)}}dt + \frac{1}{S_t^{(0)}}\int_{\mathbb{R}^{N+1}} \sigma(t)z^{(0)}\tilde{N}(dt, dz).$$

We see that $\tilde{S}$ admits a semimartingale decomposition with local martingale part

$$M_t = \int_0^t \int_{\mathbb{R}^{I+1}} z^{(0)} \tilde{N}(ds, dz).$$

Here, the exclusion of the coefficient $\frac{\sigma}{S^{(0)}} > 0$ does not affect the minimal martingale because the coefficient is finitely valued and deterministic. In addition,

$$\mathbb{E}\big[M_t^2\big] \leq t \int_{\mathbb{R}^{I+1}} (z^{(0)})^2 \nu(dz) < \infty;$$

hence, the model satisfies (SC).

We will give two different proofs of the non-existence of the minimal martingale measure. One uses stable spaces of $\mathbb{M}^2$-martingales, and the other relies on the chaos decomposition for Lévy processes (see [26] and [22]).

2.4.1 Minimal martingale density as an element of the stable subspace $\mathcal{S}(M)$

In this section we suppose, by way of contradiction, that the minimal martingale measure $\hat{\mathbb{P}}$ exists. We will derive a representation of its density and show that it fails to be positive. The set $\mathcal{S}(M)$ is defined to be the smallest closed (under the $\mathbb{M}^2$-norm) and stable subspace of $\mathbb{M}^2$ containing M. Using the finite time version of Theorem 35 in [27], we have

(2.45)
$$\mathcal{S}(M) = \Big\{ H \cdot M : H_t = H(t, \omega) \text{ is predictable and } \mathbb{E}\Big[\int_0^T H_s^2 d[M,M]_s\Big] < \infty \Big\}.$$

We note that the predictable integrand H in (2.45) cannot depend on the jump size z. On the other hand, it follows from the definition of the minimal martingale measure of M that for any square-integrable martingale $L \in \mathbb{M}^2$,

$$L \in \{M\}^{\times} \quad \Longrightarrow \quad L \text{ is a } \hat{\mathbb{P}}\text{-martingale}.$$

Here we only test $\hat{\mathbb{P}}$-martingality of the genuinely square-integrable martingales as oppose to all the locally square-integrable martingales. If we define $Z^{min} = d\hat{\mathbb{P}}/d\mathbb{P} \in \mathbb{M}^2$ to be the density for the minimal martingale measure $\hat{\mathbb{P}}$, then LZ^{min} is a (UI) $\mathbb{P}$-martingale for any $L \in \{M\}^{\times}$. In other words,

$$Z^{min} \in \{M\}^{\times\times}.$$

A direct consequence of Theorem 37 in [27] implies $\{M\}^{\times\times} = \mathcal{S}(M)$; hence, by (2.45) we can write Z^{min} as

$$Z_t^{min} = \int_0^t \int_{\mathbb{R}^{I+1}} H_s z^{(0)} \tilde{N}(ds, dz),$$

for some predictable integrand H independent of z. By comparing this to the state-price density in (2.16), we see

$$\vartheta_t^{min}(z) = \frac{H_t z^{(0)}}{Z_{t-}^{min}}.$$

Without further assumptions on the Lévy measure, condition (2.17) is violated. Therefore, Z^{min} is not almost surely positive and is not (in general) a martingale density.

2.4.2 Chaos decomposition for Lévy processes

In this section, we will introduce the chaos decomposition for Lévy process to solve for the minimal martingale measure. Although the decomposition was first used by [26], we will use its multivariate version which was extended by [22]. The decomposition holds under the following condition on the Lévy measure: There exist some $\epsilon, \lambda > 0$ such that

$$\int_{||z|| \geq \epsilon} e^{\lambda ||z||} \nu(dz) < \infty. \tag{2.46}$$

We note that the above condition is automatically satisfied if we suppose that the Assumption II.1 holds.

Next, we follow [22] for the construction of the chaos decomposition. For simplicity, we will use the multi-index notation, for instance, we write $\eta^{(p)}$ instead of $\eta^{(p_0,p_1,\ldots,p_N)}$ for $p = (p_0, \ldots, p_N)$. We denote $|p| = p_0 + p_1 + \ldots p_N$ and define an ordering on $\mathbb{N}^{I+1}$ by

$$q \prec p \text{ if } \quad \begin{cases} |q| < |p|, \text{ or} \\ |q| = |p| \text{ and } (\exists i, \forall j < i, q_j = p_j \text{ and } q_i < p_i). \end{cases}$$

We define the power jump monomial processes by

$$\eta_t^{(p)} = \sum_{0<s\leq t} \left(\Delta\eta_s^{(0)}\right)^{p_0} \left(\Delta\eta_s^{(1)}\right)^{p_1} \cdots \left(\Delta\eta_s^{(N)}\right)^{p_N}.$$

