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Motion of three-dimensional elastic films by anisotropic surface diffusion with curvature regularization

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MOTION OF THREE-DIMENSIONAL ELASTIC FILMS BY ANISOTROPIC SURFACE DIFFUSION WITH CURVATURE REGULARIZATION

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Abstract. Short time existence for a surface diffusion evolution equation with curvature regularization is proved in the context of epitaxially strained three-dimensional films. This is achieved by implementing a minimizing movement scheme, which is hinged on the H^{-1}-gradient flow structure underpinning the evolution law. Long-time behavior and Liapunov stability in the case of initial data close to a flat configuration are also addressed.

Keywords: minimizing movements, surface diffusion, gradient flows, higher order geometric flows, elastically stressed epitaxial films, volume preserving evolution, long-time behavior, Liapunov stability

Contents

1. Introduction

In this paper we study the morphologic evolution of anisotropic epitaxially strained films, driven by stress and surface mass transport in three dimensions. This can be viewed as the evolutionary counterpart of the static theory developed in [11, 23, 25, 22, 9, 15] in the two-dimensional case and in [10] in three dimensions. The two dimensional formulation of the same evolution problem has been addressed in [24] (see also [32] for the case of motion by evaporation-condensation).

The physical setting behind the evolution equation is the following. The free interface is allowed to evolve via *surface diffusion* under the influence of a chemical potential μ. Assuming that mass transport in the bulk occurs at a much faster time scale, and thus can be neglected (see [31]), we have, according to the Einstein-Nernst relation, that the evolution is governed by the *volume preserving* equation

$$V = C\Delta_{\Gamma}\mu \,, \tag{1.1}$$

where $C > 0$, V denotes the normal velocity of the evolving interface Γ, Δ_Γ stands for the tangential laplacian, and the chemical potential μ is given by the first variation of the underlying free-energy functional.

In our case, the free energy functional associated with the physical system is given by

$$\int_{\Omega_h} W(E(u))\, dz + \int_{\Gamma_h} \psi(\nu)\, d\mathcal{H}^2 \,, \tag{1.2}$$

where h is the function whose graph Γ_h describes the evolving profile of the film, Ω_h is the region occupied by the film, u is displacement of the material, which is assumed to be in (quasistatic) elastic equilibrium at each time, $E(u)$ is the symmetric part of Du, W is a positive definite quadratic form, and $\mathcal{H}^2$ denotes the two-dimensional Hausdorff measure. Finally, ψ is an anisotropic surface energy density, evaluated at the unit normal ν to Γ_h. The first variation of (1.2) can be written as the sum of three contributions: A constant Lagrange multiplier related to mass conservation, the (anisotropic) curvature of the surface, and the elastic energy density evaluated at the displacement of the solid on the profile of the film. Hence, (1.1) takes the form (assuming $C = 1$)

$$V = \Delta_\Gamma \bigl[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u))\bigr] \,, \tag{1.3}$$

where div_Γ stands for the tangential divergence along $\Gamma_{h(\cdot,t)}$, and $u(\cdot, t)$ is the elastic equilibrium in $\Omega_{h(\cdot,t)}$, i.e., the minimizer of the elastic energy under the prescribed periodicity and boundary conditions (see (1.6) below).

In the physically relevant case of a highly anisotropic non-convex interfacial energy there may exist certain directions ν at which the ellipticity condition

$$D^2\psi(\nu)[\tau, \tau] > 0 \qquad \text{for all } \tau \perp \nu,\ \tau \neq 0$$

fails, see for instance [18, 34]. Correspondingly, the above evolution equation becomes *backward parabolic* and thus ill-posed. To overcome this ill-posedness, and following the work of Herring ([29]), an additive curvature regularization to surface energy has been proposed, see [18, 28]. Here we consider the following regularized surface energy:

$$\int_{\Gamma_h} \Bigl(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\Bigr)\, d\mathcal{H}^2 \,,$$

where $p > 2$, H stands for the sum $\kappa_1 + \kappa_2$ of the principal curvatures of Γ_h, and ε is a (small) positive constant. The restriction on the range of exponents $p > 2$ is of technical nature and it is motivated by the fact that in two-dimensions the Sobolev space $W^{2,p}$ embeds into $C^{1,\frac{p-2}{p}}$ if $p > 2$. The extension of our analysis to the case $p = 2$ seems to require different ideas.

The regularized free-energy functional then reads

$$\int_{\Omega_h} W(E(u))\, dz + \int_{\Gamma_h} \Bigl(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\Bigr)\, d\mathcal{H}^2 \,, \tag{1.4}$$

and (1.1) becomes

$$V = \Delta_\Gamma \biggl[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Bigl(\Delta_\Gamma(|H|^{p-2}H) - |H|^{p-2}H\Bigl(\kappa_1^2 + \kappa_2^2 - \frac{1}{p}H^2\Bigr)\Bigr)\biggr] \,. \tag{1.5}$$

Sixth-order evolution equations of this type have already been considered in [28] for the case without elasticity. Its two-dimensional version was studied numerically in [34] for the evolution of voids in elastically stressed materials, and analytically in [24] in the context of evolving one-dimensional graphs. We also refer to [33, 12] and references therein for some numerical results in the three-dimensional case. However, to the best of our knowledge no analytical results were available in the literature prior to ours.

As in [24], in this paper we focus on evolving graphs, and to be precise on the case where (1.5) models the evolution toward equilibrium of epitaxially strained elastic films deposited over a rigid substrate. Given $Q := (0,b)^2$, $b > 0$, we look for a spatially Q-periodic solution to the following Cauchy problem:

$$\begin{cases} \dfrac{1}{J}\dfrac{\partial h}{\partial t} = \Delta_\Gamma \Big[\mathrm{div}_\Gamma (D\psi(\nu)) + W(E(u)) \\ \qquad\qquad -\varepsilon \Big(\Delta_\Gamma (|H|^{p-2}H) - |H|^{p-2}H \Big(\kappa_1^2 + \kappa_2^2 - \dfrac{1}{p}H^2 \Big) \Big) \Big] \,, \quad \text{in } \mathbb{R}^2 \times (0,T_0), \\ \mathrm{div}\, \mathbb{C}E(u) = 0 \quad \text{in } \Omega_h, \\ \mathbb{C}E(u)[\nu] = 0 \quad \text{on } \Gamma_h, \qquad u(x,0,t) = (e_0^1 x_1, e_0^2 x_2, 0)\,, \\ h(\cdot,t) \text{ and } Du(\cdot,t) \quad \text{are } Q\text{-periodic}, \\ h(\cdot,0) = h_0\,, \end{cases} \tag{1.6}$$

where, we recall, $h : \mathbb{R}^2 \times [0,T_0] \to (0,+\infty)$ denotes the function describing the two-dimensional profile Γ_h of the film,

$$J := \sqrt{1 + |D_x h|^2}\,,$$

$W(A) := \frac{1}{2}\mathbb{C}A : A$ for all $A \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}$ with $\mathbb{C}$ a positive definite fourth order tensor, $e_0 := (e_0^1, e_0^2)$, with e_0^1, $e_0^2 > 0$, is a vector that embodies the mismatch between the crystalline lattices of the film and the substrate, and $h_0 \in H^2_{loc}(\mathbb{R}^2)$ is a Q-periodic function. Note that in (1.6) the sixth-order (geometric) parabolic equation for the film profile is coupled with the elliptic system of elastic equilibrium equations in the bulk.

It was observed by Cahn and Taylor in [14] that the surface diffusion equation can be regarded as a gradient flow of the free-energy functional with respect to a suitable H^{-1}-Riemannian structure. To formally illustrate this point, consider the manifold of subsets of $Q \times (0,+\infty)$ of fixed volume d, which are subgraphs of a Q-periodic function, that is,

$$\mathcal{M} := \Big\{ \Omega_h : \, h \ Q\text{-periodic}, \ \int_Q h\, dx = d \Big\}\,,$$

where $\Omega_h := \{(x,y) : x \in Q\,, 0 < y < h(x)\}$. The tangent space $T_{\Omega_h}\mathcal{M}$ at an element Ω_h is described by the kinematically admissible normal velocities

$$T_{\Omega_h}\mathcal{M} := \Big\{ V : \Gamma_h \to \mathbb{R} : \, V \text{is } Q\text{-periodic}, \ \int_{\Gamma_h} V\, d\mathcal{H}^2 = 0 \Big\}\,,$$

where Γ_h is the graph of h over the periodicity cell Q, and it is endowed with the H^{-1} metric tensor

$$g_{\Omega_h}(V_1, V_2) := \int_{\Gamma_h} \nabla_{\Gamma_h} w_1 \nabla_{\Gamma_h} w_2 \, d\mathcal{H}^2 \qquad \text{for all } V_1,\, V_2 \in T_{\Omega_h}\mathcal{M}\,,$$

where w_i, $i = 1, 2$, is the solution to

$$\begin{cases} -\Delta_{\Gamma_h} w_i = V_i \qquad \text{on } \Gamma_h, \\ w_i \text{ is } Q\text{-periodic}, \\ \displaystyle\int_{\Gamma_h} w_i \, d\mathcal{H}^2 = 0\,. \end{cases}$$

Consider now the following *reduced free-energy functional*

$$G(\Omega_h) := \int_{\Omega_h} W(E(u_h))\, dz + \int_{\Gamma_h} \Big(\psi(\nu) + \frac{\varepsilon}{p}|H|^p \Big)\, d\mathcal{H}^2\,,$$

where u_h is the minimizer of the elastic energy in Ω_h under the boundary and periodicity conditions described above. Then, the evolution described by (1.6) is such that at each time the normal velocity V of the evolving profile $h(t)$ is the element of the tangent space $T_{\Omega_{h(t)}}\mathcal{M}$ corresponding to the steepest descent of G, i.e., (1.6) may be formally rewritten as

$$g_{\Omega_{h(t)}}(V, \tilde{V}) = -\partial G(\Omega_{h(t)})[\tilde{V}] \qquad \text{for all } \tilde{V} \in T_{\Omega_{h(t)}}\mathcal{M},$$

where $\partial G(h(t))[\tilde{V}]$ stands for the first variation of G at $\Omega_{h(t)}$ in the direction $\tilde{V}$.

In order to solve (1.6), we take advantage of this gradient flow structure and we implement a *minimizing movements scheme* (see [5]), which consists in constructing discrete time evolutions by solving iteratively suitable minimum incremental problems.

It is interesting to observe that the gradient flow of the free-energy functional G with respect to an L^2-Riemannian structure, (instead of H^{-1}) leads to a fourth order evolution equation, which describes motion by evaporation-condensation (see [14, 28] and [32], where the two-dimensional case was studied analytically).

This paper is organized as follows. In Section 2 we set up the problem and introduce the discrete time evolutions. In Section 3 we prove our main local-in-time existence result for (1.6), by showing that (up to subsequences) the discrete time evolutions converge to a weak solution of (1.6) in $[0, T_0]$ for some $T_0 > 0$ (see Theorem 3.16). By a *Q-periodic weak solution* we mean a function $h \in H^1(0, T_0; H^{-1}_\#(Q)) \cap L^\infty(0, T_0; H^2_\#(Q))$, such that (h, u_h) satisfies the system (1.6) in the distributional sense (see Definition 3.1). To the best of our knowledge, Theorem 3.16 is the first (short time) existence result for a surface diffusion type geometric evolution equation in the presence of elasticity in three-dimensions. Moreover, also the use of minimizing movements appears to be new in the context of higher order geometric flows (the only other papers we are aware of in which a similar approach is adopted, but in two-dimensions, are [24] and [32]).

Compared to mean curvature flows, where the minimizing movements algorithm is nowadays classical after the pioneering work of [3] (see also [17, 7, 16]), a major technical difference lies in the fact that no comparison principle is available in this higher order framework. The convergence analysis is instead based on subtle interpolation and regularity estimates. It is worth mentioning that for geometric surface diffusion equation without elasticity and without curvature regularization

$$V = \Delta_\Gamma H$$

(corresponding to the case $W = 0$, $\psi = 1$, and $\varepsilon = 0$) short time existence of a smooth solution was proved in [20], using semigroup techniques. See also [8, 30]. It is still an open question whether the solution constructed via the minimizing movement scheme is unique, and thus independent of the subsequence.

In Section 4 we address the Liapunov stability of the flat configuration, corresponding to an horizontal (flat) profile. Roughly speaking, we show that if the surface energy density is strictly convex and the second variation of the functional (1.2) at a given flat configuration is positive definite, then such a configuration is asymptotically stable, that is, for all initial data h_0 sufficiently close to it the corresponding evolutions constructed via minimizing movements exist for all times, and converge asymptotically to the flat configuration as $t \to +\infty$ (see Theorem 4.8). We remark that Theorem 4.8 may be regarded as an evolutionary counterpart of the static stability analysis of the flat configuration performed in [25, 9, 10]. In Theorem 4.7 we address also the case of a non-convex anisotropy and we show that if the corresponding Wulff shape contains an horizontal facet, then the Asaro-Grinfeld-Tiller instability does not occur and the flat configuration is *always* Liapunov stable (see [9, 10] for the corresponding result in the static case). Both results are completely new even in the two-dimensional case, to which they obviously apply (see Subsection 4.3). We remark that our treatment is purely variational and it is hinged on the fact that (1.4) is a Liapunov functional for the evolution.

Finally, in the Appendix, we collect several auxiliary results that are used troughout the paper.

2. Setting of the problem

Let $Q := (0,b)^2 \subset \mathbb{R}^2$, $b > 0$, $p > 2$, and let $h_0 \in W^{2,p}_{\#}(Q)$ be a positive function, describing the initial profile of the film. We recall that $W^{2,p}_{\#}(Q)$ stands for the subspace of $W^{2,p}(Q)$ of all functions whose Q-periodic extension belong to $W^{2,p}_{loc}(\mathbb{R}^2)$. Given $h \in W^{2,p}_{\#}(Q)$, with $h \geq 0$, we set

$$\Omega_h := \{(x,y) \in Q \times \mathbb{R} : 0 < y < h(x)\}$$

and we denote by Γ_h the graph of h over Q. We will identify a function $h \in W^{2,p}_{\#}(Q)$ with its periodic extension to $\mathbb{R}^2$, and denote by $\Omega_h^{\#}$ and $\Gamma_h^{\#}$ the open subgraph and the graph of such extension, respectively. Note that $\Omega_h^{\#}$ is the periodic extension of Ω_h. Set

$$\begin{aligned} LD_{\#}(\Omega_h;\mathbb{R}^3) := \big\{ u \in L^2_{\mathrm{loc}}(\Omega_h^{\#};\mathbb{R}^3) : \, u(x,y) = u(x+bk,y) \text{ for } (x,y) \in \Omega_h^{\#} \text{ and } k \in \mathbb{Z}^2 \,, \\ E(u)|_{\Omega_h} \in L^2(\Omega_h;\mathbb{R}^3) \big\} \,, \end{aligned}$$

where $E(u) := \frac{1}{2}(Du + D^T u)$, with Du the distributional gradient of u and $D^T u$ its transpose, is the strain of the displacement u. We prescribe the Dirichlet boundary condition $u(x,0) = w_0(x,0)$ for $x \in Q$, with $w_0 \in H^1(U \times (0,+\infty))$ for every bounded open subset $U \subset \mathbb{R}^2$ and such that $Dw_0(\cdot,y)$ is Q-periodic for a.e. $y > 0$. A typical choice is given by $w_0(x,y) := (e_0^1 x_1, e_0^2 x_2, 0)$, where the vector $e_0 := (e_0^1, e_0^2)$, with e_0^1, $e_0^2 > 0$, embodies the mismatch between the crystalline lattices of film and substrate. Define

$$\begin{aligned} X := \Big\{ (h,u) : \, h \in W^{2,p}_{\#}(Q),\, h \geq 0,\, u : \Omega_h^{\#} \to \mathbb{R}^3 \text{ s.t. } u - w_0 \in LD_{\#}(\Omega_h;\mathbb{R}^3) \,, \\ \text{and } u(x,0) = w_0 \text{ for all } x \in \mathbb{R}^2 \Big\} \,. \end{aligned}$$

The elastic energy density $W : \mathbb{M}^{3\times 3}_{\mathrm{sym}} \to [0,+\infty)$ takes the form

$$W(A) := \frac{1}{2}\mathbb{C}A : A \,,$$

with $\mathbb{C}$ a positive definite fourth-order tensor, so that $W(A) > 0$ for all $A \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} \setminus \{0\}$. Given $h \in W^{2,p}_{\#}(Q)$, $h \geq 0$, we denote by u_h the corresponding elastic equilibrium in Ω_h, i.e.,

$$u_h := \mathrm{argmin} \left\{ \int_{\Omega_h} W(E(u))\, dz : \, u \in w_0 + LD_{\#}(\Omega_h;\mathbb{R}^3),\, u(x,0) = w_0(x,0) \right\} .$$

Let $\psi : \mathbb{R}^3 \to [0,+\infty)$ be a positively one-homogeneous function of class C^2 away from the origin. Note that, in particular,

$$\frac{1}{c}|\xi| \leq \psi(\xi) \leq c|\xi| \qquad \text{for all } \xi \in \mathbb{R}^3 \,, \tag{2.1}$$

for some $c > 0$.

We now introduce the energy functional

$$F(h,u) := \int_{\Omega_h} W(E(u))\, dz + \int_{\Gamma_h} \Big(\psi(\nu) + \frac{\varepsilon}{p}|H|^p \Big)\, d\mathcal{H}^2 \,, \tag{2.2}$$

defined for all $(h,u) \in X$, where ν is the outer unit normal to Ω_h, $H = \mathrm{div}_{\Gamma_h}\nu$ denotes the sum of the principal curvatures of Γ_h, and ε is a positive constant. In the sequel we will often use the fact that

$$-\mathrm{div}\left(\frac{Dh}{\sqrt{1+|Dh|^2}} \right) = H \qquad \text{in } Q \,, \tag{2.3}$$

which, in turn, implies

$$\int_Q H\, dx = 0 \,. \tag{2.4}$$

Remark 2.1. Notation: In the sequel we denote by z a generic point in $Q \times \mathbb{R}$ and we write $z = (x, y)$ with $x \in Q$ and $y \in \mathbb{R}$. Moreover, given $g : \Gamma_h \to \mathbb{R}$, where Γ_h is the graph of some function h defined in Q, we denote by the same symbol g the function from Q to $\mathbb{R}$ given by $x \mapsto g(x, h(x))$. Consistently, Dg will stand for the gradient of the function from Q to $\mathbb{R}$ just defined.

2.1. **The incremental minimum problem.** In this subsection we introduce the incremental minimum problems that will be used to define the discrete time evolutions. As standing assumption throughout this paper, we start from an initial configuration $(h_0, u_0) \in X$, such that

$$h_0 \in W^{2,p}_{\#}(Q), \qquad h_0 > 0, \tag{2.5}$$

and u_0 minimizes the elastic energy in Ω_{h_0} among all u with $(h_0, u) \in X$.

Fix a sequence $\tau_n \searrow 0$ representing the discrete time increments. For $i \in \mathbb{N}$ we define inductively $(h_{i,n}, u_{i,n})$ as the solution of the minimum problem

$$\min\left\{ F(h,u) + \frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}} |D_{\Gamma_{i-1,n}} v_h|^2 \, d\mathcal{H}^2 : (h,u) \in X, \right. \\ \left. \|Dh\|_{L^\infty(Q)} \le \Lambda_0, \int_Q h\,dx = \int_Q h_0\,dx \right\}, \tag{2.6}$$

where $\Gamma_{i-1,n}$ stands for $\Gamma_{h_{i-1,n}}$, Λ_0 is a positive constant such that

$$\Lambda_0 > \|h_0\|_{C^1_{\#}(Q)}, \tag{2.7}$$

and v_h is the unique solution in $H^1_{\#}(\Gamma_{h_{i-1,n}})$ to the following problem:

$$\begin{cases} \Delta_{\Gamma_{i-1,n}} v_h = \dfrac{h - h_{i-1,n}}{\sqrt{1 + |Dh_{i-1,n}|^2}} \circ \pi, \\ \displaystyle\int_{\Gamma_{h_{i-1,n}}} v_h \, d\mathcal{H}^2 = 0, \end{cases} \tag{2.8}$$

where π is the canonical projection $\pi(x, y) = x$. For $x \in Q$ and $(i-1)\tau_n \le t \le i\tau_n$, $i \in \mathbb{N}$, we define the linear interpolation

$$h_n(x,t) := h_{i-1,n}(x) + \frac{1}{\tau_n}\big(t - (i-1)\tau_n\big)\big(h_{i,n}(x) - h_{i-1,n}(x)\big), \tag{2.9}$$

and we let $u_n(\cdot, t)$ be the *elastic equilibrium corresponding to* $h_n(\cdot, t)$, i.e.,

$$F(h_n(\cdot,t), u_n(\cdot,t)) = \min_{(h_n(\cdot,t),u)\in X} F(h_n(\cdot,t), u). \tag{2.10}$$

The remaining of this subsection is devoted to the proof of the existence of a minimizer for the minimum incremental problem (2.6).

Theorem 2.2. *The minimum problem* (2.6) *admits a solution* $(h_{i,n}, u_{i,n}) \in X$.

Proof. Let $\{(h_k, u_k)\} \subset X$ be a minimizing sequence for (2.6). Let H_k denote the sum of principal curvatures of Γ_{h_k}. Since the sequence $\{H_k\}$ is bounded in $L^p(Q)$ and $\|Dh_k\|_{L^\infty_{\#}(Q)} \le \Lambda_0$, it follows from (2.3) and Lemma 5.3 that $\|h_k\|_{W^{2,p}_{\#}(Q)} \le C$. Then, up to a subsequence (not relabelled), we may assume that $h_k \rightharpoonup h$ weakly in $W^{2,p}_{\#}(Q)$, and thus strongly in $C^{1,\alpha}_{\#}(Q)$ for some $\alpha > 0$. As a consequence, $H_k \rightharpoonup H$ in $L^p(Q)$, where H is the sum of the principal curvatures of Γ_h. In turn, the L^p-weak convergence of $\{H_k\}$ and the C^1-convergence of $\{h_k\}$ imply by lower semicontinuity that

$$\int_{\Gamma_h} \Big(\psi(\nu) + \frac{\varepsilon}{p}|H|^p\Big)\, d\mathcal{H}^2 \le \liminf_k \int_{\Gamma_{h_k}} \Big(\psi(\nu) + \frac{\varepsilon}{p}|H_k|^p\Big)\, d\mathcal{H}^2 \,. \tag{2.11}$$

Moreover, we also have that $v_{h_k} \to v_h$ strongly in $H^1(\Gamma_{i-1,n})$, and thus

$$\lim_k \frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}} |D_{\Gamma_{i-1,n}} v_{h_k}|^2\, d\mathcal{H}^2 = \frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}} |D_{\Gamma_{i-1,n}} v_h|^2\, d\mathcal{H}^2 \,. \tag{2.12}$$

Finally, since $\sup_k \int_{\Omega_{h_k}} |Eu_k|^2\, dz < +\infty$, reasoning as in [23, Proposition 2.2], from the uniform convergence of $\{h_k\}$ to h and Korn's inequality we conclude that there exists $u \in H^1_{loc}(\Omega_h^\#;\mathbb{R}^3)$ such that $(h,u) \in X$ and, up to a subsequence, $u_k \rightharpoonup u$ weakly in $H^1_{loc}(\Omega_h^\#;\mathbb{R}^3)$. Therefore, we have that

$$\int_{\Omega_h} W(E(u))\, dz \le \liminf_k \int_{\Omega_{h_k}} W(E(u_k))\, dz \,,$$

which, together with (2.11) and (2.12), allows us to conclude that (h,u) is a minimizer. □

3. Existence of the evolution

In this section we prove short time existence of a solution of the geometric evolution equation

$$V = \Delta_\Gamma\Big[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - \frac{1}{p}|H|^pH + |H|^{p-2}H|B|^2\Big)\Big] \,, \tag{3.1}$$

where V denotes the outer normal velocity of $\Gamma_{h(\cdot,t)}$, $|B|^2$ is the sum of the squares of the principal curvatures of $\Gamma_{h(\cdot,t)}$, $u(\cdot,t)$ is the elastic equilbrium in $\Omega_{h(\cdot,t)}$, and $W(E(u))$ is the trace of $W(E(u(\cdot,t)))$ on $\Gamma_{h(\cdot,t)}$. In the sequel we denote by $H^{-1}_\#(Q)$ the dual space of $H^1_\#(Q)$. Note that if $f \in H^1_\#(Q)$, then Δf can be identified with the element of $H^{-1}_\#(Q)$ defined by

$$\langle \Delta f, g\rangle := -\int_Q DfDg\, dx \qquad \text{for all } g \in H^1_\#(Q)\,.$$

Moreover, a function $f \in L^2(Q)$ can be identified with the element of $H^{-1}_\#(Q)$ defined by

$$\langle f, g\rangle := \int_Q fg\, dx \qquad \text{for all } g \in H^1_\#(Q)\,.$$

Definition 3.1. Let $T_0 > 0$. We say that $h \in L^\infty(0,T_0;W^{2,p}_\#(Q)) \cap H^1(0,T_0;H^{-1}_\#(Q))$ is a *solution of* (3.1) in $[0,T_0]$ if

(i) $\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - \frac{1}{p}|H|^pH + |H|^{p-2}H|B|^2\Big) \in L^2(0,T_0;H^1_\#(Q))$,

(ii) for a.e. $t \in (0,T_0)$

$$\frac{1}{J}\frac{\partial h}{\partial t} = \Delta_\Gamma\Big[\mathrm{div}_\Gamma(D\psi(\nu)) + W(E(u)) - \varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H) - \frac{1}{p}|H|^pH + |H|^{p-2}H|B|^2\Big)\Big] \quad \text{in } H^{-1}_\#(Q),$$

where $J := \sqrt{1+|Dh|^2}$, $u(\cdot,t)$ is the elastic equilbrium in $\Omega_{h(\cdot,t)}$, and where we wrote Γ in place of $\Gamma_{h(\cdot,t)}$.

