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A Remark on the Compactness for the Cahn-Hilliard Functional

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Abstract

In this note we prove compactness for the Cahn-Hilliard functional without assuming coercivity of the multi-well potential.

1 Introduction

The purpose of this note is to prove compactness for the Cahn–Hilliard functional (see [5], [8], [9]) without assuming coercivity of the multi-well potential $W$. Precisely, for $\varepsilon > 0$ consider the functional

$$F_\varepsilon : W^{1,2}(\Omega; \mathbb{R}^d) \to [0, \infty]$$

defined by

$$F_\varepsilon (u) := \int_{\Omega} \left( \frac{1}{\varepsilon} W(u) + \varepsilon |\nabla u|^2 \right) dx,$$

where $d \geq 1$ and the potential $W$ satisfies the following hypotheses:

$(H_1)$ $W : \mathbb{R}^d \to [0, \infty)$ is continuous, $W(z) = 0$ if and only if $z \in \{\alpha_1, \ldots, \alpha_\ell\}$ for some $\alpha_i \in \mathbb{R}^d$, $i = 1, \ldots, \ell$, with $\alpha_i \neq \alpha_j$ for $i \neq j$.

$(H_2)$ There exists $L > 0$ such that

$$\inf_{|z| \geq L} W(z) > 0.$$

Then the following result holds.

**Theorem 1.1** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded connected set with Lipschitz boundary. Assume that the multi-well potential $W$ satisfies conditions $(H_1)$ and $(H_2)$. Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}(\Omega; \mathbb{R}^d)$ be such that

$$M := \sup_{n} F_{\varepsilon_n} (u_n) < \infty \quad (1.1)$$
and
\[
\frac{1}{|\Omega|} \int_{\Omega} u_n(x) \, dx = m \quad \text{for all } n \in \mathbb{N} \quad (1.2)
\]
and for some \( m \in \mathbb{R}^d \). Then there exist \( u \in BV(\Omega; \{\alpha_1, \ldots, \alpha_\ell\}) \) and a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that
\[
\begin{align*}
\lim_{k \to \infty} u_{n_k} & \to u \text{ in } L^1(\Omega; \mathbb{R}^d). \\
\end{align*}
\]

For a two-well potential (\( \ell = 2 \)), Theorem 1.1 has been proved in the scalar case \( d = 1 \) by Modica [8] under the assumption
\[
\frac{1}{C} |z|^p \leq W(z) \leq C |z|^p
\]
for all \(|z|\) large and for some \( p > 2 \), and by Sternberg [9] for \( p \geq 2 \); while in the vectorial case \( d \geq 2 \), it has been proved by Fonseca and Tartar [4] under the assumption
\[
\frac{1}{C} |z| \leq W(z)
\]
for all \(|z|\) large. The case of a multi-well potential \( \ell \geq 3 \) has been studied by Baldo (see Propositions 4.1 and 4.2 in [2])), who proved compactness of a sequence of minimizers bounded in \( L^\infty(\Omega) \).

An example of a double-well potential satisfying \((H_1)\) and \((H_2)\) but not coercive is
\[
W(z) = \arctan \left( (z - \alpha)^2 (z - \beta)^2 \right),
\]
while an example of a potential satisfying \((H_1)\) but not \((H_2)\) is
\[
W(z) = (z - \alpha)^2 (z - \beta)^2 e^{-|z|^2}.
\]

In the one dimensional case \( N = 1 \), the hypothesis \((1.2)\) is not needed. Indeed, we have the following elementary result.

**Theorem 1.2** Assume that the multi-well potential \( W \) satisfies conditions \((H_1)\) and \((H_2)\). Let \( \varepsilon_n \to 0^+ \) and let \( \{u_n\} \subset W^{1,2}((a, b); \mathbb{R}^d) \) be such that \((1.1)\) holds. Then there exist \( u \in BV((a, b); \{\alpha_1, \ldots, \alpha_\ell\}) \) and a subsequence \( \{u_{n_k}\} \) of \( \{u_n\} \) such that
\[
\begin{align*}
\lim_{k \to \infty} u_{n_k} & \to u \text{ in } L^1((a, b); \mathbb{R}^d). \\
\end{align*}
\]

On the other hand, when \((1.2)\) holds, then condition \((H_2)\) can be weakened to:
\[
(H_3) \int_0^\infty \sqrt{V(s)} \, ds = \infty, \text{ where for every } s \geq 0,
\]
\[
V(s) := \min_{|z|=s} W(z). \quad (1.3)
\]
Note that \((H_2)\) implies that \(\sqrt{V(s)} \geq \inf_{|z| \geq L} \sqrt{W(z)} > 0\) for all \(s \geq L\), and so \((H_3)\) is satisfied. On the other hand, if

\[
W(z) \sim \frac{c}{|z|^q}
\]
as \(|z| \to \infty\) for some \(c > 0\) and \(0 < q \leq 2\), then \((H_3)\) holds but not \((H_1)\).

