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A MODEL FOR A LARGE INVESTOR TRADING AT MARKET INDIFFERENCE PRICES. II: CONTINUOUS-TIME CASE

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We develop from basic economic principles a continuous-time model for a large investor who trades with a finite number of market makers at their utility indifference prices. In this model, the market makers compete with their quotes for the investor's orders and trade among themselves to attain Pareto optimal allocations. We first consider the case of simple strategies and then, in analogy to the construction of stochastic integrals, investigate the transition to general continuous dynamics. As a result, we show that the model's evolution can be described by a nonlinear stochastic differential equation for the market makers' expected utilities.

1. Introduction. A typical financial model presumes that the prices of traded securities are not affected by an investor's buy and sell orders. From a practical viewpoint, this assumption is justified as long as his trading volume remains small enough to be easily covered by market liquidity. An opposite situation occurs, for instance, when an economic agent has to sell a large block of shares over a short period of time; see, for example, Almgren and Chriss [1] and Schied and Schöneborn [24]. This and other examples motivate the development of financial models for a "large" trader, where the dependence of market prices on his strategy, called a *price impact* or a *demand pressure*, is taken into account.

Hereafter, we assume that the interest rate is zero and, in particular, is not affected by the large investor. As usual in mathematical finance, we describe a (self-financing) strategy by a predictable process $Q = (Q_t)_{0 \le t \le T}$ where Q_t is the number of stocks held *just before* time t and T is a finite time horizon. The role of a "model" is to define a predictable process $X(Q)$ representing the evolution of the cash balance for the strategy Q. We denote by $S(Q)$ the *marginal* price process of traded stocks, that is, $S_t(Q)$ is the price at which one can trade an infinitesimal quantity of stocks at time t. Recall that in the standard model of a "small" agent

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the price S does not depend on Q and

$$X_t(Q) = \int_0^t Q_u \, dS_u - Q_t S_t.$$

In mathematical finance, a common approach is to specify the price impact of trades *exogenously*, that is, to postulate it as one of the inputs. For example, Frey and Stremme [13], Platen and Schweizer [23], Papanicolaou and Sircar [22] and Bank and Baum [4] choose a stochastic field of *reaction functions*, which explicitly state the dependence of the marginal prices on the investor's current holdings, Çetin, Jarrow and Protter in [8] start with a stochastic field of *supply curves*, which define the prices in terms of traded quantities (*changes* in holdings), and Cvitanić and Ma [10] make the drift and the volatility of the price process dependent on a trading strategy; we refer the reader to the recent survey [17] by Gökay, Roch and Soner for more details and additional references. Note that in all these models the processes $X(Q)$ and $S(Q)$, of the cash balance and of the marginal stock price, only depend on the "past" of the strategy Q, in the sense that

$$X_t(Q) = X_t(Q^t), \qquad S_t(Q) = S_t(Q^t), \tag{1.1}$$

where $Q^t \triangleq (Q_{s\wedge t})_{0\le s\le T}$ denotes the process Q "stopped" at t with $s \wedge t \triangleq \min(s,t)$.

The exogenous nature of the above models facilitates their calibration to market data; see, for example, [9] by Çetin, Jarrow and Protter. There are, however, some disadvantages. For example, the models in [4, 8, 13, 22, 23] and [9] do not satisfy the natural "closability" property for a large investor model:

$$|Q^n| \le \frac{1}{n} \quad \Longrightarrow \quad X_T(Q^n) \to 0, \qquad n \to \infty, \tag{1.2}$$

while in Cvitanić and Ma [10] the stock price is not affected by a jump in investor's holdings: $S_t(Q_t + \Delta Q_t) = S_t(Q_t)$.

In our project, we seek to derive the dependence of prices on strategies *endogenously* by relying on the framework developed in financial economics. A starting point here is the postulate that, at any given moment, a price reflects a balance between demand and supply or, more formally, it is an output of an *equilibrium*. In addition to the references cited below, we refer the reader to the book [21] by O'Hara and the survey [2] by Amihud, Mendelson and Pedersen.

To be more specific, denote by ψ the terminal price of the traded security, which we assume to be given exogenously, that is, $S_T(Q) = \psi$ for every strategy Q. Recall that in a small agent model the absence of arbitrage implies the existence of an equivalent probability measure $\mathbb{Q}$ such that

$$S_t = \mathbb{E}_{\mathbb{Q}}[\psi|\mathcal{F}_t], \qquad 0 \le t \le T, \tag{1.3}$$

where $\mathcal{F}_t$ is the σ-field describing the information available at time t. This result is often called *the fundamental theorem of asset pricing*; in full generality, it has been proved by Delbaen and Schachermayer in [11, 12]. The economic nature of this *pricing measure* $\mathbb{Q}$ does not matter in the standard, small agent, setup. However, it becomes important in an equilibrium-based construction of models for a large trader where it typically originates from a Pareto optimal allocation of wealth and is given by the expression (1.4) below.

We shall consider an economy formed by M market participants, called hereafter the *market makers*, whose preferences for terminal wealth are defined by utility functions $u_m = u_m(x)$, $m = 1, \dots, M$, and an identical subjective probability measure $\mathbb{P}$. It is well known in financial economics that the Pareto optimality of the market makers' wealth allocation $\alpha = (\alpha^m)_{m=1,\dots,M}$ yields the pricing measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = v^m u_m'(\alpha^m), \qquad m = 1, \dots, M, \tag{1.4}$$

where $v^m > 0$ is a normalizing constant.

It is natural to expect that in the case when the strategy Q is not anymore negligible an expression similar to (1.3) should still hold true for the *marginal* price process:

$$S_t(Q) = \mathbb{E}_{\mathbb{Q}_t(Q)}[\psi | \mathcal{F}_t(Q)], \qquad 0 \le t \le T. \tag{1.5}$$

This indicates that the price impact at time t described by the mapping $Q \mapsto S_t(Q)$ may be attributed to two common aspects of market's microstructure:

1. *Information*: $Q \mapsto \mathcal{F}_t(Q)$. Models focusing on information aspects naturally occur in the presence of an insider, where $\mathcal{F}_t(Q)$, the information available to the market makers at time t, is usually generated by the sum of Q and the cumulative demand process of "noise" traders; see Glosten and Milgrom [16], Kyle [20] and Back and Baruch [3], among others.
2. *Inventory*: $Q \mapsto \mathbb{Q}_t(Q)$. In view of (1.4), this reflects how $\alpha_t(Q)$, the Pareto optimal allocation of the total wealth or "inventory" induced by Q, affects the valuation of marginal trades. Note that the random variable $\alpha_t(Q)$ is measurable with respect to the terminal σ-field $\mathcal{F}_T(Q)$ [not with respect to the current σ-field $\mathcal{F}_t(Q)$!].

In our study, we shall focus on the inventory aspect of price formation and disregard the informational component. We assume that the market makers share the same exogenously given filtration $(\mathcal{F}_t)_{0\le t\le T}$ as the large trader and, in particular, their information flow is not affected by his strategy Q:

$$\mathcal{F}_t(Q) = \mathcal{F}_t, \qquad 0 \le t \le T.$$

Note that this informational symmetry is postulated only regarding the externally given random outcome. As we shall discuss below, in inventory based models, the actual form of the map $Q \mapsto \mathbb{Q}_t(Q)$, or, equivalently, $Q \mapsto \alpha_t(Q)$ is implied by game-theoretical features of the interaction between the market makers and the investor. In particular, it depends on the knowledge the market makers possess at time t about the subsequent evolution $(Q_s)_{t\le s\le T}$ of the investor's strategy, conditionally on the forthcoming random outcome on $[t,T]$.

