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Problems in Discrete Geometry and Extremal Combinatorics

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献给我的父母

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Chapter 2 is a version of the paper *A slight improvement to the colored Bárány's theorem*, which has been published in Electron. J. Combin.. Chapter 3 is a version of the paper *A bound on the number of edges in graphs without an even cycle* co-authored with Boris Bukh, which has been accepted to Comb. Probab. Comp.. Chapter 4 is a version of the paper *Bipartite algebraic graphs without*

quadrilaterals co-authored with Boris Bukh, which has been submitted. All these works were carried out under the supervision of Boris Bukh.

Abstract

We study several problems in discrete geometry and extremal combinatorics.

Discrete geometry studies the combinatorial properties of finite sets of simple geometric objects. One theme of the field is geometric Ramsey theory. Given m geometric objects, we want to select a not too small subset forming a configuration that is "regular" in some sense.

The first problem that we study is a colored variant of the point selection problem in discrete geometry. Given $d+1$ disjoint n-point sets, $P_0, \dots, P_d$, in $\mathbb{R}^d$, a colorful simplex is the convex hull of $d+1$ points each of which comes from a distinct P_i. We establish a "positive-fraction theorem" that asserts the existence of a point common to at least $\frac{2d}{(d+1)(d+1)!}n^{d+1}$ colorful simplices.

Extremal combinatorics studies the maximum or minimum size of discrete structures under given constraints. For a graph H, define the Turán number $\mathrm{ex}(n, H)$ as the maximum number of edges that an H-free graph on n vertices can have. When H is bipartite, the problem of pinning down the order of magnitude of $\mathrm{ex}(n, H)$ remains in general as one of the central open problems in combinatorics.

The second and the third problems respectively consider the Tuán number for two classes of graphs: cycles of even length and complete bipartite graphs. For cycles of even length C_{2k}, we show that for $\mathrm{ex}(n, C_{2k}) \le 80\sqrt{k \log k} \cdot n^{1+1/k} + O(n)$. For complete bipartite graphs $K_{s,t}$, motivated by the algebraic constructions of the extremal $K_{s,t}$-free graphs, we restrict our attention algebraically constructed graphs. We conjecture that every algebraic hypersurface that gives rise to a $K_{s,t}$-free graph is equivalent, in a suitable sense, to a hypersurface of low degree. We establish a version of this conjecture for $K_{2,2}$-free graphs.

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Chapter 1

Introduction

Discrete geometry studies the combinatorial properties of finite sets of points, lines, circles, planes, or other simple geometric objects. For example, one can ask, "what is the maximum number of incidences between m points and n lines?", or "what is the minimum possible number of distinct distances occurring among n points in the plane?". We refer the interested readers to the book of Matoušek [Mat02].

One theme of the field is geometric Ramsey theory. Given m geometric objects, we want to select a not too small subset forming a configuration that is "regular" in some sense. In many cases, we obtain "positive-fraction theorems": the regular configuration has size at least $\Omega(m)$. Consider n points in $\mathbb{R}^d$ in general position, and draw all the $m = \binom{n}{d+1}$ simplices with vertices at the given points. The point selection problem asks for a point of $\mathbb{R}^d$ common to as many simplices as possible. In the early eighties, Bárány [Bár82] established a "positive-fraction theorem" that asserts the existence of a point common to at least $c_d \cdot m$ of these simplices, where $c_d > 0$ is a constant depending only on d. Gromov [Gro10] made a major breakthrough in improving the lower bound of c_d by introducing a topological proof method. Later Krasev [Kar12] found a very short and elegant proof of Gromov's bound.

The problem that we study in Chapter 2 is a colored variant of the point selection problem. Given $d+1$ disjoint n-point sets, $P_0, \ldots, P_d$, in $\mathbb{R}^d$, a colorful simplex is the convex hull of $d+1$ points each of which comes from a distinct P_i. The colored point selection problem asks for a point of $\mathbb{R}^d$ covered by at least $c'_d \cdot n^{d+1}$ of the colorful simplices. As it generalizes the original point selection problem, one can show that $c_d \leq c'_d$. Karasev [Kar12] indeed proved that $c'_d \geq \frac{1}{(d+1)!}$. Based on his method that combines probabilistic and topological arguments, we improve his result to $c'_d \geq \frac{2d}{(d+1)(d+1)!}$ matching Gromov's bound for c_d.

Extremal combinatorics studies the maximum or minimum size of discrete structures under given constraints. The basic statement of extremal combinatorics is Mentel's theorem [Man07], proved in 1907, which states that any graph on n vertices with no triangle contains at most $n^2/4$ edges. The natural generalization of this theorem to clique is proved by Turán [Tur41] in 1941. Turán's theorem states that, on a given vertex set, the graph with the most edges with no clique of size t is the complete and balanced $(t-1)$-partite graph, in that the part sizes are as equal as possible.

For a graph F, define the Turán number $\mathrm{ex}(n, F)$ as the maximum number of edges that a graph on n vertices can have without containing a copy of F. For general graphs F, we still do not known how to compute the Turán number exactly, but if we are satisfied with an approximate answer, the theory becomes quite simple: Erdős and Stone [ES46] showed in 1946 that if the chromatic number $\chi(H) = t$, then $\mathrm{ex}(n, K_r) \leq \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2)$. When F is not bipartite, this gives asymptotic result for the Turán number. When F is bipartite, the problem of pinning down the order of magnitude of $\mathrm{ex}(n, F)$ remains in general as one of the central open problems in combinatorics. Most of the study of $\mathrm{ex}(n, F)$ for bipartite F has been concentrated on trees, complete bipartite graphs, and cycles of even length.

We address the cycles of even length C_{2k} in Chapter 3. A general bound of $\mathrm{ex}(n, C_{2k}) \leq \gamma_k \cdot n^{1+1/k} + O(n)$, for some unspecified constant γ_k, was asserted by Erdős [Erd64, p.33]. The first proof was by Bondy and Simonovits [BS74, Lemma 2], who showed the bound for $\gamma_k = 20k$. This was improved by Verstraëte [Ver00] to $\gamma_k = 8(k-1)$ and by Pikhurko [Pik12] to $\gamma_k = k - 1$. Pikhurko asked whether $\lim_{k\to\infty} \gamma_k/k$ can be 0. Inspired by the proof of Pikhurko, we answer this question in the positive by improving γ_k to $80\sqrt{k \log k}$.

In the final chapter, Chapter 4, we further study the Turán number for complete bipartite graphs $K_{s,t}$. So which graphs are $K_{s,t}$-free with a maximum number of edges? The question was considered by Füredi in his unpublished manuscript [Für88] asserting that every $K_{2,2}$-free graph with q vertices (for $q \geq q_0$) and $\frac{1}{2}q(q+1)^2$ edges is obtained from a projective plane via a polarity with $q+1$ absolute elements. Motivated by the algebraic constructions that match the upper bounds in the cases that $\mathrm{ex}(n, K_{s,t})$ has been solved asymptotically, we restrict our attention to algebraic bipartite graphs defined over algebraically closed fields. We conjecture that every algebraic hypersurface that gives rise to a $K_{s,t}$-free graph is equivalent, in a suitable sense, to a hypersurface of low degree. We establish a version of this conjecture for $K_{2,2}$-free graphs.

In the following chapters, backgrounds and motivations of the problems will be discussed in more details in their own introductions.

Chapter 2

A slight improvement to the colored Bárány's theorem

2.1 Introduction

Let $P \subset \mathbb{R}^d$ be a set of n points. Every $d+1$ of them span a simplex, for a total of $\binom{n}{d+1}$ simplices. The point selection problem asks for a point contained in as many simplices as possible. Boros and Füredi [BF84] showed for $d = 2$ that there always exists a point in $\mathbb{R}^2$ contained in at least $\frac{2}{9}\binom{n}{3} - O(n^2)$ simplices. A short and clever proof of this result was given by Bukh [Buk06]. Bárány [Bár82] generalized this result to higher dimensions:

Theorem 2.1 (Bárány [Bár82])**.** *There exists a point in $\mathbb{R}^d$ that is contained in at least $c_d\binom{n}{d+1} - O(n^d)$ simplices, where $c_d > 0$ is a constant depending only on the dimension d.*

This general result, the Bárány's theorem, is also known as the first selection lemma. We will henceforth denote by c_d the largest possible constant for which the Bárány's theorem holds true. Bukh, Matoušek and Nivasch [BMN10] used a specific construction called the stretched grid to prove that the constant $c_2 = \frac{2}{9}$ in the planar case found by Boros and Füredi [BF84] is the best possible. In fact, they proved that $c_d \leq \frac{d!}{(d+1)^d}$. On the other hand, Bárány's proof in [Bár82] implies that $c_d \geq (d+1)^{-d}$, and Wagner [Wag03] improved it to $c_d \geq \frac{d^2+1}{(d+1)^{d+1}}$.

Gromov [Gro10] further improved the lower bound on c_d by topological means. His method gives $c_d \geq \frac{2d}{(d+1)(d+1)!}$. Matoušek and Wagner [MW14] provided an exposition of the combinatorial component of Gromov's approach in a combinatorial language, while Karasev [Kar12] found a very elegant proof of Gromov's bound, which he described as a "decoded and refined" version of Gromov's proof.

The exact value of c_d has been the subject of ongoing research and is unknown, except for the planar case. Basit, Mustafa, Ray and Raza [BMRR10] and successively Matoušek and Wagner [MW14] improved the Bárány's theorem in $\mathbb{R}^3$. Král', Mach and Sereni [KMS12] used flag algebras from extremal combinatorics and managed to further improve the lower bound on c_3 to more than 0.07480, whereas the best upper bound known is 0.09375.

However, in this chapter, we are concerned with a colored variant of the point selection problem. Let $P_0, \dots, P_d$ be $d+1$ disjoint finite sets in $\mathbb{R}^d$. A *colorful simplex* is the convex hull of $d+1$ points each of which comes from a distinct P_i. For the colored point selection problem, we are concerned with the point(s) contained in many colorful simplices. Karasev proved:

Theorem 2.2 (Karasev [Kar12]). *Given a family of $d+1$ absolutely continuous probability measures $\mathbf{m} = (m_0, \dots, m_d)$ on $\mathbb{R}^d$, an $\mathbf{m}$-simplex*[1] *is the convex hull of $d+1$ points $v_0, \dots, v_d$ with each point v_i sampled independently according to probability measure m_i. There exists a point of $\mathbb{R}^d$ that is contained in an $\mathbf{m}$-simplex with probability $p_d \geq \frac{1}{(d+1)!}$. In addition, if two probability measures coincide, then the probability can be improved to $p_d \geq \frac{2d}{(d+1)(d+1)!}$.*

By a standard argument which we will provide immediately, a result on the colored point selection problem follows:

Corollary 2.3. *If $P_0, \dots, P_d$ each contains n points, then there exists a point that is contained in at least $\frac{1}{(d+1)!} \cdot n^{d+1}$ colorful simplices.*

Our result drops the additional assumption in theorem 2.2, hence improves corollary 2.3:

Theorem 2.4. *There is a point in $\mathbb{R}^d$ that belongs to an $\mathbf{m}$-simplex with probability $p_d \geq \frac{2d}{(d+1)(d+1)!}$.*

Corollary 2.5. *There exists a point that is contained in at least $\frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices.*

Proof of corollary 2.5 from the Theorem 2.4. Given $d+1$ sets $P_0, \dots, P_d$ in $\mathbb{R}^d$ each of which contains n points. Let $\Psi \colon \mathbb{R}^d \to \mathbb{R}$ be the *bump function* defined by $\Psi(x_1, \dots, x_d) = \prod_{i=1}^d \psi(x_i)$, where $\psi(x) = e^{-1/(1-x^2)} \mathbf{1}_{|x|<1}$, and set $\Psi_n(x_1, \dots, x_d) = n^d \Psi(nx_1, \dots, nx_d)$ for $n \in \mathbb{N}$. It is a standard fact that Ψ and Ψ_n are absolutely continuous probability measures supported on $[-1,1]^d$ and $[-1/n, 1/n]^d$ respectively.

For each $n \in \mathbb{N}$ and $0 \leq k \leq d$, define $m_k^{(n)}(x) := \frac{1}{n} \sum_{p \in P_k} \Psi_n(x-p)$ for $x \in \mathbb{R}^d$. Note that $m_k^{(n)}$ is an absolutely continuous probability measure supported on the Minkowski sum of P_k

[1] An $\mathbf{m}$-simplex is actually a simplex-valued random variable.

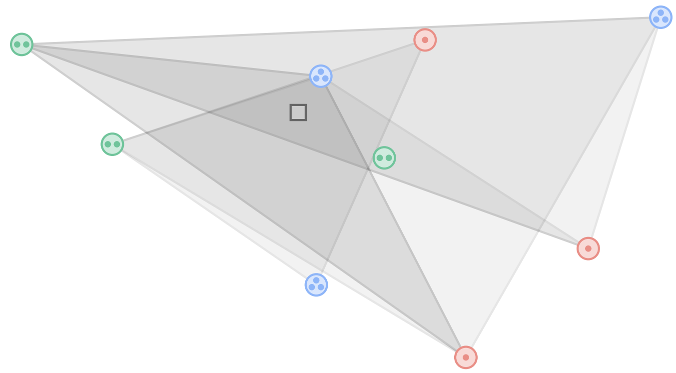

Figure 2.1: 3 red points, 3 green points and 3 blue points are placed in the plane. The point marked by a square is contained in 6 ($= \frac{2}{9} \cdot 3^3$) colorful triangles.

and $[-1/n, 1/n]^d$. Let $\mathbf{m}^{(n)}$ be the family of $d+1$ probability measures $m_0^{(n)}, \ldots, m_d^{(n)}$. By the Theorem 2.4, there is a point $p^{(n)}$ of $\mathbb{R}^d$ that belongs to an $\mathbf{m}^{(n)}$-simplex with probability at least $\frac{2d}{(d+1)(d+1)!}$.