Remark 3.2. An immediate consequence of the above definition is that the evolution is *volume preserving*, that is, $\int_Q h(x,t)\, dx = \int_Q h_0(x)\, dx$ for all $t \in [0,T_0]$. Indeed, for all $t_1, t_2 \in [0,T_0]$ and for $\varphi \in H^1_\#(Q)$ we have

$$\begin{aligned}\int_Q [h(x,t_2)-h(x,t_1)]\varphi\,dx &= \int_{t_1}^{t_2}\Big\langle \frac{\partial h}{\partial t}(\cdot,t),\varphi\Big\rangle\,dt\\ &= \int_{t_1}^{t_2}\Big\langle J\Delta_\Gamma\Big[\mathrm{div}_\Gamma(D\psi(\nu))+W(E(u))\\ &\quad -\varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H)-\frac{1}{p}|H|^pH+|H|^{p-2}H|B|^2\Big)\Big],\varphi\Big\rangle\,dt\\ &= -\int_{t_1}^{t_2}\int_\Gamma D_\Gamma\Big[\mathrm{div}_\Gamma(D\psi(\nu))+W(E(u))\\ &\quad -\varepsilon\Big(\Delta_\Gamma(|H|^{p-2}H)-\frac{1}{p}|H|^pH+|H|^{p-2}H|B|^2\Big)\Big]D_\Gamma(\varphi\circ\pi)\,d\mathcal{H}^2dt\,.\end{aligned}$$

Choosing $\varphi = 1$, we conclude that

$$\int_Q [h(x,t_2)-h(x,t_1)]\,dx = 0\,.$$

Remark 3.3. In the sequel, we consider the following equivalent norm on $H^{-1}_\#(Q)$. Given $\mu \in H^{-1}_\#(Q)$, we set

$$\|\mu\|_{H^{-1}_\#(Q)} := \sup\big\{\langle\mu,g\rangle : \, g\in H^1_\#(Q) \text{ s.t. } \big|\textstyle\int_Q g\,dx\big| + \|Dg\|_{L^2(Q)} \le 1\big\}\,.$$

Note that if $f\in L^2(Q)$, with $\int_Q f\,dx = 0$, we have

$$\|f\|_{H^{-1}_\#(Q)} = \|Dw\|_{L^2(Q)}\,,$$

where $w\in H^1_\#(Q)$ is the unique periodic solution to the problem

$$\begin{cases}\Delta w = f & \text{in } Q,\\ \displaystyle\int_Q w\,dx = 0\,.\end{cases} \tag{3.2}$$

To see this, first observe that since $\int_Q f\,dx = 0$ we have

$$\|f\|_{H^{-1}_\#(Q)} = \sup\Big\{\int_Q fg\,dx : \, g\in H^1_\#(Q) \text{ s.t. } \textstyle\int_Q g\,dx = 0 \text{ and } \|Dg\|_{L^2(Q)}\le 1\Big\}\,.$$

Thus, since by (3.2)

$$\int_Q fg\,dx = -\int_Q DwDg\,dx \le \|Dw\|_{L^2(Q)}\,,$$

we have $\|f\|_{H^{-1}_\#(Q)} \le \|Dw\|_{L^2(Q)}$. The opposite inequality follows by taking $g = -w/\|Dw\|_{L^2(Q)}$.

Theorem 3.4. *For all n, $i\in\mathbb{N}$ we have*

$$\int_0^{+\infty}\Big\|\frac{\partial h_n}{\partial t}\Big\|^2_{H^{-1}_\#(Q)}dt \le CF(h_0,u_0)\,, \tag{3.3}$$

$$F(h_{i,n},u_{i,n}) \le F(h_{i-1,n},u_{i-1,n}) \le F(h_0,u_0)\,, \tag{3.4}$$

and

$$\sup_{t\in[0,+\infty)}\|h_n(\cdot,t)\|_{W^{2,p}_\#(Q)} < +\infty \tag{3.5}$$

for some $C = C(\Lambda_0) > 0$. Moreover, up to a subsequence,

$$h_n \to h \text{ in } C^{0,\alpha}([0,T];L^2(Q)) \text{ for all } \alpha\in(0,\tfrac14),\quad h_n \rightharpoonup h \text{ weakly in } H^1(0,T;H^{-1}_\#(Q)) \tag{3.6}$$

for all $T>0$ and for some function h such that $h(\cdot,t)\in W^{2,p}_{\#}(Q)$ for every $t\in[0,+\infty)$ and

$$F(h(\cdot,t),u_{h(\cdot,t)})\le F(h_0,u_0)\qquad \textit{for all } t\in[0,+\infty). \tag{3.7}$$

Proof. By the minimality of $(h_{i,n},u_{i,n})$ (see (2.6)) we have that

$$F(h_{i,n},u_{i,n})+\frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}}|D_{\Gamma_{i-1,n}}v_{h_{i,n}}|^2\,d\mathcal{H}^2\le F(h_{i-1,n},u_{i-1,n}) \tag{3.8}$$

for all $i\in\mathbb{N}$, which yields in particular (3.4). Hence,

$$\frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}}|D_{\Gamma_{i-1,n}}v_{h_{i,n}}|^2\,d\mathcal{H}^2\le F(h_{i-1,n},u_{i-1,n})-F(h_{i,n},u_{i,n})\,,$$

and summing over i, we obtain

$$\sum_{i=1}^{\infty}\frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}}|D_{\Gamma_{i-1,n}}v_{h_{i,n}}|^2\,d\mathcal{H}^2\le F(h_0,u_0)\,. \tag{3.9}$$

Let $w_{h_{i,n}}\in H^1_{\#}(Q)$ denote the unique periodic solution to the problem

$$\begin{cases}\Delta w_{h_{i,n}}=h_{i,n}-h_{i-1,n} & \text{in } Q,\\ \displaystyle\int_Q w_{h_{i,n}}\,dx=0\,.\end{cases}$$

Note that

$$\begin{aligned}\int_Q|Dw_{h_{i,n}}|^2\,dx&=\int_Q\Delta w_{h_{i,n}}w_{h_{i,n}}\,dx=\int_{\Gamma_{i-1,n}}\frac{h_{i,n}-h_{i-1,n}}{\sqrt{1+|Dh_{i-1,n}|^2}}\circ\pi w_{h_{i,n}}\,d\mathcal{H}^2\\ &=\int_{\Gamma_{i-1,n}}\Delta_{\Gamma_{i-1,n}}v_{h_{i,n}}w_{h_{i,n}}\,d\mathcal{H}^2=-\int_{\Gamma_{i-1,n}}D_{\Gamma_{i-1,n}}v_{h_{i,n}}D_{\Gamma_{i-1,n}}w_{h_{i,n}}\,d\mathcal{H}^2\\ &\le\|D_{\Gamma_{i-1,n}}v_{h_{i,n}}\|_{L^2(\Gamma_{i-1,n})}\|D_{\Gamma_{i-1,n}}w_{h_{i,n}}\|_{L^2(\Gamma_{i-1,n})}\\ &\le C(\Lambda_0)\|D_{\Gamma_{i-1,n}}v_{h_{i,n}}\|_{L^2(\Gamma_{i-1,n})}\|Dw_{h_{i,n}}\|_{L^2(Q)}\,.\end{aligned}$$

Combining this inequality with (3.9) and recalling (2.9) and Remark 3.3, we get (3.3).

Note from (3.4) it follows that

$$\sup_{i,n}\int_{\Gamma_{i,n}}|H|^p\,d\mathcal{H}^2<+\infty\,.$$

Hence, (3.5) follows immediately by Lemma 5.3, taking into account that $\|Dh_{i,n}\|_{L^\infty(Q)}\le\Lambda_0$. Using a diagonalizing argument, it can be shown that there exist h such that $h_n\rightharpoonup h$ weakly in $H^1(0,T;H^{-1}_{\#}(Q))$ for all $T>0$. Note also that, by (3.3) and using Hölder Inequality, we have for $t_2>t_1$,

$$\|h_n(\cdot,t_2)-h_n(\cdot,t_1)\|_{H^{-1}(Q)}\le\int_{t_1}^{t_2}\Big\|\frac{\partial h_n(\cdot,t)}{\partial t}\Big\|_{H^{-1}(Q)}dt\le C(t_2-t_1)^{\frac12}\,. \tag{3.10}$$

Therefore, applying Theorem 5.4 to the solution $w\in H^1_{\#}(Q)$ of the problem

$$\begin{cases}\Delta w=h_n(\cdot,t_2)-h_n(\cdot,t_1) & \text{in } Q,\\ \displaystyle\int_Q w\,dx=0\,,\end{cases}$$

we get

$$
\begin{aligned}
\|h_n(\cdot,t_2)-h_n(\cdot,t_1)\|_{L^2(Q)} &= \|\Delta w\|_{L^2(Q)} \le C\|D^3 w\|^{\frac12}_{L^2(Q)}\|Dw\|^{\frac12}_{L^2(Q)} \\
&\le C\|Dh(\cdot,t_2)-Dh(\cdot,t_1)\|^{\frac12}_{L^2(Q)}\|h(\cdot,t_2)-h(\cdot,t_1)\|^{\frac12}_{H^{-1}(Q)} \\
&\le C(\Lambda_0)(t_2-t_1)^{\frac14}\,, \qquad (3.11)
\end{aligned}
$$

where the last inequality follows from (3.10). By the Ascoli-Arzelà theorem (see e.g. [6, Proposition 3.3.1]), we get (3.6). Finally, inequality (3.7) follows from (3.4) by lower semicontinuity, using (3.6) and (3.5). □

In what follows, $\{h_n\}$ and h are the subsequence and the function found in Theorem 3.4, respectively. The next result shows that the convergence of $\{h_n\}$ to h can be significantly improved for short time.

Theorem 3.5. *There exist $T_0>0$ and $C>0$ depending only (h_0,u_0) such that*

(i) *$h_n\to h$ in $C^{0,\beta}([0,T_0];C^{1,\alpha}_{\#}(Q))$ for every $\alpha\in(0,\frac{p-2}{p})$ and $\beta\in(0,\frac{(p-2-\alpha p)(p+2)}{16p^2})$,*

(ii) $\sup_{t\in[0,T_0]}\|Du_n(\cdot,t)\|_{C^{0,\frac{p-2}{p}}(\overline{\Omega}_{h_n(\cdot,t)})}\le C$,

(iii) *$E(u_n(\cdot,h_n))\to E(u(\cdot,h))$ in $C^{0,\beta}([0,T_0];C^{0,\alpha}_{\#}(Q))$ for every $\alpha\in(0,\frac{p-2}{p})$ and $0\le\beta<\frac{(p-2-\alpha p)(p+2)}{16p^2}$, where $u(\cdot,t)$ is the elastic equilibrium in $\Omega_{h(\cdot,t)}$.*

Moreover, $h(\cdot,t)\to h_0$ in $C^{1,\alpha}_{\#}(Q)$ as $t\to0^+$, h_n, $h\ge C_0>0$ for some positive constant C_0, and

$$
\sup_{t\in[0,T_0]}\|Dh_n(\cdot,t)\|_{L^\infty(Q)}<\Lambda_0 \qquad (3.12)
$$

for all n.

Proof. To prove assertion (i), we start by observing that by Theorem 5.6, (3.5), Theorem 5.6 again, and (3.11) we have

$$
\begin{aligned}
\Big\|Dh_n(\cdot,t_2)-Dh_n(\cdot,t_1)\Big\|_{L^\infty} &\le C\Big\|D^2h_n(\cdot,t_2)-D^2h_n(\cdot,t_1)\Big\|^{\frac{p+2}{2p}}_{L^p}\Big\|h_n(\cdot,t_2)-h_n(\cdot,t_1)\Big\|^{\frac{p-2}{2p}}_{L^p} \\
&\le C\|h_n(\cdot,t_2)-h_n(\cdot,t_1)\Big\|^{\frac{p-2}{2p}}_{L^p} \\
&\le C\bigg(\|D^2h_n(\cdot,t_2)-D^2h_n(\cdot,t_1)\Big\|^{\frac{p-2}{2p}}_{L^2}\|h_n(\cdot,t_2)-h_n(\cdot,t_1)\Big\|^{\frac{p+2}{2p}}_{L^2}\bigg)^{\frac{p-2}{2p}} \\
&\le C|t_2-t_1|^{\frac{p^2-4}{16p^2}} \qquad (3.13)
\end{aligned}
$$

for all t_1, $t_2\in[0,T_0]$. Notice that from (3.5) we have

$$
\sup_{n,t\in[0,T_0]}\|h_n(\cdot,t)\|_{C^{1,\frac{p-2}{p}}_{\#}(Q)}<+\infty\,. \qquad (3.14)
$$

Take $\alpha\in(0,\frac{p-2}{p})$ and observe that

$$
\Big[Dh_n(\cdot,t_2)-Dh_n(\cdot,t_1)\Big]_\alpha\le\Big[Dh_n(\cdot,t_2)-Dh_n(\cdot,t_1)\Big]^{\frac{\alpha p}{p-2}}_{\frac{p-2}{p}}\bigg[\operatorname*{osc}_{[0,b]}\Big(Dh_n(\cdot,t_2)-Dh_n(\cdot,t_1)\Big)\bigg]^{\frac{p-2-\alpha p}{p-2}},
$$

where $[\cdot]_\beta$ denotes the β-Hölder seminorm. From this inequality, (3.13), (3.14), and the Ascoli-Arzelà theorem [6, Proposition 3.3.1], assertion (i) follows.

Standard elliptic estimates ensure that if $h_n(\cdot,t)\in C^{1,\alpha}_{\#}(Q)$ for some $\alpha\in(0,1)$, then $Du_n(\cdot,t)$ can be estimated in $C^{0,\alpha}(\overline{\Omega}_{h_n(\cdot,t)})$ with a constant depending only on the $C^{1,\alpha}$-norm of $h_n(\cdot,t)$, see, for instance, [25, Proposition 8.9], where this property is proved in two dimensions but an entirely similar argument works in all dimensions. Hence, assertion (ii) follows from (3.14). Assertion (iii) is an immediate consequence of (i) and Lemma 5.1. Finally, (3.12) follows from (2.7) and (i). □

Remark 3.6. Note that in the previous theorem we can take

$$T_0 := \sup\{t > 0 : \|Dh_n(\cdot, s)\|_{L^\infty(Q)} < \Lambda_0 \quad \text{for all } s \in [0, t)\}.$$

In Theorem 3.16 we will show that h is a solution to (3.1) in $[0, T_0)$, in the sense of Definition 3.1.

We begin with some auxiliary results.

Proposition 3.7. *Let $h \in W^{3,q}_{\#}(Q)$ for some $q > 2$ and let Γ be its graph. Let $\Phi : Q\times\mathbb{R}\times(-1, 1) \to Q \times \mathbb{R}$ be the flow*

$$\frac{\partial \Phi}{\partial t} = X(\Phi), \qquad \Phi(\cdot, 0) = Id,$$

where X is a smooth vector field Q-periodic in the first two variables. Set $\Gamma_t := \Phi(\cdot, t)(\Gamma)$, denote by ν_t the normal to Γ_t, let H_t be the sum of principal curvatures of Γ_t, and let $|B_t|^2$ be the sum of squares of the principal curvatures of Γ_t. Then

$$\frac{d}{dt}\frac{1}{p}\int_{\Gamma_t} |H_t|^p \, d\mathcal{H}^2 = \int_{\Gamma_t} D_{\Gamma_t}(|H_t|^{p-2}H_t) D_{\Gamma_t}(X \cdot \nu_t) \, d\mathcal{H}^2 - \int_{\Gamma_t} |H_t|^{p-2}H_t\Big(|B_t|^2 - \frac{1}{p}H_t^2\Big)(X \cdot \nu_t) \, d\mathcal{H}^2 . \quad (3.15)$$

Proof. Set $\Phi_t(\cdot) := \Phi(\cdot, t)$. We can extend ν_t to a tubular neighborhood of Γ_t as the gradient of the signed distance from Γ_t. We have

$$\frac{d}{dt}\frac{1}{p}\int_{\Gamma_t} |H_t|^p \, d\mathcal{H}^2 = \frac{d}{ds}\Big(\frac{1}{p}\int_{\Gamma_{t+s}} |H_{t+s}|^p \, d\mathcal{H}^2\Big)\Big|_{s=0} = \frac{d}{ds}\Big(\frac{1}{p}\int_{\Gamma_t} |H_{t+s} \circ \Phi_s|^p J_2\Phi_s \, d\mathcal{H}^2\Big)\Big|_{s=0},$$

where J_2 denotes the two-dimensional Jacobian of Φ_s on Γ_t. Then we have

$$\frac{d}{dt}\frac{1}{p}\int_{\Gamma_t} |H_t|^p \, d\mathcal{H}^2 = \frac{1}{p}\int_{\Gamma_t} |H_t|^p \mathrm{div}_{\Gamma_t} X \, d\mathcal{H}^2 + \int_{\Gamma_t} |H_t|^{p-2}H_t \frac{d}{ds}\big(H_{t+s} \circ \Phi_s\big)\big|_{s=0} \, d\mathcal{H}^2 .$$

Concerning the last integral, we observe that

$$\frac{d}{ds}\big(H_{t+s} \circ \Phi_s\big)\big|_{s=0} = \frac{d}{ds}\Big(\mathrm{div}_{\Gamma_{t+s}}\nu_{t+s}\Big)\Big|_{s=0} + DH_t \cdot X .$$

Set

$$\dot{\nu}_t := \frac{d}{ds}\nu_{t+s}\big|_{s=0} .$$

By differentiating with respect to s the identity $D\nu_{t+s}[\nu_{t+s}] = 0$, we get $D\dot{\nu}_t[\nu_t] + D\nu_t[\dot{\nu}_t] = 0$. Multiplying this identity by ν_t and recalling that $D\nu$ is symmetric matrix we get

$$D\dot{\nu}_t[\nu_t] \cdot \nu_t = -D\nu_t[\nu_t] \cdot \dot{\nu}_t = 0 .$$

In turn, this implies that $\mathrm{div}_{\Gamma_t}\dot{\nu}_t = \mathrm{div}\dot{\nu}_t$, and so

$$\frac{d}{ds}\Big(\mathrm{div}_{\Gamma_{t+s}}\nu_{t+s}\Big)\Big|_{s=0} = \mathrm{div}_{\Gamma_t}\dot{\nu}_t .$$

In turn, see [13, Lemma 3.8-(f)],

$$\dot{\nu}_t = -(D_{\Gamma_t}X)^T[\nu_t] - D_{\Gamma_t}\nu_t[X] = -D_{\Gamma_t}(X \cdot \nu_t) .$$

Collecting the above identities, integrating by parts, and using the identity $\partial_{\nu_t} H_t = -\mathrm{trace}\big((D\nu_t)^2\big) = -|B_t|^2$ proved in [13, Lemma 3.8-(d)], we have

$$\begin{aligned}
\frac{d}{dt}\frac{1}{p}\int_{\Gamma_t}|H_t|^p\,d\mathcal{H}^2 &= \frac{1}{p}\int_{\Gamma_t}|H_t|^p \mathrm{div}_{\Gamma_t} X\,d\mathcal{H}^2 + \int_{\Gamma_t}|H_t|^{p-2}H_t\big(-\Delta_{\Gamma_t}(X\cdot\nu_t) + DH_t\cdot X\big)\,d\mathcal{H}^2 \\
&= -\int_{\Gamma_t}|H_t|^{p-2}H_t D_{\Gamma_t}H_t\cdot X\,d\mathcal{H}^2 + \frac{1}{p}\int_{\Gamma_t}|H_t|^p H_t(X\cdot\nu_t)\,d\mathcal{H}^2 \\
&\quad + \int_{\Gamma_t}|H_t|^{p-2}H_t\big(-\Delta_{\Gamma_t}(X\cdot\nu_t) + DH_t\cdot X\big)\,d\mathcal{H}^2 \\
&= \int_{\Gamma_t}|H_t|^{p-2}H_t\Big(-\Delta_{\Gamma_t}(X\cdot\nu_t) + \partial_{\nu_t}H_t(X\cdot\nu_t) + \frac{1}{p}H_t^2(X\cdot\nu_t)\Big)\,d\mathcal{H}^2 \\
&= \int_{\Gamma_t} D_{\Gamma_t}(|H_t|^{p-2}H_t)D_{\Gamma_t}(X\cdot\nu_t)\,d\mathcal{H}^2 \\
&\quad - \int_{\Gamma_t}|H_t|^{p-2}H_t\Big\{\Big(|B_t|^2 - \frac{1}{p}H_t^2\Big)(X\cdot\nu_t)\Big\}\,d\mathcal{H}^2\,. \qquad (3.16)
\end{aligned}$$

Thus (3.15) follows. ☐

Proposition 3.7 motivates the following definition.

Definition 3.8. We say that $(h, u_h) \in X$ is a *critical pair* for the functional F defined in (2.2) if $|H|^{p-2}H \in H^1(\Gamma_h)$ and

$$\varepsilon\int_{\Gamma_h} D_{\Gamma_h}(|H|^{p-2}H)D_{\Gamma_h}\phi\,d\mathcal{H}^2 + \varepsilon\int_{\Gamma_h}\Big(\frac{1}{p}|H|^pH - |H|^{p-2}H|B|^2\Big)\phi\,d\mathcal{H}^2 + \int_{\Gamma_h}\big[\mathrm{div}_{\Gamma_h}(D\psi(\nu)) + W(E(u_h))\big]\phi\,d\mathcal{H}^2 = 0$$

for all $\phi \in H^1_\#(\Gamma_h)$ with $\int_{\Gamma_h}\phi\,d\mathcal{H}^2 = 0$. We will also say that h is a *critical profile* if (h, u_h) is a critical pair.