For example, the models in Grossman and Miller [18], Garleanu, Pedersen and Poteshman [14] and German [15] rely on a setup inspired by the *Arrow–Debreu equilibrium*. Their framework implicitly assumes that right from the start the market makers have *full knowledge* of the investor's future strategy Q (of course, contingent on the unfolding random scenario). In this case, the resulting pricing measures and the Pareto allocations do not depend on time:

$$\mathbb{Q}_t(Q)=\mathbb{Q}(Q),\qquad \alpha_t(Q)=\alpha(Q),\qquad 0\le t\le T, \tag{1.6}$$

and are determined by the budget equations:

$$\mathbb{E}_{\mathbb{Q}(Q)}\big[\alpha^m(0)\big]=\mathbb{E}_{\mathbb{Q}(Q)}\big[\alpha^m(Q)\big],\qquad m=1,\dots,M,$$

and the clearing condition:

$$\sum_{m=1}^M\alpha^m(Q)=\sum_{m=1}^M\alpha^m(0)+\int_0^T Q_t\,dS_t(Q).$$

Here, $\mathbb{Q}(Q)$ and $S(Q)$ are defined in terms of $\alpha(Q)$ by (1.4) and (1.5). The positive sign in the clearing condition is due to our convention to interpret Q as the number of stocks held by the market makers. It is instructive to note that in the case of exponential utilities, when $u_m(x)=-\exp(-a_m x)$ with a risk-aversion $a_m>0$, the stock price in these models depends only on the "future" of the strategy:

$$S_t(Q)=S_t\big((Q_s)_{t\le s\le T}\big),\qquad 0\le t\le T,$$

which is just the opposite of (1.1).

In our model, the interaction between the market makers and the investor takes place according to a *Bertrand competition*; a similar framework (but with a single market maker and only in a one-period setting) was used in Stoll [25]. The key economic assumptions can be summarized as follows:

1. After every trade, the market makers can redistribute new income to form a Pareto allocation.
2. As a result of a trade, the expected utilities of the market makers do not change.

The first condition assumes that the market makers are able to find the most effective way to share among themselves the risk of the resulting total endowment, thus producing a Pareto optimal allocation. The second assumption is a consequence of a Bertrand competition which forces the market makers to quote the most aggressive prices without lowering their expected utilities; in the limit, these utilities are left unchanged.

Our framework implicitly assumes that at every time t the market makers have *no a priori knowledge* about the subsequent trading strategy $(Q_s)_{t\leq s\leq T}$ of the economic agent (even conditionally on the future random outcome). As a consequence, the marginal price process $S(Q)$ and the cash balance process $X(Q)$ are related to Q as in (1.1). Similarly, the dependence on Q of the pricing measures and of the Pareto optimal allocations is nonanticipative in the sense that

$$\mathbb{Q}_t(Q) = \mathbb{Q}_t(Q^t), \qquad \alpha_t(Q) = \alpha_t(Q^t), \qquad 0 \leq t \leq T,$$

which is quite opposite to (1.6).

In [5], we studied the model in a static, one-step, setting. The current paper deals with the general continuous-time framework. Building on the single-period case in an inductive manner, we first define *simple strategies*, where the trades occur only at a finite number of times; see Theorem 2.7. The main challenge is then to show that this construction allows for *a consistent passage* to general predictable strategies. For instance, it is an issue to verify that the cash balance process $X(Q)$ is stable with respect to uniform perturbations of the strategy Q and, in particular, that the closability property (1.2) and its generalizations stated in Questions 2.9 and 2.10 hold.

These stability questions are addressed by deriving and analyzing a nonlinear stochastic differential equation for the market makers' expected utilities; see (4.20) in Theorem 4.9. A key role is played by the fact, that together with the strategy Q, these utilities form a "sufficient statistics" in the model, that is, they uniquely determine the Pareto optimal allocation of wealth among the market makers. The corresponding functional dependencies are explicitly given as gradients of the *stochastic field of aggregate utilities* and its saddle conjugate; here we rely on our companion paper [6].

An outline of this paper is as follows. In Section 2, we define the model and study the case when the investor trades according to a simple strategy. In Section 3, we provide a conditional version of the well-known parameterization of Pareto optimal allocations and recall basic results from [6] concerning the stochastic field of aggregate utilities and its conjugate. With these tools at hand, we formally define the strategies with general continuous dynamics in Section 4. We conclude with Section 5 by showing that the construction of strategies in Section 4 is consistent with the original idea based on the approximation by simple strategies. In the last two sections, we restrict ourselves to a Brownian setting, due to convenience of references to Kunita [19].

2. Model.

2.1. *Market makers and the large investor.* We consider a financial model where $M \in \{1, 2, \dots\}$ market makers quote prices for a finite number of stocks. Uncertainty and the flow of information are modeled by a filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ satisfying the standard conditions of right-continuity and completeness; the initial σ-field $\mathcal{F}_0$ is trivial, T is a finite maturity and $\mathcal{F} = \mathcal{F}_T$.

As usual, we identify random variables differing on a set of $\mathbb{P}$-measure zero; $\mathbf{L}^0(\mathbf{R}^d)$ stands for the metric space of such equivalence classes with values in $\mathbf{R}^d$ endowed with the topology of convergence in probability; $\mathbf{L}^p(\mathbf{R}^d)$, $p \ge 1$, denotes the Banach space of p-integrable random variables. For a σ-field $\mathcal{A} \subset \mathcal{F}$ and a set $A \subset \mathbf{R}^d$ denote $\mathbf{L}^0(\mathcal{A}, A)$ and $\mathbf{L}^p(\mathcal{A}, A)$, $p \ge 1$, the respective subsets of $\mathbf{L}^0(\mathbf{R}^d)$ and $\mathbf{L}^p(\mathbf{R}^d)$ consisting of all $\mathcal{A}$-measurable random variables with values in A.

The way the market makers serve the incoming orders crucially depends on their attitude toward risk, which we model in the classical framework of expected utility. Thus, we interpret the probability measure $\mathbb{P}$ as a description of the common beliefs of our market makers (same for all) and denote by $u_m = (u_m(x))_{x\in\mathbf{R}}$ market maker m's utility function for terminal wealth.

ASSUMPTION 2.1. Each $u_m = u_m(x)$, $m = 1, \dots, M$, is a strictly concave, strictly increasing, continuously differentiable, and bounded from above function on the real line $\mathbf{R}$ satisfying

$$\lim_{x\to\infty} u_m(x) = 0. \tag{2.1}$$

The normalizing condition (2.1) is added only for notational convenience. Our main results will be derived under the following additional condition on the utility functions, which, in particular, implies their boundedness from above.

ASSUMPTION 2.2. Each utility function $u_m = u_m(x)$, $m = 1, \dots, M$, is twice continuously differentiable and its absolute risk aversion coefficient is bounded away from zero and infinity, that is, for some $c > 0$,

$$\frac{1}{c} \le a_m(x) \triangleq -\frac{u_m''(x)}{u_m'(x)} \le c, \qquad x \in \mathbf{R}.$$

The prices quoted by the market makers are also influenced by their initial endowments $\alpha_0 = (\alpha_0^m)_{m=1,\dots,M} \in \mathbf{L}^0(\mathbf{R}^M)$, where α_0^m is an $\mathcal{F}$-measurable random variable describing the terminal wealth of the mth market maker (if the large investor, introduced later, will not trade at all). We assume that the initial allocation α_0 is *Pareto optimal* in the sense of:

DEFINITION 2.3. Let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$. A vector of $\mathcal{F}$-measurable random variables $\alpha = (\alpha^m)_{m=1,\dots,M}$ is called *a Pareto optimal allocation given the information* $\mathcal{G}$ or just a $\mathcal{G}$*-Pareto allocation* if

$$\mathbb{E}[|u_m(\alpha^m)||\mathcal{G}] < \infty, \qquad m = 1, \dots, M, \tag{2.2}$$

and there is no other allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment,

$$\sum_{m=1}^{M} \beta^m = \sum_{m=1}^{M} \alpha^m, \tag{2.3}$$

leaving all market makers not worse and at least one of them better off in the sense that

$$\mathbb{E}[u_m(\beta^m)|\mathcal{G}] \geq \mathbb{E}[u_m(\alpha^m)|\mathcal{G}] \qquad \text{for all } m = 1, \dots, M \tag{2.4}$$

and

$$\mathbb{P}[\mathbb{E}[u_m(\beta^m)|\mathcal{G}] > \mathbb{E}[u_m(\alpha^m)|\mathcal{G}]] > 0 \qquad \text{for some } m \in \{1, \dots, M\}. \tag{2.5}$$

A Pareto optimal allocation given the trivial σ-field $\mathcal{F}_0$ is simply called a *Pareto allocation*.

In other words, Pareto optimality is a stability requirement for an allocation of wealth which ensures that there are no mutually beneficial trades that can be struck between market makers.

Finally, we consider an economic agent or investor who is going to trade dynamically in the financial market formed by a bank account and J stocks. We assume that the interest rate on the bank account is given *exogenously* and is not affected by the investor's trades; for simplicity of notation, we set it to be zero. The stocks pay terminal dividends $\psi = (\psi^j)_{j=1,\dots,J} \in \mathbf{L}^0(\mathbf{R}^J)$. Their prices are computed *endogenously* and depend on investor's order flow.