Because no point in a certain neighborhood of infinity is contained in any $\mathbf{m}^{(n)}$-simplex, the set $\{p^{(n)} : n \in \mathbb{N}\}$ is bounded, and consequently the set has a limit point p. Suppose p is contained in N colorful simplices. Let $\epsilon > 0$ be the distance from p to all the colorful simplices that do not contain p. Choose n large enough such that $1/n \ll \epsilon$ and $\left|p^{(n)} - p\right| \ll \epsilon$. By the choice of n, if p is not contained in a colorful simplex spanned by $v_0, \ldots, v_d$, then $p^{(n)}$ is not contained the convex hull of $v'_0, \ldots, v'_d$ for all $v'_i \in v_i + [-1/n, 1/n]^d$. This implies that the probability that $p^{(n)}$ is contained in an $\mathbf{m}^{(n)}$-simplex is at most $\frac{N}{n^{d+1}}$. Hence p is the desired point contained in $N \geq \frac{2d}{(d+1)(d+1)!} \cdot n^{d+1}$ colorful simplices. □

Readers who are familiar with Karasev's work [Kar12] would notice that our proof of the Theorem 2.4 heavily relies on his arguments. The author is deeply in debt to him.

2.2 Proof of the Theorem 2.4

In this section, we provide the proof of the Theorem 2.4. The topological terms in the proof are standard, and can be found in [Mat03]. In addition to the notion of an $\mathbf{m}$-simplex, in the proof, we will often refer to an $(m_k, \ldots, m_d)$-face which means the convex hull of $d-k+1$ points $v_k, \ldots, v_d$ with each point v_i sampled independently according to probability measure m_i. An $\mathbf{m}$-simplex and an $(m_k, \ldots, m_d)$-face are both set-valued random variables.

Proof of the Theorem 2.4. To obtain a contradiction, we suppose that for any point v in $\mathbb{R}^d$, the probability that v belongs to an $\mathbf{m}$-simplex is less than $p_d := \frac{2d}{(d+1)(d+1)!}$. Since this

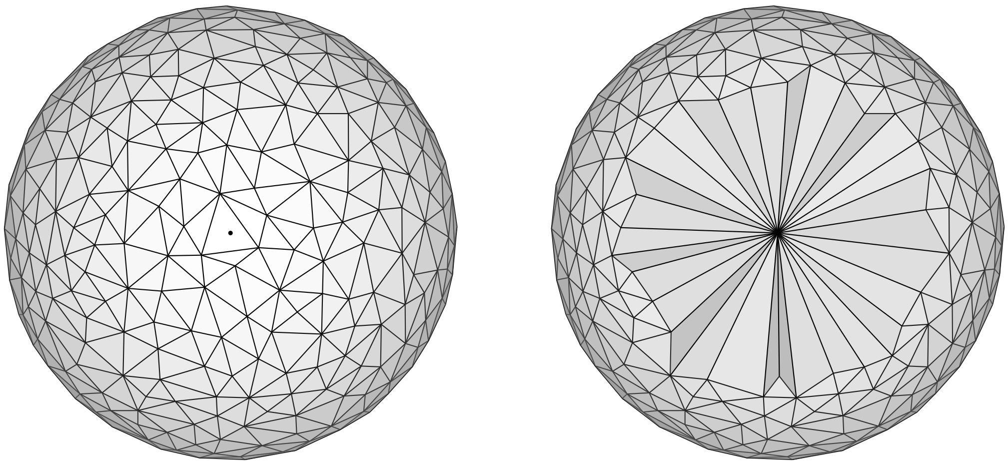

Figure 2.2: The bird's-eye view of a triangulation of S^2 with a 2-simplex containing ∞ and the cone over part of the triangulation.

probability, as a function of point v, is continuous and uniformly tends to 0 as v goes to infinity, there is an $\epsilon > 0$ such that v is contained in an $\mathbf{m}$-simples with probability at most $p_d - \epsilon$ for all v in $\mathbb{R}^d$.

Let $S^d := \mathbb{R}^d \cup \{\infty\}$ be the one-point compactification of the Euclidean space $\mathbb{R}^d$. Take $\delta = \epsilon/d$. Choose a finite triangulation[2] $\mathcal{T}$ of S^d with one of the d-simplices containing ∞ such that for $0 < k \leq d$, any k-face of $\mathcal{T}$ intersects an $(m_k, \ldots, m_d)$-face with probability less than δ and that the measure of any d-face of $\mathcal{T}$ under $(m_{d-1} + m_d)/2$ is less than δ. This can be done by taking a sufficiently fine triangulation of S^2 with one d-simplex having ∞ in its relative interior.

We use cone$(\cdot)$ as the cone functor[3] with apex O. A triangulation $\mathcal{T}$ of S^d naturally extends to a triangulation cone$(\mathcal{T})$ of cone(S^d). We denote the k-skeleton[4] of $\mathcal{T}$ and cone$(\mathcal{T})$ by $\mathcal{T}^{\leq k}$ and cone$(\mathcal{T})^{\leq k}$ respectively.

We are going to define a continuous map $f\colon \mathrm{cone}(\mathcal{T})^{\leq d} \to S^d$. Put $f(x) = x$ for all $x \in S^d = \|\mathcal{T}\| \subset \left\|\mathrm{cone}(\mathcal{T})^{\leq d}\right\|$, and set $f(O) = \infty$. We proceed to define f on cone(σ) for all

[2]A triangulation $\mathcal{T}$ of a topological space X is a simplicial complex K, homeomorphic to X, together with a homeomorphism $h\colon \|\mathsf{K}\| \to X$. Since the finite triangulation of interest is an extension of the triangulation of a d-simplex X in $\mathbb{R}^d$ and h is an identity map, we will freely use topological notions such as "a k-face (as a subset of S^d)" instead of "the image of a k-face in K under h". With such abuse of language, we can avoid going back and forth between the simplicial complex and the topological space.

[3]The cone over a space X is the quotient space cone$(X) := (X \times [0,1]) / (X \times \{1\})$. The apex is the equivalence class $\{(x,1) : x \in X\}$.

[4]The k-skeleton of a simplicial complex Δ consists of all simplices of Δ of dimension at most k.

the k-faces σ of $\mathcal{T}$ inductively on dimension k of σ while we maintain the property that the image of the boundary of cone(σ) under f, that is $f(\partial\text{cone}(\sigma))$, intersects an $(m_k, \ldots, m_d)$-face with probability at most $(k+1)!(p_d - \epsilon + k\delta)$. We say f is *economical* over a k-face σ of $\mathcal{T}^{\leq d-1}$ if f and σ satisfy the above property. Unlike Karasev [Kar12], our inductive construction of f follows the same pattern until $k = d-2$ instead of $d-1$. The main innovation of this proof is a different construction for $k = d-1$, which enables us to remove the additional assumption in theorem 2.2.

Note that for any 0-face σ in $\mathcal{T}$, $f(\partial\text{cone}(\sigma)) = f(\{\sigma, O\}) = \{\sigma, \infty\}$. According to the assumption at the beginning of the proof, $f(\partial\text{cone}(\sigma))$ intersects an $(m_0, \ldots, m_d)$-face, that is, an $\mathbf{m}$-simplex, with probability at most $p_d - \epsilon$. Therefore f is economical over 0-faces of $\mathcal{T}$. This finishes the first step.

Suppose f is already defined on cone$(\mathcal{T})^{\leq k}$ and it is economical over k-faces of $\mathcal{T}$. We are going to extend the domain of f to cone$(\mathcal{T})^{\leq k+1}$. Indeed, we only need to define f on cone(σ) for every k-face σ of $\mathcal{T}$.

Take any k-face σ of $\mathcal{T}$. Suppose convex hull of $v_k, \ldots, v_d$, denoted by conv$(v_k, \ldots, v_d)$, is an $(m_k, \ldots, m_d)$-face. Notice that the following statements are equivalent:

1. $f(\partial\text{cone}(\sigma))$ intersects conv$(v_k, \ldots, v_d)$;
2. for some $v \in f(\partial\text{cone}(\sigma))$, the ray with initial point v in the direction $\overline{v}v_kv$ intersects conv$(v_{k+1}, \ldots, v_d)$.

We call the union of such rays the *shadow* of $f(\partial\text{cone}(\sigma))$ centered at v_k. Since f is economical over σ, the probability for an $(m_k, \ldots, m_d)$-face to meet $f(\partial\text{cone}(\sigma))$ is at most $(k+1)!(p_d - \epsilon + k\delta)$, and so there exists $v_k^\sigma \in \mathbb{R}^d$ such that the shadow of $f(\partial\text{cone}(\sigma))$ centered at v_k^σ intersects conv$(v_{k+1}, \ldots, v_d)$ with probability at most $(k+1)!(p_d - \epsilon + k\delta)$.

Now, we define f on cone(σ). First, let g be the homeomorphism from cone(σ) onto the cone over ∂cone(σ) with apex c such that g is an identity on ∂cone(σ). This can be done because cone(σ) is homeomorphic to a $(k+1)$-simplex Δ and it is easy to find a homeomorphism from Δ to cone($\partial\Delta$) that keeps $\partial\Delta$ fixed.

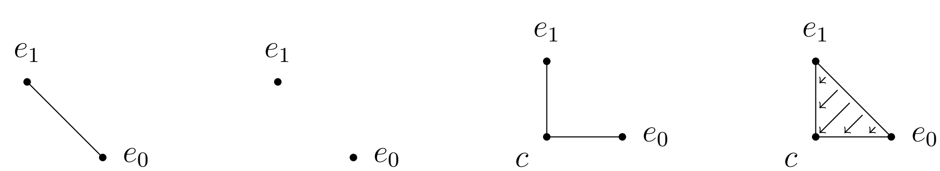

Figure 2.3: An illustration of an 1-simplex Δ, $\partial\Delta$, cone($\partial\Delta$) and a homeomorphism from Δ to cone($\partial\Delta$).

Next, note that every point w in cone(σ) except c is on a line segment $[v, c)$ for a unique point v on ∂cone(σ). If $t = \overline{vw}/\overline{wc} \in [0, \infty)$, then put $h(w) = \overline{v}f(v) + t \cdot \overline{v}v_k^\sigma f(v)$. In addition, set $h(c) = \infty$. The function h maps $[v, c)$ onto $[f(v), v_k^\sigma)$ linearly and then takes the inversion centered at v_k^σ with radius $\overline{v_k^\sigma f(v)}$ so that $[f(v), v_k^\sigma)$ gets mapped onto the ray with the initial point $f(v)$ in the direction $\overline{v} v_k^\sigma f(v)$. Evidently, h is a continuous map from $\mathrm{cone}(\partial\mathrm{cone}(\sigma))$ onto the shadow of $f(\partial\mathrm{cone})$ centered at v_k^σ that coincides with f on $\partial\mathrm{cone}(\sigma)$.

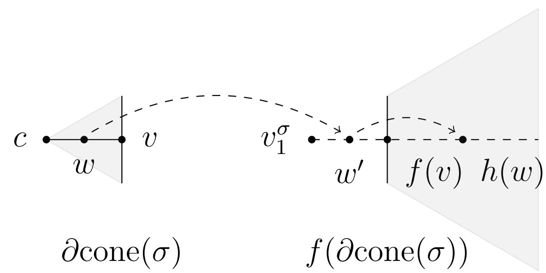

Figure 2.4: The illustration shows a cone over part of $\partial\mathrm{cone}(\sigma)$ with apex c and a point v on the boundary, and how a point w on the line segment $[v, c)$ are mapped under h.

Define f on $\mathrm{cone}(\sigma)$ to be the composition of g and h:

$$\begin{tikzcd}
\partial\mathrm{cone}(\sigma) \arrow[r, "="] \arrow[d, hook] & \partial\mathrm{cone}(\sigma) \arrow[r, "f"] \arrow[d, hook] & f(\partial\mathrm{cone}(\sigma)) \arrow[d, hook] \\
\mathrm{cone}(\sigma) \arrow[r, "g"] & \mathrm{cone}\left(\partial\mathrm{cone}(\sigma)\right) \arrow[r, "h"] & \text{the shadow of } f(\partial\mathrm{cone}(\sigma)) \text{ centered at } v_k^\sigma.
\end{tikzcd}$$

According to the commutative diagram above, f is well-defined on $\mathrm{cone}(\sigma)$ in the sense that it is compatible with its definition on $\mathrm{cone}(\mathcal{T})^{\le k}$. We use the phrase "fill in the boundary of $\mathrm{cone}(\sigma)$ against the center v_k^σ" to represent the above process that extends the domain of f from $\partial\mathrm{cone}(\sigma)$ to $\mathrm{cone}(\sigma)$.

To complete the inductive step, we must demonstrate that f is economical over $(k+1)$-faces of $\mathcal{T}$. Pick any $(k+1)$-face τ of $\mathcal{T}$. Let $\sigma_0, \ldots, \sigma_{k+1}$ be the k-faces of τ. Observing that $f(\partial\mathrm{cone}(\tau)) = f(\tau \cup \mathrm{cone}(\partial\tau)) = \tau \cup f(\mathrm{cone}(\sigma_0)) \cup \ldots \cup f(\mathrm{cone}(\sigma_{k+1}))$ and that $f(\mathrm{cone}(\sigma_i))$ is the shadow of $f(\partial\mathrm{cone}(\sigma_i))$ centered at $v_k^{\sigma_i}$ which intersects an $(m_{k+1}, \ldots, m_d)$-face with probability at most $(k+1)!(p_d - \epsilon + k\delta)$, we obtain that the probability for an $(m_{k+1}, \ldots, m_d)$-face to intersect $f(\partial\mathrm{cone}(\tau))$ is dominated by $\delta + (k+2)(k+1)!(p_d - \epsilon + k\delta) \le (k+2)!(p_d - \epsilon + (k+1)\delta)$.

We have so far defined a continuous map f on $\mathrm{cone}(\mathcal{T})^{\le d-1}$ such that for any $(d-1)$-face σ of $\mathcal{T}$ the probability for an $(m_{d-1} m_d)$-face to intersect $D := f(\partial\mathrm{cone}(\sigma))$ is at most $d!(p_d - \epsilon + (d-1)\delta)$. We write $f(X)\mathrm{mod}2 := \{y \in f(X) : |f^{-1}(y) \cap X| = 1 \pmod 2\}$ for the set of points in $f(X)$ whose fibers in X have an odd number of points. Set $\bar{m} := (m_{d-1} + m_d)/2$. We are going to define f on $\mathrm{cone}(\sigma)$ such that $\bar{m}\,(f(\mathrm{cone}(\sigma))\mathrm{mod}2)$ is less than $\frac{1-\delta}{d+1}$.