Lemma 3.9. *Let $h \in W^{2,p}_\#(Q)$ such that $|H|^{p-2}H \in W^{1,q}_\#(Q)$, for some $q > 2$. Then, there exist a sequence $\{h_j\} \subset W^{3,q}_\#(Q)$ such that $h_j \to h$ in $W^{2,p}_\#(Q)$ and $|H_j|^{p-2}H_j \to |H|^{p-2}H$ in $W^{1,q}_\#(Q)$, where H_j stands for the sum of the principal curvatures of Γ_{h_j}.*

Proof. We may assume without loss of generality that $H \neq 0$, otherwise h would have already the required regularity (see (2.3)). By the Sobolev embedding theorem it follows that $|H|^{p-2}H \in C^{0,1-\frac{2}{q}}_\#(Q)$ and, in turn, using the $\frac{1}{p-1}$ Hölder's continuity of the function $t \mapsto t^{\frac{1}{p-1}}$, $H \in C^{0,\alpha}_\#(Q)$ for $\alpha := \frac{q-2}{q(p-1)}$. Standard Schauder's estimates yield $h \in C^{2,\alpha}_\#(Q)$.

For $\delta > 0$ set

$$H_\delta := \begin{cases} H - \delta & \text{if } H \geq \delta, \\ H + \delta' & \text{if } H \leq -\delta', \\ 0 & \text{otherwise}\,, \end{cases}$$

where δ' is chosen in such a way that $\int_Q H_\delta\,dx = 0$. Observe that this choice of δ' is always possible, although not necessarily unique. Indeed, by (2.4) and the fact that $H \neq 0$, if δ is sufficiently small

$$\int_{\{H>\delta\}}(H-\delta)\,dx + \int_{\{H<0\}}H\,dx < 0 \qquad \text{and} \qquad \int_{\{H>\delta\}}(H-\delta)\,dx > 0\,.$$

By continuity it is then clear that we may find $\delta' > 0$ such that

$$\int_{\{H>\delta\}}(H-\delta)\,dx + \int_{\{H<-\delta'\}}(H+\delta')\,dx = 0\,. \qquad (3.17)$$

We now show that, independently of the choice of δ' satisfying (3.17), $\delta' \to 0$ as $\delta \to 0$. Indeed, if not, there would exist a sequence $\delta_n \to 0$ and a corresponding sequence $\delta'_n \to \delta' > 0$, such that (3.17) holds with δ and δ' replaced by δ_n and δ'_n, respectively. But then, passing to the limit as $n \to \infty$, we would get

$$\int_{\{H>0\}} H\,dx + \int_{\{H<-\delta'\}} (H+\delta')\,dx = 0\,,$$

which contradicts (2.4).

Note that $H_\delta \to H$ in $C^{0,\alpha}_\#(Q)$ as $\delta \to 0$. Moreover, we claim that $|H_\delta|^{p-2}H_\delta \to |H|^{p-2}H$ in $W^{1,q}_\#(Q)$. Indeed, observe that $H \in W^{1,q}(A_\delta)$ where $A_\delta := \{H > \delta\} \cup \{H < -\delta'\}$ for all $\delta > 0$. Hence,

$$D(|H|^{p-2}H) = \begin{cases} (p-1)|H|^{p-2}DH & \text{if } H \neq 0, \\ 0 & \text{elsewhere,} \end{cases}$$

and

$$D(|H_\delta|^{p-2}H_\delta) = \begin{cases} (p-1)|H_\delta|^{p-2}DH & \text{in } A_\delta, \\ 0 & \text{elsewhere.} \end{cases}$$

The claim follows by observing that $D(|H_\delta|^{p-2}H_\delta) \to D(|H|^{p-2}H)$ a.e. and that $|D(|H_\delta|^{p-2}H_\delta)| \leq |D(|H|^{p-2}H)|$. Observe now that $H \in W^{1,q}(A_\delta)$ implies $H_\delta \in W^{1,q}_\#(Q)$. In order to conclude the proof it is enough to show that for δ sufficiently small there exist a unique periodic solution h_δ to the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dh_\delta}{\sqrt{1+|Dh_\delta|^2}}\right) = H_\delta \\ \displaystyle\int_Q h_\delta\,dx = \int_Q h\,dx\,. \end{cases} \tag{3.18}$$

This follows from Lemma 3.10 below. □

Lemma 3.10. *Let $h \in C^{2,\alpha}_\#(Q)$ and let H denote the sum of the principal curvatures of Γ_h. Then there exist σ, $C > 0$ with the following property: for all $K \in C^{0,\alpha}_\#(Q)$, with $\int_Q K\,dx = 0$ and $\|K - H\|_{C^{0,\alpha}_\#(Q)} \leq \sigma$, there exists a unique periodic solution $k \in C^{2,\alpha}_\#(Q)$ to*

$$\begin{cases} -\mathrm{div}\left(\dfrac{Dk}{\sqrt{1+|Dk|^2}}\right) = K \\ \displaystyle\int_Q k\,dx = \int_Q h\,dx\,, \end{cases}$$

and

$$\|k - h\|_{C^{2,\alpha}_\#(Q)} \leq C\|K - H\|_{C^{0,\alpha}_\#(Q)}\,. \tag{3.19}$$

Proof. Without loss of generality we may assume that $\int_Q h\,dx = 0$.

Set $X := \{k \in C^{2,\alpha}_\#(Q) : \int_Q k\,dx = 0\}$ and $Y := \{K \in C^{0,\alpha}_\#(Q) : \int_Q K\,dx = 0\}$, and consider the operator $T\colon X \to Y$ defined by

$$T(k) := -\mathrm{div}\left(\frac{Dk}{\sqrt{1+|Dk|^2}}\right).$$

By assumption we have that $T(h) = H$. We now use the inverse function theorem (see e.g. [4, Theorem 1.2, Chap. 2]) to prove that T is invertible in a $C^{2,\alpha}$-neighborhood of h with a C^1-inverse.

To see this, note that for any $k \in X$ we have that $T'(k) : X \to Y$ is the continuous linear operator defined by

$$T'(h)[\varphi] := -\mathrm{div}\left[\frac{1}{\sqrt{1+|Dh|^2}}\left(I - \frac{Dh\otimes Dh}{1+|Dh|^2}\right)D\varphi\right].$$

It is easily checked that T' is continuous map from X to the space $\mathcal{L}(X,Y)$ of linear bounded operators from X to Y, so that $T \in C^1(X,Y)$. Finally, standard existence arguments for elliptic equations imply that for any $k \in X$ the operator $T'(k)$ is invertible. Thus we may apply the inverse function theorem to conclude that there exist $\sigma > 0$ such that for all $K \in C^{0,\alpha}_{\#}(Q)$, with $\int_Q K\,dx = 0$ and $\|K-H\|_{C^{0,\alpha}_{\#}(Q)} \le \sigma$, there exists a unique periodic function $k = T^{-1}K \in C^{2,\alpha}_{\#}(Q)$. Moreover, the continuity of T^{-1}, together with standard Schauder's estimates, implies that (3.19) holds for σ sufficiently small. □

In what follows $J_{i,n}$ stands for

$$J_{i,n} := \sqrt{1+|Dh_{i,n}|^2}\,,$$

$H_{i,n}$ is the sum of the principal curvatures of $\Gamma_{i,n}$, $|B_{i,n}|^2$ denotes the sum of the squares of the principal curvatures of $\Gamma_{i,n}$, and $\tilde{H}_n\colon Q\times[0,T_0]\to\mathbb{R}$ is the function defined as

$$\tilde{H}_n(x,t) := H_{i,n}(x,h_{i,n}(x),t) \qquad \text{if } t\in[(i-1)\tau_n, i\tau_n)\,. \tag{3.20}$$

Theorem 3.11. *Let T_0 be as in Theorem 3.5 and let $\tilde{H}_n$ be given in* (3.20). *Then there exists $C > 0$ such that*

$$\int_0^{T_0}\int_Q |D^2(|\tilde{H}_n|^{p-2}\tilde{H}_n)|^2\,dxdt \le C \tag{3.21}$$

for $n\in\mathbb{N}$.

Proof. **Step 1.** We claim that $|H_{i,n}|^{p-2}H_{i,n} \in W^{1,q}_{\#}(\Gamma_{i,n})$ for all $q \ge 1$ and that $h_{i,n} \in C^{2,\sigma}_{\#}(Q)$ for all $\sigma \in (0, \frac{1}{p-1})$.

We recall that $h_{i,n}$ is the solution to the incremental minimum problem (2.6). We are going to show that $h_{i,n} \in W^{2,q}_{\#}(Q)$ for all $q \ge 2$. Fix a function $\varphi \in C^2_{\#}(Q)$ such that $\int_Q \varphi\,dx = 0$. Then by minimality and by (3.12) we have

$$\frac{d}{ds}\left(F(h_{i,n}+s\varphi, u_{i,n}) + \frac{1}{2\tau_n}\int_{\Gamma_{i-1,n}} |D_{\Gamma_{i-1,n}} v_{h_{i,n}+s\varphi}|^2\,d\mathcal{H}^2\right)\Big|_{s=0} = 0\,,$$

where, we recall, $v_{h_{i,n}+s\varphi}$ solves (2.8) with h replaced by $h_{i,n}+s\varphi$. It can be shown that

$$\begin{aligned}
&\int_Q W(E(u_{i,n}(x,h_{i,n}(x))))\varphi\,dx + \int_Q D\psi(-Dh_{i,n},1)\cdot(-D\varphi,0)\,dx + \frac{\varepsilon}{p}\int_Q |H_{i,n}|^p\frac{Dh_{i,n}\cdot D\varphi}{J_{i,n}}\\
&\quad -\varepsilon\int_Q |H_{i,n}|^{p-2}H_{i,n}\Bigg[\Delta\varphi - \frac{D^2\varphi[Dh_{i,n},Dh_{i,n}]}{J_{i,n}^2}\\
&\qquad - \frac{\Delta h_{i,n}Dh_{i,n}\cdot D\varphi}{J_{i,n}^2} - 2\frac{D^2h_{i,n}[Dh_{i,n},D\varphi]}{J_{i,n}^2} + 3\frac{D^2h_{i,n}[Dh_{i,n},Dh_{i,n}]Dh_{i,n}\cdot D\varphi}{J_{i,n}^4}\Bigg]\,dx\\
&\quad - \frac{1}{\tau_n}\int_Q v_{h_{i,n}}\varphi\,dx = 0\,,
\end{aligned} \tag{3.22}$$

where the last integral is obtained by observing that $v_{h_{i,n}+s\varphi} = v_{h_{i,n}} + sv_\varphi$, with v_φ solving

$$\begin{cases} \Delta_{\Gamma_{i-1,n}} v_\varphi = \dfrac{\varphi}{\sqrt{1+|Dh_{i-1,n}|^2}}\circ\pi\,,\\[2ex] \displaystyle\int_{\Gamma_{h_{i-1,n}}} v_\varphi\,d\mathcal{H}^2 = 0\,. \end{cases}$$

Setting $w := |H_{i,n}|^{p-2}H_{i,n}$,

$$
\begin{aligned}
A &:= \varepsilon\Big(I - \frac{Dh_{i,n}\otimes Dh_{i,n}}{J_{i,n}^2}\Big), \qquad (3.23)\\
b &:= \pi(D\psi(-Dh_{i,n},1)) - \frac{\varepsilon}{p}|H_{i,n}|^p\frac{Dh_{i,n}}{J_{i,n}}\\
&\quad + \varepsilon w\Big[-\frac{\Delta h_{i,n}Dh_{i,n}}{J_{i,n}^2} - 2\frac{D^2h_{i,n}[Dh_{i,n}]}{J_{i,n}^2} + 3\frac{D^2h_{i,n}[Dh_{i,n},Dh_{i,n}]Dh_{i,n}}{J_{i,n}^4}\Big],\\
c &:= -W(E(u(x,h_{i,n}(x)))) + \frac{1}{\tau_n}v_{h_{i,n}}\,,
\end{aligned}
$$

we have by (3.5) and Theorem 3.5 that $A \in W^{1,p}_\#(Q;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$, $b\in L^1(Q;\mathbb{R}^2)$, $c\in C^{0,\alpha}_\#(Q)$ for some α, and we may rewrite (3.22) as

$$
\int_Q w\,AD^2\varphi\,dx + \int_Q b\cdot D\varphi + \int_Q c\varphi\,dx = 0 \qquad \text{for all } \varphi\in C^\infty_\#(Q) \text{ with } \int_Q\varphi\,dx = 0. \qquad (3.24)
$$

By Lemma 5.2 we get that $w\in L^q(Q)$ for $q\in(\frac{p}{p-1},2)$. Therefore, for any such q we have $H_{i,n}\in L^{q(p-1)}(Q)$ and thus, by Lemma 5.3, $h_{i,n}\in W^{2,q(p-1)}_\#(Q)$. In turn, using Hölder's inequality, this implies that b, $w\,\mathrm{div}A\in L^{r_0}(Q;\mathbb{R}^2)$ where $r_0 := \frac{q(p-1)}{p}$. Observe that $r_0\in(1,2)$. By applying Lemma 5.2 again, we deduce that $w\in W^{1,r_0}_\#(Q)$ and thus $w\in L^{\frac{2r_0}{2-r_0}}(Q)$. In turn, arguing as before, this implies that b, $w\,\mathrm{div}A\in L^{r_1}(Q;\mathbb{R}^2)$, where $r_1 := \frac{2r_0(p-1)}{(2-r_0)p} > r_0$. If $r_1\ge 2$, then using again Lemma 5.2 we conclude that $w\in W^{1,r_1}_\#(Q)$, which implies the claim, since $D^2h_{i,n}\in L^q(Q;\mathbb{M}^{2\times2}_{sym})$ and, in turn, b, $w\,\mathrm{div}A\in L^q(Q;\mathbb{R}^2)$ for all q. Then the conclusion follows by Lemma 5.2. Otherwise, we proceed by induction. Assume that $w\in W^{1,r_{i-1}}_\#(Q)$. If $r_{i-1}\ge 2$ then the claim follows. Otherwise, a further application of Lemma 5.2 implies that $w\in W^{1,r_i}_\#(Q)$ with $r_i := \frac{2r_{i-1}(p-1)}{(2-r_{i-1})p}$. Since $r_{i-1}<2$, we have $r_i > r_{i-1}$. We claim that there exists j such that $r_j > 2$. Indeed, if not, the increasing sequence $\{r_i\}$ would converge to some $\ell\in(1,2]$ satisfying

$$
\ell = \frac{2\ell(p-1)}{(2-\ell)p}\,.
$$

However, this is impossible since the above identity is equivalent to $\ell = \frac{2}{p} < 1$.

Finally, observe that since $|H_{i,n}|^{p-2}H_{i,n}\in W^{1,q}_\#(Q)$ for all $q\ge 1$, then $|H_{i,n}|^{p-1}\in C^{0,\alpha}_\#(Q)$ for every $\alpha\in(0,1)$. Hence $H_{i,n}\in C^{0,\sigma}_\#(Q)$ for all $\sigma\in(0,\frac{1}{p-1})$ and so, by standard Schauder's estimates, $h_{i,n}\in C^{2,\sigma}_\#(Q)$ for all $\sigma\in(0,\frac{1}{p-1})$.

Step 2. By Step 1 we may now write the Euler-Lagrange equation for $h_{i,n}$ in intrinsic form. To be precise, we claim that for all $\varphi\in C^2_\#(Q)$, with $\int_Q\varphi\,dx = 0$, we have

$$
\begin{aligned}
&\varepsilon\int_{\Gamma_{i,n}} D_{\Gamma_{i,n}}(|H_{i,n}|^{p-2}H_{i,n})D_{\Gamma_{i,n}}\phi\,d\mathcal{H}^2 - \varepsilon\int_{\Gamma_{i,n}}|H_{i,n}|^{p-2}H_{i,n}\Big(|B_{i,n}|^2 - \frac{1}{p}H_{i,n}^2\Big)\phi\,d\mathcal{H}^2\\
&+\int_{\Gamma_{i,n}}\big[\mathrm{div}_{\Gamma_{i,n}}(D\psi(\nu_{i,n})) + W(E(u_{i,n}))\big]\phi\,d\mathcal{H}^2 - \frac{1}{\tau_n}\int_{\Gamma_{i,n}} v_{h_{i,n}}\phi\,d\mathcal{H}^2 = 0\,, \qquad (3.25)
\end{aligned}
$$

where $\phi := \frac{\varphi}{J_{i,n}}\circ\pi$. To see this, fix $h\in W^{3,q}_\#(Q)$ for some $q>2$, denote by Γ and Γ_t the graphs of h and $h+t\varphi$, respectively, and by H and H_t the corresponding sums of the principal curvatures.

Then by Proposition 3.7 and arguing as in the proof of (3.22), we have

$$\int_\Gamma D_\Gamma(|H|^{p-2}H)D_\Gamma\phi\,d\mathcal{H}^2 - \int_\Gamma |H|^{p-2}H\Big(|B|^2 - \frac{1}{p}H^2\Big)\phi\,d\mathcal{H}^2$$

$$= \frac{1}{p}\int_Q |H|^p\frac{Dh\cdot D\varphi}{J} - \int_Q |H|^{p-2}H\Big[\Delta\varphi - \frac{D^2\varphi[Dh,Dh]}{J^2}$$

$$- \frac{\Delta h Dh\cdot D\varphi}{J^2} - 2\frac{D^2h[Dh,D\varphi]}{J^2} + 3\frac{D^2h[Dh,Dh]Dh\cdot D\varphi}{J^4}\Big]\,dx\,,$$

where ϕ stands for $\frac{\varphi}{J}\circ\pi$ and $J := \sqrt{1+|Dh|^2}$. By the approximation Lemma 3.9, this identity still holds if $h \in C^{2,\alpha}_\#(Q)$ and thus (3.25) follows from (3.22), recalling that by Step 1, $h_{i,n} \in C^{2,\sigma}_\#(Q)$ for some $\sigma > 0$.

In order to show (3.21), observe that Lemma 5.3, together with the bound $\|Dh_{i,n}\|_{L^\infty} \le \Lambda_0$, implies that

$$\|D^2h_{i,n}\|_{L^q(Q)} \le C(q,\Lambda_0)\|H_{i,n}\|_{L^q(Q)}\,. \tag{3.26}$$

Moreover, since $\Gamma_{i,n}$ is of class $C^{2,\sigma}$, equation (3.25) yields that $|H_{i,n}|^{p-2}H_{i,n} \in H^2(\Gamma_{i,n})$, and in turn $|H_{i,n}|^{p-2}H_{i,n} \in H^2(Q)$ (see Remark 2.1).

As before, setting $w := |H_{i,n}|^{p-2}H_{i,n}$, by approximation we may rewrite (3.25) as

$$\int_Q A(x)DwD\Big(\frac{\varphi}{J_{i,n}}\Big)J_{i,n}\,dx - \varepsilon\int_Q w\varphi\Big(|B_{i,n}|^2 - \frac{1}{p}H^2_{i,n}\Big)\,dx$$

$$+ \int_Q\big[\mathrm{div}_{\Gamma_{i,n}}(D\psi(\nu_{i,n})) + W(E(u_{i,n}))\big]\varphi\,dx - \frac{1}{\tau_n}\int_Q v_{h_{i,n}}\varphi\,dx = 0\,, \tag{3.27}$$

for all $\varphi \in H^1_\#(Q)$, with $\int_Q\varphi\,dx = 0$, where A, defined as in (3.23), is an elliptic matrix with ellipticity constants depending only on Λ_0. Recall that $w \in H^2(Q)$. We now choose $\varphi = D_s\eta$, with $\eta \in H^2_\#(Q)$, and observe that integrating by parts twice yields

$$\int_Q ADwD\Big(\frac{D_s\eta}{J_{i,n}}\Big)J_{i,n}\,dx = -\int_Q AD(D_sw)D\Big(\frac{\eta}{J_{i,n}}\Big)J_{i,n}\,dx - \int_Q D_s(AJ_{i,n})DwD\Big(\frac{\eta}{J_{i,n}}\Big)\,dx$$

$$+ \int_Q ADwD\Big(\frac{\eta D_sJ_{i,n}}{J^2_{i,n}}\Big)J_{i,n}\,dx$$

$$= -\int_Q AD(D_sw)D\Big(\frac{\eta}{J_{i,n}}\Big)J_{i,n}\,dx - \int_Q D_s(AJ_{i,n})DwD\Big(\frac{\eta}{J_{i,n}}\Big)\,dx$$

$$- \int_Q AD^2w\frac{\eta D_sJ_{i,n}}{J_{i,n}}\,dx - \int_Q D(AJ_{i,n})Dw\frac{\eta D_sJ_{i,n}}{J^2_{i,n}}\,dx\,.$$

Therefore, recalling (3.27), and by density we may conclude that for every $\eta \in H^1_\#(Q)$

$$\int_Q AD(D_sw)D\Big(\frac{\eta}{J_{i,n}}\Big)J_{i,n}\,dx = -\int_Q D_s(AJ_{i,n})DwD\Big(\frac{\eta}{J_{i,n}}\Big)\,dx$$

$$- \int_Q AD^2w\frac{\eta D_sJ_{i,n}}{J_{i,n}}\,dx - \int_Q D(AJ_{i,n})Dw\frac{\eta D_sJ_{i,n}}{J^2_{i,n}}\,dx$$

$$- \varepsilon\int_Q wD_s\eta\Big(|B_{i,n}|^2 - \frac{1}{p}H^2_{i,n}\Big)\,dx$$

$$+ \int_Q\big[\mathrm{div}_{\Gamma_{i,n}}(D\psi(\nu_{i,n})) + W(E(u_{i,n}))\big]D_s\eta\,dx - \frac{1}{\tau_n}\int_Q v_{h_{i,n}}D_s\eta\,dx\,.$$

Finally, choosing $\eta = D_s w J_{i,n}$, we obtain

$$\begin{aligned}\int_Q AD(D_s w)D(D_s w)J_{i,n}\,dx = &-\int_Q D_s(AJ_{i,n})DwD(D_s w)\,dx\\ &-\int_Q AD^2wD_swD_sJ_{i,n}\,dx - \int_Q D(AJ_{i,n})Dw\frac{D_swD_sJ_{i,n}}{J_{i,n}}\,dx\\ &-\varepsilon\int_Q wD_s(D_swJ_{i,n})\Big(|B_{i,n}|^2-\frac{1}{p}H_{i,n}^2\Big)\,dx\\ &+\int_Q\big[\mathrm{div}_{\Gamma_{i,n}}(D\psi(\nu_{i,n}))+W(E(u_{i,n}))\big]D_s(D_swJ_{i,n})\,dx\\ &-\frac{1}{\tau_n}\int_Q v_{h_{i,n}}D_s(D_swJ_{i,n}),dx\,.\end{aligned}$$

Summing the resulting equations for $s = 1, 2$, estimating $D(AJ_{i,n})$ by $D^2h_{i,n}$, and using several times Young's Inequality, we deduce

$$\begin{aligned}\int_Q |D^2w|^2\,dx \le C\int_Q\Big(&|Dw|^2|D^2h_{i,n}|^2\,dx + |H_{i,n}|^{2p+2}\\ &+|H_{i,n}|^{2p-2}|D^2h_{i,n}|^4+\frac{v_{i,n}^2}{(\tau_n)^2}+1\Big)\,dx \end{aligned}\tag{3.28}$$

for some constant C depending only on Λ_0, $D^2\psi$, and on the $C^{1,\alpha}$ bounds on $u_{i,n}$ provided by Theorem 3.5. Note that by Young's Inequality and (3.26), we have

$$\int_Q |H_{i,n}|^{2p-2}|D^2h_{i,n}|^4\,dx \le C\int_Q\big(|H_{i,n}|^{2p+2}+|D^2h_{i,n}|^{2p+2}\big)\,dx \le C\int_Q |H_{i,n}|^{2p+2}\,dx\,.$$