**Theorem 1.3** Assume that the multi-well potential \(W\) satisfies conditions \((H_1)\) and \((H_3)\). Let \(\varepsilon_n \to 0^+\) and let \(\{u_n\} \subset W^{1,2}((a,b) ; \mathbb{R}^d)\) be such that (1.1) and (1.2) hold. Then there exist \(u \in BV((a,b) ; \{\alpha_1, \ldots, \alpha_l\})\) and a subsequence \(\{u_{n_k}\}\) of \(\{u_n\}\) and such that

\[
u_{n_k} \rightharpoonup u \text{ in } L^1((a,b) ; \mathbb{R}^d).
\]

The next simple example shows that compactness fails without (1.2) or \((H_2)\).

**Example 1.4** If condition \((H_2)\) does not hold, then there exists \(\{z_n\} \subset \mathbb{R}^d\) such that \(|z_n| \to \infty\) and

\[
\lim_{n \to \infty} W(z_n) = 0.
\]

Find a sequence \(\varepsilon_n \to 0\) such that

\[
\frac{1}{\varepsilon_n} W(z_n) \to 0,
\]
(e.g. \(\varepsilon_n := \sqrt{W(z_n)}\)) and consider the sequence of functions \(u_n(x) := z_n\).

Then

\[
F_{\varepsilon_n}(u_n) = \frac{1}{\varepsilon_n} W(z_n)(b-a) \to 0
\]
but no subsequence of \(\{u_n\}\) converge in \(L^1((a,b))\).

**Remark 1.5** I have not been able to determine if Theorems 1.2 and 1.3 hold in dimension \(N \geq 2\) or if \((H_3)\) is needed in Theorem 1.3.

### 2 Proof of Theorems 1.1 and 1.2

The proof of Theorem 1.1 will make use of the following auxiliary results. For a proof of the following theorem see, e.g., Proposition 16.21 in [6].

**Theorem 2.1** Let \(u \in W^{1,1}(\mathbb{R}^N)\), \(N \geq 2\). Then

\[
\sup_{s>0} s \left[ L^N \left( \{x \in \mathbb{R}^N : |u(x)| \geq s\} \right) \right]_{N-1}^{\alpha_N} \leq \frac{1}{\alpha_N} \int_{\mathbb{R}^N} |\nabla u(x)| \, dx.
\]

For a proof of the next theorem, see Lemma 2.6 in [1].
Theorem 2.2 Let $A, \Omega \subset \mathbb{R}^N$ be open sets and let $1 \leq p < \infty$. Assume that $A$ is bounded and that $\Omega$ is connected and has Lipschitz boundary at each point of $\partial \Omega \cap \overline{A}$. Then there exists a linear and continuous operator $T : W^{1,p}(\Omega) \to W^{1,p}(A)$ such that, for every $u \in W^{1,p}(\Omega)$,

$$T(u)(x) = u(x) \quad \text{for } \mathcal{L}^N \text{ a.e. } x \in \Omega \cap A,$$

$$\int_A |T(u)(x)|^p \, dx \leq C \int_\Omega |u(x)|^p \, dx,$$

$$\int_A |\nabla T(u)(x)|^p \, dx \leq C \int_\Omega |\nabla u(x)|^p \, dx,$$

where $C > 0$ depends only on $N, p, A,$ and $\Omega$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of (1.1) and $(H_2)$ for every $n \in \mathbb{N}$, we have

$$M \geq \frac{1}{2} \int_\Omega \sqrt{W(u_n(x))} |\nabla u_n(x)| \, dx \quad (2.1)$$

$$\geq c \int_{\{|u_n| \geq L\}} |\nabla u_n(x)| \, dx,$$

where $c := \frac{1}{2} \inf_{|z| \geq L} \sqrt{W(z)} > 0$. Construct a $C^1$ function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(z) = z$ if $|z| \geq 2L$ and $f(z) = 0$ if $|z| < L$. By the chain rule, for every $n \in \mathbb{N}$ the function $v_n := f \circ u_n$ belongs to $W^{1,2}(\Omega; \mathbb{R}^d)$ and for all $i = 1, \ldots, N$ and for $\mathcal{L}^N$-a.e. $x \in \Omega$,