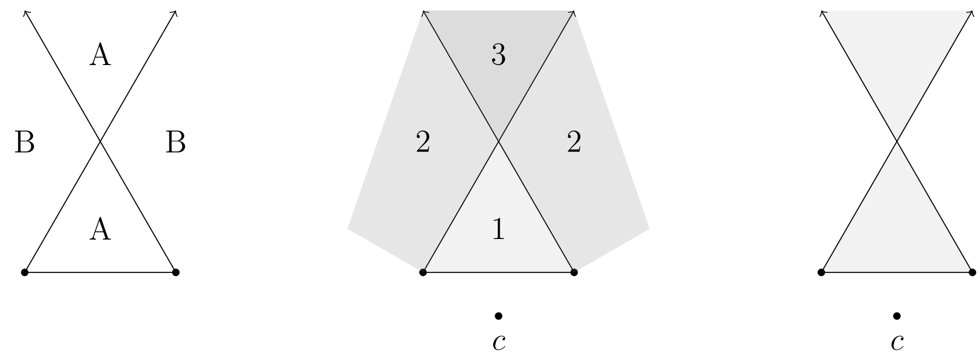

Figure 2.5: An illustration of the partition, the result of filling in against c, and $f(\mathrm{cone}(\sigma))\mathrm{mod}2$.

Fix a point s in $\mathbb{R}^d \setminus D$. For any point t in $\mathbb{R}^d \setminus D$, if a generic piecewise linear path from s to t intersects with D an odd number of times, then put t in B, otherwise put it in A. Here the number of intersections of a piecewise linear path L and D might not be the cardinality of $L \cap D$. Instead, the number of intersections is precisely $\sum_{x \in L \cap D} |f^{-1}(x) \cap \partial\mathrm{cone}(\sigma)|$, that is, it takes the multiplicity into account. Thus we have partitioned $\mathbb{R}^d \setminus D$ into A and B such that any generic piecewise linear path from a point in A to a point in B meets D an odd number of times. Suppose $a := m_{d-1}(A)$, $b := m_d(A)$ and $x := \bar{m}(A) = (a+b)/2$. The probability that an $(m_{d-1}m_d)$-face intersects with D is at least $a(1-b)+(1-a)b$. Hence $a(1-b)+(1-a)b < d!(p_d - \epsilon + (d-1)\delta) < 2\left(\frac{1-\delta}{d+1}\right)\left(1 - \frac{1-\delta}{d+1}\right)$. Because $a(1-b)+(1-a)b = (a+b) - 2ab \geq (a+b) - (a+b)^2/2 = 2x(1-x)$, either x or $1-x$ is less than $\frac{1-\delta}{d+1}$. In other words, one of $\bar{m}(A)$ and $\bar{m}(B)$ is less than $\frac{1-\delta}{d+1}$. We may assume that $\bar{m}(B) < \frac{1-\delta}{d+1}$.

Fix a point $c \in A$. Again, we fill in the boundary of cone(σ) against the center c. For any generic point $x \in A$, the line segment $[c, x]$ intersects with D an even number of times. For every v on $\partial\mathrm{cone}(\sigma)$, the ray with the initial point $f(v)$ in the direction $\overline{v}cf(v)$ covers x once if and only if the line segment $[c, x]$ intersects with D at $f(v)$. Because $f(\mathrm{cone}(\sigma))$ is the union of such rays, the number of times that x is covered by $f(\mathrm{cone}(\sigma))$ is exactly the number of intersections between $[c, x]$ and D. This implies that x is not in $f(\mathrm{cone}(\sigma))\mathrm{mod}2$. Therefore $f(\mathrm{cone}(\sigma))\mathrm{mod}2$ is a subset of $B \cup D$ almost surely. Noticing that $\bar{m}(D) = 0$, the extension of f has the desired property $\bar{m}\left(f(\mathrm{cone}(\sigma))\mathrm{mod}2\right) < \frac{1-\delta}{d+1}$.

Pick any d-face τ of $\mathcal{T}$. Suppose the $(d-1)$-faces of τ are $\sigma_0, \ldots, \sigma_d$. By a parity argument, we have

$$\begin{aligned} f(\partial\mathrm{cone}(\tau))\mathrm{mod}2 &= [\tau \cup f(\mathrm{cone}(\sigma_0)) \cup \ldots \cup f(\mathrm{cone}(\sigma_d))]\,\mathrm{mod}2 \\ &\subset \tau \cup f(\mathrm{cone}(\sigma_0))\mathrm{mod}2 \cup \ldots \cup f(\mathrm{cone}(\sigma_d))\mathrm{mod}2. \end{aligned}$$

Therefore $\bar{m}\left(f(\partial\text{cone}(\tau))\text{mod}2\right)$ is less than $\delta + (d+1)\frac{1-\delta}{d+1} = 1$, and so the degree of f on $\partial\text{cone}(\tau)$, denoted by $\deg\left(f, \partial\text{cone}(\tau)\right)$, is even. Because

$$\sum_{\tau} \deg\left(f, \partial\text{cone}(\tau)\right) = 2\sum_{\sigma} \deg\left(f, \text{cone}(\sigma)\right) + \deg\left(f, \mathcal{T}\right) = \deg\left(f, \mathcal{T}\right) \pmod 2,$$

where the first sum and the second sum are over all d-faces and all $(d-1)$-faces of $\mathcal{T}$ respectively, we know that $\deg\left(f, \mathcal{T}\right)$ is even, which contradicts with the fact that f is identity on $\mathcal{T}$. $\square$

Chapter 3

A bound on the number of edges in graphs without an even cycle

3.1 Introduction

Let $\mathrm{ex}(n, F)$ be the largest number of edges in an n-vertex graph that contains no copy of a fixed graph F. The systematic study of $\mathrm{ex}(n, F)$ was started by Turán [Tur41] over 70 years ago, and it has developed into a central problem in extremal graph theory (see surveys [FS13, Kee11, Sid95]).

The function $\mathrm{ex}(n, F)$ exhibits a dichotomy: if F is not bipartite, then $\mathrm{ex}(n, F)$ grows quadratically in n, and is fairly well understood; if F is bipartite, $\mathrm{ex}(n, F)$ is subquadratic, and for very few F the order of magnitude is known. The simplest classes of bipartite graphs are trees, complete bipartite graphs, and cycles of even length. Most of the study of $\mathrm{ex}(n, F)$ for bipartite F has been concentrated on these classes. In this chapter, we address the even cycles. For an overview of the status of $\mathrm{ex}(n, F)$ for complete bipartite graphs see [BBK13]. For a thorough survey on bipartite Turán problems see [FS13].

The first bound on the problem is due to Erdős [Erd38] who showed that $\mathrm{ex}(n, C_4) = O(n^{3/2})$. Thanks to the works of Erdős and Rényi [ER62], Brown [Bro66, Section 3], and Kövari, Sós and Turán [KST54] it is now known that

$$\mathrm{ex}(n, C_4) = (1/2 + o(1))n^{3/2}.$$

The current best bounds for $\mathrm{ex}(n, C_6)$ for large values of n are

$$0.5338n^{4/3} < \mathrm{ex}(n, C_6) \leq 0.6272n^{4/3}$$

due to Füredi, Naor and Verstraëte [FNV06].

A general bound of $\mathrm{ex}(n, C_{2k}) \le \gamma_k n^{1+1/k}$, for some unspecified constant γ_k, was asserted by Erdős [Erd64, p. 33]. The first proof was by Bondy and Simonovits [BS74, Lemma 2], who showed that $\mathrm{ex}(n, C_{2k}) \le 20kn^{1+1/k}$ for all sufficiently large n. This was improved by Verstraëte [Ver00] to $8(k-1)n^{1+1/k}$ and by Pikhurko [Pik12] to $(k-1)n^{1+1/k} + O(n)$. The principal result of the chapter is an improvement of these bounds:

Theorem 3.1. *If G is an n-vertex graph that contains no C_{2k} and $n \ge (2k)^{8k^2}$, then*

$$\mathrm{ex}(n, C_{2k}) \le 80\sqrt{k \log k} \cdot n^{1+1/k} + 10k^2 n.$$

It is our duty to point out that the improvement offered by the Theorem 3.1 is of uncertain value because we still do not know if $n^{1+1/k}$ is the correct order of magnitude for $\mathrm{ex}(n, C_{2k})$. Only for $k = 2, 3, 5$ are constructions of C_{2k}-free graphs with $\Omega(n^{1+1/k})$ edges known [Ben66, Wen91, LU95, MM05]. We stress again that the situation is completely different for odd cycles, where the value of $\mathrm{ex}(n, C_{2k+1})$ is known exactly for all large n [Sim68].

Our proof is inspired by that of Pikhurko [Pik12]. Apart from a couple of lemmas that we quote from [Pik12], the present chapter is self-contained. However, we advise the readers to at least skim [Pik12] to see the main idea in a simpler setting.

Pikhurko's proof builds a breadth-first search tree, and then argues that a pair of adjacent levels of the tree cannot contain a Θ-graph[1]. It is then deduced that each level must be at least $\delta/(k-1)$ times larger than the previous, where δ is the (minimum) degree. The bound on $\mathrm{ex}(n, C_{2k})$ then follows. The estimate of $\delta/(k-1)$ is sharp when one restricts one's attention to a pair of levels.

In our proof, we use three adjacent levels. We find a Θ-graph satisfying an extra technical condition that permits an extension of Pikhurko's argument. Annoyingly, this extension requires a bound on the *maximum degree*. To achieve such a bound we use a modification of breadth-first search that avoids the high-degree vertices.

What we really prove in this chapter is the following:

Theorem 3.2. *Suppose $k \ge 4$. If G is a biparite n-vertex graph of minimum degree at least $2d + 5k^2$, where*

$$d \ge \max(20\sqrt{k \log k} \cdot n^{1/k}, (2k)^{8k}), \tag{3.1}$$

then G contains C_{2k}.

Theorem 3.1 follows from Theorem 3.2 and two well-known facts: every graph contains a bipartite subgraph with half of the edges, and every graph of average degree d_{avg} contains a subgraph of minimum degree at least $\frac{1}{2}d_{\mathrm{avg}}$.

[1] We recall the definition of a Θ-graph in Section 3.3

The rest of the chapter is organized as follows. We present our modification of breadth-first search in Section 3.2. In Section 3.3, which is the heart of the chapter, we explain how to find Θ-graphs in triples of consecutive levels. Finally, in Section 3.4 we assemble the pieces of the proof.

3.2 Graph exploration

Our aim is to have vertices of degree at most Δd for some $k \ll \Delta \ll d^{1/k}$. The particular choice is fairly flexible; we choose to use $\Delta := k^3$.

Let G be a graph, and let x be any vertex of G. We start our exploration with the set $V_0 = \{x\}$, and mark the vertex x as explored. Suppose $V_0, V_1, \dots, V_{i-1}$ are the sets explored in the first i steps respectively. We then define V_i as follows:

1. Let V_i' consist of those neighbors of V_{i-1} that have not yet been explored. Let Bg_i be the set of those vertices in V_i' that have more than Δd unexplored neighbors, and let $\mathrm{Sm}_i = V_i' \setminus \mathrm{Bg}_i$.
2. Define
$$V_i = \begin{cases} V_i' & \text{if } |\mathrm{Bg}_i| > \frac{1}{2k} |V_i'|, \\ \mathrm{Sm}_i & \text{if } |\mathrm{Bg}_i| \leq \frac{1}{2k} |V_i'|. \end{cases}$$
The vertices of V_i are then marked as explored.

We call sets $V_0, V_1, \dots$ *levels* of G. A level V_i is *big* if $|\mathrm{Bg}_i| > \frac{1}{2k} |V_i'|$, and is *normal* otherwise.

Lemma 3.3. *If $\delta \leq \Delta d$, and G is a bipartite graph of minimum degree at least δ, then each $v \in V_{i+1}$ has at least δ neighbors in $V_i \cup V_{i+2}'$.*

Proof. Fix a vertex $v \in V(G)$. We will show, by induction on i, that if $v \notin V_1 \cup \dots \cup V_i$, then v has at least δ neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$. The base case $i = 1$ is clear. Suppose $i > 1$. If $v \in \mathrm{Bg}_i$, then v has $\Delta d \geq \delta$ neighbors in the required set. Otherwise, v is not in V_i' and hence has no neighbors in V_{i-1}. Hence, v has as many neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$ as in $V(G) \setminus (V_1 \cup \dots \cup V_{i-2})$, and our claim follows from the induction hypothesis.

If $v \in V_{i+1}$, then the neighbors of v are a subset of $V_1 \cup \dots \cup V_i \cup V_{i+2}'$. Hence, at least δ of these neighbors lie in $V_i \cup V_{i+2}'$. □

3.2.1 Trilayered graphs

We abstract out the properties of a triple of consecutive levels into the following definition. A *trilayered graph* with layers V_1, V_2, V_3 is a graph G on a vertex set $V_1 \cup V_2 \cup V_3$ such that the only edges in G are between V_1 and V_2, and between V_2 and V_3. If $V_1' \subset V_1$,

$V_2' \subset V_2$ and $V_3' \subset V_3$, then we denote by $G[V_1', V_2', V_3']$ the trilayered subgraph induced by three sets V_1', V_2', V_3'. Because the graph G that has been explored is bipartite, there are no edges inside each level. Therefore any three sets V_{i-1}, V_i, V_{i+1}' from the exploration process naturally form a trilayered graph; these graphs and their subgraphs are the only trilayered graphs that appear in this chapter.

We say that a trilayered graph has *minimum degree* at least $[A : B, C : D]$ if each vertex in V_1 has at least A neighbors in V_2, each vertex in V_2 has at least B neighbors in V_1, each vertex in V_2 has at least C neighbors in V_3, and each vertex in V_3 has at least D neighbors in V_2.