Combining the last estimate with (3.28), we therefore have

$$\int_Q |D^2w|^2\,dx \le C_0\int_Q\Big(|D^2h_{i,n}|^2|Dw|^2+|w|^{\frac{2(p+1)}{p-1}}+\frac{v_{i,n}^2}{(\tau_n)^2}+1\Big)\,dx\,. \tag{3.29}$$

To deal with the first term on the right-hand side, we use Hölder's inequality, (3.26) and Theorem 5.6 twice to get

$$\begin{aligned}C_0\int_Q |D^2h_{i,n}|^2|Dw|^2\,dx &\le C_0\Big(\int_Q |D^2h|^{2(p-1)}\,dx\Big)^{\frac{1}{p-1}}\Big(\int_Q |Dw|^{\frac{2(p-1)}{p-2}}\,dx\Big)^{\frac{p-2}{p-1}}\\ &\le C\|w\|_2^{\frac{2}{p-1}}\|Dw\|^2_{\frac{2(p-1)}{p-2}} \le C\|w\|_2^{\frac{2}{p-1}}\Big(\|D^2w\|_2^{\frac{p}{2(p-1)}}\|w\|_2^{\frac{p-2}{2(p-1)}}\Big)^2\\ &= C\|D^2w\|_2^{\frac{p}{p-1}}\|w\|_2^{\frac{p}{p-1}} \le C\|D^2w\|_2^{\frac{p}{p-1}}\Big(\|D^2w\|_{\frac{p}{p-1}}^{\frac{p-2}{2p}}\|w\|_{\frac{p}{p-1}}^{\frac{p+2}{2p}}\Big)^{\frac{p}{p-1}}\\ &\le C\|D^2w\|_2^{\frac{3p-2}{2(p-1)}}\|w\|_{\frac{p}{p-1}}^{\frac{p+2}{2(p-1)}} \le \frac{1}{4}\|D^2w\|_2^2+C\,,\end{aligned}$$

where in the last inequality we used the fact that $\frac{3p-2}{2(p-1)} < 2$ and that $\|w\|_{\frac{p}{p-1}} = \|H_{i,n}\|_p^{p-1}$ is uniformly bounded with respect to i, n. Using again Theorem 5.6, we also have

$$C_0\int_Q |w|^{\frac{2(p+1)}{p-1}}\,dx \le C\|D^2w\|_{\frac{p}{p-1}}^{\frac{p+2}{p}}\|w\|_{\frac{p}{p-1}}^{\frac{p^2+p+2}{p(p-1)}} \le \frac{1}{4}\|D^2w\|_2^2+C\,,$$

where as before we used the fact that $\frac{p+2}{p} < 2$ and $\|w\|_{\frac{p}{p-1}}$ is uniformly bounded. Inserting the two estimates above in (3.29), we then get

$$\int_Q |D^2w|^2\,dx \le C\int_Q\Big(1+\frac{v_{i,n}^2}{(\tau_n)^2}\Big)\,dx\,. \tag{3.30}$$

Integrating the last inequality with respect to time and using (3.9) we conclude the proof of the theorem. □

Remark 3.12. The same argument used in Step 1 of the proof of Theorem 3.11 and in the proof of (3.25) shows that if $(h, u_h) \in X$ satisfies

$$\int_Q W(E(u_h(x,h(x))))\varphi\,dx + \int_Q D\psi(-Dh,1)\cdot(-D\varphi,0)\,dx + \frac{\varepsilon}{p}\int_Q |H|^p \frac{Dh\cdot D\varphi}{J}$$
$$-\varepsilon\int_Q |H|^{p-2}H\Big[\Delta\varphi - \frac{D^2\varphi[Dh,Dh]}{J^2}$$
$$-\frac{\Delta h Dh\cdot D\varphi}{J^2} - 2\frac{D^2h[Dh,D\varphi]}{J^2} + 3\frac{D^2h[Dh,Dh]Dh\cdot D\varphi}{J^4}\Big]\,dx = 0$$

for all $\varphi \in C^2_{\#}(Q)$ such that $\int_Q \varphi\,dx = 0$, then (h, u_h) is a critical pair for the functional F.

Lemma 3.13. *Let T_0 and $\tilde{H}_n$ be as in Theorem 3.5. Then $|\tilde{H}_n|^p$ is a Cauchy sequence in $L^1(0,T_0;L^1(Q))$. Moreover, $|\tilde{H}_n|^{p-2}\tilde{H}_n$ is a Cauchy sequence in $L^1(0,T_0;L^2(Q))$.*

For the proof of the lemma we need the following inequality.

Lemma 3.14. *Let $p > 1$. There exists $c_p > 0$ such that*

$$\frac{1}{c_p}(x^{p-1}+y^{p-1}) \le \frac{|x^p-y^p|}{|x-y|} \le c_p(x^{p-1}+y^{p-1})\,.$$

Proof. By homogeneity it is enough to assume $y = 1$ and $x > 1$ and to observe that

$$\lim_{x\to+\infty}\frac{x^p-1}{(x-1)(x^{p-1}+1)} = 1 \qquad \lim_{x\to 1}\frac{x^p-1}{(x-1)(x^{p-1}+1)} = \frac{p}{2}\,.$$

□

Proof of Lemma 3.13. We split the proof into two steps.
Step 1. We start by showing that $|\tilde{H}_n|^p$ is a Cauchy sequence in $L^1(0,T_0;L^1(Q))$. Set $k := [p]$, where $[\cdot]$ denotes the integer part. Note that $k \ge 2$ since $p > 2$. From Lemma 3.14 we get

$$\begin{aligned}\int_0^{T_0}\int_Q \big||\tilde{H}_n|^p - |\tilde{H}_m|^p\big|\,dxdt &= \int_0^{T_0}\int_Q \big||\tilde{H}_n^k|^{\frac{p}{k}} - |\tilde{H}_m^k|^{\frac{p}{k}}\big|\,dxdt \\ &\le c\int_0^{T_0}\int_Q \big||\tilde{H}_n^k| - |\tilde{H}_m^k|\big|\big(|\tilde{H}_n|^k + |\tilde{H}_m|^k\big)^{\frac{p}{k}-1}\,dxdt \\ &\le c\int_0^{T_0}\Big(\int_Q |\tilde{H}_n^k - \tilde{H}_m^k|^2\,dx\Big)^{\frac{1}{2}}(\|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{p-k}dt \\ &\le c\int_0^{T_0}\big(\|\tilde{H}_n^k - \tilde{H}_m^k - M_{m,n}\|_2 + |M_{m,n}|\big)(\|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{p-k}dt\,, \end{aligned} \tag{3.31}$$

where $M_{m,n} := \int_Q(\tilde{H}_n^k - \tilde{H}_m^k)\,dx$. Set

$$w_n := |\tilde{H}_n|^{p-2}\tilde{H}_n \tag{3.32}$$

and observe that $\tilde{H}_n^k = (w_n^+)^{\frac{k}{p-1}} + (-1)^k(w_n^-)^{\frac{k}{p-1}}$. Thus,

$$|D\tilde{H}_n^k| \le \big|D(w_n^+)^{\frac{k}{p-1}}\big| + \big|D(w_n^-)^{\frac{k}{p-1}}\big| \le c|Dw_n||w_n|^{\frac{k}{p-1}-1} = c|Dw_n||\tilde{H}_n|^{k-p+1}\,. \tag{3.33}$$

From Lemma 5.7 and inequalities (3.31), (3.33) we get

$$\begin{aligned}
&\int_0^{T_0}\int_Q \big||\tilde H_n|^p-|\tilde H_m|^p\big|\,dxdt\\
&\quad\le c\int_0^{T_0}\big(\|\tilde H_n^k-\tilde H_m^k-M_{m,n}\|_{H^{-1}}^{\frac12}\|D\tilde H_n^k-D\tilde H_m^k\|_2^{\frac12}+|M_{m,n}|\big)(\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{p-k}\,dt\\
&\quad\le c\int_0^{T_0}\|\tilde H_n^k-\tilde H_m^k-M_{m,n}\|_{H^{-1}}^{\frac12}(\|Dw_n\|_2+\|Dw_m\|_2)^{\frac12}(\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{\frac{p-k+1}{2}}\,dt\\
&\qquad+\int_0^{T_0}|M_{m,n}|(\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{p-k}\,dt\,. \qquad (3.34)
\end{aligned}$$

Fix n, $m\in\mathbb{N}$. We now estimate the H^{-1}-norm of $\tilde H_n^k-\tilde H_m^k-M_{m,n}$. Recall that, in view of Remark 3.3,

$$\|\tilde H_n^k-\tilde H_m^k-M_{m,n}\|_{H^{-1}}=\|Du\|_2\,, \qquad (3.35)$$

where u is the unique Q-periodic solution of

$$\begin{cases}-\Delta u=\tilde H_n^k-\tilde H_m^k-M_{m,n} & \text{in } Q,\\ \int_Q u\,dx=0\,.\end{cases} \qquad (3.36)$$

Thus

$$\int_Q|Du|^2\,dx=\int_Q u(\tilde H_n^k-\tilde H_m^k-M_{m,n})\,dx=\int_Q u(\tilde H_n-\tilde H_m)\sum_{i=0}^{k-1}\tilde H_n^{k-1-i}\tilde H_m^i\,dx\,, \qquad (3.37)$$

where we used also the fact that $\int_Q u\,dx=0$. Fix $\delta\in(0,1)$ (to be chosen) and let $T^\delta(t):=(t\vee-\delta)\wedge\delta$. Then

$$\tilde H_n=[(\tilde H_n-\delta)^++\delta]+T^\delta(\tilde H_n)-[(-\tilde H_n-\delta)^++\delta] \qquad (3.38)$$

and (see (3.32))

$$(\tilde H_n-\delta)^++\delta=\begin{cases}w_n^{\frac{1}{p-1}} & \text{if } w_n\ge\delta^{p-1},\\ \delta & \text{otherwise.}\end{cases}$$

Hence

$$|D[(\tilde H_n-\delta)^++\delta]|\le c\frac{|Dw_n|}{\delta^{p-2}}\,, \qquad (3.39)$$

and a similar estimates holds for $D[(-\tilde H_n-\delta)^++\delta]$. We now set

$$V_{n,\delta}:=[(\tilde H_n-\delta)^++\delta]-[(-\tilde H_n-\delta)^++\delta]\,. \qquad (3.40)$$

From (3.37) we have

$$\begin{aligned}
&\int_Q|Du|^2\,dx\\
&=\int_Q u(\tilde H_n-\tilde H_m)\sum_{i=0}^{k-1}\sum_{r=0}^{k-1-i}\sum_{s=0}^{i}\binom{k-1-i}{r}\binom{i}{s}V_{n,\delta}^{k-1-i-r}V_{m,\delta}^{i-s}\big[T^\delta(\tilde H_n)\big]^r\big[T^\delta(\tilde H_m)\big]^s\,dx\\
&=\int_Q u(\tilde H_n-\tilde H_m)\sum_{i=0}^{k-1}V_{n,\delta}^{k-1-i}V_{m,\delta}^{i}\,dx\\
&+\int_Q u(\tilde H_n-\tilde H_m)\sum_{i=0}^{k-1}\sum_{(r,s)\neq(0,0)}\binom{k-1-i}{r}\binom{i}{s}V_{n,\delta}^{k-1-i-r}V_{m,\delta}^{i-s}\big[T^\delta(\tilde H_n)\big]^r\big[T^\delta(\tilde H_m)\big]^s\,dx\\
&=:L+M\,. \qquad (3.41)
\end{aligned}$$

We start by estimating the last term in the previous chain of equalities:

$$
\begin{aligned}
|M| &\le c\int_Q |u||\tilde H_n-\tilde H_m|\sum_{i=0}^{k-1}\sum_{(r,s)\neq(0,0)}\delta^{r+s}V_{n,\delta}^{k-1-i-r}V_{m,\delta}^{i-s}\,dx\\
&\le c\int_Q |u|(|\tilde H_n|+|\tilde H_m|)\sum_{\ell=1}^{k-1}\delta^\ell\big[V_{n,\delta}^{k-1-\ell}+V_{m,\delta}^{k-1-\ell}\big]\,dx\\
&\le c\delta\int_Q |u|(|\tilde H_n|+|\tilde H_m|)\big(1+V_{n,\delta}^{k-2}+V_{m,\delta}^{k-2}\big)\,dx\\
&\le c\delta\Big(\int_Q u^2\,dx\Big)^{\frac12}(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{k-1}\\
&\le \frac16\int_Q|Du|^2\,dx+c\delta^2(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{2(k-1)}\,,
\end{aligned}
\tag{3.42}
$$

where we used (3.40) and the Poincaré and Young inequalities. To deal with L, we integrate by parts and use (2.3) and the periodicity of u, $\tilde h_n$, and $\tilde h_m$ to get

$$
L=\int_Q\Big(\frac{D\tilde h_n}{\tilde J_n}-\frac{D\tilde h_m}{\tilde J_m}\Big)Du\sum_{i=0}^{k-1}V_{n,\delta}^{k-1-i}V_{m,\delta}^{i}\,dx+\int_Q\Big(\frac{D\tilde h_n}{\tilde J_n}-\frac{D\tilde h_m}{\tilde J_m}\Big)u\sum_{i=0}^{k-1}D\big(V_{n,\delta}^{k-1-i}V_{m,\delta}^{i}\big)\,dx\,,
$$

where

$$
\tilde h_n(x,t):=h_{i,n}(x)\quad\text{if } t\in[(i-1)\tau_n,i\tau_n)\quad\text{and}\quad \tilde J_n(x.t):=\sqrt{1+|D\tilde h_n(x,t)|^2}\,. \tag{3.43}
$$

From the equality above, recalling (3.32), (3.39), and (3.40), and setting

$$
\varepsilon_{n,m}:=\sup_{t\in[0,T_0]}\Big\|\frac{D\tilde h_n}{\tilde J_n}(\cdot,t)-\frac{D\tilde h_m}{\tilde J_m}(\cdot,t)\Big\|_\infty\,,
$$

we may estimate

$$
\begin{aligned}
|L| &\le c\varepsilon_{n,m}\int_Q|Du|\big(1+|\tilde H_n|^{k-1}+|\tilde H_m|^{k-1}\big)\,dx\\
&\quad+c\varepsilon_{n,m}\int_Q|u|\sum_{i=0}^{k-1}\big[|DV_{n,\delta}^{k-1-i}|V_{m,\delta}^{i}+|DV_{m,\delta}^{i}|V_{n,\delta}^{k-1-i}\big]\,dx\\
&\le\frac16\int_Q|Du|^2\,dx+c\varepsilon_{n,m}^2(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{2(k-1)}\\
&\quad+c\varepsilon_{n,m}\int_Q|u|\frac{|Dw_n|}{\delta^{p-2}}\sum_{i=0}^{k-2}V_{n,\delta}^{k-2-i}V_{m,\delta}^{i}\,dx+c\varepsilon_{n,m}\int_Q|u|\frac{|Dw_m|}{\delta^{p-2}}\sum_{i=0}^{k-2}V_{m,\delta}^{i-1}V_{n,\delta}^{k-i-1}\,dx\\
&\le\frac16\int_Q|Du|^2\,dx+c\varepsilon_{n,m}^2(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{2(k-1)}\\
&\quad+c\frac{\varepsilon_{n,m}}{\delta^{p-2}}\int_Q|u|(|Dw_n|+|Dw_m|)(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{k-2}\,dx\\
&\le\frac13\int_Q|Du|^2\,dx+c\varepsilon_{n,m}^2(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{2(k-1)}\\
&\quad+c\frac{\varepsilon_{n,m}^2}{\delta^{2(p-2)}}\int_Q(|Dw_n|+|Dw_m|)^2(1+\|\tilde H_n\|_\infty+\|\tilde H_m\|_\infty)^{2(k-2)}\,dx\,.
\end{aligned}
$$

From this estimate, (3.35), (3.36), (3.41), and (3.42), choosing $\delta^{2(p-2)} = \varepsilon_{n,m}$, with n, m so large that $\varepsilon_{n,m} < 1$ (see Theorem 3.5(i)), we obtain

$$\|\tilde{H}_n^k - \tilde{H}_m^k - M_{m,n}\|_{H^{-1}}^2 \leq c\varepsilon_{n,m}^{\alpha}\big[(1 + \|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{2(k-1)} + (\|Dw_n\|_2 + \|Dw_m\|_2)^2(1 + \|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{2(k-2)}\big], \quad (3.44)$$

where $\alpha := \min\{1, \frac{1}{p-2}\}$.

We now estimate $M_{m,n}$. Since

$$M_{m,n} = \int_Q (\tilde{H}_n^k - \tilde{H}_m^k)\, dx = \int_Q (\tilde{H}_n - \tilde{H}_m) \sum_{i=0}^{k-1} \tilde{H}_n^{k-1-i} \tilde{H}_m^i \, dx\,,$$

using the same argument with $u \equiv 1$ (see (3.44)) gives

$$|M_{m,n}| \leq c\varepsilon_{n,m}^{\frac{\alpha}{2}}\big[(1 + \|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{(k-1)} + (\|Dw_n\|_2 + \|Dw_m\|_2)(1 + \|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^{(k-2)}\big]\,.$$

From this inequality, recalling (3.32), (3.34), and (3.44), we deduce

$$\begin{aligned}\int_0^{T_0}\int_Q \big||\tilde{H}_n|^p - |\tilde{H}_m|^p\big|\, dxdt &\leq c(\varepsilon_{n,m})^{\frac{\alpha}{4}} \int_0^{T_0} (\|Dw_n\|_2 + \|Dw_m\|_2)^{\frac{1}{2}}(1 + \|w_n\|_\infty + \|w_m\|_\infty)^{\frac{p}{2(p-1)}}\, dt \\ &+ c(\varepsilon_{n,m})^{\frac{\alpha}{4}} \int_0^{T_0} (\|Dw_n\|_2 + \|Dw_m\|_2)(1 + \|w_n\|_\infty + \|w_m\|_\infty)^{\frac{1}{2}}\, dt \\ &+ c(\varepsilon_{n,m})^{\frac{\alpha}{2}} \int_0^{T_0} (1 + \|w_n\|_\infty + \|w_m\|_\infty)\, dt \\ &+ c(\varepsilon_{n,m})^{\frac{\alpha}{2}} \int_0^{T_0} (\|Dw_n\|_2 + \|Dw_m\|_2)(\|w_n\|_\infty + \|w_m\|_\infty)^{\frac{p-2}{p-1}}\, dt\,.\end{aligned}$$

Observe now that by (3.5) and (3.20) there exists $C > 0$ such that $\int_Q |w_n|\, dx \leq \|\tilde{H}_n\|_{p-1}^{p-1} \leq C$ for all n and thus, using the embedding of $H^2(Q)$ into $C(\overline{Q})$ and Poincaré's inequality,

$$\|Dw_n\|_2 + \|w_n\|_\infty \leq C(1 + \|D^2 w_n\|_2)\,. \quad (3.45)$$

Therefore, from the above inequalities and using also the fact that $\frac{1}{2} + \frac{p}{2(p-1)} < 2$ and that $1 + \frac{p-2}{p-1} < 2$, we conclude

$$\int_0^{T_0}\int_Q \big||\tilde{H}_n|^p - |\tilde{H}_m|^p\big|\, dxdt \leq c(\varepsilon_{n,m})^{\frac{\alpha}{4}} \int_0^{T_0} \big(1 + \|D^2 w_n\|_2 + \|D^2 w_m\|_2\big)^2\, dt \leq c(\varepsilon_{n,m})^{\frac{\alpha}{4}}\,,$$

where the last inequality follows from (3.21). This proves that the sequence $|\tilde{H}_n|^p$ is a Cauchy sequence in $L^1(0, T_0; L^1(Q))$. Note also that using Lemma 3.14 we have

$$\begin{aligned}\int_0^{T_0}\int_Q \big||\tilde{H}_n| - |\tilde{H}_m|\big|^p\, dxdt &\leq c\int_0^{T_0}\int_Q \big||\tilde{H}_n| - |\tilde{H}_m|\big|\big(|\tilde{H}_n| + |\tilde{H}_m|\big)^{p-1}\, dxdt \\ &\leq c\int_0^{T_0}\int_Q \big||\tilde{H}_n|^p - |\tilde{H}_m|^p\big|\, dxdt\,. \quad (3.46)\end{aligned}$$

Step 2. We now conclude the proof by showing that w_n is a Cauchy sequence in $L^1(0,T_0;L^2(Q))$. To this purpose, we use Lemma 5.7 to obtain

$$\begin{aligned}\int_0^{T_0}\|w_n-w_m\|_2\,dt&\le\int_0^{T_0}\|w_n-w_m-N_{m,n}\|_2\,dt+\int_0^{T_0}|N_{m,n}|\,dt\\&\le c\int_0^{T_0}\|w_n-w_m-N_{m,n}\|_{H^{-1}}^{\frac{2}{3}}\|D^2w_n-D^2w_m\|_2^{\frac13}\,dt+\int_0^{T_0}|N_{m,n}|\,dt\,,\end{aligned}\tag{3.47}$$

where $N_{m,n}:=\fint_Q(w_n-w_m)\,dx$. As observed in (3.35) and (3.36) , $\|w_n-w_m-N_{m,n}\|_{H^{-1}}=\|Dv\|_2$, where v is the unique Q-periodic solution of

$$\begin{cases}-\Delta v=w_n-w_m-N_{m,n} & \text{in } Q,\\ \int_Q v\,dx=0\,.\end{cases}$$

As in (3.37), using the fact that $\int_Q v\,dx=0$, we have

$$\begin{aligned}\int_Q|Dv|^2\,dx&=\int_Q(w_n-w_m-N_{m,n})v=\int_Q\big(|\tilde H_n|^{p-2}\tilde H_n-|\tilde H_m|^{p-2}\tilde H_m\big)v\,dx\\&=\int_Q\big(|\tilde H_n|^{p-2}-|\tilde H_m|^{p-2}\big)\tilde H_n v\,dx+\int_Q(\tilde H_n-\tilde H_m)|\tilde H_m|^{p-2}v\,dx\\&=:\tilde L+\tilde M\,.\end{aligned}\tag{3.48}$$

Now, by Hölder's inequality twice and the Sobolev embedding theorem,

$$\begin{aligned}|\tilde L|&\le\int_Q\big|(|\tilde H_n|^p)^{\frac{p-2}{p}}-(|\tilde H_m|^p)^{\frac{p-2}{p}}\big||\tilde H_n||v|\,dx\le\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|^{\frac{p-2}{p}}|\tilde H_n||v|\,dx\\&\le\|v\|_p\|\tilde H_n\|_\infty\Big(\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|^{\frac{p-2}{p-1}}\,dx\Big)^{\frac{p-1}{p}}\le c\|Dv\|_2\|\tilde H_n\|_\infty\||\tilde H_n|^p-|\tilde H_m|^p\|_1^{\frac{p-2}{p}}\\&\le\frac16\int_Q|Dv|^2\,dx+c\|\tilde H_n\|_\infty^2\||\tilde H_n|^p-|\tilde H_m|^p\|_1^{\frac{2(p-2)}{p}}\,.\end{aligned}\tag{3.49}$$

To estimate $\tilde M$, arguing as in the previous step (see (3.38)) and observing that $(-|\tilde H_m|^{p-2}-\delta)^+=0)$, we write

$$\begin{aligned}\tilde M&=\int_Q(\tilde H_n-\tilde H_m)\big[(|\tilde H_m|^{p-2}-\delta)^++\delta\big]v\,dx+\int_Q(\tilde H_n-\tilde H_m)\big[T^\delta(|\tilde H_m|^{p-2})-\delta\big]v\,dx\\&=\int_Q\Big(\frac{D\tilde h_n}{\tilde J_n}-\frac{D\tilde h_m}{\tilde J_m}\Big)Dv\big[(|\tilde H_m|^{p-2}-\delta)^++\delta\big]\,dx\\&\quad+\int_Q\Big(\frac{D\tilde h_n}{\tilde J_n}-\frac{D\tilde h_m}{\tilde J_m}\Big)vD\big[(|\tilde H_m|^{p-2}-\delta)^++\delta\big]\,dx+\int_Q(\tilde H_n-\tilde H_m)\big[T^\delta(|\tilde H_m|^{p-2})-\delta\big]v\,dx\,.\end{aligned}$$

Similarly to what we proved in (3.39), we have

$$|D[(|\tilde H_m|^{p-2}-\delta)^++\delta]|\le c\frac{|Dw_m|}{\delta^{\frac{1}{p-2}}}\,.$$

Therefore, arguing as in the previous step, we obtain

$$
\begin{aligned}
|\tilde{M}| &\le \frac{1}{6}\int_Q |Dv|^2\,dx + c\varepsilon_{n,m}^2(1+\|\tilde{H}_m\|_\infty)^{2(p-2)} \\
&\quad + c\varepsilon_{n,m}\int_Q |v|\frac{|Dw_m|}{\delta^{\frac{1}{p-2}}}\,dx + c\delta\int_Q |v|(\|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)\,dx \\
&\le \frac{1}{3}\int_Q |Dv|^2\,dx + c\varepsilon_{n,m}^2(1+\|\tilde{H}_m\|_\infty)^{2(p-2)} + c\frac{\varepsilon_{n,m}^2}{\delta^{\frac{2}{p-2}}}\|Dw_m\|_2^2 + c\delta^2(\|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty)^2\,,
\end{aligned}
$$

where in the last line we used the Young and Poincaré inequalities. Choosing δ so that $\delta^{\frac{2}{p-2}} = \varepsilon_{n,m}$ and recalling (3.48) and (3.49), we conclude that

$$
\begin{aligned}
\|w_n - w_m - N_{m,n}\|_{H^{-1}} &\le c\|\tilde{H}_n\|_\infty \||\tilde{H}_n|^p - |\tilde{H}_m|^p\|_1^{\frac{p-2}{p}} \\
&\quad + c(\varepsilon_{n,m})^{\frac{\beta}{2}}(1 + \|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty + \|\tilde{H}_m\|_\infty^{p-2} + \|Dw_m\|_2)\,, \qquad (3.50)
\end{aligned}
$$

where $\beta = \min\{1, p-2\}$.