As the result of trading with the investor, up to and including time $t \in [0, T]$, the total endowment of the market makers may change from $\Sigma_0 \triangleq \sum_{m=1}^{M} \alpha_0^m$ to

$$\Sigma(\xi, \theta) \triangleq \Sigma_0 + \xi + \langle \theta, \psi \rangle = \Sigma_0 + \xi + \sum_{j=1}^{J} \theta^j \psi^j, \tag{2.6}$$

where $\xi \in \mathbf{L}^0(\mathcal{F}_t, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathcal{F}_t, \mathbf{R}^J)$ are, respectively, the cash amount and the number of assets *acquired* by the market makers from the investor; they are $\mathcal{F}_t$-measurable random variables with values in $\mathbf{R}$ and $\mathbf{R}^J$, respectively. Our model will assume that $\Sigma(\xi, \theta)$ is allocated among the market makers in the form of an $\mathcal{F}_t$-Pareto allocation. For this to be possible, we have to impose:

ASSUMPTION 2.4. For every $x \in \mathbf{R}$ and $q \in \mathbf{R}^J$, there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total random endowment $\Sigma(x, q)$ defined in (2.6) such that

$$\mathbb{E}[u_m(\beta^m)] > -\infty, \qquad m = 1, \dots, M. \tag{2.7}$$

See (3.15) for an equivalent reformulation of this assumption in terms of the aggregate utility function. For later use, we verify its conditional version.

LEMMA 2.5. *Under Assumptions 2.1 and 2.4, for every σ-field $\mathcal{G} \subset \mathcal{F}$ and random variables $\xi \in \mathbf{L}^0(\mathcal{G}, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathcal{G}, \mathbf{R}^J)$ there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total endowment $\Sigma(\xi, \theta)$ such that*

$$\mathbb{E}[u_m(\beta^m)|\mathcal{G}] > -\infty, \qquad m = 1, \dots, M. \tag{2.8}$$

PROOF. Clearly, it is sufficient to verify (2.8) on each of the $\mathcal{G}$-measurable sets

$$A_n \triangleq \{\omega \in \Omega : |\xi(\omega)| + |\theta(\omega)| \le n\}, \qquad n \ge 1,$$

which shows that without loss of generality we can assume ξ and θ to be bounded when proving (2.8). Then (ξ, θ) can be written as a convex combination of finitely many points $(x_k, q_k) \in \mathbf{R}^{1+J}$, $k = 1, \dots, K$ with $\mathcal{G}$-measurable weights $\lambda^k \ge 0$, $\sum_{k=1}^K \lambda^k = 1$. By Assumption 2.4, for each $k = 1, \dots, K$ there is an allocation β_k with the total endowment $\Sigma(x_k, q_k)$ such that

$$\mathbb{E}[u_m(\beta_k^m)] > -\infty, \qquad m = 1, \dots, M.$$

Thus, the allocation

$$\beta \triangleq \sum_{k=1}^K \lambda^k \beta_k$$

has the total endowment $\Sigma(\xi, \theta)$ and, by the concavity of the utility functions, satisfies (2.7), and hence, also (2.8). $\square$

2.2. *Simple strategies*. An investment strategy of the agent is described by a predictable J-dimensional process $Q = (Q_t)_{0 \le t \le T}$, where $Q_t = (Q_t^j)_{j=1,\dots,J}$ is the cumulative number of the stocks *sold* by the investor through his transactions up to time t. For a strategy to be self-financing we have to complement Q by a corresponding predictable process $X = (X_t)_{0 \le t \le T}$ describing the cumulative amount of cash *spent* by the investor. Hereafter, we shall call such an X a *cash balance* process.

REMARK 2.6. Our description of a trading strategy follows the standard practice of mathematical finance except for the sign: positive values of Q or X now mean *short* positions for the investor in stocks or cash, and hence total *long* positions for the market makers. This convention makes future notation more simple and intuitive.

To facilitate the understanding of the economic assumptions behind our model, we consider first the case of a simple strategy Q where trading occurs only at a finite number of times, that is,

$$Q_t = \sum_{n=1}^N \theta_n 1_{(\tau_{n-1}, \tau_n]}(t), \qquad 0 \le t \le T, \tag{2.9}$$

with stopping times $0 = \tau_0 \le \cdots \le \tau_N = T$ and random variables $\theta_n \in \mathbf{L}^0(\mathcal{F}_{\tau_{n-1}}, \mathbf{R}^J)$, $n = 1, \dots, N$. It is natural to expect that, for such a strategy Q, the cash balance process X has a similar form:

$$X_t = \sum_{n=1}^{N} \xi_n 1_{(\tau_{n-1}, \tau_n]}(t), \qquad 0 \le t \le T, \tag{2.10}$$

with $\xi_n \in \mathbf{L}^0(\mathcal{F}_{\tau_{n-1}}, \mathbf{R})$, $n = 1, \dots, N$. In our model, these cash amounts will be determined by (forward) induction along with a sequence of conditionally Pareto optimal allocations $(\alpha_n)_{n=1,\dots,N}$ such that each α_n is an $\mathcal{F}_{\tau_{n-1}}$-Pareto allocation with the total endowment

$$\Sigma(\xi_n, \theta_n) = \Sigma_0 + \xi_n + \langle \theta_n, \psi \rangle.$$

Recall that at time 0, before any trade with the investor has taken place, the market makers have the initial Pareto allocation α_0 and the total endowment Σ_0. After the first transaction of θ_1 stocks and ξ_1 in cash, the total random endowment becomes $\Sigma(\xi_1, \theta_1)$. The central assumptions of our model, which will allow us to identify the cash amount ξ_1 uniquely, are that, as a result of the trade:

1. The random endowment $\Sigma(\xi_1, \theta_1)$ is redistributed between the market makers to form a new *Pareto* allocation α_1.
2. The market makers' expected utilities *do not change*:

$$\mathbb{E}[u_m(\alpha_1^m)] = \mathbb{E}[u_m(\alpha_0^m)], \qquad m = 1, \dots, M.$$

Proceeding by induction, we arrive at the re-balance time τ_n with the economy characterized by an $\mathcal{F}_{\tau_{n-1}}$-Pareto allocation α_n of the random endowment $\Sigma(\xi_n, \theta_n)$. We assume that after exchanging $\theta_{n+1} - \theta_n$ securities and $\xi_{n+1} - \xi_n$ in cash the market makers will hold an $\mathcal{F}_{\tau_n}$-Pareto allocation α_{n+1} of $\Sigma(\xi_{n+1}, \theta_{n+1})$ satisfying the key condition of the preservation of expected utilities:

$$\mathbb{E}[u_m(\alpha_{n+1}^m) | \mathcal{F}_{\tau_n}] = \mathbb{E}[u_m(\alpha_n^m) | \mathcal{F}_{\tau_n}], \qquad m = 1, \dots, M. \tag{2.11}$$

The fact that this inductive procedure indeed works is ensured by the following result, established in a single-period framework in [5], Theorem 2.6.

THEOREM 2.7. *Under Assumptions* 2.1 *and* 2.4, *every sequence of stock positions* $(\theta_n)_{n=1,\dots,N}$ *as in* (2.9) *yields a unique sequence of cash balances* $(\xi_n)_{n=1,\dots,N}$ *as in* (2.10) *and a unique sequence of allocations* $(\alpha_n)_{n=1,\dots,N}$ *such that, for each* $n = 1, \dots, N$, α_n *is an* $\mathcal{F}_{\tau_{n-1}}$*-Pareto allocation of* $\Sigma(\xi_n, \theta_n)$ *preserving the market makers' expected utilities in the sense of* (2.11).