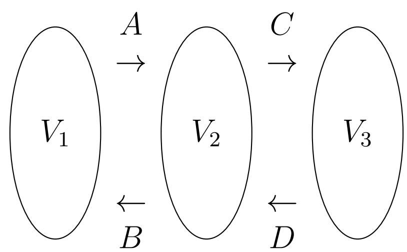

Figure 3.1: A schematic drawing of a tyilayered graph.

3.3 Θ-graphs

A Θ-*graph* is a cycle of length at least $2k$ with a chord. We shall use several lemmas from the previous works.

Lemma 3.4 (Lemma 2.1 in [Pik12], also Lemma 2 in [Ver00])**.** *Let F be a* Θ*-graph and $1 \leq l \leq |V(F)| - 1$. Let $V(F) = W \cup Z$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in F of length l that begins in W necessarily ends in W. Then F is bipartite with parts W and Z.*

Lemma 3.5 (Lemma 2.2 in [Pik12])**.** *Let $k \geq 3$. Any bipartite graph H of minimum degree at least k contains a* Θ*-graph.*

Corollary 3.6. *Let $k \geq 3$. Any bipartite graph H of average degree at least $2k$ contains a* Θ*-graph.*

For a graph G and a set $Y \subset V(G)$, let $G[Y]$ denote the graph induced on Y. For disjoint $Y, Z \subset V(G)$, let $G[Y, Z]$ denote the bipartite subgraph of G that is induced by the bipartition $Y \cup Z$.

3.3.1 Well-placed Θ-graphs

Suppose G is a trilayered graph with layers V_1, V_2, V_3. We say that a Θ-graph $F \subset G$ is *well-placed* if each vertex of $V(F) \cap V_2$ is adjacent to some vertex in $V_1 \setminus V(F)$. The condition ensures that, for each vertex v of F in V_2 there exists a path from the root x to v that avoids F.

Lemma 3.7. *Suppose G is a trilayered graph with layers V_1, V_2, V_3 such that the degree of every vertex in V_2 is at least $2d + 5k^2$, and no vertex in V_2 has more than Δd neighbors in V_3. Suppose t is a nonnegative integer, and let $F = d \cdot e(V_1, V_2)/8k\,|V_3|$. Assume that*

$$F \geq 2, \tag{3.2a}$$

$$e(V_1, V_2) \geq 2kF\,|V_1|\,, \tag{3.2b}$$

$$e(V_1, V_2) \geq 8k(t+1)^2(2\Delta k)^{2k-1}\,|V_1|\,, \tag{3.2c}$$

$$e(V_1, V_2) \geq 8(et/F)^t k\,|V_2|\,, \tag{3.2d}$$

$$e(V_1, V_2) \geq 20(t+1)^2\,|V_2|\,. \tag{3.2e}$$

Then at least one of the following holds:

1. *There is a Θ-graph in $G[V_1, V_2]$.*
2. *There is a well-placed Θ-graph in $G[V_1, V_2, V_3]$.*

The proof of Lemma 3.7 is in two parts: finding trilayered subgraph of large minimum degree (Lemmas 3.8 and 3.9), and finding a well-placed Θ-graph inside that trilayered graph (Lemma 3.10).

3.3.2 Finding a trilayered subgraph of large minimum degree

The disjoint union of two bipartite graphs shows that a trilayered graph with many edges need not contain a trilayered subgraph of large minimum degree. We show that, in contrast, if a trilayered graph contains no Θ-graph between two of its levels, then it must contain a subgraph of large minimum degree. The next lemma demonstrates a weaker version of this claim: it leaves open a possibility that the graph contains a denser trilayered subgraph. In that case, we can iterate inside that subgraph, which is done in Lemma 3.9.

Lemma 3.8. *Let a, A, B, C, D be positive real numbers. Suppose G is a trilayered graph with layers V_1, V_2, V_3 and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Assume also that*

$$a \cdot e(V_1, V_2) \geq (A + k + 1)\,|V_1| + B\,|V_2|\,. \tag{3.3}$$

Then one of the following holds:

1. There is a Θ-graph in $G[V_1, V_2]$.

2. There exist non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$.

3. There is a subset $\widetilde{V}_2 \subset V_2$ such that $e(V_1, \widetilde{V}_2) \geq (1-a)e(V_1, V_2)$, and $\left|\widetilde{V}_2\right| \leq D\,|V_3|\,/d$.

Proof. We suppose that alternative 1 does not hold. Then, by Corollary 3.6, the average degree of every subgraph of $G[V_1, V_2]$ is at most $2k$.

Consider the process that aims to construct a subgraph satisfying 2. The process starts with $V_1' = V_1$, $V_2' = V_2$ and $V_3' = V_3$, and at each step removes one of the vertices that violate the minimum degree condition on $G[V_1', V_2', V_3']$. The process stops when either no vertices are left, or the minimum degree of $G[V_1', V_2', V_3']$ is at least $[A : B, C : D]$. Since in the latter case we are done, we assume that this process eventually removes every vertex of G.

Let R be the vertices of V_2 that were removed because at the time of removal they had fewer than C neighbors in V_3'. Put

$$E' := \{uv \in E(G) : u \in V_2,\ v \in V_3,\ \text{and } v \text{ was removed before } u\},$$
$$S := \{v \in V_2 : v \text{ has at least } 4k^2 \text{ neighbors in } V_1\}.$$

Note that $|E'| \leq D\,|V_3|$. We cannot have $|S| \geq |V_1|\,/k$, for otherwise the average degree of the bipartite graph $G[V_1, S]$ would be at least $\frac{4k}{1+1/k} \geq 2k$. So $|S| \leq |V_1|\,/k$.

The average degree condition on $G[V_1, S]$ implies that

$$e(V_1, S) \leq k(|V_1| + |S|) \leq (k+1)\,|V_1|\,.$$

Let u be any vertex in $R \setminus S$. Since it is connected to at least $(d + 4k^2 + C) - 4k^2 = d + C$ vertices of V_3, it must be adjacent to at least d edges of E'. Thus,

$$|R \setminus S| \leq |E'|\,/d \leq D\,|V_3|\,/d.$$

Assume that the conclusion 3 does not hold with $\widetilde{V}_2 = R \setminus S$. Then $e(V_1, R \setminus S) < (1-a)e(V_1, V_2)$. Since the total number of edges between V_1 and V_2 that were removed due to the minimal degree conditions on V_1 and V_2 is at most $A\,|V_1|$ and $B\,|V_2|$ respectively, we conclude that

$$\begin{aligned} e(V_1, V_2) &\leq e(V_1, S) + e(V_1, R \setminus S) + A\,|V_1| + B\,|V_2| \\ &< (k+1)\,|V_1| + (1-a)e(V_1, V_2) + A\,|V_1| + B\,|V_2|\,, \end{aligned}$$

implying that

$$a \cdot e(V_1, V_2) < (A + k + 1)\,|V_1| + B\,|V_2|\,.$$

The contradiction with (3.3) completes the proof. $\square$

Remark 3.1. The next lemma can be eliminated at the cost of obtaining the bound $\mathrm{ex}(n, C_{2k}) = O(k^{2/3} n^{1+1/k})$ in place of $\mathrm{ex}(n, C_{2k}) = O(\sqrt{k \log k} \cdot n^{1+1/k})$. To do that, we can set $B \approx k^{2/3}$, $D \approx k^{1/3}$ and $a = 1/2$. One can then show that when applied to trilayered graphs arising from the exploration process the alternative 3 leads to a subgraph of average degree $2k$. The two remaining alternatives are dealt by Corollary 3.6 and Lemma 3.10. However, it is possible to obtain a better bound by iterating the preceding lemma.

Lemma 3.9. *Let C be a positive real number. Suppose G is a trilayered graph with layers V_1, V_2, V_3, and the degree of every vertex in V_2 is at least $d+4k^2+C$. Let $F = d \cdot e(V_1, V_2)/8k\,|V_3|$, and assume that F and $e(V_1, V_2)$ satisfy (3.2) for some integer $t \geq 1$. Then one of the following holds:*

1. *There is a Θ-graph in $G[V_1, V_2]$.*
2. *There exist numbers A, B, D and non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$, with the following inequalities that bind A, B, and D:*

$$B \geq 5, \quad (B-4)D \geq 2k, \quad A \geq 2k(\Delta D)^{D-1}. \tag{3.4}$$

Proof. Assume, for the sake of contradiction, that neither 1 nor 2 hold. With hindsight, set $a_j = \frac{1}{t-j+1}$ for $j = 0, \ldots, t-1$. We shall define a sequence of sets $V_2 = V_2^{(0)} \supseteq V_2^{(1)} \supseteq \cdots \supseteq V_2^{(t)}$ inductively. We denote by

$$d_i := e(V_1, V_2^{(i)}) / \left| V_2^{(i)} \right|$$

the average degree from $V_2^{(i)}$ into V_1. The sequence $V_2^{(0)}, V_2^{(1)}, \ldots, V_2^{(t)}$ will be constructed so as to satisfy

$$e(V_1, V_2^{(i+1)}) \geq (1-a_i) e(V_1, V_2^{(i)}), \tag{3.5}$$

$$d_{i+1} \geq d_i \cdot F a_i \prod_{j=0}^{i} (1-a_j). \tag{3.6}$$

Note that (3.5) and the choice of $a_0, \ldots, a_i$ imply that

$$e(V_1, V_2^{(i)}) \geq \frac{e(V_1, V_2)}{t+1}. \tag{3.7}$$

The sequence starts with $V_2^{(0)} = V_2$. Assume $V_2^{(i)}$ has been defined. We proceed to define $V_2^{(i+1)}$. Put

$$A = \frac{a_i e(V_1, V_2^{(i)})}{2\,|V_1|} - k - 1, \quad B = \tfrac{1}{4} a_i d_i + 5, \quad D = \min\left(2k, \frac{8k}{a_i d_i}\right).$$

With help of (3.7) and (3.2c) it is easy to check that the inequalities (3.4) hold for this choice of constants.

In addition,

$$
\begin{aligned}
(A+k+1)\left|V_1\right| + B\left|V_2^{(i)}\right| &= \tfrac{3}{4}a_i e(V_1, V_2^{(i)}) + 5\left|V_2^{(i)}\right| \\
&\overset{(3.2e)}{\leq} \tfrac{3}{4}a_i e(V_1, V_2^{(i)}) + \tfrac{1}{4(t+1)^2} e(V_1, V_2) \\
&\overset{(3.7)}{\leq} a_i e(V_1, V_2^{(i)}).
\end{aligned}
$$

So, the condition (3.3) of Lemma 3.8 is satisfied for the graph $G\left[V_1, V_2^{(i)}, V_3\right]$. By Lemma 3.8 there is a subset $V_2^{(i+1)} \subset V_2^{(i)}$ satisfying (3.5) and $\left|V_2^{(i+1)}\right| \leq D\left|V_3\right|/d$.

Next we show that the set $V_2^{(i+1)}$ satisfies inequality (3.6). Indeed, we have

$$
\begin{aligned}
d_{i+1} = \frac{e(V_1, V_2^{(i+1)})}{\left|V_2^{(i+1)}\right|} &\geq \frac{(1-a_i)e(V_1, V_2^{(i)})}{D\left|V_3\right|/d} \geq (1-a_i)a_i d_i \frac{d}{8k\left|V_3\right|} e(V_1, V_2^{(i)}) \\
&\overset{(3.5)}{\geq} (1-a_i)a_i d_i \frac{d \cdot e(V_1, V_2)}{8k\left|V_3\right|} \prod_{j=0}^{i-1}(1-a_j) = d_i \cdot F a_i \prod_{j=0}^{i}(1-a_j).
\end{aligned}
$$

Iterative application of (3.6) implies

$$
d_t \geq d_0 F^t \prod_{j=0}^{t-1} a_j (1-a_j)^{t-j} \geq d_0 F^t \prod_{j=0}^{t-1} \frac{e^{-1}}{t-j+1} = d_0 \frac{(F/e)^t}{(t+1)!}. \tag{3.8}
$$

If we have $\left|V_2^{(t)}\right| < \left|V_1\right|$, then the average degree of induced subgraph $G\left[V_1, V_2^{(t)}\right]$ is greater than

$$
e(V_1, V_2^{(t)})/\left|V_1\right| \overset{(3.7)}{\geq} e(V_1, V_2)/(t+1)\left|V_1\right| \overset{(3.2c)}{\geq} 2k,
$$

which by Corollary 3.6 leads to outcome 1.

If $\left|V_2^{(t)}\right| \geq \left|V_1\right|$ and $d_t \geq 4k$, then the average degree of $G\left[V_1, V_2^{(t)}\right]$ is at least $d_t/2 \geq 2k$ because d_t is the average degree of $V_2^{(t)}$ into V_1, again leading to the outcome 1. So, we may assume that $d_t < 4k$. Since $(t+1)! \leq 2t^t$ we deduce from (3.8) that

$$
d_0 < 4k(t+1)!(e/F)^t \leq 8k(et/F)^t.
$$

This contradicts (3.2d), and so the proof is complete. $\square$

3.3.3 Locating well-placed Θ-graphs in trilayered graphs

We come to the central argument of the chapter. It shows how to embed well-placed Θ-graphs into trilayered graphs of large minimum degree. Or rather, it shows how to embed well-placed Θ-graphs into regular trilayered graphs; the contortions of the previous two lemmas, and the factor of $\sqrt{\log k}$ in the final bound, come from authors' inability to deal with irregular graphs.

Lemma 3.10. *Let A, B, D be positive real numbers. Let G be a trilayered graph with layers V_1, V_2, V_3 of minimum degree at least $[A : B, d + k : D]$. Suppose that no vertex in V_2 has more than Δd neighbors in V_3. Assume also that*

$$B \geq 5 \tag{3.9}$$

$$(B-4)D \geq 2k-2 \tag{3.10}$$

$$A \geq 2k(\Delta D)^{D-1}. \tag{3.11}$$

Then G contains a well-placed Θ-graph.