Since by (3.32),

$$
N_{m,n} = \int_Q (w_n - w_m)\,dx = \int_Q \big(|\tilde{H}_n|^{p-2} - |\tilde{H}_m|^{p-2}\big)\tilde{H}_n\,dx + \int_Q (\tilde{H}_n - \tilde{H}_m)|\tilde{H}_m|^{p-2}\,dx\,,
$$

the same argument used to estimate the last two integrals in (3.48) (with $v \equiv 1$) gives

$$
\begin{aligned}
|N_{m,n}| &\le c\|\tilde{H}_n\|_\infty \||\tilde{H}_n|^p - |\tilde{H}_m|^p\|_1^{\frac{p-2}{p}} \\
&\quad + c(\varepsilon_{n,m})^{\frac{\beta}{2}}(\|\tilde{H}_n\|_\infty + \|\tilde{H}_m\|_\infty + \|\tilde{H}_m\|_\infty^{p-2} + \|Dw_m\|_2)\,.
\end{aligned}
$$

From this estimate, recalling (3.32), (3.47) and (3.50), we have that

$$
\begin{aligned}
&\int_0^{T_0} \|w_n - w_m\|_2\,dt \le c\int_0^{T_0} \||\tilde{H}_n|^p - |\tilde{H}_m|^p\|_1^{\frac{2(p-2)}{3p}} \|w_n\|_\infty^{\frac{2}{3(p-1)}}(\|D^2w_n\|_2 + \|D^2w_m\|_2)^{\frac{1}{3}}\,dt \\
&+ c(\varepsilon_{n,m})^{\frac{\beta}{3}}\int_0^{T_0} \big(1 + \|w_n\|_\infty^{\frac{1}{p-1}} + \|w_m\|_\infty^{\frac{1}{p-1}} + \|w_m\|_\infty^{\frac{p-2}{p-1}} + \|Dw_m\|_2\big)^{\frac{2}{3}}(\|D^2w_n\|_2 + \|D^2w_m\|_2)^{\frac{1}{3}}\,dt \\
&+ c\int_0^{T_0} \|w_n\|_\infty^{\frac{1}{p-1}} \||\tilde{H}_n|^p - |\tilde{H}_m|^p\|_1^{\frac{p-2}{p}}\,dt \\
&+ c(\varepsilon_{n,m})^{\frac{\beta}{2}}\int_0^{T_0} (\|w_n\|_\infty^{\frac{1}{p-1}} + \|w_m\|_\infty^{\frac{1}{p-1}} + \|w_m\|_\infty^{\frac{p-2}{p-1}} + \|Dw_m\|_2)\,dt\,.
\end{aligned}
$$

Using (3.45) and Hölder's inequality, we can bound the right-hand side of this inequality by

$$
\begin{aligned}
&\le c\int_0^{T_0} \||\tilde H_n|^p-|\tilde H_m|^p\|_1^{\frac{2(p-2)}{3p}}(1+\|D^2w_n\|_2+\|D^2w_m\|_2)^{\frac13+\frac{2}{3(p-1)}}\,dt\\
&\quad+c(\varepsilon_{n,m})^{\frac{\beta}{3}}\int_0^{T_0}\big(1+\|D^2w_n\|_2+\|D^2w_m\|_2\big)^{\frac23}(\|D^2w_n\|_2+\|D^2w_m\|_2)^{\frac13}\,dt\\
&\quad+c\int_0^{T_0}\big(1+\|D^2w_n\|_2\big)^{\frac{1}{p-1}}\||\tilde H_n|^p-|\tilde H_m|^p\|_1^{\frac{p-2}{p}}\,dt+c(\varepsilon_{n,m})^{\frac{\beta}{2}}\int_0^{T_0}(1+\|D^2w_n\|_2+\|D^2w_m\|_2)\,dt\\
&\le c\Big(\int_0^{T_0}\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|\,dxdt\Big)^{\frac{2(p-2)}{3p}}\Big[\int_0^{T_0}\big(\|D^2w_n\|_2+\|D^2w_m\|_2\big)^{\frac{p(p+1)}{(p-1)(p+4)}}\Big]^{\frac{p+4}{3p}}\\
&\quad+c\Big(\int_0^{T_0}\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|\,dxdt\Big)^{\frac{(p-2)}{2}}\Big[\int_0^{T_0}\big(1+\|D^2w_n\|_2\big)^{\frac{p}{2(p-1)}}\Big]^{\frac2p}\\
&\quad+c(\varepsilon_{n,m})^{\frac{\beta}{3}}\int_0^{T_0}(1+\|D^2w_n\|_2+\|D^2w_m\|_2)\,dt\,.
\end{aligned}
$$

Since $\frac{p(p+1)}{(p-1)(p+4)}<2$ and $\frac{p}{2(p-1)}<2$, recalling (3.21), we finally have

$$
\begin{aligned}
\int_0^{T_0}\|w_n-w_m\|_2\,dt\le c\Big(\int_0^{T_0}\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|\,dxdt\Big)^{\frac{2(p-2)}{3p}}&\\
+c\Big(\int_0^{T_0}\int_Q\big||\tilde H_n|^p-|\tilde H_m|^p\big|\,dxdt\Big)^{\frac{(p-2)}{2}}&+c(\varepsilon_{n,m})^{\frac{\beta}{3}}\,.
\end{aligned}
$$

The conclusion follows from Step 1. □

Corollary 3.15. *Let $\tilde H_n$ be the functions defined in* (3.20)*, let h be the limiting function provided by Theorem 3.5, and set*

$$H:=-\mathrm{div}\Big(\frac{Dh}{1+|Dh|^2}\Big)\,.$$

Then,

$$|\tilde H_n|^p\to|H|^p \text{ in } L^1(0,T_0;L^1(Q)) \quad\text{and}\quad |\tilde H_n|^{p-2}\tilde H_n\to|H|^{p-2}H \text{ in } L^1(0,T_0;L^2(Q)). \tag{3.51}$$

Proof. Let $\tilde h_n$ and $\tilde J_n$ be as in the proof of Lemma 3.13. From Theorem 3.5(i) we get that for all $t\in(0,T_0)$ and for all $\varphi\in C^1_\#(Q)$ we have

$$\int_Q\tilde H_n\varphi\,dx=\int_Q\frac{D\tilde h_n}{\tilde J_n}\cdot D\varphi\,dx\to\int_Q\frac{Dh}{J}\cdot D\varphi\,dx=\int_Q\tilde H\varphi\,dx\,,$$

where $J=\sqrt{1+|Dh|^2}$. Since for every t, $\tilde H_n(\cdot,t)$ is bounded in $L^p(Q)$, we deduce that for all $t\in(0,T_0)$,

$$\tilde H_n(\cdot,t)\rightharpoonup H(\cdot,t)\qquad\text{weakly in } L^p(Q). \tag{3.52}$$

On the other hand, from Lemma 3.13 we know that there exist a subsequence n_j and two functions z, w such that for a.e. t,

$$|\tilde H_{n_j}(\cdot,t)|^p\to z(\cdot,t)\quad\text{in } L^1(Q)\qquad\text{and}\qquad(|\tilde H_{n_j}|^{p-2}\tilde H_{n_j})(\cdot,t)\to w(\cdot,t)\quad\text{in } L^2(Q). \tag{3.53}$$

Moreover, for any such t there exists a further subsequence, not relabelled, (depending on t) such that $|\tilde H_{n_j}(x,t)|^p$, $|\tilde H_{n_j}(x,t)|^{p-2}\tilde H_{n_j}(x,t)$, and thus $\tilde H_{n_j}(x,t)$ converge for a.e. x. By (3.52) $\tilde H_{n_j}(x,t)\to H(\cdot,t)$ for a.e. x. Thus, we conclude that $z=|H|^p$ and $w=|H|^{p-2}H$. □

We now prove short time existence for (3.1).

Theorem 3.16. *Let $h_0 \in W^{2,p}_{\#}(Q)$, let h be the function given in Theorem 3.4, and let $T_0 > 0$ be as in Theorem 3.5. Then h is a solution of* (3.1) *in* $[0, T_0]$ *in the sense of Definition 3.1 with initial datum h_0. Moreover, there exists a non increasing g such that*

$$F(h(\cdot,t), u_h(\cdot,t)) = g(t) \quad \text{for } t \in [0,T_0] \setminus Z_0 \,, \tag{3.54}$$

where Z_0 is a set of zero measure, and

$$F(h(\cdot,t), u_h(\cdot,t)) \leq g(t+) \quad \text{for } t \in Z_0 \,. \tag{3.55}$$

This result motivates the following definition.

Definition 3.17. We say that a solution to (3.1) is *variational* if it is the limit of a subsequence of the minimizing movements scheme as in Theorem 3.5(i).

Proof of Theorem 3.16. Let $\tilde{H}_n$, $\tilde{h}_n$, $\tilde{J}_n$ be the functions given in (3.20), and (3.43). Set $\tilde{W}_n(x,t) := W(E(u_{i,n})(x, h_{i,n}(x)))$ and $\tilde{v}_n(x,t) := v_{h_{i,n}}(x)$ for $t \in [(i-1)\tau_n, i\tau_n)$. Moreover, define $\hat{v}_n := \frac{\tilde{v}_n}{\tau_n}$. Note that for all t, $\hat{v}_n(\cdot,t)$ is the unique Q-periodic solution to

$$\begin{cases} \Delta_{\Gamma_{\tilde{h}_n(\cdot,t-\tau_n)}} w = \dfrac{1}{\tilde{J}_n(\cdot,t-\tau_n)} \dfrac{\partial h_n(\cdot,t)}{\partial t} \\[2mm] \displaystyle\int_{\Gamma_{\tilde{h}_n(\cdot,t-\tau_n)}} w \, d\mathcal{H}^2 = 0 \,. \end{cases} \tag{3.56}$$

Fix $t \in (0, T_0)$ and a sequence (i_k, n_k) such that $t_k := i_k \tau_{n_k} \to t$. Summing (3.22) from $i = 1$ to $i = i_k$, we get

$$\begin{aligned} &\int_0^{t_k}\!\!\int_Q \tilde{W}_{n_k} \varphi \, dxdt + \int_0^{t_k}\!\!\int_Q D\psi(-D\tilde{h}_{n_k}, 1)\cdot(-D\varphi, 0)\, dxdt + \frac{\varepsilon}{p}\int_0^{t_k}\!\!\int_Q |\tilde{H}_{n_k}|^p \frac{D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}_{n_k}}\, dxdt \\ &\quad - \varepsilon \int_0^{t_k}\!\!\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\Big[\Delta\varphi - \frac{D^2\varphi[D\tilde{h}_{n_k}, D\tilde{h}_{n_k}]}{\tilde{J}^2_{n_k}} \\ &\qquad - \frac{\Delta\tilde{h}_{n_k} D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^2_{n_k}} - 2\frac{D^2\tilde{h}_{n_k}[D\tilde{h}_{n_k}, D\varphi]}{\tilde{J}^2_{n_k}} + 3\frac{D^2\tilde{h}_{n_k}[D\tilde{h}_{n_k}, D\tilde{h}_{n_k}]D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^4_{n_k}}\Big]\, dxdt \\ &\quad - \int_0^{t_k}\!\!\int_Q \hat{v}_{n_k}\varphi\, dxdt = 0\,. \end{aligned} \tag{3.57}$$

We claim that we can pass to the limit in the above equation to get

$$\begin{aligned} &\int_0^t\!\!\int_Q W(E(u(x,h(x,s),s)))\varphi\, dxds + \int_0^t\!\!\int_Q D\psi(-Dh,1)\cdot(-D\varphi,0)\, dxds \\ &\quad + \frac{\varepsilon}{p}\int_0^t\!\!\int_Q |H|^p \frac{Dh\cdot D\varphi}{J}\, dxds \\ &\quad - \varepsilon\int_0^t\!\!\int_Q |H|^{p-2}H\Big[\Delta\varphi - \frac{D^2\varphi[Dh,Dh]}{J^2} \\ &\qquad - \frac{\Delta h Dh\cdot D\varphi}{J^2} - 2\frac{D^2h[Dh,D\varphi]}{J^2} + 3\frac{D^2h[Dh,Dh]Dh\cdot D\varphi}{J^4}\Big]\, dxds \\ &\quad - \int_0^t\!\!\int_Q \hat{v}\varphi\, dxds = 0\,, \end{aligned} \tag{3.58}$$

where $\hat{v}(\cdot,t)$ is the unique periodic solution in $H^1_\#(\Gamma(t))$ to

$$\begin{cases} \Delta_{\Gamma_{h(\cdot,t)}} w = \dfrac{1}{J(\cdot,t)}\dfrac{\partial h(\cdot,t)}{\partial t}\,, \\ \displaystyle\int_{\Gamma_{h(\cdot,t)}} w\, d\mathcal{H}^2 = 0 \end{cases} \tag{3.59}$$

for a.e. $t \in (0,T_0)$. To prove the claim, observe that the convergence of the first two integrals in (3.57) immediately follows from (i) and (iii) of Theorem 3.5. The convergence of the third integral (3.57) follows from (3.51) and (i) of Theorem 3.5. Similarly (3.51) and (i) of Theorem 3.5 imply that

$$\int_0^{t_k}\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\Big[\Delta\varphi - \frac{D^2\varphi[D\tilde{h}_{n_k}, D\tilde{h}_{n_k}]}{\tilde{J}^2_{n_k}}\Big]\,dxdt \to \int_0^t\int_Q |H|^{p-2}H\Big[\Delta\varphi - \frac{D^2\varphi[Dh,Dh]}{J^2}\Big]\,dxds\,.$$

Next we show the convergence of

$$\int_0^{t_k}\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\Big[-\frac{\Delta\tilde{h}_{n_k} D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^2_{n_k}} - 2\frac{D^2\tilde{h}_{n_k}[D\tilde{h}_{n_k}, D\varphi]}{\tilde{J}^2_{n_k}} + 3\frac{D^2\tilde{h}_{n_k}[D\tilde{h}_{n_k}, D\tilde{h}_{n_k}]D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^4_{n_k}}\Big]\,dxdt$$

to the corresponding term in (3.58). To this purpose, we only show that

$$\int_0^{t_k}\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\frac{\Delta\tilde{h}_{n_k} D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^2_{n_k}}\,dxdt \to \int_0^t\int_Q |H|^{p-2}H\frac{\Delta h Dh\cdot D\varphi}{J^2}\,dxds \tag{3.60}$$

since the convergence of the other terms can be shown in a similar way. To prove (3.60), we first observe that by (3.5) and Theorem 3.5(i) we have $\Delta\tilde{h}_{n_k}(\cdot,t) \rightharpoonup \Delta h(\cdot,t)$ in $L^p(Q)$ for all $t\in(0,T_0)$. On the other hand, (3.51) yields that for a.e. $t\in(0,T_0)$ we have $(\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k})(\cdot,t) \to (|H|^{p-2}H)(\cdot,t)$ in $L^2(Q)$. Therefore, for a.e. $t\in(0,T_0)$

$$\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\frac{\Delta\tilde{h}_{n_k} D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^2_{n_k}}\,dx \to \int_Q |H|^{p-2}H\frac{\Delta h Dh\cdot D\varphi}{J^2}\,dx\,.$$

The conclusion then follows by applying the Lebesgue dominated convergence theorem after observing that by (2.9) and (3.5),

$$\Big|\int_Q |\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\frac{\Delta\tilde{h}_{n_k} D\tilde{h}_{n_k}\cdot D\varphi}{\tilde{J}^2_{n_k}}\,dx\Big| \le C\|\Delta\tilde{h}_{n_k}\|_{L^2(Q)}\||\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\|_{L^2(Q)} \le C\||\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\|_{L^2(Q)}$$

and that $\||\tilde{H}_{n_k}|^{p-2}\tilde{H}_{n_k}\|_{L^2(Q)}$ converges in $L^1(0,T_0)$ thanks to (3.51).

Note (3.51) implies that for a.e. $t\in(0,T_0)$ we have $\|\tilde{H}_{n_k}(\cdot,t)\|_{L^p(Q)} \to \|H(\cdot,t)\|_{L^p(Q)}$. Since $\tilde{H}_{n_k}(\cdot,t) \rightharpoonup H(\cdot,t)$ in $L^p(Q)$ (see (3.52)), we may conclude that $\tilde{H}_{n_k}(\cdot,t)\to H(\cdot,t)$ in $L^p(Q)$ for a.e. $t\in(0,T_0)$. Therefore, by (2.3) and [1, Lemma 7.2], we also have $\tilde{h}_{n_k}(\cdot,t)\to h(\cdot,t)$ in $W^{2,p}_\#(Q)$ for a.e. $t\in(0,T_0)$. Thus, by (2.9) and (3.5) and the Lebesgue dominated convergence theorem we infer that

$$\int_0^{T_0}\int_Q |D^2\tilde{h}_{n_k} - D^2 h|^p\,dxdt \to 0\,. \tag{3.61}$$

This, together with the fact that $h_n \rightharpoonup h$ weakly in $H^1(0,T_0;H^{-1}_\#(Q))$ (see (3.6)), implies that

$$\frac{1}{\tilde{J}_{n_k}(\cdot,\cdot-\tau_{n_k})}\frac{\partial h_{n_k}}{\partial t} \rightharpoonup \frac{1}{J}\frac{\partial h}{\partial t} \quad \text{in } L^2(0,T_0;H^{-1}_\#(Q)). \tag{3.62}$$

Indeed, for any $\varphi \in L^2(0,T_0;H^1_\#(Q))$,

$$
\begin{aligned}
&\Big|\int_0^{T_0}\int_Q\Big(\frac{1}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})}\frac{\partial h_{n_k}}{\partial t}-\frac{1}{J}\frac{\partial h}{\partial t}\Big)\varphi\,dxdt\Big|\\
&\le\Big|\int_0^{T_0}\int_Q\Big(\frac{1}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})}-\frac{1}{J}\Big)\frac{\partial h_{n_k}}{\partial t}\varphi\,dxdt\Big|+\Big|\int_0^{T_0}\int_Q\Big(\frac{\partial h_{n_k}}{\partial t}-\frac{\partial h}{\partial t}\Big)\frac{\varphi}{J}\,dxdt\Big|\\
&\le\int_0^{T_0}\int_Q\Big\|\frac{\partial h_{n_k}}{\partial t}\Big\|_{H^{-1}}\Big\|\frac{\varphi}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})}-\frac{\varphi}{J}\Big\|_{H^1}dxdt+\Big|\int_0^{T_0}\int_Q\Big(\frac{\partial h_{n_k}}{\partial t}-\frac{\partial h}{\partial t}\Big)\frac{\varphi}{J}\,dxdt\Big|.
\end{aligned}
\tag{3.63}
$$

Since $H^1_\#(Q)$ is embedded in $L^q(Q)$ for all $q\ge 1$, we deduce from (3.61) that $\frac{\varphi}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})}\to\frac{\varphi}{J}$ in $L^2(0,T_0;H^1_\#(Q))$. This convergence together with (3.3) shows that the second last integral in (3.63) vanishes in the limit. On the other hand, also the last integral in (3.63) vanishes in the limit since $h_{n_k}\rightharpoonup h$ weakly in $H^1(0,T_0;H^{-1}_\#(Q))$. Thus, (3.62) follows.