PROOF. The proof follows from Lemma 2.5 above, Lemma 2.8 below and a standard induction argument. □

LEMMA 2.8. *Let Assumption 2.1 hold and consider a σ-field $\mathcal{G} \subset \mathcal{F}$ and random variables $\gamma \in \mathbf{L}^0(\mathcal{G}, (-\infty, 0)^M)$ and $\Sigma \in \mathbf{L}^0(\mathbf{R})$. Suppose there is an allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ which has the total endowment Σ and satisfies the integrability condition (2.8).*

Then there are a unique $\xi \in \mathbf{L}^0(\mathcal{G}, \mathbf{R})$ and a unique $\mathcal{G}$-Pareto allocation α with the total endowment $\Sigma + \xi$ such that

$$\mathbb{E}[u_m(\alpha^m)|\mathcal{G}] = \gamma^m, \qquad m = 1, \dots, M.$$

PROOF. The uniqueness of such ξ and α is a consequence of the definition of the $\mathcal{G}$-Pareto optimality and the strict concavity and monotonicity of the utility functions. Indeed, let $\widetilde{\xi}$ and $\widetilde{\alpha}$ be another such pair. The allocation

$$\beta^m \triangleq \left(\widetilde{\alpha}^m + \frac{\xi - \widetilde{\xi}}{M}\right) 1_{\{\widetilde{\xi} < \xi\}} + \alpha^m 1_{\{\widetilde{\xi} \geq \xi\}}, \qquad m = 1, \dots, M,$$

has the same total endowment $\Sigma + \xi$ as α. If the $\mathcal{G}$-measurable set $\{\widetilde{\xi} < \xi\}$ is not empty, then because the utility functions (u_m) are strictly increasing, β dominates α in the sense of Definition 2.3 and we get a contradiction with the $\mathcal{G}$-Pareto optimality of α. Hence, $\widetilde{\xi} \geq \xi$ and then, by symmetry, $\widetilde{\xi} = \xi$. In this case, the allocation $\widetilde{\beta} \triangleq (\alpha + \widetilde{\alpha})/2$ has the same total endowment as α and $\widetilde{\alpha}$. If $\widetilde{\alpha} \neq \alpha$ then, in view of the strict concavity of the utility functions, $\widetilde{\beta}$ dominates both α and $\widetilde{\alpha}$, contradicting their $\mathcal{G}$-Pareto optimality.

To verify the existence, we shall use a conditional version of the argument from the proof of Theorem 2.6 in [5]. To facilitate references, we assume hereafter that $|\gamma| \triangleq \sqrt{\sum_{m=1}^M (\gamma^m)^2}$ is integrable, that is, $\gamma \in \mathbf{L}^1(\mathcal{G}, (-\infty, 0)^M)$. This extra condition does not restrict any generality as, if necessary, we can replace the reference probability measure $\mathbb{P}$ with the equivalent measure $\mathbb{Q}$ such that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \text{const} \frac{1}{1 + |\gamma|}.$$

Note that because γ is $\mathcal{G}$-measurable this change of measure does not affect $\mathcal{G}$-Pareto optimality.

For $\eta \in \mathbf{L}^0(\mathcal{G}, \mathbf{R})$, denote by $\mathcal{B}(\eta)$ the family of allocations $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with total endowments less than or equal to $\Sigma + \eta$ such that

$$\mathbb{E}[u_m(\beta^m)|\mathcal{G}] \geq \gamma^m, \qquad m = 1, \dots, M.$$

Since the utility functions $u_m = u_m(x)$ are increasing and converge to 0 as $x \to \infty$ and because there is an allocation β of Σ satisfying (2.8), the set

$$\mathcal{H} \triangleq \{\eta \in \mathbf{L}^0(\mathcal{G}, \mathbf{R}) : \mathcal{B}(\eta) \neq \varnothing\}$$

is nonempty. For instance, it contains the random variable

$$\widetilde{\eta} \triangleq M \sum_{n=1}^{\infty} n(1_{A_n} - 1_{A_{n-1}}),$$

where, for $n = 0, 1, \dots,$

$$A_n \triangleq \{\omega \in \Omega : \mathbb{E}[u_m(\beta^m + n)|\mathcal{G}](\omega) \geq \gamma^m(\omega), m = 1, \dots, M\}.$$

Indeed, by construction, $\widetilde{\eta}$ is $\mathcal{G}$-measurable and, as $A_n \uparrow \Omega$,

$$\mathbb{E}[u_m(\beta^m + \widetilde{\eta}/M)|\mathcal{G}] \geq \gamma^m, \qquad m = 1, \dots, M.$$

Hence, the allocation $(\beta^m + \widetilde{\eta}/M)_{m=1,\dots,M}$ belongs to $\mathcal{B}(\widetilde{\eta})$.

If $\eta \in \mathcal{H}$, then the set $\mathcal{B}(\eta) \in \mathbf{L}^0(\mathbf{R}^M)$ is convex (even with respect to $\mathcal{G}$-measurable weights) by the concavity of the utility functions. Moreover, this set is bounded in $\mathbf{L}^0(\mathbf{R}^M)$:

$$\lim_{z\to\infty} \sup_{\beta\in\mathcal{B}(\eta)} \mathbb{P}[|\beta| \geq z] = 0.$$

Indeed, from the properties of utility functions in Assumption 2.1 we deduce that

$$x^- \triangleq \max(0, -x) \leq -\frac{u_m(x)}{u_m'(0)}, \qquad x \in \mathbf{R}.$$

Hence, for $\beta \in \mathcal{B}(\eta)$,

$$\mathbb{E}[(\beta^m)^-] \leq \frac{1}{u_m'(0)}\mathbb{E}[-u_m(\beta^m)] \leq \frac{1}{u_m'(0)}\mathbb{E}[-\gamma^m] < \infty,$$

implying that the set $\{((\beta^m)^-)_{m=1,\dots,M} : \beta \in \mathcal{B}(\eta)\}$ is bounded in $\mathbf{L}^1(\mathbf{R}^M)$. The boundedness of $\mathcal{B}(\eta)$ in $\mathbf{L}^0(\mathbf{R}^M)$ then follows after we recall that

$$\sum_{m=1}^M \beta^m \leq \Sigma + \eta, \qquad \beta \in \mathcal{B}(\eta).$$

Observe that if the random variables $(\eta_i)_{i=1,2}$ belong to $\mathcal{H}$, then so does their minimum $\eta_1 \wedge \eta_2$. It follows that there is a decreasing sequence $(\eta_n)_{n\geq1}$ in $\mathcal{H}$ such that its limit ξ is less than or equal to every element of $\mathcal{H}$. Let $\beta_n \in \mathcal{B}(\eta_n)$, $n \geq 1$. As $\beta_n \in \mathcal{B}(\eta_1)$, the family of all possible convex combinations of $(\beta_n)_{n\geq1}$ is bounded in $\mathbf{L}^0(\mathbf{R}^M)$. By Lemma A1.1 in Delbaen and Schachermayer [11], we can then choose convex combinations ζ_n of $(\beta_k)_{k\geq n}$, $n \geq 1$, converging almost surely to a random variable $\alpha \in \mathbf{L}^0(\mathbf{R}^M)$. It is clear that

$$\sum_{m=1}^M \alpha^m \leq \Sigma + \xi. \tag{2.12}$$

Since the utility functions are bounded above and, by the convexity of $\mathcal{B}(\eta_n)$, $\zeta_n \in \mathcal{B}(\eta_n)$, an application of Fatou's lemma yields

$$\mathbb{E}[u_m(\alpha^m)|\mathcal{G}] \geq \limsup_{n\to\infty} \mathbb{E}[u_m(\zeta_n^m)|\mathcal{G}] \geq \gamma^m, \qquad m = 1, \dots, M. \tag{2.13}$$

It follows that $\alpha \in \mathcal{B}(\xi)$. The minimality property of ξ then immediately implies that in (2.12) and (2.13) we have, in fact, equalities and that α is a $\mathcal{G}$-Pareto allocation. □

In Section 4, we shall prove a more constructive version of Theorem 2.7, namely, Theorem 4.1, where the cash balances ξ_n and the Pareto allocations α_n will be given as explicit functions of their predecessors and of the new position θ_n.

The main goal of this paper is to extend the definition of the cash balance processes X from simple to general predictable strategies Q. This task has a number of similarities with the construction of a stochastic integral with respect to a semimartingale. In particular, we are interested in the following questions.

QUESTION 2.9. For simple strategies $(Q^n)_{n\geq 1}$ that converge to another simple strategy Q in ucp, that is, such that

$$(Q^n - Q)^*_T \triangleq \sup_{0\leq t\leq T} |Q^n_t - Q_t| \to 0, \tag{2.14}$$

do the corresponding cash balance processes converge in ucp as well:

$$(X^n - X)^*_T \to 0?$$

QUESTION 2.10. For every sequence of simple strategies $(Q^n)_{n\geq 1}$ converging in ucp to a predictable process Q, does the sequence $(X^n)_{n\geq 1}$ of their cash balance processes converge to a predictable process X in ucp?