The compensated power jump processes are defined by

$$\tilde{\eta}_t^{(p)} = \eta_t^{(p)} - \mathbb{E}\left[\eta_t^{(p)}\right],$$

where

$$\mathbb{E}\left[\eta_t^{(p)}\right] = \int_{\mathbb{R}^{I+1}} \left(z^{(0)}\right)^{p_0} \cdots \left(z^{(I)}\right)^{p_I} \nu(dz)t.$$

The compensated power jump processes are genuine martingales but they are not orthogonal. In fact, the cross characteristics is

$$\langle \tilde{\eta}^{(p)}, \tilde{\eta}^{(q)} \rangle_t = \int_{\mathbb{R}^{I+1}} z^p z^q \nu(dz)t,$$

which is non-zero when $p \neq q$. The following is the standard Gram-Schmidt process to construct an orthogonal basis $\{H^{(p)} : p \in \mathbb{N}^{I+1}\}$. We define

$$H^{(1,0,\ldots,0)} = \tilde{\eta}^{(1,0,\ldots,0)},$$

$$H^{(p)} = \tilde{\eta}^{(p)} + \sum_{q\prec p} c_{(p,q)} \tilde{\eta}^{(q)}, \quad \forall p \succ (1, 0, \cdots, 0),$$

where the constants $c_{(p,q)}$ are chosen so that $\langle H^{(p)}, H^{(l)} \rangle = 0$ for all $l \prec p$. The chaos decomposition theorem is stated below.

Theorem II.21 (Predictable Representation Property)**.** *Under Condition* (2.46)*, every random variable F in $\mathbb{L}^2(\Omega, \mathcal{F})$ has a representation of the form*

$$F = \mathbb{E}\left[F\right] + \sum_{d=1}^{\infty} \sum_{|p|=d} \int_0^{\infty} \Phi_s^{(p)} dH_s^{(p)},$$

where $\Phi^{(p)}$ is predictable function of (t, ω) and does not depend on the jump size z.

The chaos decomposition enables us to identify all the processes strongly orthogonal to M and in turn find their common martingale-preserving measure. To see this we suppose $Z_T^{min} \in \mathbb{L}^2$ is the minimal martingale measure which decomposes into

$$Z_t^{min} = 1 + \sum_{d=1}^{\infty} \sum_{|p|=d} \int_0^{t} \Phi_s^{(p)} dH_s^{(p)}, \quad 0 \leq t \leq T, \tag{2.47}$$

for some predictable processes $\Phi^{(p)}$. On the other hand, the martingale part of discounted stock price M is identified in the chaotic decomposition as

$$M_t = H_t^{(1,0,\ldots,0)}.$$

We quickly note that for a process $L \in \mathbb{M}^2$ to be a $\hat{\mathbb{P}}$-martingale, we must have that L is strongly orthogonal to Z^{min}. We then solve for the predictable processes $\Phi^{(p)}$ in (2.47) by testing the strong orthogonality with different M-orthogonal martingales. For instant, the process $L_t^{(p)}$ defined by

$$L_t^{(p)} = \int_0^t \Phi_s^{(p)} dH_s^{(p)}, \quad p \neq (1, 0, \ldots, 0),$$

due to the construction of the orthogonal basis $\{H^{(p)} : p \in \mathbb{N}^{I+1}\}$, must satisfy

$$\langle L^{(p)}, M \rangle_t = \langle L^{(p)}, H^{(1,0,\ldots,0)} \rangle_t = 0.$$

Then, by definition of the minimal martingale measure, we have

$$0 = \langle L^{(p)}, M^{min} \rangle_t = \mathbb{E} \int_0^t \left(\Phi_s^{(p)}\right)^2 d\langle H \rangle_s.$$

Consequently, $\Phi^{(p)} = 0$, for $p \succ (1, 0, \ldots 0)$, and the minimal martingale density Z^{min} satisfies

$$Z_t^{min} = \int_0^t \Phi_s^{(1,0,\ldots,0)} dH_s^{(1,0,\ldots,0)} = \int_0^t \int_{\mathbb{R}^{I+1}} \Phi_s^{(1,0,\ldots,0)} z^{(0)} d\tilde{N}(ds, dz).$$

Finally, we argue that the martingale Z^{min} above cannot be a martingale measure as it fails to be strictly positive a.s. We see this as

$$\vartheta_t^{min}(z) = \frac{\Phi_t^{(1,0,\ldots,0)}}{M_{t-}^{min}} z^{(0)} \not> -1,$$

especially when η's jump on the $z^{(0)}$-direction is unbounded. We conclude that the minimal martingale measure does not exist as the only 'minimal' candidate may not be a positive martingale measure.

BIBLIOGRAPHY

[1] R.M. Anderson and R. C. Raimondo (2008): *Equilibrium in continuous-time financial markets: Endogenously dynamically complete markets*, Econometrica **76**, 841–907.

[2] D. Applebaum (2009): *Lévy processes and stochastic calculus*, Cambridge university press.

[3] L. Ballotta, E. Bonfiglioli (2014). Multivariate asset models using Lévy processes and applications. The European Journal of Finance.

[4] D.S. Bates (1996): *Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche mark options*, Review of Financial Studies **9**, 69–107.