Arguing as in the proof of Theorem 3.11 and integrating with respect to t, we have from (3.56),

$$
\int_0^t\int_Q A_{n_k}D\hat v_n\cdot D\varphi\,dxds=\int_0^t\int_Q\frac{1}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})}\frac{\partial h_{n_k}}{\partial t}\varphi\,dxds \tag{3.64}
$$

for all $\varphi\in L^2(0,T_0;H^1_\#(Q))$, where

$$
A_{n_k}(x,t):=\bigg(I-\frac{D\tilde h_{n_k}(\cdot,\cdot-\tau_{n_k})\otimes D\tilde h_{n_k}(\cdot,\cdot-\tau_{n_k})}{\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})^2}\bigg)\tilde J_{n_k}(\cdot,\cdot-\tau_{n_k})\,.
$$

Note that (3.12) implies that $A_{n_k}(x,t)$ is an elliptic matrix with ellipticity constants depending only on Λ_0 for all (x,t). Therefore, (3.64) immediately implies that

$$
\int_0^{T_0}\int_Q|D\hat v_{n_k}|^2\,dxdt\le c\int_0^{T_0}\Big\|\frac{\partial h_{n_k}}{\partial t}\Big\|^2_{H^{-1}}dt\le c
$$

thanks to (3.3). Since $A_{n_k}\to A:=(I-\frac{Dh\otimes Dh}{J^2})J$ in $L^\infty(0,T_0;L^\infty(Q))$ by Theorem 3.5(i), from the estimate above and recalling (3.62) and (3.64), we conclude that

$$
\hat v_{n_k}\rightharpoonup\hat v\quad\text{weakly in } L^2(0,T_0;H^1_\#(Q)),
$$

where $\hat v$ satisfies

$$
\int_0^t\int_Q AD\hat v\cdot D\varphi\,dxds=\int_0^t\int_Q\frac{1}{J}\frac{\partial h}{\partial t}\varphi\,dxds
$$

for all $\varphi\in L^2(0,T_0;H^1_\#(Q))$ and for all $t\in(0,T_0)$. In turn, letting φ vary in a countable dense subset of $H^1_\#(Q)$ and differentiating the above equation with respect to t, we conclude that for a.e. $t\in(0,T_0)$ $\hat v(\cdot,t)$ is the unique solution in $H^1_\#(\Gamma_{h(\cdot,t)})$ to (3.59) for a.e. $t\in(0,T_0)$. This shows that the last integral in (3.57) converges and thus (3.58) holds. Again letting φ vary in a countable dense subset of $H^1_\#(Q)$ and differentiating (3.58) with respect to t we obtain

$$
\begin{aligned}
&\int_Q W(E(u(x,h(x,t),t)))\varphi\,dx+\int_Q D\psi(-Dh,1)\cdot(-D\varphi,0)\,dx+\frac{\varepsilon}{p}\int_Q|H|^p\frac{Dh\cdot D\varphi}{J}\,dx\\
&\quad-\varepsilon\int_Q|H|^{p-2}H\bigg[\Delta\varphi-\frac{D^2\varphi[Dh,Dh]}{J^2}\\
&\qquad-\frac{\Delta hDh\cdot D\varphi}{J^2}-2\frac{D^2h[Dh,D\varphi]}{J^2}+3\frac{D^2h[Dh,Dh]Dh\cdot D\varphi}{J^4}\bigg]dx\\
&\quad-\int_Q\hat v\varphi\,dx=0
\end{aligned}
\tag{3.65}
$$

for all $\varphi \in H^1_\#(Q)$. Since, by (3.21), $|H|^{p-2}H \in L^2(0,T_0; H^2_\#(Q))$, arguing as in Step 2 of the proof of Theorem 3.11, we have that the above equation is equivalent to

$$\varepsilon \int_{\Gamma_h} D_{\Gamma_h}(|H|^{p-2}H) D_{\Gamma_h}\phi \, d\mathcal{H}^2 - \varepsilon \int_{\Gamma_h} |H|^{p-2}H\Big(|B|^2 - \frac{1}{p}H^2\Big)\phi \, d\mathcal{H}^2$$
$$+ \int_{\Gamma_h} \big[\mathrm{div}_{\Gamma_h}(D\psi(\nu)) + W(E(u))\big]\phi \, d\mathcal{H}^2 - \int_{\Gamma_h} \hat{v}\phi \, d\mathcal{H}^2 = 0$$

for a.e. $t \in (0, T_0)$, where $\phi := \frac{\varphi}{J}$. This equation, together with (3.59), implies that h is a solution to (3.1) in the sense of Definition 3.1.

Next, to show that the energy decreases during the evolution, we observe first that for every n the map $t \mapsto F(\tilde{h}_n(\cdot,t), \tilde{u}_n(\cdot,t))$ is non increasing, as shown in (3.4). Note also that thanks to (3.51) we may assume, up to extracting a further subsequence, that for a.e. t, $\tilde{H}_n \to H$ in $L^p(Q)$. This fact, together with (i) and (iii) of Theorem 3.5, implies that for all such t, $F(\tilde{h}_n(\cdot,t), \tilde{u}_n(\cdot,t)) \to F(h(\cdot,t), u(\cdot,t))$. Thus also (3.54) follows. Let $t \in Z_0$ and choose $t_n \to t^+$, with $t_n \notin Z_0$ for every n. Finally, since $h(\cdot,t_n) \rightharpoonup h(\cdot,t)$ weakly in $W^{2,p}_\#(Q)$ by (3.5), by lower semicontinuity we get that

$$F(h(\cdot,t), u(\cdot,t)) \le \liminf_n F(h(\cdot,t_n), u(\cdot,t_n)) = \lim_n g(t_n) = g(t+).$$

□

4. Liapunov stability of the flat configuration

In this section we are going to study the Liapunov stability of an admissible flat configuration. Take $h(x) \equiv d > 0$ and let u_d denote the corresponding elastic equilibrium. Throughout this section we assume that the Dirichlet datum w_0 is affine, i.e., of the form $w_0(x,y) = (A[\,x\,], 0)$ for some $A \in \mathbb{M}^{2\times 2}$. As already mentioned, a tipical choice is given by $w_0(x,y) := (e_0^1 x_1, e_0^2 x_2, 0)$, where the vector $e_0 := (e_0^1, e_0^2)$, with e_0^1, $e_0^2 > 0$, embodies the mismatch between the crystalline lattices of film and substrate.

A detailed analysis of the so-called Asaro-Tiller-Grinfeld morphological stability/instability was undertaken in [10, 25]. It was shown that if d is sufficiently small, then the flat configuration (d, u_d) is a volume constrained local minimizer for the functional

$$G(h,u) := \int_{\Omega_h} W(E(u))\,dz + \int_{\Gamma_h} \psi(\nu)\,d\mathcal{H}^2. \tag{4.1}$$

To be precise, it was proved that if d is small enough, then the second variation $\partial^2 G(d, u_d)$ is positive definite and that, in turn, this implies the local minimality property. In order to state the results of this section, we need to introduce some preliminary notation. In the following, given $h \in C^2_\#(Q)$, $h \ge 0$, ν will denote the unit vector field coinciding with the gradient of the signed distance from $\Omega_h^\#$, which is well defined in a sufficiently small tubular neighborhood of $\Gamma_h^\#$. Moreover, for every $x \in \Gamma_h$ we set

$$\mathbb{B}(x) := D\nu(x). \tag{4.2}$$

Note that the bilinear form associated with $\mathbb{B}(x)$ is symmetric and, when restricted to $T_x\Gamma_h \times T_x\Gamma_h$, it coincides with the *second fundamental form of* Γ_h *at* x. Here $T_x\Gamma_h$ denotes the tangent space to Γ_h at x. For $x \in \Gamma_h$ we also set $H(x) := \mathrm{div}\nu(x) = \mathrm{trace}\,\mathbb{B}(x)$, which is the *sum of the principal curvatures* of Γ_h at x. Given a (sufficiently) smooth and positively 1-homogeneous function $\omega : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$, we consider the *anisotropic second fundamental form* defined as

$$\mathbb{B}^\omega := D(D\omega \circ \nu)$$

and we set

$$H^\omega := \mathrm{trace}\,\mathbb{B}^\omega = \mathrm{div}\,(D\omega \circ \nu). \tag{4.3}$$

We also introduce the following space of periodic displacements

$$A(\Omega_h) := \{u \in LD_\#(\Omega_h;\mathbb{R}^3) :\ u(x,0)=0\}\,. \tag{4.4}$$

Given a regular configuration $(h,u_h)\in X$ with $h\in C^2_\#(Q)$ and $\varphi\in \widetilde{H}^1_\#(Q)$, where

$$\widetilde{H}^1_\#(Q) := \left\{\varphi\in H^1_\#(Q) :\ \int_Q \varphi\,dx = 0\right\}, \tag{4.5}$$

we recall that the second variation of G at (h,u_h) with respect to the direction φ is

$$\frac{d^2}{dt^2}G(h+t\varphi,u_{h+t\varphi})|_{t=0}\,,$$

where, as usual, $u_{h+t\varphi}$ denotes the elastic equilibrium in $\Omega_{h+t\varphi}$. It turns out (see [10, Theorem 4.1]) that

$$\begin{aligned}\frac{d^2}{dt^2}G(h+t\varphi,u_{h+t\varphi})|_{t=0} &= \partial^2 G(h,u_h)[\varphi]\\ &\quad -\int_{\Gamma_h}(W(E(u_h))+H^\psi)\,\mathrm{div}_{\Gamma_h}\left[\left(\frac{(Dh,|Dh|^2)}{\sqrt{1+|Dh|^2}}\circ\pi\right)\phi^2\right]d\mathcal{H}^2\,, \end{aligned} \tag{4.6}$$

where $\partial^2 G(h,u_h)[\varphi]$ is the (non local) quadratic form defined as

$$\begin{aligned}\partial^2 G(h,u_h)[\varphi] :=& -2\int_{\Omega_h} W(E(v_\phi))\,dz + \int_{\Gamma_h} D^2\psi(\nu)[D_{\Gamma_h}\phi, D_{\Gamma_h}\phi]\,d\mathcal{H}^2\\ &+\int_{\Gamma_h}(\partial_\nu[W(E(u_h)] - \mathrm{trace}(\mathbb{B}^\psi\mathbb{B}))\,\phi^2\,d\mathcal{H}^2\,,\end{aligned} \tag{4.7}$$

$$\phi := \frac{\varphi}{\sqrt{1+|Dh|^2}}\circ\pi\,,$$

and v_ϕ the unique solution in $A(\Omega_h)$ to

$$\int_{\Omega_h}\mathbb{C}E(v_\phi) : E(w)\,dz = \int_{\Gamma_h}\mathrm{div}_{\Gamma_h}(\phi\,\mathbb{C}E(u_h))\cdot w\,d\mathcal{H}^2 \qquad \text{for all } w\in A(\Omega_h)\,. \tag{4.8}$$

Note that if (h,u_h) is a critical pair of G (see Definition 3.8 with $\varepsilon=0$), then the integral in (4.6) vanishes so that

$$\frac{d^2}{dt^2}G(h+t\varphi,u_{h+t\varphi})|_{t=0} = \partial^2 G(h,u_h)[\varphi]\,.$$

Throughout this section α will denote a fixed number in the interval $(0,1-\frac{2}{p})$. The next result is a simple consequence of [10, Theorem 6.6].

Theorem 4.1. *Assume that the surface density ψ is of class C^3 away from the origin, it satisfies* (2.1)*, and the following convexity condition holds: for every $\xi\in S^2$,*

$$D^2\psi(\xi)[w,w]>0 \qquad \text{for all } w\perp\xi,\ w\neq 0. \tag{4.9}$$

If

$$\partial^2 G(d,u_d)[\varphi]>0 \qquad \text{for all } \varphi\in\widetilde{H}^1_\#(Q)\setminus\{0\}\,, \tag{4.10}$$

then there exists $\delta>0$ such that

$$G(d,u_d)<G(k,v)$$

for all $(k,v)\in X$, with $|\Omega_k|=|\Omega_d|$, $0<\|k-d\|_{C^{1,\alpha}_\#(Q)}\le\delta$.

Proof. By condition (4.10) and [10, Theorem 6.6] there exists $\delta_0 > 0$ such that if $0 < \|k-d\|_{C^1_\#(Q)} \le \delta_0$ and $\|D\eta\|_\infty \le 1 + \|Du_d\|_\infty$, with $(k,\eta) \in X$, then

$$G(d, u_d) < G(k, \eta)\,. \tag{4.11}$$

Note that we may choose $0 < \delta < \delta_0$ such that if $\|k-d\|_{C^{1,\alpha}_\#(Q)} \le \delta$ and u_k is the elastic equilibrium corresponding to k, by elliptic regularity (see also Lemma 5.1) we have that $\|Du_k\|_\infty \le 1+\|Du_d\|_\infty$. Therefore, using (4.11) with $\eta := u_k$, we may conclude that

$$G(d, u_d) < G(k, u_k) \le G(k, v)\,,$$

where in the last inequality we used the minimality of u_k, and the result follows. □

Remark 4.2. It can be shown that Theorem 4.1 continues to hold if (4.9) is replaced by the weaker condition

$$D^2\psi(e_3)[w,w] > 0 \qquad \text{for all } w \perp e_3,\ w \neq 0. \tag{4.12}$$

Indeed, (4.12) implies that (4.9) holds for all ξ belonging to a suitable neighborhood $U \subset S^2$ of e_3. In turn, by choosing δ sufficiently small we can ensure that the outer unit normals to Γ_k lie in U, provided $\|k-d\|_{C^{1,\alpha}_\#(Q)} < \delta$. A careful inspection of the proof of [10, Theorem 6.6] shows that under these circumstances condition (4.9) is only needed to hold at vectors $\xi \in U$.

Remark 4.3. Under assumption (4.9), it can be shown that (4.10) is equivalent to having (see [10, Corollary 4.8])

$$\inf\{\partial^2 G(d,u_d)[\varphi] :\ \varphi \in \widetilde{H}^1_\#(Q),\ \|\varphi\|_{H^1_\#(Q)} = 1\} =: m_0 > 0\,, \tag{4.13}$$

i.e.,

$$\partial^2 G(d,u_d)[\varphi] \ge m_0 \|\varphi\|^2_{H^1_\#(Q)} \qquad \text{for all } \varphi \in \widetilde{H}^1_\#(Q)\,.$$

Remark 4.4. Note that if the profile $h \equiv d$ is flat, then the corresponding elastic equilibrium u_d is affine. It immediately follows that (d, u_d) is a critical pair in the sense of Definition 3.8.

We now consider the case of a non-convex surface energy density ψ, and introduce the "relaxed" functional defined for all $(h,u) \in X$ as

$$\overline{G}(h,u) := \int_{\Omega_h} W(E(u))\,dz + \int_{\Gamma_h} \psi^{**}(\nu)\,d\mathcal{H}^2\,, \tag{4.14}$$

where ψ^{**} is the convex envelope of ψ. It turns out that if the boundary of the Wulff shape W_ψ associated with the nonconvex density ψ contains a flat horizontal facet, then the flat configuration is always an isolated volume-constrained local minimizer, irrespectively of the value of d. We recall that the Wulff shape W_ψ is given by (see [21, Definition 3.1])

$$W_\psi := \{z \in \mathbb{R}^3 :\ z\cdot\nu < \psi(\nu) \text{ for all } \nu \in S^2\}\,.$$

The following result can be easily obtained from [10, Theorem 7.5 and Remark 7.6] arguing as in the last part of the proof of Theorem 4.1.

Theorem 4.5. *Let $\psi : \mathbb{R}^3 \to [0,+\infty)$ be a Lipschitz positively one-homogeneous function, satisfying (2.1), and let $\{(x,y) \in \mathbb{R}^3 :\ |x| \le \alpha, y = \beta\} \subset \partial W_\psi$ for some α, $\beta > 0$. Then there exists $\delta > 0$ such that*

$$\overline{G}(d,u_d) < \overline{G}(k,v)$$

for all $(k,v) \in X$, with $|\Omega_k| = |\Omega_d|$, $0 < \|k-d\|_{C^{1,\alpha}_\#(Q)} \le \delta$.

In the next two subsections we use the previous theorems to study the Liapunov stability of the flat configuration both in the convex and nonconvex case.

Definition 4.6. We say that the flat configuration (d, u_d) is *Liapunov stable* if for every $\sigma > 0$, there exists $\delta(\sigma) > 0$ such that if $(h_0, u_0) \in X$ with $|\Omega_{h_0}| = |\Omega_d|$ and $\|h_0 - d\|_{W^{2,p}_\#(Q)} \le \delta(\sigma)$, then every variational solution h to (3.1) according to Definition 3.17, with initial datum h_0, exists for all times and $\|h(\cdot,t) - d\|_{W^{2,p}_\#(Q)} \le \sigma$ for all $t > 0$.

4.1. **The case of a non-convex surface density.** In this subsection will show that if the boundary of the Wulff shape W_ψ associated with ψ contains a flat horizontal facet, then the flat configuration is always Liapunov stable.

Theorem 4.7. *Let $\psi : \mathbb{R}^3 \to [0, +\infty)$ be a positively one-homogeneous function of class C^2 away from the origin, such that (2.1) holds, and let $\{(x, y) \in \mathbb{R}^3 : |x| \le \alpha, y = \beta\} \subset \partial W_\psi$ for some α, $\beta > 0$. Then for every $d > 0$ the flat configuration (d, u_d) is Liapunov stable (according to Definition 4.6).*

Proof. We start by observing that from the assumptions on ψ, e_3 is normal to boundary ∂W_ψ of the Wulff shape W_ψ associated with ψ. Thus, by [21, Proposition 3.5-(iv)] it follows that $\psi(e_3) = \psi^{**}(e_3)$. In turn, by Theorem 4.5, we may find $\delta > 0$ such that

$$F(d, u_d) = \overline{G}(d, u_d) < \overline{G}(k, v) \le F(k, v) \tag{4.15}$$

for all $(k, v) \in X$, with $|\Omega_k| = |\Omega_d|$ and $0 < \|k - d\|_{C^{1,\alpha}_\#(Q)} \le \delta$. Fix $\sigma > 0$ and choose $\delta_0 \in (0, \min\{\delta, \sigma/2\})$ so small that

$$\|h - d\|_{C^{1,\alpha}_\#(Q)} \le \delta_0 \Longrightarrow \|Dh\|_\infty < \Lambda_0 \,, \tag{4.16}$$

where Λ_0 is as in (2.6). For every $\tau > 0$ set

$$\omega(\tau) := \sup\bigl\{\|k - d\|_{C^{1,\alpha}_\#(Q)}\bigr\}$$

where the supremum is taken over all $(k, v) \in X$ such that

$$|\Omega_k| = |\Omega_d| \,, \quad \|k - d\|_{C^{1,\alpha}_\#(Q)} \le \delta \,, \quad \text{and } F(k, v) - F(d, u_d) \le \tau \,.$$

Clearly, $\omega(\tau) > 0$ for $\tau > 0$. We claim that $\omega(\tau) \to 0$ as $\tau \to 0^+$. Indeed, to see this we assume by contradiction that there exists a sequence $(k_n, v_n) \in X$, with $|\Omega_{k_n}| = |\Omega_d|$, such that

$$\liminf_n F(k_n, v_n) \le F(d, u_d) \quad \text{and} \quad 0 < c_0 \le \|k_n - d\|_{C^{1,\alpha}_\#(Q)} \le \delta \tag{4.17}$$

for some $c_0 > 0$. By Lemma 5.3, up to a subsequence, we may assume that $k_n \rightharpoonup k$ in $W^{2,p}_\#(Q)$ and that $v_n \rightharpoonup v$ in $H^1_{\mathrm{loc}}(\Omega_k; \mathbb{R}^3)$ for some $(k, v) \in X$ satisfying $\delta \ge \|k - d\|_{C^{1,\alpha}_\#(Q)} \ge c_0$, since $W^{2,p}_\#(Q)$ is compactly embedded in $C^{1,\alpha}_\#(Q)$. By lower semicontinuity we also have that

$$F(k, v) \le \liminf_n F(k_n, v_n) \le F(d, u_d) \,,$$

which contradicts (4.15).

Let $\delta(\sigma)$ so small that if $\|h_0 - d\|_{W^{2,p}_\#(Q)} \le \delta(\sigma)$ then

$$\|h_0 - d\|_{C^{1,\alpha}_\#(Q)} < \delta_0 \quad \text{and } F(h_0, u_0) - F(d, u_d) \le \omega^{-1}(\delta_0/2) \,,$$

where ω^{-1} is the generalized inverse of ω defined as $\omega^{-1}(s) := \sup\{\tau > 0 : \omega(\tau) \le s\}$ for all $s > 0$. Note that since $\omega(\tau) > 0$ for $\tau > 0$ and $\omega(\tau) \to 0$ as $\tau \to 0+$ we have that $\omega^{-1}(s) \to 0$ as $s \to 0+$. Let h be a variational solution as in Theorem 3.4 (see Definition 3.17). Let

$$T_1 := \sup\{t > 0 : \|h(\cdot, s) - d\|_{C^{1,\alpha}_\#(Q)} \le \delta_0 \quad \text{for all } s \in (0, t)\} \,.$$

Note that by Theorem 3.5, $T_1 > 0$. We claim that $T_1 = +\infty$. Indeed, if T_1 were finite, then, recalling (3.7), we would get for all $s \in [0, T_1]$

$$F(h(\cdot, T_1), u_{h(\cdot,T_1)}) - F(d, u_d) \leq F(h_0, u_0) - F(d, u_d) \leq \omega^{-1}(\delta_0/2)\,, \tag{4.18}$$

which implies $\|h(\cdot, T_1)-d\|_{C^{1,\alpha}_{\#}(Q)} \leq \delta_0/2$ by the definition of ω. Then, equation (4.16), Remark 3.6, and Theorem 3.5 would imply that there exists $T > T_1$ such that $\|h(\cdot, t) - d\|_{C^{1,\alpha}_{\#}(Q)} \leq \delta_0$ for all $t \in (T_1, T)$, thus giving a contradiction. We conclude that $T_1 = +\infty$ and that $\|h(\cdot, t)-d\|_{C^{1,\alpha}_{\#}(Q)} \leq \delta_0$ for all $t > 0$. Therefore, (4.16) implies that $\|Dh(\cdot, t)\|_\infty < \Lambda_0$ for all times, which, together with Remark 3.6, gives that h is a solution to (3.1) for all times. Moreover, by (4.18) we have also shown that $F(h(\cdot, t), u_{h(\cdot,t)}) - F(d, u_d) \leq \omega^{-1}(\delta_0/2)$ for all $t > 0$, which by (4.15) implies that

$$\varepsilon \int_{\Gamma_{h(\cdot,t)}} |H|^p \, d\mathcal{H}^2 \leq \omega^{-1}(\delta_0/2)\,.$$

Using elliptic regularity (see (2.3)), this inequality and the fact that $\|h(\cdot, t) - d\|_\infty \leq \sigma/2$ for all $t > 0$ imply that $\|h(\cdot, t) - d\|_{W^{2,p}_{\#}(Q)} \leq \sigma$ provided that δ_0 and in turn $\delta(\sigma)$ are chosen sufficiently small. □

4.2. **The case of a convex surface density.** In this section we will show that, under the convexity assumption (4.9), the condition $\partial^2 G(d, u_d) > 0$ implies that (d, u_d) is asymptotically stable for the regularized evolution equation (3.1) (see Theorem 4.14 below). We start by addressing the Liapunov stability (see Definition 4.6).

Theorem 4.8. *Assume that the surface density ψ satisfies the assumptions of Theorem 4.1 and that the flat configuration (d, u_d) satisfies* (4.10)*. Then (d, u_d) is Liapunov stable.*

Proof. Since (4.15) still holds with $\overline{G}$ replaced by G in view of Theorem 4.1, we can conclude as in the proof of Theorem 4.7. □

Remark 4.9 (Stability of the flat configuration for small volumes)**.** If the surface density ψ satisfies the assumptions of Theorem 4.1, then there exists $d_0 > 0$ (depending only on Dirichlet boundary datum w_0) such that (4.10) holds for all $d \in (0, d_0)$ (see [10, Proposition 7.3]).

Definition 4.10. We say that flat configuration (d, u_d) is *asymptotically stable* if there exists $\delta > 0$ such that if $(h_0, u_0) \in X$, with $|\Omega_{h_0}| = |\Omega_d|$ and $\|h_0 - d\|_{W^{2,p}_{\#}(Q)} \leq \delta$, then every variational solution h to (3.1) according to Definition 3.17, with initial datum h_0, exists for all times and $\|h(\cdot, t) - d\|_{W^{2,p}_{\#}(Q)} \to 0$ as $t \to +\infty$.

We start by showing that if a variational solution to (3.1) exists for all times, then there exists a sequence $\{t_n\} \subset (0, +\infty)$, with $t_n \to \infty$, such that $h(\cdot, t_n)$ converges to a critical profile (see Definition 3.8).