Naturally, when we have an affirmative answer to Question 2.10, the process X should be called the cash balance process for the strategy Q. Note that a predictable process Q can be approximated by simple processes as in (2.14) if and only if it has LCRL (left-continuous with right limits) trajectories.

The construction of cash balance processes X and processes of Pareto allocations for general strategies Q will be accomplished in Section 4, while the answers to Questions 2.9 and 2.10 will be given in Section 5. These results rely on the parameterization of Pareto allocations in Section 3.1 and the properties of sample paths of the stochastic field of aggregate utilities established in [6] and recalled in Section 3.2.

3. Random fields associated with Pareto allocations. Let us collect in this section some notation and results which will allow us to work efficiently with conditional Pareto allocations. We first recall some terminology. For a set $A \subset \mathbf{R}^d$ a map $\xi : A \to \mathbf{L}^0(\mathbf{R}^n)$ is called a *random field*; ξ is continuous, convex, etc., if its *sample paths* $\xi(\omega) : A \to \mathbf{R}^n$ are continuous, convex, etc., for all $\omega \in \Omega$. A random field $X : A \times [0, T] \to \mathbf{L}^0(\mathbf{R}^n)$ is called a *stochastic field* if, for $t \in [0, T]$, $X_t \triangleq X(\cdot, t) : A \to \mathbf{L}^0(\mathcal{F}_t, \mathbf{R}^n)$, that is, the random variable X_t is $\mathcal{F}_t$-measurable.

3.1. *Parameterization of Pareto allocations*. We begin by recalling the results and notation from [5] concerning the classical parameterization of Pareto allocations. As usual in the theory of such allocations, a key role is played by *the aggregate utility function*

$$r(v,x) \triangleq \sup_{x^1+\cdots+x^M=x} \sum_{m=1}^{M} v^m u_m(x^m), \qquad v \in (0,\infty)^M, x \in \mathbf{R}. \tag{3.1}$$

We shall rely on the properties of this function stated in Section 3 of [6]. In particular, r is continuously differentiable and the upper bound in (3.1) is attained at the unique vector $\widehat{x} = \widehat{x}(v,x)$ in $\mathbf{R}^M$ determined by either

$$v^m u_m'(\widehat{x}^m) = \frac{\partial r}{\partial x}(v,x), \qquad m = 1,\dots,M, \tag{3.2}$$

or, equivalently,

$$u_m(\widehat{x}^m) = \frac{\partial r}{\partial v^m}(v,x), \qquad m = 1,\dots,M. \tag{3.3}$$

Following [5], we denote by

$$\mathbf{A} \triangleq (0,\infty)^M \times \mathbf{R} \times \mathbf{R}^J, \tag{3.4}$$

the parameter set of Pareto allocations in our economy. An element $a \in \mathbf{A}$ will often be represented as $a = (v,x,q)$. Here, $v \in (0,\infty)^M$ is a Pareto weight and $x \in \mathbf{R}$ and $q \in \mathbf{R}^J$ stand for, respectively, a cash amount and a number of stocks owned collectively by the market makers.

According to Lemma 3.2 in [5], for $a = (v,x,q) \in \mathbf{A}$, the random vector $\pi(a) \in \mathbf{L}^0(\mathbf{R}^M)$ defined by

$$v^m u_m'(\pi^m(a)) = \frac{\partial r}{\partial x}(v, \Sigma(x,q)), \qquad m = 1,\dots,M, \tag{3.5}$$

forms a Pareto allocation and, conversely, for $(x,q) \in \mathbf{R} \times \mathbf{R}^J$, every Pareto allocation of the total endowment $\Sigma(x,q)$ is given by (3.5) for some $v \in (0,\infty)^M$. Moreover, $\pi(v_1,x,q) = \pi(v_2,x,q)$ if and only if $v_1 = cv_2$ for some constant $c > 0$ and, therefore, (3.5) defines a one-to-one correspondence between the Pareto allocations with total endowment $\Sigma(x,q)$ and the set

$$\mathbf{S}^M \triangleq \left\{ w \in (0,1)^M : \sum_{m=1}^{M} w^m = 1 \right\},$$

the interior of the simplex in $\mathbf{R}^M$. Following [5], we denote by

$$\pi : \mathbf{A} \to \mathbf{L}^0(\mathbf{R}^M),$$

the random field of Pareto allocations given by (3.5). Clearly, the sample paths of this random field are continuous. From the equivalence of (3.2) and (3.3), we deduce that the Pareto allocation $\pi(a)$ can be equivalently defined by

$$u_m(\pi^m(a)) = \frac{\partial r}{\partial v^m}(v, \Sigma(x,q)), \qquad m = 1, \dots, M. \tag{3.6}$$

In Corollary 3.2 below, we provide the description of the conditional Pareto allocations in our economy, which is analogous to (3.5). The proof of this corollary relies on the following general and well-known fact, which is a conditional version of Theorem 3.1 in [5].

THEOREM 3.1. *Consider the family of market makers with utility functions* $(u_m)_{m=1,\dots,M}$ *satisfying Assumption 2.1. Let* $\mathcal{G} \subset \mathcal{F}$ *be a* σ*-field and* $\alpha \in \mathbf{L}^0(\mathbf{R}^M)$. *Then the following statements are equivalent*:

1. *The allocation* α *is* $\mathcal{G}$*-Pareto optimal.*
2. *Integrability condition* (2.2) *holds and there is a* $\mathcal{G}$*-measurable random variable* λ *with values in* $\mathbf{S}^M$ *such that*

$$\lambda^m u_m'(\alpha^m) = \frac{\partial r}{\partial x}(\lambda, \Sigma), \qquad m = 1, \dots, M, \tag{3.7}$$

where $\Sigma \triangleq \sum_{m=1}^M \alpha^m$ *and the function* $r = r(v,x)$ *is defined in* (3.1).

Moreover, such a random variable λ *is defined uniquely in* $\mathbf{L}^0(\mathcal{G}, \mathbf{S}^M)$.

PROOF. $1 \Longrightarrow 2$: It is enough to show that

$$\frac{u_m'(\alpha^m)}{u_1'(\alpha^1)} \in \mathbf{L}^0(\mathcal{G}, (0,\infty)), \qquad m = 1, \dots, M. \tag{3.8}$$

Indeed, in this case, define

$$\lambda^m \triangleq \frac{1/u_m'(\alpha^m)}{\sum_{k=1}^M 1/u_k'(\alpha^k)}, \qquad m = 1, \dots, M,$$

and observe that, as u_m' are strictly decreasing functions, (α^m) is the only allocation of Σ such that

$$\lambda^m u_m'(\alpha^m) = \lambda^1 u_1'(\alpha^1), \qquad m = 1, \dots, M.$$

However, in view of (3.2), an allocation with such property is provided by (3.7).

Clearly, every $\lambda \in \mathbf{L}^0(\mathcal{G}, \mathbf{S}^M)$ obeying (3.7) also satisfies the equality above and, hence, is defined uniquely.

Suppose (3.8) fails to hold for some index m, for example, for $m = 2$. Then we can find a random variable ξ such that

$$|\xi| \le 1, \qquad \big(u_1'(\alpha^1 - 1) + u_2'(\alpha^2 - 1)\big)|\xi| \in \mathbf{L}^1(\mathbf{R}), \tag{3.9}$$

and the set

$$A \triangleq \{\omega \in \Omega : \mathbb{E}[u_1'(\alpha^1)\xi|\mathcal{G}](\omega) < 0 < \mathbb{E}[u_2'(\alpha^2)\xi|\mathcal{G}](\omega)\}$$

has positive probability. For instance, we can take

$$\xi \triangleq \frac{\zeta - \widetilde{\mathbb{E}}[\zeta|\mathcal{G}]}{1 + u_1'(\alpha^1 - 1) + u_2'(\alpha^2 - 1)},$$

where

$$\zeta \triangleq \frac{u_2'(\alpha^2)}{u_1'(\alpha^1) + u_2'(\alpha^2)}$$

and $\widetilde{\mathbb{E}}$ is the expectation under the probability measure $\widetilde{\mathbb{P}}$ with the density

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \text{const}\,\frac{u_1'(\alpha^1) + u_2'(\alpha^2)}{1 + u_1'(\alpha^1 - 1) + u_2'(\alpha^2 - 1)}.$$

Indeed, in this case, (3.9) holds easily, while, as direct computations show

$$A = \{\omega \in \Omega : \widetilde{\mathbb{E}}[(\zeta - \widetilde{\mathbb{E}}[\zeta|\mathcal{G}])^2|\mathcal{G}](\omega) > 0\}$$

and $\mathbb{P}[A] > 0$ because ζ is not $\mathcal{G}$-measurable.