Proof. Assume, for the sake of contradiction, that G contains no well-placed Θ-graphs. Leaning on this assumption, we shall build an arbitrary long path P of the form

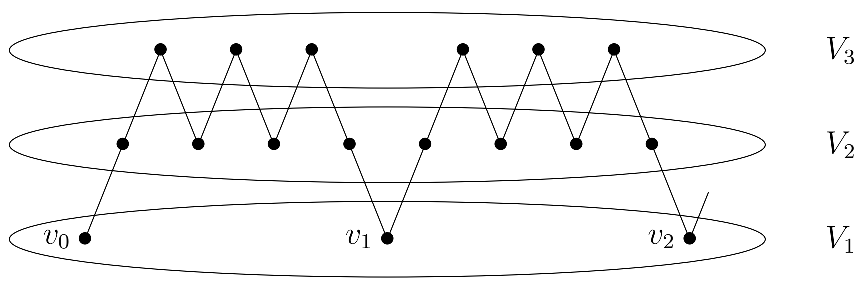

Figure 3.2: An arbitrary long path P.

where, for each i, vertices v_i and v_{i+1} are joined by a path of length $2D$ that alternates between V_2 and V_3. Since the graph is finite, this would be a contradiction.

While building the path, we maintain the following property:

$$\text{Every } v \in P \cap V_2 \text{ has at least one neighbor in } V_1 \setminus P. \tag{3.12}$$

We call a path satisfying (3.12) *good*.

We construct the path inductively. We begin by picking v_0 arbitrarily from V_1. Suppose a good path $P = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1}$ has been constructed, and we wish to find a path extension $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow v_l$.

There are at least about A ways to extend the path by a single vertex. The idea of the following argument shows that many of these extensions can be extended to another vertex, and then another, and so on.

For each $i = 1, 2, \dots, 2D-1$ we shall define a family $\mathcal{Q}_i$ of good paths that satisfy

1. Each path in $\mathcal{Q}_i$ is of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u$, where $v_{l-1} \leftrightsquigarrow u$ is a path of length i that alternates between V_2 and V_3. The vertex u is called a *terminal* of the path. The set of terminals of the paths in $\mathcal{Q}_i$ is denoted by $T(\mathcal{Q}_i)$. Note that $T(\mathcal{Q}_i) \subset V_2$ for odd i and $T(\mathcal{Q}_i) \subset V_3$ for even i.
2. For each i, the paths in $\mathcal{Q}_i$ have distinct terminals.
3. For odd-numbered indices, we have the inequality

$$|\mathcal{Q}_{2i+1}| \geq -3k + A\left(\frac{1}{\Delta}\right)^i \prod_{j \leq i}\left(1 - \frac{j}{D}\right). \tag{3.13}$$

4. For even-numbered indices, we have the inequality

$$e(T(\mathcal{Q}_{2i}), V_2) \geq d\,|\mathcal{Q}_{2i-1}|\,. \tag{3.14}$$

Let $t := \lceil B/2 \rceil$. We will repeatedly use the following straightforward fact, which we call the *small-degree argument*: whenever Q is a good path and $u \in V_2 \setminus Q$ is adjacent to the terminal of Q, then u is adjacent to fewer than t vertices in $V_1 \cap Q$. Indeed, if vertex u were adjacent to $v_{j_1}, v_{j_2}, \dots, v_{j_t} \in V_1 \cap Q$ with $j_1 < j_2 < \ldots < j_k$, then $v_{j_2} \leftrightsquigarrow u$ (along path Q) and the edge uv_{j_2} would form a cycle of total length at least

$$2D(t-2) + 2 \geq 2D(B/2 - 2) + 2 \overset{(3.10)}{\geq} 2k.$$

As uv_{j_3} is a chord of the cycle, and u is adjacent to v_{j_1} that is not on the cycle, that would contradict the assumption that G contains no well-placed Θ-graph.

The set $\mathcal{Q}_1$ consists of all paths of the form Pu for $u \in V_2 \setminus P$. Let us check that the preceding conditions hold for $\mathcal{Q}_1$. Vertex v_{l-1} cannot be adjacent to k or more vertices in $P \cap V_2$, for otherwise G would contain a well-placed Θ-graph with a chord through v_{l-1}. So, $|\mathcal{Q}_1| \geq A - k$. Next, consider any $u \in V_2 \setminus P$ that is a neighbor of v_{l-1}. By the small-degree argument vertex u cannot be adjacent to t or more vertices of $P \cap V_1$, and Pu is good.

Suppose $\mathcal{Q}_{2i-1}$ has been defined, and we wish to define $\mathcal{Q}_{2i}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i-1}$. Vertex u cannot have k or more neighbors in $Q \cap V_3$, for otherwise G would contain a well-placed Θ-graph with a chord through u. Hence, there are at least d edges of the form uw, where $w \in V_3 \setminus Q$. As we vary u we obtain a family of at least $d\,|\mathcal{Q}_{2i-1}|$ paths. We let $\mathcal{Q}_{2i}$ consist of any maximal subfamily of such paths with distinct terminals. The condition (3.14) follows automatically as each vertex of $T(\mathcal{Q}_{2i-1})$ has at least d neighbors in $T(\mathcal{Q}_{2i})$.

Suppose $\mathcal{Q}_{2i}$ has been defined, and we wish to define $\mathcal{Q}_{2i+1}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i}$. An edge uw is called *long* if $w \in P$, and w is at a distance exceeding $2k$ from u along path Q. If uw is a long edge, then from u to Q there is only one edge, namely the edge to the predecessor of u on Q, for otherwise there is a well-placed Θ-graph. Also, at most i neighbors of u lie on the path $v_{l-1} \leftrightsquigarrow u$. Since $\deg u \geq D$, it follows that there are at least $(1 - i/D) \deg u$ short edges from u that miss $v_{l-1} \leftrightsquigarrow u$. Thus there is a set $\mathcal{W}$ of at least $(1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)$ walks (not necessarily paths!) of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw$ such that $v_{l-1} \leftrightsquigarrow uw$ is a path and w occurs only among the last $2k$ vertices of the walk.

From the maximum degree condition on V_2 it follows that walks in $\mathcal{W}$ have at least $(1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d$ distinct terminals. A walk fails to be a path only if the terminal vertex lies on P. However, since the edge uw is short, this can happen for at most $2k$ possible terminals. Hence, there is a $\mathcal{Q}_{2i+1} \subset \mathcal{W}$ of size

$$|\mathcal{Q}_{2i+1}| \geq (1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d - 2k \tag{3.15}$$

that consists of paths with distinct terminals. It remains to check that every path in $\mathcal{Q}_{2i+1}$ is good. The only way that $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw \in \mathcal{Q}_{2i+1}$ may fail to be good is if w has no neighbors in $V_1 \setminus Q$. By the small-degree argument w has fewer than t neighbors in V_1. Since w has at least B neighbors in V_1 and $B \geq t + 2$, we conclude that w has at least *two* neighbors in V_1 outside the path. Of course, the same is true for every terminal of a path in $\mathcal{Q}_{2i+1}$. The condition (3.13) for $\mathcal{Q}_{2i+1}$ follows from (3.15), (3.14) and from validity of (3.13) for $\mathcal{Q}_{2i-1}$.

Note that $\mathcal{Q}_{2D-1}$ is non-empty. Let $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2D-1}$ be an arbitrary path. Note that since $2D - 1$ is odd, $u \in V_2$. By the property of terminals of V_i (odd i) that we noted in the previous paragraph, there are two vertices in $V_1 \setminus Q$ that are neighbors of u. Let v_l be any of them, and let the new path be $Qv_l = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uv_l$. This path can fail to be good if there is a vertex w on the path Q that is good in Q, but is bad in Qv_l. By the small-degree argument, w is adjacent to fewer than t vertices in $Q \cap V_1$ that precede w in Q. The same argument applied to the reversal of the path Qv_l shows that w is adjacent to fewer than t vertices in $Q \cap V_1$ that succeeds w in Q. Since $2t - 2 < B$, the path Qv_l is good.

Hence, it is possible to build an arbitrarily long path in G. This contradicts the finiteness of G. □

Lemma 3.7 follows from Lemmas 3.9 and 3.10 by setting $C = d + k$, in view of inequality $4k^2 + k \leq 5k^2$. We lose $k^2 - k$ here for cosmetic reason: $5k^2$ is tidier than $4k^2 + k$.

3.4 Proof of Theorem 3.2

Suppose that G is a bipartite graph of minimum degree at least $2d + 5k^2$ and contains no C_{2k}. Pick a root vertex x arbitrarily, and let $V_0, V_1, \dots, V_{k-1}$ be the levels obtained from the exploration process in Section 3.2.

Lemma 3.11. *For $1 \leq i \leq k-1$, the graph $G[V_{i-1}, V_i, V_{i+1}]$ contains no well-placed Θ-graph.*

Proof. The following proof is almost an exact repetition of the proof of Claim 3.1 from [Pik12] (which is also reproduced as Lemma 3.12 below).

Suppose, for the sake of contradiction, that a well-placed Θ-graph $F \subset G[V_{i-1}, V_i, V_{i+1}]$ exists. Let $Y = V_i \cap V(F)$. Since F is well-placed, for every vertex of Y there is a path avoiding $V(F)$ of length i to the vertex x. The union of these paths forms a tree T with x as a root. Let y be the vertex farthest from x such that every vertex of Y is a T-descendant of y. Paths that connect x to Y branch at y. Pick one such branch, and let $W \subset Y$ be the set of all the T-descendants of that branch. Let $Z = V(F) \setminus W$. From $W \neq V_i \cap V(F)$ it follows that Z is not an independent set of F, and so $W \cup Z$ is not a bipartition of F.

Let ℓ be the distance between x and y. We have $\ell < i$ and $2k - 2i + 2\ell < 2k \leq |V(F)|$. By Lemma 3.4 in F there is a path P of length $2k - 2i + 2\ell$ that starts at some $w \in W$ and ends in $z \in Z$. Since the length of P is even, $z \in Y$. Let P_w and P_z be unique paths in T that connect y to respectively w and z. They intersect only at y. Each of P_w and P_z has length $i - \ell$. The union of paths P, P_w, P_z forms a $2k$-cycle in G. □

The same argument (with a different Y) also proves the next lemma.

Lemma 3.12 (Claim 3.1 in [Pik12])**.** *For $1 \leq i \leq k-1$, neither of $G[V_i]$ and $G[V_i, V_{i+1}]$ contains a bipartite Θ-graph.*

The next step is to show that the levels $V_0, V_1, V_2, \dots$ increase in size. We shall show by induction on i that

$$e(V_i, V_{i+1}) \geq d\,|V_i|\,, \tag{3.16}$$

$$e(V_i, V_{i+1}) \leq 2k\,|V_{i+1}|\,, \tag{3.17}$$

$$e(V_i, V'_{i+1}) \leq 2k\,\left|V'_{i+1}\right|\,, \tag{3.18}$$

$$|V_{i+1}| \geq (2k)^{-1} d\,|V_i|\,, \tag{3.19}$$

$$|V_{i+1}| \geq \tfrac{d^2}{400k \log k}\,|V_{i-1}|\,. \tag{3.20}$$

To prove Theorem 3.2, we only need (3.20); the remaining inequalities play auxiliary roles in derivation of (3.20). Clearly, these inequalities hold for $i = 0$ since each vertex of V_1 sends only one edge to V_0.

Proof of (3.16). By Lemma 3.3 the degree of every vertex in V_i is at least $2d + 4k$, and so

$$e(V_i, V'_{i+1}) \geq (2d+4k)\,|V_i| - e(V_{i-1}, V_i) \overset{\text{induc.}}{\geq} (2d+2k)\,|V_i|\,.$$

We next distinguish two cases depending on whether V_{i+1} is big (in the sense of the definition from Section 3.2). If V_{i+1} is big, then $e(V_i, V_{i+1}) = e(V_i, V'_{i+1})$, and (3.16) follows. If V_{i+1} is normal, then Corollary 3.6 and Lemma 3.12 imply that

$$e(V_i, \mathrm{Bg}_{i+1}) \leq k(|V_i| + \left|\mathrm{Bg}_{i+1}\right|) \leq k\left(|V_i| + \tfrac{1}{2k}\left|V'_{i+1}\right|\right) \leq k\,|V_i| + \tfrac{1}{2}e(V_i, V'_{i+1})$$

and so

$$e(V_i, V_{i+1}) = e(V_i, V'_{i+1}) - e(V_i, \mathrm{Bg}_{i+1}) \geq \tfrac{1}{2}e(V_i, V'_{i+1}) - k\,|V_i| \geq d\,|V_i|$$

implying (3.16). □

Proof of (3.17). Consider the graph $G[V_i, V_{i+1}]$. Inequality (3.16) asserts that the average degree of V_i is at least $d \geq 2k$. If (3.17) does not hold, then the average degree of V_{i+1} is at least $2k$ as well, contradicting Corollary 3.6 and Lemma 3.12. □

Proof of (3.18). The argument is the same as for (3.17) with $G[V_i, V'_{i+1}]$ in place of $G[V_i, V_{i+1}]$. □

Proof of (3.19). This follows from (3.17) and (3.16). □

Proof of (3.20) *in the case V_i is a normal level.* We assume that (3.20) does not hold and will derive a contradiction. Consider the trilayered graph $G[V_{i-1}, V_i, V'_{i+1}]$. Let $t = 2\log k$. Suppose momentarily that the inequalities (3.2) in Lemma 3.7 hold. Then since V_i is normal, each vertex in V_i has at most Δd neighbors in V'_{i+1}, and so Lemma 3.7 applies. However, the lemma's conclusion contradicts Lemmas 3.11 and 3.12. Hence, to prove (3.20) it suffices to verify inequalities (3.2a–3.2d) with $F = d \cdot e(V_{i-1}, V_i)/8k\left|V'_{i+1}\right|$.

We may assume that

$$F \geq 2e^2 \log k, \tag{3.21}$$

and in particular that (3.2a) holds. Indeed, if (3.21) were not true, then inequality (3.16) would imply $\left|V'_{i+1}\right| \geq (d^2/16e^2 k \log k)\,|V_{i-1}|$, and thus

$$|V_{i+1}| \geq (1 - \tfrac{1}{k})\left|V'_{i+1}\right| \geq (d^2/32e^2 k \log k)\,|V_{i-1}|\,,$$

and so (3.20) would follow in view of $32e^2 \leq 400$.