[5] D. Breeden (1979): *An intertemporal asset pricing model with stochastic consumption and investment opportunities*, Journal of Financial Economics **7**, 265–96.

[6] J.Y. Campbell (2000): *Asset pricing at the Millennium*, Journal of Finance **LV**, 1515–1567.

[7] P. Carr, H. Geman, D.B. Madan, and M. Yor (2002): *The Fine Structure of asset returns: an empirical investigation*, Journal of Business **75**, 305-332.

[8] J.H. Choi and K. Larsen (2015): *Taylor approximation of incomplete Radner equilibrium models*, Finance and Stochastics **19**, 653–679.

[9] P.O. Christensen, K. Larsen, and C. Munk (2012): *Equilibrium in securities markets with heterogeneous investors and unspanned income risk*, Journal of Economic Theory **147**, 1035–1063.

[10] P.O. Christensen and K. Larsen (2014): *Incomplete continuous-time securities markets with stochastic income volatility*, Review Asset Pricing Studies **4**, 247–285.

[11] M.M. Christensen and E. Platen (2007): *Sharpe ratio maximization and expected utility when asset prices have jumps*, International Journal of Theoretical and Applied Finance **10**, 1339–1364.

[12] D. Duffie (2001): *Dynamical asset pricing theory*, 3rd Ed., Princeton University Press.

[13] H. Föllmer and M. Schweizer (1991): *Hedging of contingent claims under incomplete information*, M. H. A. Davis and R. J. Elliott (eds.), Applied Stochastic Analysis, Stochastics Monographs **5**, 389–414.

[14] H. Föllmer and M. Schweizer (2010): *The minimal martingale measure*, Encyclopedia of Quantitative Finance, Wiley, 1200-1204.

[15] S.J. Grossman and R. Shiller (1982): *Consumption correlatedness and risk measurement in economies with non-traded assets and heterogeneous information*, Journal of Financial Economics **10**, 195–210.

[16] F. Herzberg and F. Riedel (2013): *Existence of financial equilibria in continuous time with potentially complete markets*, Journal of Mathematical Economics **49**, 398–404.

[17] C. Huang and R. Litzenberger (1988): *Foundations for financial economics*, Prentice-Hall.

[18] J. Hugonnier, S. Malamud, and E. Trubowitz (2012): *Endogenous completeness of diffusion driven equilibrium markets*, Econometrica **80**, 1249–1270.

[19] C. Kardaras, H. Xing, and G. Žitković (2015): *Incomplete stochastic equilibria with exponential utilities close to Pareto optimality*, working paper, available online `http://arxiv.org/abs/1505.07224`.

[20] D. Kramkov (2015): *Existence of an endogenously complete equilibrium driven by a diffusion*, Finance and Stochastics **19**, 1–22.

[21] K. Larsen and T. Sae-Sue (2016):*Radner equilibrium in incomplete Lévy models*, Mathematics and Financial Economics (forthcoming).

[22] J. Lin (2011): *Chaotic and predictable representations for multidimensional Lévy processes*, arXiv preprint.

[23] D.B. Madan, P.P. Carr, and E.C. Chang (1998): *The variance gamma process and option pricing*, European Finance Review **2**, 79–105.

[24] R. Mehra and E.C. Prescott (1985): *The equity premium: A puzzle*, Journal of Monetary Economics **15**, 145–162.

[25] R.C. Merton (1976): *Option pricing when underlying stock returns are discontinuous*, Journal of Financial Economics **3**, 125–144.

[26] D. Nualart and W. Schoutens (2000): *Chaotic and predictable representations for Lévy processes*, Stochastic Processes and their Applications **90**, 109-122.

[27] P.E. Protter (2004): *Stochastic integration and differential equations*, Springer.

[28] P.E. Protter and K. Shimbo (2008): *No arbitrage and general semimartingales*, In Markov processes and related topics: a Festschrift for Thomas G. Kurtz, Institute of Mathematical Statistics, 267–283.

[29] L.C.G. Rogers and D. Williams (2000): *Diffusions, Markov processes and martingales, Volume 2, Itô calculus*, Cambridge university press.

[30] K.-I. Sato (1999): *Lévy processes and infinitely divisible distributions*, Cambridge university press.

[31] N. Wang (2004): *Precautionary saving and partially observed income*, Journal of Monetary Economics **51**, 1645–1681.

[32] N. Wang (2006): *Generalizing the permanent-income hypothesis: Revisiting Friedman's conjecture on consumption*, Journal of Monetary Economics **53**, 737–752.

[33] P. Weil (1989): *The equity premium and the risk-free rate puzzle*, Journal of Monetary Economics **24**, 401–421.

[34] Y. Zhao (2012): *Stochastic equilibria in a general class of incomplete Brownian market environments*, Ph.D. thesis, available online `https://repositories.lib.utexas.edu/bitstream/handle/2152/ETD-UT-2012-05-5064/ZHAO-DISSERTATION.pdf?sequence=1`

[35] G. Žitković (2012): *An example of a stochastic equilibrium with incomplete markets*, Finance and Stochastics **16**, 177–206.