$$\frac{\partial v_n}{\partial x_i}(x) = \sum_{j=1}^d \frac{\partial f}{\partial z^j}(u_n(x)) \frac{\partial (u_n)^{(j)}}{\partial x_i}(x),$$

where we write $z = (z^{(1)}, \ldots, z^{(d)})$. Since $\frac{\partial f}{\partial z^{(j)}}(z) = 0$ if $|z| < L$, it follows that

$$\int_\Omega |\nabla v_n(x)| \, dx = \int_{\{|u_n| \geq L\}} |\nabla v_n(x)| \, dx \quad (2.2)$$

$$\leq \text{Lip } f \int_{\{|u_n| \geq L\}} |\nabla u_n(x)| \, dx \leq c^{-1} M \text{Lip } f.$$

Let $r > 0$ be so large that $\overline{\Omega} \subset B(0, r)$ and set $A := B(0, 2r)$. By Theorem 2.2 we may extend each function $v_n$ to a function in $W^{1,1}(A; \mathbb{R}^d)$, still denoted $v_n$, in such a way that

$$\int_A |v_n(x)| \, dx \leq C \int_\Omega |v_n(x)| \, dx, \quad (2.3)$$

$$\int_A |\nabla v_n(x)| \, dx \leq C \int_\Omega |\nabla v_n(x)| \, dx \leq Cc^{-1} M \text{Lip } f, \quad (2.4)$$

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where $C$ depends only on $r$, $N$, and $\Omega$. By the Poincaré inequality,
$$
\int_A \left| v_n(x) - c_n \right| \, dx \leq C \int_A |\nabla v_n(x)| \, dx,
$$
where $c_n := \frac{1}{|\Omega|} \int_\Omega v_n(x) \, dx$ and again $C$ depends only on $r$, $N$, and $\Omega$. Note that, since $f(z) = z$ if $|z| \geq 2L$,
$$
|c_n| = \frac{1}{|\Omega|} \left| \int_\Omega f \circ u_n \, dx \right| = \frac{1}{|\Omega|} \left| \int_{\{|u_n| > 2L\}} u_n \, dx + \int_{\{|u_n| \leq 2L\}} f \circ u_n \, dx \right|
= |m + \frac{1}{|\Omega|} \int_{\{|u_n| \leq 2L\}} (f \circ u_n - u_n) \, dx | \leq |m| + 4L.
$$

Consider a cut-off function $\varphi \in C_0^\infty (A; [0,1])$ such that $\varphi = 1$ in $B(0,r)$ and define
$$
w_n := \varphi (v_n - c_n).
$$
Then $w_n \in W^{1,1} (\mathbb{R}^N)$ and by (2.5),
$$
\int_{\mathbb{R}^N} |\nabla w_n(x)| \, dx \leq \text{Lip } \varphi \int_A |v_n - c_n| \, dx + \int_A |\nabla v_n(x)| \, dx
\leq (C \text{ Lip } \varphi + 1) \int_A |\nabla v_n(x)| \, dx.
$$

Applying Theorem 2.1 to $|w_n|$, we obtain
$$
\sup_{s>0} \left( \mathcal{L}^N \left( \{ x \in \mathbb{R}^N : |w_n|(x) \geq s \} \right) \right)^\frac{N-1}{N} \leq \frac{1}{\alpha_N^N} \int_{\mathbb{R}^N} |\nabla |w_n|(x)| \, dx
\leq C_1 \int_{\{|u_n| \geq 4L\}} |\nabla u_n(x)| \, dx \leq C_2,
$$
where we have used (2.2), (2.4), and (2.6).