From the continuity of the first derivatives of the utility functions, we deduce the existence of $0 < \varepsilon < 1$ such that the set

$$B \triangleq \{\omega \in \Omega : \mathbb{E}[u_1'(\alpha^1 - \varepsilon\xi)\xi|\mathcal{G}](\omega) < 0 < \mathbb{E}[u_2'(\alpha^2 + \varepsilon\xi)\xi|\mathcal{G}](\omega)\}$$

also has positive probability. Denoting $\eta \triangleq \varepsilon\xi 1_B$ and observing that, by the concavity of utility functions,

$$u_1(\alpha^1) \le u_1(\alpha^1 - \eta) + u_1'(\alpha^1 - \eta)\eta,$$
$$u_2(\alpha^2) \le u_2(\alpha^2 + \eta) - u_2'(\alpha^2 + \eta)\eta,$$

we obtain that the allocation

$$\beta^1 = \alpha^1 - \eta, \qquad \beta^2 = \alpha^2 + \eta, \qquad \beta^m = \alpha^m, \qquad m = 3, \ldots, M,$$

satisfies (2.3), (2.4) and (2.5), thus contradicting the $\mathcal{G}$-Pareto optimality of α.

$2 \Longrightarrow 1$: For every allocation $\beta \in \mathbf{L}^0(\mathbf{R}^M)$ with the same total endowment Σ as α, we have

$$\sum_{m=1}^{M} \lambda^m u_m(\beta^m) \le r(\lambda, \Sigma) = \sum_{m=1}^{M} \lambda^m u_m(\alpha^m), \tag{3.10}$$

where the last equality is equivalent to (3.7) in view of (3.2). Granted integrability as in (2.2), this clearly implies the $\mathcal{G}$-Pareto optimality of α. $\square$

From Theorem 3.1 and the definition of the random field $\pi = \pi(a)$ in (3.5), we obtain the following corollary.

COROLLARY 3.2. *Let Assumptions 2.1 and 2.4 hold and consider a σ-field $\mathcal{G} \subset \mathcal{F}$ and random variables $\xi \in \mathbf{L}^0(\mathcal{G}, \mathbf{R})$ and $\theta \in \mathbf{L}^0(\mathcal{G}, \mathbf{R}^J)$.*

Then for every $\lambda \in \mathbf{L}^0(\mathcal{G}, (0, \infty)^M)$ the random vector $\pi(\lambda, \xi, \theta)$ forms a $\mathcal{G}$-Pareto allocation. Conversely, every $\mathcal{G}$-Pareto allocation of the total endowment $\Sigma(\xi, \theta)$ is given by $\pi(\lambda, \xi, \theta)$ for some $\lambda \in \mathbf{L}^0(\mathcal{G}, (0, \infty)^M)$.

PROOF. The only delicate point is to show that the allocation

$$\alpha^m \triangleq \pi^m(\lambda, \xi, \theta), \qquad m = 1, \dots, M,$$

satisfies the integrability condition (2.2). Lemma 2.5 implies the existence of an allocation β of $\Sigma(\xi, \theta)$ satisfying (2.8). The result now follows from inequality (3.10) which holds true by the properties of $r = r(v, x)$. $\square$

3.2. *Stochastic field of aggregate utilities and its conjugate.* A key role in the construction of the general investment strategies will be played by the stochastic field F of aggregate utilities and its saddle conjugate stochastic field G given by

$$F_t(a) \triangleq \mathbb{E}[r(v, \Sigma(x, q))|\mathcal{F}_t], \qquad a = (v, x, q) \in \mathbf{A}, \tag{3.11}$$

$$G_t(b) \triangleq \sup_{v \in (0,\infty)^M} \inf_{x \in \mathbf{R}} [\langle v, u \rangle + xy - F_t(v, x, q)], \qquad b = (u, y, q) \in \mathbf{B}, \tag{3.12}$$

where $t \in [0, T]$, the aggregate utility function $r = r(v, x)$ is given by (3.1), the parameter set $\mathbf{A}$ is defined in (3.4), and

$$\mathbf{B} \triangleq (-\infty, 0)^M \times (0, \infty) \times \mathbf{R}^J.$$

These stochastic fields are studied in [6]. For the convenience of future references, we recall below some of their properties.

First, we need to introduce some notation. For a nonnegative integer m and an open subset U of $\mathbf{R}^d$ denote by $\mathbf{C}^m = \mathbf{C}^m(U)$ the Fréchet space of m-times continuously differentiable maps $f : U \to \mathbf{R}$ with the topology generated by the semi-norms

$$\|f\|_{m,C} \triangleq \sum_{0 \le |k| \le m} \sup_{x \in C} |D^k f(x)|. \tag{3.13}$$

Here, C is a compact subset of U, $k = (k_1, \dots, k_d)$ is a multi-index of nonnegative integers, $|k| \triangleq \sum_{i=1}^d k_i$, and

$$D^k \triangleq \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}. \tag{3.14}$$

In particular, for $m = 0$, D^0 is the identity operator and $\|f\|_{0,C} \triangleq \sup_{x \in C} |f(x)|$.

For a metric space $\mathbf{X}$, we denote by $\mathbf{D}([0,T],\mathbf{X})$ the space of RCLL (right-continuous with left limits) maps of $[0,T]$ to $\mathbf{X}$.

Suppose now that Assumptions 2.1 and 2.4 hold. Note that in [6] instead of Assumption 2.4 we used the equivalent condition:

$$\mathbb{E}[r(v, \Sigma(x,q))] > -\infty, \qquad (v,x,q) \in \mathbf{A}; \tag{3.15}$$

see Lemma 3.2 in [5] for the proof of equivalence. Theorem 4.1 and Corollary 4.3 in [6] describe in detail the properties of the sample paths of the stochastic fields F and G. In particular, these sample paths belong to $\mathbf{D}([0,T],\mathbf{C}^1)$ and for every $t \in [0,T]$, $a = (w,x,q) \in \mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J$, and $b = (u,1,q)$ with $u \in (-\infty,0)^M$ we have the invertibility relations

$$w = \frac{\partial G_t}{\partial u}\left(\frac{\partial F_t}{\partial v}(a,t),1,q\right) \Big/ \left(\sum_{m=1}^{M} \frac{\partial G_t}{\partial u^m}\left(\frac{\partial F_t}{\partial v}(a),1,q\right)\right), \tag{3.16}$$

$$x = G_t\left(\frac{\partial F_t}{\partial v}(a),1,q\right), \tag{3.17}$$

$$\begin{aligned} u &= \frac{\partial F_t}{\partial v}\left(\frac{\partial G_t}{\partial u}(b), G(b), q\right) \\ &= \frac{\partial F_t}{\partial v}\left(\frac{\partial G_t}{\partial u}(b) \Big/ \left(\sum_{m=1}^{M} \frac{\partial G}{\partial u^m}(b)\right), G(b), q\right). \end{aligned} \tag{3.18}$$

Moreover, the left-limits $F_{t-}(\cdot)$ and $G_{t-}(\cdot)$ are conjugate to each other in a sense analogous to (3.12) and they also satisfy the corresponding versions of the invertibility relations (3.16)–(3.18).

Theorem 4.1 in [6] also states that

$$\frac{\partial F_t}{\partial a^i}(a) = \mathbb{E}\left[\frac{\partial F_T}{\partial a^i}(a)\Big|\mathcal{F}_t\right], \qquad t \in [0,T], a \in \mathbf{A},$$

which, in view of (3.6), implies that the derivatives of F with respect to v equal to the *expected utilities* of the market markers given the Pareto allocation $\pi(a)$:

$$\frac{\partial F_t}{\partial v^m}(a) = \mathbb{E}[u_m(\pi^m(a))|\mathcal{F}_t], \qquad m = 1,\dots,M. \tag{3.19}$$

By (3.17), the random variable $G_t(u,1,q)$ then defines the *collective cash amount* of the market makers at time t when their current expected utilities are given by u and they jointly own q stocks.