Inequality (3.2b) is implied by (3.19). Indeed,

$$e(V_{i-1}, V_i) = 8k \left|V'_{i+1}\right| F/d \geq 8k \left|V_{i+1}\right| F/d \overset{(3.19)}{\geq} 4F \left|V_i\right| \overset{(3.19)}{\geq} 2k^{-1}dF \left|V_{i-1}\right|,$$

and $d \geq k^2$ by the definition of d from (3.1).

Inequality (3.2c) is implied by (3.1) and (3.16).

Next, suppose (3.2d) were not true. Since $F/t \geq e^2$ by (3.21), we would then conclude

$$\begin{aligned} \left|V_{i+1}\right| &\overset{(3.19)}{\geq} (2k)^{-1}d \left|V_i\right| \geq d(16k^2)^{-1}(F/et)^t e(V_{i-1}, V_i) \\ &\geq d(16k^2)^{-1}e^{2\log k} e(V_{i-1}, V_i) \overset{(3.16)}{\geq} \tfrac{1}{16}d^2 \left|V_{i-1}\right|, \end{aligned}$$

and so (3.20) would follow.

Finally, (3.2e) is a consequence of (3.16). □

Proof of (3.20) *in the case V_i is a big level.* We have

$$\begin{aligned} \left|V_{i+1}\right| &\geq \tfrac{1}{2}\left|V'_{i+1}\right| \overset{(3.18)}{\geq} (4k)^{-1}e(V_i, V'_{i+1}) \geq (4k)^{-1}e(\mathrm{Bg}_i, V'_{i+1}) \geq (4k)^{-1}\Delta d \left|\mathrm{Bg}_i\right| \\ &\geq (8k^2)^{-1}\Delta d \left|V_i\right| \overset{(3.19)}{\geq} (16k^3)^{-1}\Delta d^2 \left|V_{i-1}\right| = \tfrac{1}{16}d^2 \left|V_{i-1}\right|, \end{aligned}$$

and so (3.20) holds. □

Proof of Theorem 3.2. If k is even, then $k/2$ applications of (3.20) yield

$$\left|V_k\right| \geq \frac{d^k}{(400k\log k)^{k/2}}.$$

If k is odd, then $(k-1)/2$ applications of (3.20) yield

$$\left|V_k\right| \geq \frac{d^{k-1}}{(400k\log k)^{(k-1)/2}} \left|V_1\right| \geq \frac{d^k}{(400k\log k)^{(k-1)/2}}.$$

Either way, since $\left|V_k\right| < n$ we conclude that $d < 20\sqrt{k\log k} \cdot n^{1/k}$. □

Chapter 4

Bipartite algebraic graphs without quadrilaterals

4.1 Introduction

The Turán number $\mathrm{ex}(n, F)$ is the maximum number of edges in an F-free graph[1] on n vertices. The first systematic study of $\mathrm{ex}(n, F)$ was initiated by Turán [Tur41], who solved the case when $F = K_t$ is a complete graph on t vertices. Turán's theorem states that, on a given vertex set, the K_t-free graph with the most edges is the complete and balanced $(t-1)$-partite graph, in that the part sizes are as equal as possible.

For general graphs F, we still do not know how to compute the Turán number exactly, but if we are satisfied with an approximate answer, the theory becomes quite simple: Erdős and Stone [ES46] showed that if the chromatic number $\chi(F) = t$, then $\mathrm{ex}(n, F) = \mathrm{ex}(n, K_t) + o(n^2) = \left(1 - \frac{1}{t-1}\right)\binom{n}{2} + o(n^2)$. When F is not bipartite, this gives an asymptotic result for the Turán number. On the other hand, for all but few bipartite graphs F, the order of $\mathrm{ex}(n, F)$ is not known. Most of the research on this problem focused on two classes of graphs: complete bipartite graphs and cycles of even length. A comprehensive survey is by Füredi and Simonovits [FS13].

Suppose G is a $K_{s,t}$-free graph with $s \le t$. The Kövari–Sós–Turán theorem [KST54] implies an upper bound $\mathrm{ex}(n, K_{s,t}) \le \frac{1}{2}\sqrt[s]{t-1} \cdot n^{2-1/s} + o(n^{2-1/s})$, which was improved by Füredi [Für96b] to

$$\mathrm{ex}(n, K_{s,t}) \le \frac{1}{2}\sqrt[s]{t-s+1} \cdot n^{2-1/s} + o(n^{2-1/s}).$$

Despite the lack of progress on the Turán problem for complete bipartite graphs, there are certain complete bipartite graphs for which the problem has been solved asymptotically, or

[1] We say a graph is F-free if it does not have a subgraph isomorphic to F.

even exactly. The constructions that match the upper bounds in these cases are all similar to one another. Each of the constructions is a bipartite graph G based on an algebraic hypersurface[2] H. Both partite sets of G are $\mathbb{F}_p^s$ and the edge set is defined by: $\overline{u} \sim \overline{v}$ if and only if $(\overline{u}, \overline{v}) \in H$. In short, $G = \big(\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p)\big)$, where $H(\mathbb{F}_p)$ denotes the $\mathbb{F}_p$-points of H. Note that G has $n := 2p^s$ vertices.

In the previous works of Erdős, Rényi and Sós [ERS66], Brown [Bro66], Füredi [Für96a], Kollár, Rónyai and Szabó [KRS96] and Alon, Rónyai and Szabó [ARS99], various hypersurfaces were used to define $K_{s,t}$-free graphs. Their equations were

$$x_1y_1 + x_2y_2 = 1, \quad \text{for } K_{2,2}; \tag{4.1a}$$

$$(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 = 1, \quad \text{for } K_{3,3}; \tag{4.1b}$$

$$(N_s \circ \pi_s)(x_1 + y_1, x_2 + y_2, \dots, x_s + y_s) = 1, \quad \text{for } K_{s,t} \text{ with } t \geq s! + 1; \tag{4.1c}$$

$$(N_{s-1} \circ \pi_{s-1})(x_2 + y_2, x_3 + y_3, \dots, x_s + y_s) = x_1y_1, \quad \text{for } K_{s,t} \text{ with } t \geq (s-1)! + 1, \tag{4.1d}$$

where $\pi_s \colon \mathbb{F}_p^s \to \mathbb{F}_{p^s}$ is an $\mathbb{F}_p$-linear isomorphism and $N_s(\alpha)$ is the field norm, $N_s(\alpha) := \alpha^{(p^s-1)/(p-1)}$.

Clearly, the coefficients in (4.1a) and (4.1b) are integers and even independent of p. With some work, one can show that both (4.1c) and (4.1d) are polynomial equations of degree $\leq s$ with coefficients in $\mathbb{F}_p$. Therefore each equation in (4.1) can be written as $F(\overline{x}, \overline{y}) := F(x_1, \dots, x_s, y_1, \dots, y_s) = 0$ for some $F(\overline{x}, \overline{y}) \in \mathbb{F}_p[\overline{x}, \overline{y}]$ of bounded degree. The previous works directly count the number of $\mathbb{F}_p$ solutions to $F(\overline{x}, \overline{y}) = 0$ and yield $|H(\mathbb{F}_p)| = \Theta(p^{2s-1}) = \Theta(n^{2-1/s})$, for each prime[3] p.

Definition 1. *Given two sets P_1 and P_2, a set $V \subset P_1 \times P_2$ is said to contain an (s,t)-grid if there exist $S \subset P_1, T \subset P_2$ such that $s = |S|$, $t = |T|$ and $S \times T \subset V$. Otherwise, we say that V is (s,t)-grid-free.*

Observe that every $F(\overline{x}, \overline{y})$ derived from (4.1) is symmetric in x_i and y_i for all i. We know that $(\overline{u}, \overline{v}) \in H$ if and only if $(\overline{v}, \overline{u}) \in H$ for all $\overline{u}, \overline{v} \in \mathbb{F}_p^s$. The resulting bipartite graph $G = \big(\mathbb{F}_p^s, \mathbb{F}_p^s, H(\mathbb{F}_p)\big)$ would be an extremal $K_{s,t}$-free graph if $H(\mathbb{F}_p)$ had been (s,t)-grid-free.

So which graphs are $K_{s,t}$-free with a maximum number of edges? The question was considered by Zoltán Füredi in his unpublished manuscript [Für88] asserting that every $K_{2,2}$-free graph with q vertices (for $q \geq q_0$) and $\frac{1}{2}q(q+1)^2$ edges is obtained from a projective

[2]An algebraic hypersurface in a space of dimension n is an algebraic subvariety of dimension $n-1$. The terminology from algebraic geometry used throughout the article is standard, and can be found in [Sha13].

[3]We need $p \equiv 3 \pmod 4$ for (4.1b) to get the correct number of $\mathbb{F}_p$ points on H. If $p \equiv 1 \pmod 4$, then the right hand side of (4.1b) should be replaced by a quadratic non-residue in $\mathbb{F}_p$.

plane via a polarity with $q+1$ absolute elements. This loosely amounts to saying that all extremal $K_{2,2}$-free graphs are defined by generalization of (4.1a).

However, classification of all extremal $K_{s,t}$-free graphs seems out of reach. We restrict our attention to algebraically constructed graphs. Given a field $\mathbb{F}$ and a hypersurface H defined over $\mathbb{F}$, it is natural to ask when $H(\mathbb{F})$ is (s,t)-grid-free. Because the general case is difficult, we work with algebraically closed fields $\mathbb{K}$ in this chapter. Denote by $\mathbb{P}^s(\mathbb{K})$ the s-dimensional projective space over $\mathbb{K}$. We are interested in hypersurface H in $\mathbb{P}^s(\mathbb{K})\times\mathbb{P}^s(\mathbb{K})$.

Since standard machinery from model theory, to be discussed in Section 4.5, allows us to transfer certain results over $\mathbb{C}$ (the field of complex numbers) to algebraically closed fields of large characteristic, our focus will be on the $\mathbb{K}=\mathbb{C}$ case. We use $\mathbb{P}^s$ for the s-dimensional complex projective space and $\mathbb{A}^s := \mathbb{P}^s \setminus \{x_0 = 0\}$ for the s-dimensional complex affine space.

Note that even if H contains (s,t)-grids, one may remove a few points from the projective space to destroy all (s,t)-grids in H. For example, the homogenization of (4.1b) is

$$(x_1y_0 - x_0y_1)^2 + (x_2y_0 - x_0y_2)^2 + (x_3y_0 - x_0y_3)^2 = x_0^2y_0^2.$$

The equation defines hypersurface H in $\mathbb{P}^3 \times \mathbb{P}^3$. Let $V := \{x_0 = x_1^2 + x_2^2 + x_3^2 = 0\}$ be a variety in $\mathbb{P}^3$. Since $V \times \mathbb{P}^3 \subset H$, H contains a lot of $(3,3)$-grids. However, $H \cap (\mathbb{A}^2 \times \mathbb{A}^2)$ is $(3,3)$-grid-free.

Definition 2. *A set $V \subset \mathbb{P}^s \times \mathbb{P}^s$ is almost-(s,t)-grid-free if there are two nonempty Zariski-open sets $X, Y \subset \mathbb{P}^s$ such that $V \cap (X \times Y)$ is (s,t)-grid-free.*

Suppose the defining equation of H, say $F(\overline{x},\overline{y})$, is of low degree in $\overline{y}$. Heuristically, for generic distinct $\overline{u}_1,\ldots,\overline{u}_s \in \mathbb{P}^s$, by Bézout's theorem, one would expect $\{F(\overline{u}_1,\overline{y}) = \cdots = F(\overline{u}_s,\overline{y}) = 0\}$ to have few points. So we conjecture the following.

Informal conjecture. *Every almost-(s,t)-grid-free hypersurface is equivalent, in a suitable sense, to a hypersurface whose degree in $\overline{y}$ is bounded by some constant $d := d(s,t)$.*

The right equivalence notion depends on X and Y in Definition 2. We shall discuss possible notions of equivalence in Section 4.2, and make three specific conjectures. Results in support of these conjectures can be found in Section 4.3 and Section 4.4.

Before we make our conjectures precise, we note that an analogous situation occurs for C_{2t}-free graphs. The upper bound $\mathrm{ex}(n, C_{2t}) = O(n^{1+1/t})$ first established by Bondy–Simonovits [BS74] has been matched only for $t = 2,3,5$. The $t = 2$ case was already mentioned above because $C_4 = K_{2,2}$. The constructions for $t = 3,5$ are also algebraic (see [Ben66, FNV06] for $t = 3$ and [Ben66, Wen91] for $t = 5$). Also, a conjecture in a similar spirit about algebraic graphs of girth eight was made by Dmytrenko, Lazebnik and Williford [DLW07]. It was recently resolved by Hou, Lappano and Lazebnik [HLL15].

The chapter is organized as follows. In Section 4.2 we flesh out the informal conjecture above, in Section 4.3 we briefly discuss the $s = 1$ case, in Section 4.4 we partially resolve the $s = t = 2$ case, and finally in Section 4.5, we consider algebraically closed fields of large characteristic.

4.2 Conjectures on the (s,t)-grid-free case

Given a field $\mathbb{F}$, we denote by $\mathbb{F}[\overline{x}]$ the set of homogeneous polynomials in $\mathbb{F}[\overline{x}]$ and by $\mathbb{F}_{\text{hom}}[\overline{x},\overline{y}]$ the set of polynomials in $\mathbb{F}[\overline{x},\overline{y}]$ that are separately homogeneous in $\overline{x}$ and $\overline{y}$.

We might be tempted to guess the following instance of the informal conjecture.

False conjecture A. *If H is almost-(s,t)-grid-free, then there exists $F(\overline{x},\overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x},\overline{y}]$ of degree $\le d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H = \{F = 0\}$.*

Unfortunately, Conjecture A is false because of the following example.

Example 4.1. Consider $H_0 := \{x_0y_0 + x_1y_1 + x_2y_2 = 0\}$ and H_1 defined by

$$x_0y_0^d + x_1y_0^{d-1}y_1 + x_2\left(y_0^{d-1}y_2 + y_0^d f(y_1/y_0)\right) = 0, \tag{4.2}$$

where f is a polynomial of degree d. One can check that both H_0 and $H_1 \setminus \{y_0 = 0\}$ are $(2,2)$-grid-free, whereas equation (4.2) can be of arbitrary large degree in $\overline{y}$.