Fix $s_1 > 2(|m| + 4L) + 1$. Using the facts that $\varphi = 1$ in $B(0,r)$, that $f(z) = z$ if $|z| \geq 2L$, and that $|c_n| \leq |m| + 4L$, for $s \geq s_1$ we have
$$
\{ x \in \Omega : |u_n(x)| \geq s \} = \{ x \in \Omega : |v_n(x)| \geq s \} \subset \{ x \in \Omega : |v_n(x) - c_n| \geq \frac{s}{2} \}
\subset \{ x \in \mathbb{R}^N : |w_n|(x) \geq \frac{s}{2} \},
$$
and so
$$
\mathcal{L}^N (\{ x \in \Omega : |u_n(x)| \geq s \}) \leq \frac{C}{s^{N-1}}
$$
for all $s \geq s_1$. Hence,
$$
\int_{\{|u_n| > s_1\}} |u_n(x)| \, dx = \int_{s_1}^\infty \mathcal{L}^N (\{ x \in \Omega : |u_n(x)| \geq s \}) \, ds
\leq C \int_{s_1}^\infty \frac{1}{s^{N-1}} \, ds = \frac{N-1}{s_1^{N-1}},
$$
which shows that \( \{u_n\} \) is bounded in \( L^1(\Omega; \mathbb{R}^d) \) and equi-integrable.

In view of Vitali’s convergence theorem, it remains to show that a subsequence converges in measure to some function \( u \in BV(\Omega; \{\alpha_1, \ldots, \alpha_\ell\}) \). This is classical (see e.g. [2] or [4]). ■

**Remark 2.3** Theorem 1.1 continues to hold if in place of (1.2) we assume that
\[
u_n = g \quad \text{on} \partial \Omega
\] (2.7)
for all \( n \in \mathbb{N} \) and for some function \( g \in L^1(\partial \Omega; \{\alpha_1, \ldots, \alpha_\ell\}) \). In this case, by Gagliardo’s trace theorem (see, e.g. Theorem 15.10 in [6]) there exists a function \( w \in W^{1,1} (\mathbb{R}^N \setminus \Omega; \mathbb{R}^d) \) such that \( w = g \) on \( \partial \Omega \). Extend each \( u_n \) to be zero outside \( \Omega \). We can now apply Theorem 2.1 directly to \( f u_n \in W^{1,1} (\mathbb{R}^N; \mathbb{R}^d) \) without introducing the constants \( c_n \), the function \( \varphi \), and without using Theorem 2.2.

We now turn to the proof of Theorem 1.2. The argument below is likely well-known. We present it for the convenience of the reader.

**Proof of Theorem 1.2.** Without loss of generality, we can assume that each function \( u_n \) is absolutely continuous. Since the set \( A_n := \{x \in (a, b) : |u_n(x)| > L\} \) is open, we may write it as
\[A_n = \bigcup_k (a_{k,n}, b_{k,n}).\]
Moreover, by (1.1) and \((H_2)\), for every \( n \in \mathbb{N} \), we have
\[M e_n \geq \int_a^b W(u_n(x)) \, dx \geq |A_n| \inf_{|z| \geq L} W(z),\]
and so its complement \((a, b) \setminus A_n\) is nonempty for all \( n \) sufficiently large. Fix any such \( n \). If \( A_n \) is empty, then \(|u_n(x)| \leq L\) for all \( x \in (a, b) \). Otherwise, let \( x \in (a_{k,n}, b_{k,n}) \). Then at least one of the endpoints, say \( a_{k,n} \), is not an endpoint of \((a, b)\) and so \(|u_n(a_{k,n})| = L\). By the fundamental theorem of calculus,
\[u_n(x) = u_n(a_{k,n}) + \int_{a_{k,n}}^x u'_n(t) \, dt.\]
Hence,
\[\sup_{x \in (a_{k,n}, b_{k,n})} |u_n(x)| \leq L + \int_{|u_n| \geq L} |u'_n(t)| \, dt \leq L + c^{-1} M,\]
where we have used (2.1). This shows that \( \{u_n\} \) is bounded in \( L^\infty((a, b); \mathbb{R}^d) \). We can now continue as in Lemma 6.2 of [3]. ■

Finally, we prove Theorem 1.3.

**Proof of Theorem 1.3.** Without loss of generality, we can assume that each function \( u_n \) is absolutely continuous. In view of (1.1) and (1.3), for every \( n \in \mathbb{N} \) we have
\[M \geq \frac{1}{2} \int_a^b \sqrt{W(u_n(x)) |u'_n(x)|} \, dx \geq \frac{1}{2} \int_a^b \sqrt{V(|u_n(x)|)} |u_n(x)| \, dx.\]
Using the area formula for absolutely continuous functions (see, e.g., Theorem 3.65 in [6]), we obtain

\[
M \geq \frac{1}{2} \int_a^b \sqrt{V(|u_n|(x))} |u_n'(x)| \, dx = \frac{1}{2} \int_{\mathbb{R}} \sqrt{V(s)} \text{card} |u_n|^{-1}(\{s\}) \, ds
\]

\[
\geq \frac{1}{2} \int_{\min|u_n|}^{\max|u_n|} \sqrt{V(s)} \, ds,
\]

where \(\text{card}\) is the cardinality and \(|u_n|^{-1}(\{s\}) = \{x \in (a, b) : |u_n(x)| = s\}\). By (1.2) and the intermediate value theorem, there exists \(x_n \in (a, b)\) such that

\[
u_n(x_n) = \frac{1}{b-a} \int_a^b u_n(x) \, dx = \frac{m}{b-a}.
\]