If Assumption 2.2 holds as well, then by Theorem 4.2 in [6], the sample paths of F and G get an extra degree of smoothness; they now belong to $\mathbf{D}([0,T],\mathbf{C}^2)$.

4. Continuous-time strategies. We proceed now with the main topic of the paper, which is the construction of trading strategies with general continuous-time dynamics. Recall that the key economic assumption of our model is that the large investor can re-balance his portfolio *without changing the expected utilities of the market makers*.

4.1. *Simple strategies revisited.* To facilitate the transition from the discrete evolution in Section 2.2 to the continuous dynamics below, we begin by revisiting the case of a simple strategy

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(\tau_{n-1},\tau_n]}(t), \qquad 0 \le t \le T, \tag{4.1}$$

with stopping times $0 = \tau_0 \le \cdots \le \tau_N = T$ and random variables $\theta_n \in \mathbf{L}^0(\mathcal{F}_{\tau_{n-1}}, \mathbf{R}^J)$, $n = 1, \dots, N$.

The following result is an improvement over Theorem 2.7 in the sense that the forward induction for cash balances and Pareto optimal allocations is now made explicit through the use of the parameterization $\pi = \pi(a)$ of Pareto allocations from (3.5) and the stochastic fields $F = F_t(a) = F(a,t)$ and $G = G_t(b) = G(b,t)$ defined in (3.11) and (3.12).

Denote by $\lambda_0 \in \mathbf{S}^M$ the weight of the initial Pareto allocation α_0. This weight is uniquely determined by Theorem 3.1.

THEOREM 4.1. *Let Assumptions 2.1 and 2.4 hold and consider a simple strategy Q given by (4.1). Then the sequence of conditionally Pareto optimal allocations $(\alpha_n)_{n=0,\dots,N}$ constructed in Theorem 2.7 takes the form*

$$\alpha_n = \pi(\zeta_n), \qquad n = 0, \dots, N, \tag{4.2}$$

where $\zeta_0 \triangleq (\lambda_0, 0, 0)$ and the random vectors $\zeta_n \triangleq (\lambda_n, \xi_n, \theta_n) \in \mathbf{L}^0(\mathbf{S}^M \times \mathbf{R} \times \mathbf{R}^J, \mathcal{F}_{\tau_{n-1}})$, $n = 1, \dots, N$, with λ_n and ξ_n uniquely determined by

$$\begin{aligned}\lambda_n = {} & \frac{\partial G}{\partial u}\left(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1}\right) \\ & \bigg/ \left(\sum_{m=1}^{M} \frac{\partial G}{\partial u^m}\left(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1}\right)\right),\end{aligned} \tag{4.3}$$

$$\xi_n = G\left(\frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), 1, \theta_n, \tau_{n-1}\right). \tag{4.4}$$

PROOF. The recurrence relations (4.3) and (4.4) clearly determine λ_n and ξ_n, $n = 1, \dots, N$, uniquely. In view of the identity (3.19), for conditionally Pareto optimal allocations $(\alpha_n)_{n=0,\dots,N}$ defined by (4.2) the indifference condition (2.11) can be expressed as

$$\frac{\partial F}{\partial v}(\zeta_n, \tau_{n-1}) = \frac{\partial F}{\partial v}(\zeta_{n-1}, \tau_{n-1}), \qquad n = 1, \dots, N, \tag{4.5}$$

which, by the invertibility relations (3.16) and (3.17) and the fact that λ_n has values in $\mathbf{S}^M$, is, in turn, equivalent to (4.3) and (4.4). $\square$

In the setting of Theorem 4.1, let $A \triangleq (W, X, Q)$, where

$$W_t = \lambda_0 1_{[0]}(t) + \sum_{n=1}^{N} \lambda_n 1_{(\tau_{n-1},\tau_n]}(t), \tag{4.6}$$

$$X_t = \sum_{n=1}^{N} \xi_n 1_{(\tau_{n-1},\tau_n]}(t). \tag{4.7}$$

Then A is a simple predictable process with values in $\mathbf{A}$:

$$A_t = \zeta_0 1_{[0]}(t) + \sum_{n=1}^{N} \zeta_n 1_{(\tau_{n-1},\tau_n]}(t), \qquad 0 \le t \le T, \tag{4.8}$$

with ζ_n belonging to $\mathbf{L}^0(\mathcal{F}_{\tau_{n-1}}, \mathbf{A})$ and defined in Theorem 4.1. It was shown in the proof of this theorem that the main condition (2.11) of the preservation of expected utilities is equivalent to (4.5). Observe now that (4.5) can also be expressed as

$$\frac{\partial F}{\partial v}(A_t, t) = \frac{\partial F}{\partial v}(A_0, 0) + \int_0^t \frac{\partial F}{\partial v}(A_s, ds), \qquad 0 \le t \le T, \tag{4.9}$$

where, for a *simple* process A as in (4.8),

$$\int_0^t \frac{\partial F}{\partial v}(A_s, ds) \triangleq \sum_{n=1}^{N} \left(\frac{\partial F}{\partial v}(\zeta_n, \tau_n \wedge t) - \frac{\partial F}{\partial v}(\zeta_n, \tau_{n-1} \wedge t) \right)$$

denotes its nonlinear stochastic integral against the random field $\frac{\partial F}{\partial v}$. Note that, contrary to (2.11) and (4.5), the condition (4.9) also makes sense for predictable processes A which are not necessarily simple, provided that the nonlinear stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is well defined. This will be key for extending our model to general predictable strategies in the next section.

4.2. *Extension to general predictable strategies.* For a general predictable process A, the construction of $\int \frac{\partial F}{\partial v}(A_s, ds)$ requires additional conditions on the stochastic field $\frac{\partial F}{\partial v} = \frac{\partial F}{\partial v}(a, t)$; see, for example, Sznitman [26] and Kunita [19], Section 3.2. We choose to rely on [19], where the corresponding theory of stochastic integration is developed for continuous semi-martingales. To simplify notation, we shall work in a finite-dimensional Brownian setting. We assume that, for every $a \in \mathbf{A}$, the martingale $F(a)$ of (3.11) admits an integral representation of the form

$$F_t(a) = F_0(a) + \int_0^t H_s(a)\, dB_s, \qquad 0 \le t \le T, \tag{4.10}$$

where B is a d-dimensional Brownian motion and $H(a)$ is a predictable process with values in $\mathbf{R}^d$. Of course, the integral representation (4.10) holds automatically if the filtration $(\mathcal{F}_t)_{0\le t\le T}$ is generated by B. To use the construction of the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ from [19], we have to impose an additional regularity condition on the integrand H with respect to the parameter a.

ASSUMPTION 4.2. There exists a stochastic field $H = H_t(a)$ such that for every $a \in \mathbf{A}$ the process $H(a) = (H_t(a))_{t\in[0,T]}$ is predictable and satisfies the integral representation (4.10). In addition, for every $t \in [0, T]$, the random field $H_t(\cdot)$ has sample paths in $\mathbf{C}^1(\mathbf{A}, \mathbf{R}^d)$, and for every compact set $C \subset \mathbf{A}$

$$\int_0^T \|H_t\|_{1,C}^2 \, dt < \infty,$$

where the semi-norm $\|\cdot\|_{m,C}$ is given by (3.13).

See Remark 4.10 below regarding the verification of this assumption in terms of the primal inputs to our model.

Hereafter, we shall work under Assumptions 2.1, 2.4 and 4.2. For convenience of future references, we formulate an easy corollary of the properties of the sample paths of F and G stated in Section 3.2. For a metric space $\mathbf{X}$ denote by $\mathbf{C}([0, T], \mathbf{X})$, the space of continuous maps of $[0, T]$ to $\mathbf{X}$. Recall the definition of the Fréchet space $\mathbf{C}^m$ from Section 3.2.

LEMMA 4.3. *Under Assumptions* 2.1, 2.4 *and* 4.2, *the stochastic fields* $F = F_t(a)$ *and* $G = G_t(b)$ *have sample paths in* $\mathbf{C}([0, T], \mathbf{C}^1)$. *If, in addition, Assumption* 2.2 *holds, then* F *and* G *have sample paths in* $\mathbf{C}([0, T], \mathbf{C}^2)$.