Behind Example 4.1 is the birational automorphism $\sigma\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$\sigma(y_0 : y_1 : y_2) := \left(y_0^d : y_0^{d-1}y_1 : y_0^{d-1}y_2 + y_0^d f(y_1/y_0)\right).$$

Note that $\mathrm{id}\times\sigma$ is a biregular map[4] from $H_1\setminus\{y_0 = 0\}$ to $H_0\setminus\{y_0 = 0\}$. Composition with the automorphism increased the degree of H_0 in $\overline{y}$ while preserving almost-$(2,2)$-grid-freeness. Here is another example illustrating the relationship between birational automorphisms and (s,t)-grid-free hypersurfaces.

Example 4.2. Define $H_2 := \{x_0y_1y_2 + x_1y_0y_2 + x_2y_0y_1 = 0\}$. One can also check that $H_2 \setminus \{y_0y_1y_2 = 0\}$ is $(2,2)$-grid-free. Behind this example is *the standard quadratic transformation* σ from $\mathbb{P}^2$ to itself given by $\sigma(y_0 : y_1 : y_2) = (y_1y_2 : y_0y_2 : y_0y_1)$. Note that $\mathrm{id}\times\sigma$ is a biregular map from $H_2 \setminus \{y_0y_1y_2 = 0\}$ to $H_0 \setminus \{y_0y_1y_2 = 0\}$.

Let $\mathrm{Cr}(\mathbb{P}^s)$ be the group of birational automorphisms on $\mathbb{P}^s$, also known as *the Cremona group*. Evidently, the almost-(s,t)-grid-freeness is invariant under $\mathrm{Cr}(\mathbb{P}^s) \times \mathrm{Cr}(\mathbb{P}^s)$.

[4] A biregular map is a regular map whose inverse is also regular.

Proposition 4.1. *If $V_1 \subset \mathbb{P}^s \times \mathbb{P}^s$ is an almost-(s,t)-grid-free set, then so is $V_2 := (\sigma_X \times \sigma_Y)V_1$ for all $\sigma_X, \sigma_Y \in \mathrm{Cr}(\mathbb{P}^s)$.*

Remark 4.1. Though little is known about the structure of the Cremona group in 3 dimensions and higher, the classical Noether–Castelnuovo theorem says that the Cremona group $\mathrm{Cr}(\mathbb{P}^2)$ is generated by the group of projective linear transformations and the standard quadratic transformation. The proof of this theorem, which is very delicate, can be found in [AC02, Chapter 8].

We say that sets $V_1, V_2 \subset \mathbb{P}^s \times \mathbb{P}^s$ are *almost equal* if there exist nonempty Zariski-open sets $X, Y \subset \mathbb{P}^s$ such that $V_1 \cap (X \times Y) = V_2 \cap (X \times Y)$. We believe that the only obstruction to Conjecture A is the Cremona group.

Conjecture B. *Suppose H is an irreducible hypersurface in $\mathbb{P}^s \times \mathbb{P}^s$. If H is almost-(s,t)-grid-free, then there exist $\sigma \in \mathrm{Cr}(\mathbb{P}^s)$ and $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\mathrm{hom}}[\overline{x}, \overline{y}]$ of degree $\le d$ in $\overline{y}$ for some $d = d(s,t)$ such that H is almost equal to $\{F \circ (\mathrm{id} \times \sigma) = 0\}$.*

Remark 4.2. The conjecture is false if the irreducibility of H is dropped. Take H_0 and H_1 from Example 4.1 and set $f(y) = y^d$ in (4.2), where d can be arbitrarily large. Because both H_0 and H_1 are almost-$(2,2)$-grid-free, we know that $H_0 \cup H_1$ is almost-$(2,3)$-grid-free. However, one can show[5] that for any $\sigma \in \mathrm{Cr}(\mathbb{P}^s)$, the degree of $(\mathrm{id} \times \sigma)(H_0 \cup H_1)$ in $\overline{y}$ is $\ge d$.

In fact, we believe in an even stronger conjecture.

Conjecture C. *Suppose H is an irreducible hypersurface in $\mathbb{P}^s \times \mathbb{P}^s$. Let X, Y be nonempty Zariski-open subsets of $\mathbb{P}^s$. If $H \cap (X \times Y)$ is (s,t)-grid-free, then there exist $Y' \subset \mathbb{P}^s$, a biregular map $\sigma \colon Y \to Y'$ and $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\mathrm{hom}}[\overline{x}, \overline{y}]$ of degree $\le d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H \cap (X \times Y) = \{F \circ (\mathrm{id} \times \sigma) = 0\} \cap (X \times Y)$.*

We prove Conjecture C if $s = 1$ and if $s = t = 2, Y = \mathbb{P}^2$ (see Section 4.3 and 4.4 respectively).

One special case is when $H \cap (\mathbb{A}^s \times \mathbb{A}^s)$ is (s,t)-grid-free. In this case, H can be seen as an affine algebraic hypersurface in $2s$-dimensional affine space. The group of automorphisms of $\mathbb{A}^s$, denoted by $\mathrm{Aut}(\mathbb{A}^s)$, is a subgroup of the Cremona group. In this special case, we make a stronger conjecture.

[5]Let $R_1 = P_1/Q_1, R_2 = P_2/Q_2$ be rational functions such that $\sigma^{-1} = (1 : R_1 : R_2)$ and set $d_1 = \deg P_1 = \deg Q_1, d_2 = \deg P_2 = \deg Q_2$. On the one hand, $H_0' := (\mathrm{id} \times \sigma)H_0$ is defined by $x_0 + x_1 R_1 + x_2 R_2 = 0$, and so $\deg_{\overline{y}} H_0' \ge d_2$. On the other hand, H_1' is defined by $x_0 + x_1 R_1 + x_2(R_2 + R_1^d) = 0$ and so $\deg_{\overline{y}} H_1' = dd_1 + d_2 - \deg G$, where $G = \gcd\left(Q_1^d Q_2, P_1 Q_1^{d-1} Q_2, Q_1^d P_2 + P_1^d Q_2\right)$. Since $G = \gcd\left(Q_1^{d-1} Q_2, Q_1^d P_2 + P_1^d Q_2\right)$, it follows that G divides Q_2^2. Hence we estimate that $\deg G \le 2d_2$ and $\deg_{\overline{y}} H_1' \ge dd_1 - d_2 \ge d - d_2$. So, $\deg_{\overline{y}}(H_0' \cup H_1') \ge d$.

Conjecture D. *Suppose H is an irreducible affine hypersurface in $\mathbb{A}^s \times \mathbb{A}^s$. If H is (s,t)-grid-free, then there exist $\sigma \in \operatorname{Aut}(\mathbb{A}^s)$ and $F(\overline{x},\overline{y}) \in \mathbb{C}[\overline{x},\overline{y}]$ of degree $\leq d$ in $\overline{y}$ for some $d = d(s,t)$ such that $H = \{F \circ (\mathrm{id} \times \sigma) = 0\}$.*

Remark 4.3. An automorphism $\sigma \in \operatorname{Aut}(\mathbb{A}^s)$ is *elementary* if it has a form

$$\sigma\colon (x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_s) \mapsto (x_1,\dots,x_{i-1},cx_i+f,x_{i+1},\dots,x_s),$$

where $0 \neq c \in \mathbb{C}, f \in \mathbb{C}[x_1,\dots,x_{i-1},x_{i+1},\dots,x_s]$. The *tame subgroup* is the subgroup of $\operatorname{Aut}(\mathbb{A}^s)$ generated by all the elementary automorphisms, and the elements from this subgroup are called *tame automorphisms*, while non-tame automorphisms are called *wild*. In Example 4.1, we used a tame automorphism to make a counterexample to Conjecture A. It is known [Jun42, vdK53] that all the elements of $\operatorname{Aut}(\mathbb{A}^2)$ are tame. However, in the case of 3 dimensions, the following automorphism constructed by Nagata (see [Nag72]):

$$\sigma(x,y,z) = \big(x + (x^2-yz)z, y + 2(x^2-yz)x + (x^2-yz)z, z\big)$$

was shown [SU03, SU04] to be wild. See also [Kur10]. We note that the question on the existence of wild automorphisms remains open for higher dimensions.

4.3 Results on the $(1,t)$-grid-free case

As for the $s = 1$ case, one is able to fully characterize $(1,t)$-grid-free hypersurfaces. We always assume that H is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ and X, Y are nonempty Zariski-open subsets of $\mathbb{P}^1$ throughout this section.

Theorem 4.2. *Suppose $H = \{F = 0\}$, where $F(\overline{x},\overline{y}) \in \mathbb{C}_{\mathrm{hom}}[\overline{x},\overline{y}]$. Let*

$$F(\overline{x},\overline{y}) = f(\overline{x})g(\overline{y})h_1(\overline{x},\overline{y})^{r_1}h_2(\overline{x},\overline{y})^{r_2}\dots h_n(\overline{x},\overline{y})^{r_n} \tag{4.3}$$

be the factorization of F such that $h_1,h_2,\dots,h_n$ are distinct irreducible polynomials depending on both $\overline{x}$ and $\overline{y}$. Let d_i be the degree of h_i in $\overline{y}$. Then $H \cap (X \times Y)$ is $(1,t)$-grid-free if and only if $\{f = 0\} \cap X = \emptyset$ and $|\{g = 0\} \cap Y| + d_1 + d_2 + \dots + d_n < t$.

Proof. Clearly, if $H \cap (X \times Y)$ is $(1,t)$-grid-free, then $\{f = 0\} \cap X$ is empty. For fixed $\overline{u} \in \mathbb{P}^1$, consider the following $n+n+n+n+\binom{n}{2}$ systems of equations in $\overline{y}$:

$$\begin{aligned}
\deg h_i(\overline{u},\overline{y}) < d_i, &\quad i = 1,2,\dots,n;\\
\overline{y} \in \mathbb{P}^1 \setminus Y \text{ and } h_i(\overline{u},\overline{y}) = 0, &\quad i = 1,2,\dots,n;\\
h_i(\overline{u},\overline{y}) = \partial_y h_i(\overline{u},\overline{y}) = 0, &\quad i = 1,2,\dots,n;\\
h_i(\overline{u},\overline{y}) = g(\overline{y}) = 0, &\quad i = 1,2,\dots,n;\\
h_i(\overline{u},\overline{y}) = h_j(\overline{u},\overline{y}) = 0, &\quad i \neq j.
\end{aligned}$$

Since h_i's are irreducible and distinct, Bézout's theorem tells us that each of these systems has no solution in $\mathbb{P}^1$ for a generic $\overline{u}$. So for a generic $\overline{u} \in \mathbb{P}^1$, $F(\overline{u}, \overline{y}) = 0$ has exactly $M := |\{g = 0\} \cap Y| + d_1 + d_2 + \cdots + d_n$ distinct solutions in Y. The conclusion follows as M is the maximal number of distinct solutions. □

The informal conjecture thus holds when $s = 1$ as Theorem 4.2 implies:

Corollary 4.3. *If $H \cap (X \times Y)$ is $(1,t)$-grid-free, then there exists $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}]$ of degree $< t$ in $\overline{y}$ such that $H \cap (X \times Y) = \{F = 0\} \cap (X \times Y)$.*

Proof. Let $H = \{F = 0\}$, and let f, g and $h_1, h_2, \ldots, h_n$ be the factors of F as in (4.3). Suppose $m := |\{g = 0\} \cap Y|$ and $\{g = 0\} \cap Y = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_m\}$. Let $g_i(\overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{y}]$ be linear such that $\{g_i = 0\} = \{\overline{v}_i\}$. By Theorem 4.2,

$$\tilde{F}(\overline{x}, \overline{y}) := g_1(\overline{y})g_2(\overline{y}) \ldots g_m(\overline{y})h_1(\overline{x}, \overline{y})h_2(\overline{x}, \overline{y}) \ldots h_n(\overline{x}, \overline{y})$$

is of degree $m + d_1 + d_2 + \ldots d_n < t$ in $\overline{y}$. Clearly, $\tilde{F} = F$ on $X \times Y$. □

Conjectures B, C and D follow from the corollary in the $s = 1$ case. The birational map σ becomes trivial in those conjectures since $\mathrm{Cr}\,(\mathbb{P}^1)$ consists only of projective linear transformations.

4.4 Results on the $(2, 2)$-grid-free case

Throughout the section we assume that H is a hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ and X is a nonempty Zariski-open subset of $\mathbb{P}^2$.

Theorem 4.4. *If $H \cap (X \times \mathbb{P}^2)$ is $(2, 2)$-grid-free, then there exists $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}]$ of degree ≤ 2 in $\overline{y}$ such that $H \cap (X \times \mathbb{P}^2) = \{F = 0\} \cap (X \times \mathbb{P}^2)$.*

The theorem resolves Conjecture C for $s = t = 2, Y = \mathbb{P}^2$. Note that the birational map $\sigma : Y \to Y'$ in the conjecture becomes trivial since biregular automorphisms of $\mathbb{P}^2$ are linear.

Our argument uses a reduction to an intersection problem of plane algebraic curves. The key ingredient is a theorem by Moura [Mou04] on the intersection multiplicity of plane algebraic curves.

Theorem 4.5 (Moura [Mou04]). *Denote by $I_{\overline{v}}(C_1, C_2)$ the intersection multiplicity of algebraic curves C_1 and C_2 at $\overline{v}$. For a generic point $\overline{v}$ on an irreducible algebraic curve C_1 of degree d_1 in $\mathbb{P}^2$,*

$$\max_{C_2} \{I_{\overline{v}}(C_1, C_2) : C_1 \not\subset C_2, \deg C_2 \leq d_2\} = \begin{cases} \frac{1}{2}(d_2^2 + 3d_2) & \text{if } d_1 > d_2; \\ d_1 d_2 - \frac{1}{2}(d_1^2 - 3d_1 + 2) & \text{if } d_1 \leq d_2. \end{cases}$$

Corollary 4.6. *For a generic point $\overline{v}$ on an algebraic curve C in $\mathbb{P}^2$, any algebraic curve C' with $\overline{v} \in C'$ intersects with C at another point unless C is irreducible of degree ≤ 2.*

Proof. Suppose C has more than one irreducible components. Let C_1 and C_2 be any two of them. Since $C_1 \cap C_2$ is finite, we can pick a generic point $\overline{v}$ on $C_1 \setminus C_2$. Now any algebraic curve C' containing $\overline{v}$ intersects C at another point on C_2. So, C is irreducible.