Hence, \(|u_n(x_n)| = \frac{|m|}{b-a}\), which implies that

\[
M \geq \frac{1}{2} \int_{\frac{|m|}{b-a}}^{\max|u_n|} \sqrt{V(s)} \, ds.
\]

By \((H_3)\) there exists \(R > 0\) such that \(\int_{\frac{|m|}{b-a}}^R \sqrt{V(s)} \, ds > 2M\). In turn, \(|u_n(x)| < R\) for all \(x \in (a, b)\) and all \(n \in \mathbb{N}\). This shows that \(\{u_n\}\) is bounded in \(L^1((a, b); \mathbb{R}^d)\).

**Remark 2.4** Observe that in Theorems 1.2 and 1.3 we can replace \((H_1)\) with the weaker hypothesis

\((H_4)\) \(W : \mathbb{R}^d \to [0, \infty)\) is continuous and for every \(r > 0\) the set

\[
\{z \in B(0, r) : W(z) = 0\}
\]

has finitely many elements.

Indeed, if \(\{u_n\} \subset W^{1,2}((a, b) ; \mathbb{R}^d)\) is such that (1.1) holds, then by Theorem 1.2 or 1.3, there exists \(R > 0\) such that \(|u_n(x)| < R\) for all \(x \in (a, b)\) and all \(n \in \mathbb{N}\). Find \(S \in (R, 2R)\) such that \(V(S) > 0\). Note that such \(S\) exists, since otherwise we would have \(V(s) = 0\) for all \(s \in (R, 2R)\), which would imply that \(\{z \in B(0, 2R) : W(z) = 0\}\) has infinitely many elements and would contradict \((H_4)\). Define

\[
W_1(z) := \begin{cases} W(z) & \text{if } |z| < S, \\ \frac{W(z)}{|z|} S & \text{if } |z| \geq S. \end{cases}
\]

Since \(|u_n(x)| < R < S\) for all \(x \in (a, b)\) and all \(n \in \mathbb{N}\), we have that

\[
M \geq F_{\varepsilon_n}(u_n) = \int_a^b \left( \frac{1}{\varepsilon_n} W_1(u_n) + \varepsilon_n |u_n'|^2 \right) \, dx.
\]
The function $W_1$ satisfies hypotheses $(H_1)$ and $(H_2)$. Hence, we may now apply Theorem 1.2 to find $u \in BV((a,b);\{\alpha_1,\ldots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that
\[ u_{n_k} \to u \text{ in } L^1((a,b);\mathbb{R}^d). \]
Here $\{\alpha_1,\ldots,\alpha_\ell\}$ are the zeros of $W$ in $B(0,s)$.

In view of the previous remark, we can prove a compactness result for $N = 1$ and bounded domains for the functional studied in the classical paper of Modica and Mortola [7].

**Corollary 2.5** Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}((a,b);\mathbb{R}^d)$ be such that
\[ \int_a^b \left( \frac{1}{\varepsilon_n} \sin^2(\pi u_n) + \varepsilon_n |u_n'(x)|^2 \right) dx \leq M \]
and (1.2) hold. Then there exist $u \in BV((a,b);\{\alpha_1,\ldots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that
\[ u_{n_k} \to u \text{ in } L^1(a,b). \]
Here, $\{\alpha_1,\ldots,\alpha_\ell\} \subset \mathbb{Z}$.

**Proof.** It is enough to observe that the function $W(z) = \sin^2(\pi z)$ satisfies $(H_3)$ and $(H_4)$. 

**Remark 2.6** I am not aware of any compactness result for $N \geq 2$ for the functional
\[ \int_{\Omega} \left( \frac{1}{\varepsilon} \sin^2(\pi u) + \varepsilon |\nabla u|^2 \right) dx, \]
when (1.2) holds. Note that $W(z) = \sin^2(\pi z)$ satisfies $(H_3)$ and $(H_4)$ but not $(H_1)$ and $(H_2)$.

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