Proposition 4.11. *Assume that for a certain initial datum $h_0 \in W^{2,p}_{\#}(Q)$ there exists a global in time variational solution h. Then there exist a sequence $\{t_n\} \subset (0, +\infty) \setminus Z_0$, where Z_0 is the set in* (3.54)*, and a critical profile $\bar{h}$ for F such that $t_n \to \infty$ and $h(\cdot, t_n) \to \bar{h}$ strongly in $W^{2,p}_{\#}(Q)$.*

Proof. From equation (3.3), by lower semicontinuity we have that

$$\int_0^\infty \Big\| \frac{\partial h}{\partial t} \Big\|^2_{H^{-1}(Q)} dt \leq C F(h_0, u_0)\,.$$

Since the set Z_0 has measure zero, we may find a sequence $\{t_n\} \subset (0, +\infty) \setminus Z_0$, $t_n \to \infty$, such that $\big\| \frac{\partial h(\cdot,t_n)}{\partial t} \big\|_{H^{-1}(Q)} \to 0$. Since $h \in L^\infty(0, \infty; W^{2,p}_{\#}(Q)) \cap H^1(0, \infty; H^{-1}_{\#}(Q))$, setting $h^n = h(\cdot, t_n)$, we may also assume that there exists $\bar{h} \in W^{2,p}_{\#}(Q)$ such that $h^n \rightharpoonup \bar{h}$ weakly in $W^{2,p}_{\#}(Q)$. In turn, denoting by u_{h^n} the corresponding elastic equilibria, by elliptic regularity (see also Lemma 5.1) we have that $u_{h^n}(\cdot, h^n(\cdot)) \to u_{\bar h}(\cdot, \bar h(\cdot))$ in $C^{1,\alpha}_\#(Q;\mathbb{R}^3)$. Let $\hat v^n$ be the unique Q-periodic solution to (3.59) with $t = t_n$ and note that $\hat v^n \to 0$ in $H^1_\#(Q)$ since $\big\|\frac{\partial h(\cdot,t_n)}{\partial t}\big\|_{H^{-1}(Q)} \to 0$. Writing the equation satisfied by h^n as in (3.22), we have for all $\varphi \in C^2_\#(Q)$, with $\int_Q \varphi\,dx = 0$,

$$
\begin{aligned}
\int_Q W(E(u_{h^n}(x,h^n(x))))\varphi\,dx &+ \int_Q D\psi(-Dh^n,1)\cdot(-D\varphi,0)\,dx + \frac{\varepsilon}{p}\int_Q |H^n|^p\frac{Dh^n\cdot D\varphi}{J^n}\\
&- \varepsilon\int_Q |H^n|^{p-2}H^n\bigg[\Delta\varphi - \frac{D^2\varphi[Dh^n,Dh^n]}{(J^n)^2}\\
&- \frac{\Delta h^n Dh^n\cdot D\varphi}{(J^n)^2} - 2\frac{D^2h^n[Dh^n,D\varphi]}{(J^n)^2} + 3\frac{D^2h^n[Dh^n,Dh^n]Dh^n\cdot D\varphi}{(J^n)^4}\bigg]\,dx\\
&- \int_Q \hat v^n\varphi\,dx = 0\,,
\end{aligned}
\tag{4.19}
$$

where H^n stands for the sum of the principal curvatures of h^n and $J^n = \sqrt{1+|Dh^n|^2}$. Arguing exactly as in the proof of Theorem 3.11(see (3.30)) we deduce that

$$
\int_Q |D^2(|H^n|^{p-2}H^n)|^2\,dx \le C\int_Q (1+(\hat v^n)^2)\,dx \tag{4.20}
$$

for some constant C independent of n. Thus, passing to a subsequence, if necessary, we may also assume that there exists $w \in H^2_\#(Q)$ such that $|H^n|^{p-2}H^n \rightharpoonup w$ weakly in $H^2_\#(Q)$ and $|H^n|^{p-2}H^n \to w$ strongly in $H^1_\#(Q)$. Since $H^1_\#(Q)$ is continuously embedded in $L^q(Q)$ for every $1 \le q < \infty$ by the Sobolev embedding theorem, there exists $z \in L^1(Q)$ such that $|H^n|^p \to z$ in $L^1(Q)$. The same argument used at the end of the proof of Corollary 3.15 shows that $z = |\bar H|^p$ and $w = |\bar H|^{p-2}\bar H$, where $\bar H$ is the sum of the principal curvatures of $\bar h$.

Using all the convergences proved above, and arguing as in the proof of Theorem 3.16 we may pass to the limit in equation (4.19), thus getting that $\bar h$ is a critical profile by Remark 3.12. □

Lemma 4.12. *Assume that* (4.9) *and* (4.10) *hold. Then there exist* $\sigma > 0$ *and* $c_0 > 0$ *such that*

$$
\partial^2 G(h,u_h)[\varphi] \ge c_0\|\varphi\|^2_{H^1_\#(Q)} \qquad \text{for all } \varphi \in \widetilde H^1_\#(Q)\,,
$$

provided $\|h-d\|_{C^{2,\alpha}_\#(Q)} \le \sigma$, *where* $\widetilde H^1_\#(Q)$ *is defined in* (4.5).

Proof. Let m_0 be the positive constant defined in (4.13). We claim that there exists $\sigma > 0$ such that

$$
\inf\{\partial^2 G(h,u_h)[\varphi] : \varphi \in \widetilde H^1_\#(Q),\ \|\varphi\|_{H^1_\#(Q)} = 1\} \ge \frac{m_0}{2}\,,
$$

whenever $\|h-d\|_{C^{2,\alpha}_\#(Q)} \le \sigma$. Indeed, if not, then there exist two sequences $\{h_n\} \subset C^{2,\alpha}_\#(Q)$, with $h_n \to d$ in $C^{2,\alpha}_\#(Q)$, and $\{\varphi_n\} \subset \widetilde H^1_\#(Q)$, with $\|\varphi_n\|_{H^1_\#(Q)} = 1$, such that

$$
\partial^2 G(h_n,u_{h_n})[\varphi_n] < \frac{m_0}{2}\,. \tag{4.21}
$$

Set

$$
\phi_n := \frac{\varphi_n}{\sqrt{1+|Dh_n|^2}}\circ\pi\,, \tag{4.22}
$$

where we recall that $\pi(x,y) = x$. Let v_{ϕ_n} be the unique solution in $A(\Omega_{h_n})$, see (4.4), to

$$
\int_{\Omega_{h_n}} \mathbb{C}E(v_{\phi_n}) : E(w)\,dz = \int_{\Gamma_{h_n}} \mathrm{div}_{\Gamma_{h_n}}(\phi_n\,\mathbb{C}E(u_{h_n}))\cdot w\,d\mathcal{H}^2 \qquad \text{for all } w \in A(\Omega_{h_n}) \tag{4.23}
$$

and let v_{φ_n} be the unique solution in $A(\Omega_d)$ to

$$\int_{\Omega_d} \mathbb{C}E(v_{\varphi_n}) : E(w)\,dz = \int_{\Gamma_d} \mathrm{div}_{\Gamma_d}(\varphi_n\,\mathbb{C}E(u_d))\cdot w\,d\mathcal{H}^2 \qquad \text{for all } w\in A(\Omega_d)\,. \tag{4.24}$$

Observe that (see, e.g., Lemma 5.1)

$$\|\mathrm{div}_{\Gamma_{h_n}}(\phi_n\,\mathbb{C}E(u_{h_n}))\|_{L^2(\Gamma_{h_n})} \le C\|\varphi_n\|_{H^1_\#(Q)}$$

for some constant $C>0$ depending only on

$$\sup_n(\|\mathbb{C}E(u_{h_n})\|_{C^1(\Gamma_{h_n})} + \|h_n\|_{C^2_\#(Q)})$$

and thus independent of n. Therefore, choosing $w = v_{\phi_n}$ in (4.23), and using Korn's inequality, we deduce that

$$\sup_n \|v_{\phi_n}\|_{H^1(\Omega_{h_n})} < +\infty\,. \tag{4.25}$$

The same bound holds for the sequence $\{v_{\varphi_n}\}$.

Next we show that

$$\int_{\Omega_{h_n}} W(E(v_{\phi_n}))\,dz - \int_{\Omega_d} W(E(v_{\varphi_n}))\,dz \to 0 \tag{4.26}$$

as $n\to\infty$. Consider a sequence $\{\Phi_n\}$ of diffeomorphisms $\Phi_n : \Omega_d \to \Omega_{h_n}$ such that $\Phi_n - Id$ is Q-periodic with respect to x, $\Phi_n(x,y) = (x, y+d-h_n(x))$ in a neighborhood of Γ_d, and $\|\Phi_n - Id\|_{C^{2,\alpha}(\overline{\Omega}_d;\mathbb{R}^3)} \le C\|h_n - d\|_{C^{2,\alpha}_\#(Q)} \to 0$. Set $w_n := v_{\phi_n}\circ\Phi_n$. Changing variables, we get that $w_n \in A(\Omega_d)$ satisfies

$$\int_{\Omega_d} A_n Dw_n : Dw\,dz = \int_{\Gamma_d} \big(\mathrm{div}_{\Gamma_{h_n}}(\phi_n\,\mathbb{C}E(u_{h_n}))\circ\Phi_n\big)\cdot w\,J_{\Phi_n}\,d\mathcal{H}^2 \tag{4.27}$$

for every $w\in A(\Omega_d)$, where J_{Φ_n} stands for the $(N-1)$-Jacobian of Φ_n and the fourth order tensor valued functions A_n satisfy $A_n\to\mathbb{C}$ in $C^{1,\alpha}(\overline{\Omega}_d)$. We claim that

$$\int_{\Omega_d} W(E(w_n - v_{\varphi_n}))\,dz \to 0 \tag{4.28}$$

as $n\to\infty$. Note that this would immediately imply $\int_{\Omega_d} W(E(w_n))\,dz - \int_{\Omega_d} W(E(v_{\varphi_n}))\,dz \to 0$ and in turn, taking also into account that $A_n\to\mathbb{C}$ uniformly and that $\frac12\int_{\Omega_d} A_n Dw_n : Dw_n\,dz = \int_{\Omega_{h_n}} W(E(v_{\phi_n}))\,dz$, claim (4.26) would follow. In order to prove (4.28), we write

$$\begin{aligned}
&\int_{\Omega_d} \mathbb{C}D(v_{\varphi_n} - w_n) : D(v_{\varphi_n} - w_n)\,dz \\
&\quad= \int_{\Omega_d} \mathbb{C}Dv_{\varphi_n} : D(v_{\varphi_n} - w_n)\,dz - \int_{\Omega_d} (\mathbb{C} - A_n)Dw_n : D(v_{\varphi_n} - w_n)\,dz \\
&\qquad - \int_{\Omega_d} A_n Dw_n : D(v_{\varphi_n} - w_n)\,dz \\
&\quad= \int_{\Gamma_d} \mathrm{div}_{\Gamma_d}(\varphi_n\mathbb{C}E(u_d))\cdot(v_{\varphi_n} - w_n)\,d\mathcal{H}^2 - \int_{\Omega_d} (\mathbb{C} - A_n)Dw_n : D(v_{\varphi_n} - w_n)\,dz \\
&\qquad - \int_{\Gamma_d} \big(\mathrm{div}_{\Gamma_{h_n}}(\phi_n\mathbb{C}E(u_{h_n}))\circ\Phi_n\big)\cdot(v_{\varphi_n} - w_n)J_{\Phi_n}\,d\mathcal{H}^2 \\
&\quad=: I_1 - I_2 - I_3\,,
\end{aligned}$$

where we used (4.24) and (4.27). From (4.25), the analogous bound for the sequence $\{v_{\varphi_n}\}$, and the uniform convergence of A_n to $\mathbb{C}$ we deduce that I_2 tends to 0.

Fix $\eta = (\eta_1, \eta_2, \eta_3) \in C^1_\#(\Gamma_d; \mathbb{R}^3) \simeq C^1_\#(Q; \mathbb{R}^3)$. Using the fact that $\Phi_n^{-1}(x, y) = (x, y - h_n(x) + d)$ in a neighborhood of Γ_{h_n} we have

$$D_{\Gamma_{h_n}}(\eta_j \circ \Phi_n^{-1}) = \big(I - \nu_{h_n} \otimes \nu_{h_n}\big) D_{\Gamma_d}\eta_j \circ \Phi_n^{-1}\,,$$

where we set $\nu_{h_n} := \frac{(-Dh_n, 1)}{\sqrt{1+|Dh_n|^2}}$. Using this fact, we then have by repeated integrations by parts and changes of variables,

$$\begin{aligned}
&\int_{\Gamma_d} \big(\mathrm{div}_{\Gamma_{h_n}}(\phi_n \mathbb{C}E(u_{h_n})) \circ \Phi_n\big) \cdot \eta\, J_{\Phi_n} d\mathcal{H}^2 \\
&\qquad = \int_{\Gamma_{h_n}} \mathrm{div}_{\Gamma_{h_n}}(\phi_n \mathbb{C}E(u_{h_n})) \cdot \eta \circ \Phi_n^{-1}\, d\mathcal{H}^2 \\
&\qquad = -\int_{\Gamma_{h_n}} \phi_n \mathbb{C}E(u_{h_n}) : D_{\Gamma_{h_n}}(\eta \circ \Phi_n^{-1})\, d\mathcal{H}^2 \\
&\qquad = -\int_{\Gamma_{h_n}} \big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\phi_n \mathbb{C}E(u_{h_n}) : D_{\Gamma_d}\eta \circ \Phi_n^{-1}\ \ d\mathcal{H}^2 \\
&\qquad = -\int_{\Gamma_d} \big[\big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\phi_n \mathbb{C}E(u_{h_n})\big] \circ \Phi_n : D_{\Gamma_d}\eta\, J_{\Phi_n}\, d\mathcal{H}^2 \\
&\qquad = \int_{\Gamma_d} \mathrm{div}_{\Gamma_d}\Big[\big[\big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\phi_n \mathbb{C}E(u_{h_n})\big] \circ \Phi_n J_{\Phi_n}\Big] \cdot \eta\, d\mathcal{H}^2.
\end{aligned}$$

Hence, we may rewrite

$$I_1 - I_3 = \int_{\Gamma_d} \mathrm{div}_{\Gamma_d} g_n \cdot (v_{\varphi_n} - w_n)\, d\mathcal{H}^2, \tag{4.29}$$

where by (4.22),

$$\begin{aligned}
g_n &:= \varphi_n \mathbb{C}E(u_d) - \big[\big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\phi_n \mathbb{C}E(u_{h_n})\big] \circ \Phi_n J_{\Phi_n} \\
&= \varphi_n \left[\mathbb{C}E(u_d) - \big[\big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\mathbb{C}E(u_{h_n})\big] \circ \Phi_n \frac{J_{\Phi_n}}{\sqrt{1+|Dh_n|^2}}\right].
\end{aligned}$$

Since $h_n \to d$ in $C^{2,\alpha}_\#(Q)$, by standard Schauder's estimates for the elastic displacements u_{h_n}, we get

$$\mathbb{C}E(u_d) - \big[\big(I - \nu_{h_n} \otimes \nu_{h_n}\big)\mathbb{C}E(u_{h_n})\big] \circ \Phi_n \frac{J_{\Phi_n}}{\sqrt{1+|Dh_n|^2}} \to 0 \qquad \text{in } C^{1,\alpha}(\Gamma_d)\,.$$

Therefore, by (4.29) and the equiboundedness of $\{v_{\phi_n}\}$ and $\{w_n\}$ we have that $I_1 - I_3 \to 0$. This concludes the proof of (4.28) and, in turn, of (4.26).

Finally, again from the $C^{2,\alpha}$-convergence of $\{h_n\}$ to d and the fact that

$$\partial_\nu[W(E(u_{h_n})] \circ \Phi_n \to \partial_\nu[W(E(u_d))] \quad \text{in } C^{0,\alpha}_\#(\Gamma_d)$$

by standard Schauder's elliptic estimates, recalling (4.7) we easily infer that

$$\begin{aligned}
&\left(\partial^2 G(h_n, u_{h_n})[\varphi_n] + 2\int_{\Omega_{h_n}} W(E(v_{\phi_n}))\, dz\right) \\
&\qquad\qquad - \left(\partial^2 G(d, u_d)[\varphi_n] + 2\int_{\Omega_d} W(E(v_{\varphi_n}))\, dz\right) \to 0
\end{aligned} \tag{4.30}$$

as $n \to \infty$. Thus, recalling (4.26), we also have

$$\partial^2 G(h_n, u_{h_n})[\varphi_n] - \partial^2 G(d, u_d)[\varphi_n] \to 0$$

and, in turn, by (4.21)

$$\limsup \partial^2 G(d, u_d)[\varphi_n] \le \frac{m_0}{2}\,,$$

which is a contradiction to (4.13). This concludes the proof of the lemma. □

Next we prove that (d, u_d) is an isolated critical pair.

Proposition 4.13. *Assume that* (4.9) *and* (4.10) *hold. Then there exists* $\sigma > 0$ *such that if* $(h, u_h) \in X$ *with* $|\Omega_h| = |\Omega_d|$ *and* $0 < \|h - d\|_{W^{2,p}_{\#}(Q)} \le \sigma$, *then* (h, u_h) *is not a critical pair.*

Proof. Assume by contradiction that there exists a sequence $h_n \to d$ in $W^{2,p}_{\#}(Q)$, with $h_n \neq d$ and $|\Omega_{h_n}| = |\Omega_d|$, such that (h_n, u_{h_n}) is a critical pair. Using the Euler-Lagrange equation and arguing as in the proof of Theorem 3.11, one can show that

$$\int_Q |D^2(|H_n|^{p-2}H_n)|^2\,dx \le C\int_Q \Big(|D^2h_n|^2|D(|H_n|^{p-2}H_n)|^2 + |H_n|^{2(p+1)} + 1\Big)\,dx\,.$$

Indeed, this can obtained as (3.29), taking into account that there is no contribution from the time derivative. From this inequality, arguing exactly as in the final part of the proof of Theorem 3.11 we deduce that

$$\int_Q |D^2(|H_n|^{p-2}H_n)|^2\,dx \le C$$

for some C independent of n. In particular, by the Sobolev embedding theorem, $\{|H_n|^{p-2}H_n\}$ is bounded in $C^{0,\beta}_{\#}(Q)$ for every $\beta \in (0,1)$. Hence, $\{H_n\}$ is bounded in $C^{0,\beta}_{\#}(Q)$ for all $\beta \in (0, 1/(p-1))$. In turn, by (2.3) and standard elliptic regularity this implies that $\{h_n\}$ is bounded in $C^{2,\beta}_{\#}(Q)$ for all $\beta \in (0, 1/(p-1))$ and thus $h_n \to d$ in $C^{2,\beta}(Q)$ for all such β. Since (d, u_d) is a critical pair (see Remark 4.4), $\frac{d}{ds}F(d + s(h_n - d), u_{d+s(h_n-d)})|_{s=0} = 0$, and so by (4.6) to reach a contradiction it is enough to show that for n large

$$\begin{aligned}\frac{d^2}{ds^2}F(d + s(h_n - d), u_{d+s(h_n-d)})|_{s=t} &= \partial^2 G(h_{n,t}, u_{h_{n,t}})[h_n - d]\\ &- \int_{\Gamma_{h_{n,t}}} (W(E(u_{h_{n,t}})) + H^{\psi}_{h_{n,t}})\,\mathrm{div}_{\Gamma_{h_{n,t}}}\left(\frac{(Dh_{n,t}, |Dh_{n,t}|^2)(h_{n,t} - d)^2}{(1 + |Dh_{n,t}|)^{\frac{3}{2}}}\circ\pi\right) d\mathcal{H}^2\\ &+ \varepsilon\frac{d^2}{ds^2}\mathcal{W}_p(d + s(h_n - d))|_{s=t} > 0\end{aligned}$$

for all $t \in (0,1)$, where $h_{n,t} := d + t(h_n - d)$, $H^{\psi}_{h_{n,t}}$ is defined as in (4.3) with h replaced by $h_{n,t}$, and

$$\mathcal{W}_p(h) := \int_{\Gamma_h} |H|^p\,d\mathcal{H}^2\,.$$

To this purpose, note that since $h_n \to d$ in $C^{2,\beta}$, by Lemma 5.1 we have

$$\sup_{t\in(0,1)} \|W(E(u_{h_{n,t}})) + H^{\psi} - W_d\|_{L^\infty(\Gamma_{h_{n,t}})} \to 0$$

as $n \to \infty$, where W_d is the constant value of $W(E(u_d))$ on Γ_d (see Remark 4.4). Therefore, also by Lemma 4.12, we deduce that

$$\begin{aligned}&\partial^2 G(h_{n,t}, u_{h_{n,t}})[h_n - d]\\ &\quad - \int_{\Gamma_{h_{n,t}}} (W(E(u_{h_{n,t}})) + H^{\psi}_{h_{n,t}})\,\mathrm{div}_{\Gamma_{h_{n,t}}}\left(\frac{(Dh_{n,t}, |Dh_{n,t}|^2)(h_{n,t} - d)^2}{(1 + |Dh_{n,t}|)^{\frac{3}{2}}}\circ\pi\right) d\mathcal{H}^2\\ &= \partial^2 G(h_{n,t}, u_{h_{n,t}})[h_n - d]\\ &\quad - \int_{\Gamma_{h_{n,t}}} (W(E(u_{h_{n,t}})) + H^{\psi}_{h_{n,t}} - W_d)\,\mathrm{div}_{\Gamma_{h_{n,t}}}\left(\frac{(Dh_{n,t}, |Dh_{n,t}|^2)(h_{n,t} - d)^2}{(1 + |Dh_{n,t}|)^{\frac{3}{2}}}\circ\pi\right) d\mathcal{H}^2\\ &\ge c_0\|h_n - d\|^2_{H^1_{\#}(Q)} - C\|W(E(u_{h_{n,t}})) + H^{\psi}_{h_{n,t}} - W_d\|_{L^\infty(\Gamma_{h_{n,t}})}\|h_n - d\|^2_{H^1_{\#}(Q)} \ge \frac{c_0}{2}\|h_n - d\|^2_{H^1_{\#}(Q)}\end{aligned}$$

for n large and for some constant $c_0 > 0$ independent of n, where we used the facts that

$$\int_{\Gamma_{h_{n,t}}} \left\| \operatorname{div}_{\Gamma_{h_{n,t}}} \left(\frac{(Dh_{n,t}, |Dh_{n,t}|^2)(h_{n,t} - d)^2}{(1 + |Dh_{n,t}|)^{\frac{3}{2}}} \circ \pi \right) \right\| d\mathcal{H}^2 \leq C \|h_n\|_{C^2_\#(Q)} \|h_n - d\|^2_{H^1_\#(Q)}$$

and that $h_n \to d$ in $C^{2,\beta}(Q)$.

Since

$$\mathcal{W}_p(d + t(h_n - d)) = t^p \int_Q \left| \operatorname{div} \frac{Dh_n}{\sqrt{1 + t^2 |Dh_n|^2}} \right|^p dx =: f_n(t)\,,$$

in order to conclude it is enough to show that $f_n''(t) \geq 0$ for all $t \in (0, 1)$. Set

$$g_n(x, t) := \left| \operatorname{div} \frac{Dh_n(x)}{\sqrt{1 + t^2 |Dh_n(x)|^2}} \right|^2$$

so that

$$f_n'' = \int_Q \Big[p(p-1) t^{p-2} g_n^{\frac{p}{2}} + p^2 t^{p-1} g_n^{\frac{p-2}{2}} \partial_t g_n + \tfrac{p}{2} t^p \big((\tfrac{p}{2} - 1) g_n^{\frac{p-4}{2}} (\partial_t g_n)^2 + g_n^{\frac{p-2}{2}} \partial_{tt} g_n \big) \Big]\, dx\,. \quad (4.31)$$

On the other hand, observe that

$$g_n = \frac{|\Delta h_n|^2}{1 + t^2 |Dh_n|^2} + t^2 \frac{|D^2 h_n[Dh_n, Dh_n]|^2}{(1 + t^2 |Dh_n|^2)^3} - 2t \frac{D^2 h_n[Dh_n, Dh_n] \Delta h_n}{(1 + t^2 |Dh_n|^2)^2}$$

so that for n large

$$g_n \geq \frac{1}{2} |\Delta h_n|^2 - C |D^2 h_n| |Dh_n|^2 \quad \text{and} \quad |\partial_t g_n| + |\partial_{tt} g_n| \leq C |D^2 h_n| |Dh_n|\,.$$

We then deduce from (4.31) that there exist C_0, $C_1 > 0$ independent of n and $t \in (0, 1)$ such that

$$f_n''(t) \geq C_0 \int_Q |\Delta h_n|^p\, dx - C_1 \|Dh_n\|^p_\infty \int_Q |D^2 h_n|^p\, dx\,.$$

Since $\|Dh_n\|_\infty \to 0$, by Lemma 5.3 we conclude that the right-hand side in the above inequality is non-negative for n large, thus concluding the proof of the proposition. □

Finally, we prove the main result of this section, namely, the asymptotic stability of the flat configuration (see Definition 4.10).