Let d and d' be the degrees of C and C' respectively. By Theorem 4.5, one can check that $I_{\overline{v}}(C, C') < dd'$ for a generic point $\overline{v} \in C$ for all $d > 2$. From Bézout's theorem, we deduce that C intersects C' at another point unless $d \leq 2$. $\square$

In our proof of Theorem 4.4, we think of H as a family of algebraic curves in $\mathbb{P}^2$, each of which is indexed by $\overline{u} \in X$ and is defined by $C(\overline{u}) := \{\overline{v} \in \mathbb{P}^2 : (\overline{u}, \overline{v}) \in H\}$. We call algebraic curve $C(\overline{u})$ the *section* of H at $\overline{u}$. A hypersurface H is $(2,2)$-grid-free if and only if $C(\overline{u})$ and $C(\overline{u}')$ intersect at ≤ 1 point for all distinct $\overline{u}, \overline{u}' \in X$. The last piece that we need for our proof is a technical lemma on generic sections of irreducible hypersurfaces.

Lemma 4.7. *Suppose H_1 and H_2 are two different irreducible hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$ defined by $h_1(\overline{x}, \overline{y}), h_2(\overline{x}, \overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}] \setminus (\mathbb{C}_{\text{hom}}[\overline{x}] \cup \mathbb{C}_{\text{hom}}[\overline{y}])$ respectively. Denote the section of H_i at $\overline{u}$ by $C_i(\overline{u})$ for $i = 1, 2$. For generic $\overline{u} \in \mathbb{P}^2$, $C_1(\overline{u})$ and $C_2(\overline{u})$ share no common irreducible components, and moreover, each $C_i(\overline{u})$ is a reduced[6] algebraic curve.*

Proof of Theorem 4.4 assuming Lemma 4.7. Suppose $H \cap (X \times \mathbb{P}^2)$ is $(2,2)$-grid-free. Take an arbitrary $\overline{u} \in X$ and consider algebraic curve $C(\overline{u})$ in $\mathbb{P}^2$. We claim that every $\overline{v} \in C(\overline{u})$ is an intersection of $C(\overline{u})$ and $C(\overline{u}')$ for some $\overline{u}' \in X \setminus \{\overline{u}\}$. Define $D(\overline{v}) := \{F(\overline{x}, \overline{v}) = 0\} \cap X$. Since $\mathbb{P}^2 \setminus X$ is Zariski-closed, the set $D(\overline{v})$ is either empty or infinite. However, $\overline{u} \in D(\overline{v})$ and the claim is equivalent to $|D(\overline{v})| \geq 2$.

Now pick a generic $\overline{v} \in C(\overline{u})$. We know that point $\overline{v}$ is an intersection of $C(\overline{u})$ and $C(\overline{u}')$ for some $\overline{u}' \in X \setminus \{\overline{u}\}$ and it is the only intersection because $H \cap (X \times \mathbb{P}^2)$ is $(2,2)$-grid-free. We apply Corollary 4.6 to $C(\overline{u})$ and $C(\overline{u}')$ and get that $C(\overline{u})$ is irreducible of degree ≤ 2.

Suppose H is defined by $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}]$ and

$$F(\overline{x}, \overline{y}) = f(\overline{x}) g(\overline{y}) h_1(\overline{x}, \overline{y})^{r_1} h_2(\overline{x}, \overline{y})^{r_2} \ldots h_n(\overline{x}, \overline{y})^{r_n}$$

is the factorization of F such that $h_1, h_2, \ldots, h_n$ are distinct irreducible polynomials in $\mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}] \setminus (\mathbb{C}_{\text{hom}}[\overline{x}] \cup \mathbb{C}_{\text{hom}}[\overline{y}])$. The set $\{f = 0\} \cap X$ is either empty or infinite. So, for $H \cap (X \times \mathbb{P}^2)$ to be $(2,2)$-grid-free we must have $\{f = 0\} \cap X = \emptyset$. Similarly, we know that $\{g = 0\} = \emptyset$, that is, $g(\overline{y})$ is a nonzero constant.

[6] The algebraic curve $C_i(\overline{u})$ is reduced in the sense that its defining equation $h_i(\overline{u}, \overline{y})$ is square-free.

Without loss of generality, we may assume that $f(\overline{x}) = g(\overline{y}) = 1$ and that $F(\overline{x}, \overline{y})$ is square-free, that is, $r_1 = r_2 = \cdots = r_n = 1$. Let $C_i(\overline{u})$ be the section of $H_i := \{h_i = 0\}$ at $\overline{u}$ for $i = 1, 2, \ldots, n$. From Lemma 4.7, we know that, for a generic $\overline{u} \in X$, $C_i(\overline{u})$ and $C_j(\overline{u})$ have no common irreducible components for all $i \neq j$. Therefore $C(\overline{u}) = \cup_{i=1}^{n} C_i(\overline{u})$ has at least n irreducible components, and so $n = 1$. Now $C(\overline{u}) = C_1(\overline{u}) = \{h_1(\overline{u}, \overline{y}) = 0\}$ for all $\overline{u} \in X$. By Lemma 4.7, $h_1(\overline{u}, \overline{y})$ is square-free for generic $\overline{u}$. This and the fact that $C(\overline{u})$ is irreducible of degree ≤ 2 imply that $\deg h_1(\overline{u}, \overline{y}) \leq 2$ for a generic $\overline{u} \in X$, and so $\deg_{\overline{y}} h_1(\overline{x}, \overline{y}) \leq 2$. □

Proof of Lemma 4.7. Let d_1, d_2 be the degrees of h_1, h_2 in $\overline{y}$ respectively. Suppose on the contrary that $C_1(\overline{u})$ and $C_2(\overline{u})$ share common irreducible components for a generic $\overline{u} \in \mathbb{P}^2$. So, $h_1(\overline{u}, \overline{y})$ and $h_2(\overline{u}, \overline{y})$ have a common divisor in $\mathbb{C}[\overline{y}]$. Therefore there exist two nonzero polynomials $g_1^{\overline{u}}(\overline{y}) \in \mathbb{C}_{\mathrm{hom}}[\overline{y}]$ of degree $< d_2$ and $g_2^{\overline{u}}(\overline{y}) \in \mathbb{C}_{\mathrm{hom}}[\overline{y}]$ of degree $< d_1$ such that

$$h_1(\overline{u}, \overline{y}) g_1^{\overline{u}}(\overline{y}) + h_2(\overline{u}, \overline{y}) g_2^{\overline{u}}(\overline{y}) = 0. \tag{4.4}$$

By treating the coefficients of $g_1^{\overline{u}}(\overline{y})$ and $g_2^{\overline{u}}(\overline{y})$ as variables, we can view equation (4.4) as a homogeneous system of $M := \binom{d_1+d_2+1}{2}$ linear equations involving $N := \binom{d_1+1}{2} + \binom{d_2+1}{2}$ variables. Note that the coefficient in the ith equation of the jth variable, say c_{ij}, is a polynomial of $\overline{u}$, that is, $c_{ij} = c_{ij}(\overline{u})$ for some $c_{ij}(\overline{x}) \in \mathbb{C}[\overline{x}]$ that depends on h_1, h_2 only. Because the system of linear equations has a nontrivial solution and clearly $M > N$, the rank of its coefficient matrix $(c_{ij}(\overline{u}))$ is $< N$. Using the determinants of all $N \times N$ minors of matrix $(c_{ij}(\overline{u}))$, we can rewrite the statement that matrix $(c_{ij}(\overline{u}))$ is of rank $< N$ as $L := \binom{M}{N}$ polynomial equations of entries in the matrix, say

$$P_k(c_{ij}(\overline{u})) = 0, \quad \text{for all } k \in [L], \tag{4.5}$$

where $P_k(c_{ij}(\overline{x}))$ is a polynomial of $\overline{x}$ independent of $\overline{u}$. Since (4.5) holds for a generic $\overline{u} \in \mathbb{P}^2$, we have

$$P_k(c_{ij}(\overline{x})) = 0 \text{ in } \mathbb{C}[\overline{x}], \quad \text{for all } k \in [L], \tag{4.6}$$

Reversing the argument above, we can deduce that the rank of matrix $(c_{ij}(\overline{x}))$, over the quotient field $\mathbb{C}(\overline{x})$, is $< N$, and so there exist two nonzero polynomials $g_1^{\overline{x}}(\overline{y}) \in \mathbb{C}(\overline{x})_{\mathrm{hom}}[\overline{y}]$ of degree $< d_2$ and $g_2^{\overline{x}}(\overline{y}) \in \mathbb{C}(\overline{x})_{\mathrm{hom}}[\overline{y}]$ of degree $< d_1$ such that

$$h_1(\overline{x}, \overline{y}) g_1^{\overline{x}}(\overline{y}) + h_2^{\overline{x}}(\overline{x}, \overline{y}) g_2(\overline{y}) = 0. \tag{4.7}$$

Multiplying (4.7) by the common denominator of $g_1^{\overline{x}}(\overline{y})$ and $g_2^{\overline{x}}(\overline{y})$, we get two nonzero polynomials $g_1(\overline{x}, \overline{y}) \in \mathbb{C}[\overline{x}, \overline{y}]$ of degree $< d_2$ in $\overline{y}$ and $g_2(\overline{x}, \overline{y}) \in \mathbb{C}[\overline{x}, \overline{y}]$ of degree $< d_1$ in $\overline{y}$ such that

$$h_1(\overline{x}, \overline{y}) g_1(\overline{x}, \overline{y}) + h_2(\overline{x}, \overline{y}) g_2(\overline{x}, \overline{y}) = 0, \tag{4.8}$$

which is impossible as $\gcd(h_1, h_2) = 1$ and $\deg_{\overline{y}} h_1(\overline{x}, \overline{y}) = d_1 > \deg_{\overline{y}} g_2(\overline{x}, \overline{y})$.

It remains to prove that $C_1(\overline{u})$ is reduced for generic $\overline{u}$. Because $h_1(\overline{x}, \overline{y}) \notin \mathbb{C}_{\text{hom}}[\overline{x}]$, the polynomial $h_1'(\overline{x}, \overline{y}) := \partial h_1(\overline{x}, \overline{y})/\partial y_0$ might be assumed to be nonzero. Again, we assume, on the contrary, that $h_1(\overline{u}, \overline{y})$ is not square-free for a generic $\overline{u} \in \mathbb{P}^2$. This implies that $h_1(\overline{u}, \overline{y})$ and $h_1'(\overline{u}, \overline{y})$ have a common divisor. The same linear-algebraic argument, applied to h_1 and h_1' instead of h_1 and h_2, then yields a contradiction. $\square$

We can adapt the proof of Theorem 4.4 to the case when $\mathbb{P}^2 \setminus Y$ is finite. In this case, we obtain a weaker result though.

Proposition 4.8. *Suppose $\mathbb{P}^2 \setminus Y = \{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n\}$. If $H \cap (X \times Y)$ is $(2,2)$-grid-free, then either*

1. *there exists $F(\overline{x}, \overline{y}) \in \mathbb{C}_{\text{hom}}[\overline{x}, \overline{y}]$ of degree ≤ 2 in $\overline{y}$ such that $H \cap (X \times Y) = \{F = 0\} \cap (X \times Y)$,*
2. *or there exists $i \in [n]$ such that $\mathbb{P}^2 \times \{\overline{v}_i\} \subset H$.*

Sketch of a proof. We follow the proof of Theorem 4.4 up to the point where we apply Corollary 4.6. Note that $X_i := \{\overline{u} \in \mathbb{P}^2 : \overline{v}_i \in C(\overline{u})\}$ is Zariski-closed for all $i \in [n]$. If none of those X_i's equals $\mathbb{P}^2$, then for a generic $\overline{u} \in \mathbb{P}^2$, $C(\overline{u})$ does not pass through any of the points in $\mathbb{P}^2 \setminus Y$. The rest of the proof of Theorem 4.4 still holds and we end in the first case. Otherwise $X_i = \mathbb{P}^2$ for some $i \in [n]$, which corresponds to the second case. $\square$

4.5 Fields of finite characteristic

A standard model-theoretic argument allows us to transfer statements over fields of characteristic 0 to the fields of large characteristic.

Theorem 4.9. *Let ϕ be a sentence in the language of rings. The following are equivalent.*

1. *ϕ is true in complex numbers.*
2. *ϕ is true in every algebraically closed field of characteristic zero.*
3. *ϕ is true in all algebraically closed fields of characteristic p for all sufficiently large prime p.*

The theorem is an application of the compactness theorem and the completeness of the theory of algebraically closed field of fixed characteristic. We refer the readers to [Mar02, Section 2.1] for further details of the theorem and related notions.

As quantifiers over all polynomials are not part of the language of rings, one has to limit the degree of hypersurface H and the complexity of the open set X in Theorem 4.4. We now formulate the analog over the fields of large characteristic.

Theorem 4.10. *Let $\mathbb{K}$ be an algebraically closed field of large characteristic, let H be a hypersurface in $\mathbb{P}^2(\mathbb{K}) \times \mathbb{P}^2(\mathbb{K})$ of bounded degree, and let X be a Zariski-open subset of $\mathbb{P}^2(\mathbb{K})$ of bounded complexity (i.e. X is a Zariski-open subset of $\mathbb{P}^2(\mathbb{K})$ that can be described by some first order predicate in the language of rings of bounded length). If H is $(2,2)$-grid-free in $X \times \mathbb{P}^2(\mathbb{K})$, then there exists $F(\overline{x}, \overline{y}) \in \mathbb{K}_{\text{hom}}[\overline{x}, \overline{y}]$ of degree ≤ 2 in $\overline{y}$ such that $H \cap (X \times \mathbb{P}^2) = \{F = 0\} \cap (X \times \mathbb{P}^2)$.*

The proof essentially rewrites Theorem 4.4 as a sentence in the language of rings to which Theorem 4.9 is applicable. We skip the tedious but routine proof.

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