Theorem 4.14. *Under the assumptions of Theorem 4.8, (d, u_d) is asymptotically stable.*

Proof. By Proposition 4.13 there exists $\sigma > 0$ such that if h is a critical profile, with $|\Omega_h| = |\Omega_d|$ and $\|h - d\|_{W^{2,p}_\#(Q)} \leq \sigma$, then $h = d$. In view of Theorem 4.1 we may take σ so small that

$$F(d, u_d) < F(k, u_k) \qquad \text{for all } (k, u_k) \in X \text{ with } 0 < \|k - d\|_{W^{2,p}_\#(Q)} \leq \sigma. \quad (4.32)$$

Since (d, u_d) is Liapunov stable by Theorem 4.8, for every fixed $(h_0, u_0) \in X$ with $|\Omega_{h_0}| = |\Omega_d|$ and $\|h_0 - d\|_{W^{2,p}_\#(Q)} \leq \delta(\sigma)$, we have

$$\|h(\cdot, t) - d\|_{W^{2,p}_\#(Q)} \leq \sigma \qquad \text{for all } t > 0. \quad (4.33)$$

Here $\delta(\sigma)$ is the number given in Definition 4.6. We claim that

$$F(h(\cdot, t), u_h(\cdot, t)) \to F(d, u_d) \qquad \text{as } t \to +\infty. \quad (4.34)$$

By Proposition 4.11 there exists a sequence $\{t_n\} \subset (0, +\infty) \setminus Z_0$ such that $t_n \to +\infty$ and $\{h(\cdot, t_n)\}$ converges to a critical profile in $W^{2,p}_\#(Q)$, where Z_0 is the set in (3.54). In view of the choice of σ and by (4.33), we conclude that $h(\cdot, t_n) \to d$ in $W^{2,p}_\#(Q)$.

In particular $F(h(\cdot,t_n),u_{h(\cdot,t_n)})\to F(d,u_d)$. In turn, by (3.54), this implies that $F(h(\cdot,t),u_h(\cdot,t))\to F(d,u_d)$ as $t\to+\infty$, $t\notin Z_0$. On the other hand, by (3.55) for $t\in Z_0$ we have that $F(h(\cdot,t),u_h(\cdot,t))\leq F(h(\cdot,\tau),u_h(\cdot,\tau))$ for all $\tau<t$, $\tau\notin Z_0$. Therefore

$$\limsup_{t\to+\infty,t\in Z_0} F(h(\cdot,t),u_h(\cdot,t))\leq F(d,u_d).$$

Recalling (4.32), we finally obtain (4.34). In turn, reasoning as in the proof of Theorem 4.7 (see (4.17)), it follows from (4.32) and (4.33) that for every sequence $\{s_n\}\subset(0,+\infty)$, with $s_n\to+\infty$, there exists a subsequence such that $\{h(\cdot,s_n)\}$ converges to d in $W^{2,p}_{\#}(Q)$. This implies that $h(\cdot,t)\to d$ in $W^{2,p}_{\#}(Q)$ as $t\to+\infty$ and concludes the proof. □

4.3. **The two-dimensional case.** As remarked in the introduction, the arguments presented in the previous subsections apply to the two-dimensional version of (3.1), with $p=2$, studied in [24], with

$$V=\Bigl((g_{\theta\theta}+g)k+W(E(u))-\varepsilon\bigl(k_{\sigma\sigma}+\frac{1}{2}k^3\bigr)\Bigr)_{\sigma\sigma}. \tag{4.35}$$

Here V denotes the outer normal velocity of $\Gamma_{h(\cdot,t)}$, k is its curvature, $W(E(u))$ is the trace of $W(E(u(\cdot,t)))$ on $\Gamma_{h(\cdot,t)}$, with $u(\cdot,t)$ the elastic equilibrium in $\Omega_{h(\cdot,t)}$, under the conditions that $Du(\cdot,y)$ is b-periodic and $u(x,0)=e_0(x,0)$, for some $e_0>0$; and $(\cdot)_\sigma$ stands for tangential differentiation along $\Gamma_{h(\cdot,t)}$. The constant $e_0>0$ measures the lattice mismatch between the elastic film and the (rigid) substrate. Moreover, $g:[0,2\pi]\to(0,+\infty)$ is defined as

$$g(\theta)=\psi(\cos\theta,\sin\theta) \tag{4.36}$$

and is evaluated at $\arg(\nu(\cdot,t))$, where $\nu(\cdot,t)$ is the outer normal to $\Gamma_{h(\cdot,t)}$. The underlying energy functional is then given by

$$F(h,u):=\int_{\Omega_h}W(E(u))\,dz+\int_{\Gamma_h}\Bigl(\psi(\nu)+\frac{\varepsilon}{2}k^2\Bigr)\,d\mathcal{H}^1.$$

In the two-dimensional framework, given $b>0$, we search for for b-periodic solutions to (4.35).

A local-in-time *b-periodic weak solution* to (4.35) is a function $h\in H^1\bigl(0,T_0;H^{-1}_{\#}(0,b)\bigr)\cap L^\infty(0,T_0;H^2_{\#}(0,b))$ such that:

(i) $(g_{\theta\theta}+g)k+W(E(u))-\varepsilon\bigl(k_{\sigma\sigma}+\frac{1}{2}k^3\bigr)\in L^2(0,T_0;H^1_{\#}(0,b))$,

(ii) for almost every $t\in[0,T_0]$,

$$\frac{\partial h}{\partial t}=J\Bigl((g_{\theta\theta}+g)k+Q(E(u))-\varepsilon\bigl(k_{\sigma\sigma}+\frac{1}{2}k^3\bigr)\Bigr)_{\sigma\sigma}\qquad\text{in }H^{-1}_{\#}(0,b).$$

Given (h_0,u_0), with $h_0\in H^2_{\#}(0,b)$, $h_0>0$, and u_0 the corresponding elastic equilibrium, local-in-time existence of a *unique* weak solution with initial datum (h_0,u_0) has been established in [24]. The Liapunov and asymptotic stability analysis of the flat configuration established in Subsections 4.1 and 4.2 extends to the two-dimensional case, where, in addition, the range of d's under which (4.10) holds can be analytically determined for isotropic elastic energies of the form

$$W(\xi):=\mu|\xi|^2+\frac{\lambda}{2}(\operatorname{trace}\xi)^2.$$

In the above formula the *Lamé coefficients* μ and λ are chosen to satisfy the ellipticity conditions $\mu>0$ and $\mu+\lambda>0$, see [25, 9]. The stability range of the flat configuration depends on μ, λ, and the mismatch constant e_0 appearing in the Dirichlet condition $u(x,0)=e_0(x,0)$. For the reader's convenience, we recall the results. Consider the *Grinfeld function* K defined by

$$K(y):=\max_{n\in\mathbb{N}}\frac{1}{n}J(ny),\quad y\geq 0, \tag{4.37}$$

where

$$J(y) := \frac{y + (3-4\nu_p)\sinh y \cosh y}{4(1-\nu_p)^2 + y^2 + (3-4\nu_p)\sinh^2 y}\,,$$

and ν_p is the *Poisson modulus* of the elastic material, i.e.,

$$\nu_p := \frac{\lambda}{2(\lambda+\mu)}\,. \tag{4.38}$$

It turns out that K is strictly increasing and continuous, $K(y) \le Cy$, and $\lim_{y\to+\infty} K(y) = 1$, for some positive constant C. We also set, as in the previous subsections,

$$G(h,u) := \int_{\Omega_h} W(E(u))\,dz + \int_{\Gamma_h} \psi(\nu)\,d\mathcal{H}^1\,.$$

Combining [25, Theorem 2.9] and [9, Theorem 2.8] with the results of the previous subsection, we obtain the 2D asymptotic stability of the flat configuration.

Theorem 4.15. *Assume $\partial^2_{11}\psi(0,1) > 0$. Let $d_{\mathrm{loc}} : (0,+\infty) \to (0,+\infty]$ be defined as $d_{\mathrm{loc}}(b) := +\infty$, if $0 < b \le \frac{\pi}{4}\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}$, and as the solution to*

$$K\Big(\frac{2\pi d_{\mathrm{loc}}(b)}{b}\Big) = \frac{\pi}{4}\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\frac{1}{b}\,, \tag{4.39}$$

otherwise. Then the second variation of G at (d,u_d) is positive definite, i.e.,

$$\partial^2 G(d,u_d)[\varphi] > 0 \qquad \text{for all } \varphi \in H^1_\#(0,b)\setminus\{0\},\ \text{with } \int_0^b \varphi\,dx = 0\,,$$

if and only if $0 < d < d_{\mathrm{loc}}(b)$. In particular, for all $d \in (0, d_{\mathrm{loc}}(b))$ the flat configuration (d,u_d) is asymptotically stable.

5. Appendix

5.1. **Regularity results.** In this subsection we collect a few regularity results that have been used in the previous sections. We start with the following elliptic estimate, whose proof is essentially contained in [24, Lemma 6.10].

Lemma 5.1. *Let $M > 0$, $c_0 > 0$. Let h_1, $h_2 \in C^{1,\alpha}_\#(Q)$ for some $\alpha \in (0,1)$, with $\|h_i\|_{C^{1,\alpha}_\#(Q)} \le M$ and $h_i \ge c_0$, $= i = 1,2$, and let u_1 and u_2 be the corresponding elastic equilibria in Ω_{h_1} and Ω_{h_2}, respectively. Then,*

$$\|E(u_1(\cdot,h_1(\cdot)) - E(u_2(\cdot,h_2(\cdot))\|_{C^{1,\alpha}_\#(Q)} \le C\|h_1 - h_2\|_{C^{1,\alpha}_\#(Q)} \tag{5.1}$$

for some constant $C > 0$ depending only on M, c_0, and α.

The following lemma is probably well-known to the experts, however for the reader's convenience we provide a proof.

Lemma 5.2. *Let $p > 2$, $u \in L^{\frac{p}{p-1}}(Q)$ such that*

$$\int_Q u\,AD^2\varphi\,dx + \int_Q b\cdot D\varphi + \int_Q c\varphi\,dx = 0 \qquad \text{for all } \varphi \in C^\infty_\#(Q) \text{ with } \int_Q \varphi\,dx = 0,$$

where $A \in W^{1,p}_\#(Q;\mathbb{M}^{2\times 2}_{\mathrm{sym}})$ satisfies standard uniform ellipticity conditions (see (5.6) below), $b \in L^1(Q;\mathbb{R}^2)$, and $c \in L^1(Q)$. Then $u \in L^q(Q)$ for all $q \in (1,2)$. Moreover, if b, $u\,\mathrm{div}A \in L^r(Q;\mathbb{R}^2)$ and $c \in L^r(Q)$ for some $r > 1$, then $u \in W^{1,r}_\#(Q)$.

Proof. We only prove the first assertion, since the other one can be proven using similar arguments. Denote by A_ε, u_ε, b_ε, and c_ε the standard mollifications of A, u, b, and c, and let $v_\varepsilon \in C^\infty_\#(Q)$ be the unique solution to the following problem

$$\begin{cases} \displaystyle\int_Q \big(A_\varepsilon Dv_\varepsilon + u_\varepsilon \mathrm{div} A_\varepsilon - b_\varepsilon\big)\cdot D\varphi\, dx - \int_Q c_\varepsilon \varphi\, dx = 0 \quad \text{for all } \varphi \in C^1_\#(Q)\,, \int_Q \varphi\, dx = 0, \\ \displaystyle\int_Q v_\varepsilon\, dx = \int_Q u\, dx\,. \end{cases}$$

Denoting by G_ε the Green's function associated with the elliptic operator

$$-\mathrm{div}\,(A_\varepsilon Du)$$

it is known, [19, equation (3.66)] and [27, equation (1.6)], that for all $q \in [1,2)$ and for all $x \in Q$ we have

$$\|D_y G_\varepsilon(x,\cdot)\|_{L^q(Q)} \le C\,,$$

with C depending only on the ellipticity constants and q and not on ε. Since

$$\begin{aligned} v_\epsilon(x) &= \int_Q G_\epsilon(x,y)\big[-\mathrm{div}(u_\varepsilon \mathrm{div} A_\varepsilon - b_\varepsilon) + c_\epsilon\big]\, dy \\ &= \int_Q \big[\big(u_\varepsilon \mathrm{div} A_\varepsilon - b_\varepsilon\big)\cdot D_y G_\varepsilon(x,y) + G_\epsilon(x,y)c_\epsilon\big]\, dy\,, \end{aligned}$$

it follows by standard properties of convolution that for all $q > 1$ there exists $C > 0$ depending only on q and the L^1-norms of $u_\varepsilon \mathrm{div} A_\varepsilon$, b_ε, c_ε, hence on the L^1-norms of b, c, the $L^{\frac{p}{p-1}}$ norm of u, and the $W^{1,p}$ norm of A, such that $\|v_\varepsilon\|_{L^q(Q)} \le C$ for ε sufficiently small. Thus, we may assume (up to subsequences) that $v_\varepsilon \rightharpoonup v$ weakly in $L^q(Q)$, where v solves

$$\int_Q vAD^2\varphi\, dx + \int_Q \big(v\,\mathrm{div}A - u\,\mathrm{div}A + b\big)\cdot D\varphi\, dx + \int_Q c\varphi\, dx = 0 \tag{5.2}$$

for all $\varphi \in C^2_\#(Q)$, with $\int_Q \varphi\, dx = 0$, and satisfies

$$\int_Q v\, dx = \int_Q u\, dx\,. \tag{5.3}$$

Since by assumption u solves the problem (5.2)-(5.3), it is enough to show that the problem admits a unique solution. Let v_1 and v_2 be two solutions and set $w := v_2 - v_1$. Then, we have

$$\int_Q wAD^2\varphi\, dx + \int_Q w\,\mathrm{div}A\cdot D\varphi\, dx = 0 \tag{5.4}$$

for all $\varphi \in C^2_\#(Q)$, with $\int_Q \varphi\, dx = 0$. Let $g \in C^1_\#(Q)$, with $\int_Q g\, dx = 0$ and denote by φ_g the unique solution to the equation $\mathrm{div}(A[D\varphi_g]) = g$ such that $\int_Q \varphi_g\, dx = 0$. Hence, from (5.4) we deduce that $\int_Q wg\, dx = 0$ for all $g \in C^1_\#(Q)$, with $\int_Q g\, dx = 0$. This implies that w is constant and, in turn, $w \equiv 0$ since $\int_Q w\, dx = 0$. □

In the next lemma we denote by Lu an elliptic operator of the form

$$Lu := \sum_{ij} a_{ij}(x) D_{ij} u + \sum_i b_i(x) D_i u\,, \tag{5.5}$$

where all the coefficients are Q-periodic functions, the a_{ij}'s are continuous, and the b_i are bounded. Moreover, there exist λ, $\Lambda > 0$ such that

$$\Lambda|\xi|^2 \ge \sum_{ij} a_{ij}(x)\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^2, \qquad \sum_i |b_i| \le \Lambda\,. \tag{5.6}$$

Lemma 5.3. *Let $p \geq 2$. Then, there exists $C > 0$ such that for all $u \in W^{2,p}_{\#}(Q)$ we have*

$$\|D^2 u\|_{L^p(Q)} \leq C\|Lu\|_{L^p(Q)}\,,$$

where L is the differential operator defined in (5.5). *The constant C depends only on p, λ, Λ and the moduli of continuity of the coefficients a_{ij}.*

Proof. We argue by contradiction assuming that there exists a sequence $\{u_h\} \subset W^{2,p}_{\#}(Q)$, a modulus of continuity ω, and a sequence of operators $\{L_h\}$ as in (5.5), with periodic coefficients a^h_{ij}, b^h_i satisfying (5.6) and

$$|a^h_{ij}(x_1) - a^h_{ij}(x_2)| \leq \omega(|x_1 - x_2|)$$

for all x_1, $x_2 \in Q$, such that

$$\|D^2 u_h\|_{L^p(Q)} \geq h\|L_h u_h\|_{L^p(Q)}\,.$$

By homogeneity we may assume that

$$\|D^2 u_h\|_{L^p(Q)} = 1 \qquad \text{for all } h \in \mathbb{N}. \tag{5.7}$$

Recall that by periodicity

$$\int_Q Du_h\,dx = 0\,.$$

Moreover, by adding a constant if needed, we may also assume that $\int_Q u_h\,dx = 0$. Therefore, by Poincaré inequality and up to a subsequence, $u_h \rightharpoonup u$ weakly in $W^{2,p}_{\#}(Q)$. Moreover, we may also assume that there exist a_{ij} and b_i satisfying (5.6), such that

$$a^h_{ij} \to a_{ij} \quad \text{uniformly in } Q \qquad \text{and} \qquad b^h_i \overset{*}{\rightharpoonup} b_i \quad \text{weakly* in } L^\infty(Q).$$

Since $\|L_h u_h\|_{L^p(Q)} \to 0$, we have that u is a periodic function satisfying $Lu = 0$, where L is the operator associated with the coefficients a_{ij} and b_i. Thus, by the Maximum Principle ([26, Theorem 9.6]) u is constant, and thus $u = 0$. On the other hand, by elliptic regularity (see [26, Theorem 9.11]) there exists a constant $C > 0$ depending on p, λ, Λ, and ω such that

$$\|D^2 u_h\|_{L^p(Q)} \leq C(\|u_h\|_{W^{1,p}(Q)} + \|L_h u_h\|_{L^p(Q)})\,.$$

Since the right-hand side vanishes, we reach a contradiction to (5.7). □

5.2. **Interpolation results.**

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let $1 \leq p \leq \infty$ and j, m be two integers such that $0 \leq j \leq m$ and $m \geq 1$. Then there exists $C > 0$ such that*

$$\|D^j f\|_{L^p(\Omega)} \leq C\big(\|D^m f\|^{\frac{j}{m}}_{L^p(\Omega)}\|f\|^{\frac{m-j}{m}}_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\big) \tag{5.8}$$

for all $f \in W^{m,p}(\Omega)$. Moreover, if Ω is a cube, $f \in W^{m,p}_{\#}(\Omega)$, and if either f vanishes at the boundary or $\int_\Omega f\,dx = 0$, then (5.8) *holds in the stronger form*

$$\|D^j f\|_{L^p(\Omega)} \leq C\|D^m f\|^{\frac{j}{m}}_{L^p(\Omega)}\|f\|^{\frac{m-j}{m}}_{L^p(\Omega)}\,. \tag{5.9}$$

Proof. Inequality (5.8) follows by combining inequalities (1) and (3) in [2, Theorem 5.2]. If Ω is a cube, f is periodic and if either f vanishes at the boundary or $\int_\Omega f\,dx = 0$, then inequality (5.9) follows by observing that

$$\|f\|_{W^{m,p}(\Omega)} \leq C\|D^m f\|_{L^p(\Omega)}\,,$$

as a straightforward application of the Poincaré inequality. □

The next interpolation result is obtained by combining [2, Theorem 5.8] with (5.8).

Theorem 5.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. If $mp > n$, let $1 \le p \le q \le \infty$; if $mp = n$ let $1 \le p \le q < \infty$; if $mp < n$ let $1 \le p \le q \le np/(n - mp)$. Then there exists $C > 0$ such that*

$$\|f\|_{L^q(\Omega)} \le C\big(\|D^m f\|^{\theta}_{L^p(\Omega)}\|f\|^{1-\theta}_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\big) \tag{5.10}$$

for all $f \in W^{m,p}(\Omega)$, where $\theta := \frac{n}{mp} - \frac{n}{mq}$. Moreover, if Ω is a cube, $f \in W^{m,p}_{\#}(\Omega)$, and if either f vanishes at the boundary or $\int_\Omega f\,dx = 0$, then (5.10) holds in the stronger form

$$\|f\|_{L^q(\Omega)} \le C\|D^m f\|^{\theta}_{L^p(\Omega)}\|f\|^{1-\theta}_{L^p(\Omega)}\,. \tag{5.11}$$

Combining Theorems 5.4 and 5.5, and arguing as in the proof of [24, Theorem 6.4], we have the following theorem.

Theorem 5.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the cone condition. Let s, j, and m be integers such that $0 \le s \le j \le m$. Let $1 \le p \le q < \infty$ if $(m-j)p \ge n$, and let $1 \le p \le q \le \infty$ if $(m-j)p > n$. Then, there exists $C > 0$ such that*

$$\|D^j f\|_{L^q(\Omega)} \le C\big(\|D^m f\|^{\theta}_{L^p(\Omega)}\|D^s f\|^{1-\theta}_{L^p(\Omega)} + \|D^s f\|_{L^p(\Omega)}\big) \tag{5.12}$$

for all $f \in W^{m,p}(\Omega)$, where

$$\theta := \frac{1}{m-s}\left(\frac{n}{p} - \frac{n}{q} + j - s\right).$$

Moreover, if Ω is a cube, $f \in W^{m,p}_{\#}(\Omega)$, and if either f vanishes at the boundary or $\int_\Omega f\,dx = 0$, then (5.12) holds in the stronger form

$$\|D^j f\|_{L^q(\Omega)} \le C\|D^m f\|^{\theta}_{L^p(\Omega)}\|D^s f\|^{1-\theta}_{L^p(\Omega)}\,. \tag{5.13}$$

Finally, we conclude with an interpolation estimate involving the H^{-1}-norm, see Remark 3.3.

Lemma 5.7. *There exists $C > 0$ such that for all $f \in H^1_{\#}(Q)$, with $\int_Q f\,dx = 0$, we have*

$$\|f\|_{L^2(Q)} \le C\|Df\|^{\frac{1}{2}}_{L^2(Q)}\|f\|^{\frac{1}{2}}_{H^{-1}_{\#}(Q)}\,.$$

Similarly, there exists $C > 0$ such that for all $f \in H^2_{\#}(Q)$, with $\int_Q f\,dx = 0$, we have

$$\|f\|_{L^2(Q)} \le C\|D^2 f\|^{\frac{1}{3}}_{L^2(Q)}\|f\|^{\frac{2}{3}}_{H^{-1}_{\#}(Q)}\,.$$

Proof. Let w be the unique Q-periodic solution to

$$\begin{cases} -\Delta w = f & \text{in } Q\,,\\ \int_Q w\,dx = 0\,. \end{cases}$$

Combining Lemma 5.3 with (5.9) we obtain

$$\begin{aligned}\|f\|_{L^2(Q)} = \|\Delta w\|_{L^2(Q)} \le C\|D^2 w\|_{L^2(Q)} &\le C\|D^3 w\|^{\frac{1}{2}}_{L^2(Q)}\|Dw\|^{\frac{1}{2}}_{L^2(Q)}\\ &\le C\|\Delta(Dw)\|^{\frac{1}{2}}_{L^2(Q)}\|Dw\|^{\frac{1}{2}}_{L^2(Q)} = C\|Df\|^{\frac{1}{2}}_{L^2(Q)}\|f\|^{\frac{1}{2}}_{H^{-1}_{\#}(Q)}\,.\end{aligned}$$

The second inequality of the statement is proven similarly. □

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