PROOF. As we recalled in Section 3.2, Theorem 4.1 in [6] implies that under Assumptions 2.1 and 2.4 the stochastic fields F and G have sample paths in the space $\mathbf{D}([0, T], \mathbf{C}^1)$ of RCLL maps and that their left-limits satisfy conjugacy relations analogous to (3.12). Moreover, under the additional Assumption 2.2, Theorem 4.2 in [6] implies that the sample paths of F and G belong to $\mathbf{D}([0, T], \mathbf{C}^2)$. These results readily imply the assertions of the lemma as soon as we observe that, in view of (4.10), for every $a \in \mathbf{A}$, the trajectories of the martingale $F(a)$ are continuous. □

We also need the following elementary fact. Recall that if ξ and η are stochastic fields on A then η is a *modification* of ξ if $\xi(x) = \eta(x)$ for every $x \in A$.

LEMMA 4.4. *Let* m *be a nonnegative integer,* U *be an open set in* $\mathbf{R}^n$, *and* $\xi : U \to \mathbf{L}^0(\mathbf{R})$ *be a random field with sample paths in* $\mathbf{C}^m = \mathbf{C}^m(U)$ *such that for every compact set* $C \subset U$

$$\mathbb{E}\big[\|\xi\|_{m,C}\big] < \infty. \tag{4.11}$$

Assume also that there are a Brownian motion B with values in $\mathbf{R}^d$ and a stochastic field $H = H_t(x) : U \times [0,T] \to \mathbf{R}^d$ such that for every $t \in [0,T]$ the random field $H_t(\cdot)$ has sample paths in $\mathbf{C}^m(U, \mathbf{R}^d)$ and such that for every $x \in U$ the process $H(x)$ is predictable with

$$M_t(x) \triangleq \mathbb{E}[\xi(x)|\mathcal{F}_t] = M_0(x) + \int_0^t H_s(x)\, dB_s. \tag{4.12}$$

Suppose finally that for every compact set $C \subset U$

$$\int_0^T \|H_t\|_{m,C}^2\, dt < \infty. \tag{4.13}$$

Then M has a modification with sample paths in $\mathbf{C}([0,T], \mathbf{C}^m(U))$ and for $t \in [0,T]$, $x \in U$, and a multi-index $k = (k_1, \dots, k_n)$ with $|k| \le m$,

$$D^k M_t(x) = D^k M_0(x) + \int_0^t D^k H_s(x)\, dB_s, \tag{4.14}$$

where the differential operator D^k is given by (3.14).

PROOF. Observe first that (4.11) implies that M has a modification with sample paths in $\mathbf{D}([0,T], \mathbf{C}^m(U))$; see Lemma C.1 in [6]. We shall work with this modification. As, for every $x \in U$, the martingale $M(x)$ is continuous, we deduce that the sample paths of M belong to $\mathbf{C}([0,T], \mathbf{C}^m(U))$.

To verify (4.14), it is sufficient to consider the case $m = 1$ and $k = (1, 0, \dots, 0)$. Denote $e_1 \triangleq (1, 0, \dots, 0) \in \mathbf{R}^n$. By (4.11),

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{1}{\varepsilon}\left|\xi(x + \varepsilon e_1) - \xi(x) - \varepsilon \frac{\partial \xi}{\partial x_1}(x)\right|\right] = 0$$

and then, by Doob's inequality,

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(M(x + \varepsilon e_1) - M(x) - \varepsilon \frac{\partial M}{\partial x_1}(x)\right)_T^* = 0,$$

where $X_T^* \triangleq \sup_{t \in [0,T]} |X_t|$. Observe also that by (4.13)

$$\lim_{\varepsilon \to 0} \int_0^T \left|\frac{1}{\varepsilon}\left(H(x + \varepsilon e_1) - H(x) - \varepsilon \frac{\partial H}{\partial x_1}(x)\right)\right|^2 dt = 0.$$

The result now follows from the fact that for a sequence of continuous local martingales $(N^n)_{n \ge 1}$ its maximal elements $(N^n)_T^* \triangleq \sup_{t \in [0,T]} |N_t^n|$ converge to 0 in probability if and only if the initial values N_0^n and the quadratic variations $\langle N^n \rangle_T$ converge to 0 in probability. $\square$

REMARK 4.5. If the filtration is generated by a d-dimensional Brownian motion B, then the integral representation (4.12) holds automatically. In this case, the results of [26], based on Sobolev's embeddings and Itô's isometry, show that (4.13) with $m = m_1$ follows from (4.11) with $m = m_2$ provided that $m_1 < m_2 - d/2$; see our companion paper [7].

From Lemma 4.4, we deduce

$$\frac{\partial F_t}{\partial v}(a) = \frac{\partial F_0}{\partial v}(a) + \int_0^t \frac{\partial H_s}{\partial v}(a)\, dB_s.$$

Following Section 3.2 in [19], we say that a predictable process A with values in $\mathbf{A}$ is *integrable* with respect to the *kernel* $\frac{\partial F}{\partial v}(\cdot, dt)$ or, equivalently, that the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is *well defined* if

$$\int_0^T \left| \frac{\partial H_t}{\partial v}(A_t) \right|^2 dt < \infty.$$

In this case, we set

$$\int_0^t \frac{\partial F}{\partial v}(A_s, ds) \triangleq \int_0^t \frac{\partial H_s}{\partial v}(A_s)\, dB_s, \qquad 0 \le t \le T.$$

We are now in a position to give a formal definition of a general trading strategy. Recall that processes X and Y are *indistinguishable* if $(X - Y)_T^* \triangleq \sup_{t \in [0,T]} |X_t - Y_t| = 0$.

DEFINITION 4.6. A predictable process Q with values in $\mathbf{R}^J$ is called a *strategy* if there are unique (in the sense of indistinguishability) predictable processes W and X with values in $\mathbf{S}^M$ and $\mathbf{R}$, respectively, such that, for $A \triangleq (W, X, Q)$, the initial Pareto allocation is given by

$$\alpha_0 = \pi(A_0), \tag{4.15}$$

the stochastic integral $\int \frac{\partial F}{\partial v}(A_s, ds)$ is well defined and (4.9) holds.

REMARK 4.7. From now on, the term "strategy" will always be used in the sense of Definition 4.6. Note that, at this point, it is still an open question whether a simple predictable process Q is a (valid) strategy, as in Theorem 4.1 the uniqueness of W and X, such that $A \triangleq (W, X, Q)$ solves (4.9), was proved only in the class of *simple* processes. The affirmative answer to this question will be given in Theorem 4.19 below, where in addition to the standing Assumptions 2.1, 2.4 and 4.2, we shall also require Assumptions 2.2 and 4.15.

The predictable processes W and X in Definition 4.6 will be called the *Pareto weights* and *cash balance* processes for the strategy Q. We remind the reader, that the bookkeeping in our model is done from the collective point of view of the market makers; see Remark 2.6. In other words, for a strategy Q, the number of shares and the amount of cash *owned by the large investor* at time t are given by $-Q_t$ and $-X_t$.

Accounting for (3.19), we call

$$U_t \triangleq \frac{\partial F_t}{\partial v}(A_t), \qquad 0 \le t \le T, \tag{4.16}$$

the process of *expected utilities* for the market makers. Observe that, as $U < 0$ and $U - U_0$ is a stochastic integral with respect to a Brownian motion, U is a local martingale and a (global) sub-martingale. The invertibility relations (3.16) and (3.17) imply the following expressions for W and X in terms of U and Q:

$$W_t = \frac{\partial G_t}{\partial u}(U_t, 1, Q_t) \Big/ \left(\sum_{m=1}^{M} \frac{\partial G_t}{\partial u^m}(U_t, 1, Q_t) \right), \tag{4.17}$$

$$X_t = G_t(U_t, 1, Q_t). \tag{4.18}$$

We also call

$$V_t \triangleq -G_t(U_t, 1, 0) = -G_t\left(\frac{\partial F_t}{\partial v}(A_t), 1, 0 \right), \qquad 0 \le t \le T, \tag{4.19}$$

the *cumulative gain* process for the large trader. This term is justified as, by (4.18), V_t represents the cash amount the agent will hold at t if he liquidates his position in stocks. Of course, at maturity

$$V_T = -\big(X_T + \langle Q_T, \psi \rangle\big).$$

It is interesting to observe that, contrary to the standard, small agent, model of mathematical finance, no further "admissibility" conditions on a strategy Q are needed to exclude an arbitrage.

LEMMA 4.8. *Let Assumptions 2.1, 2.4 and 4.2 hold and Q be a strategy such that the terminal gain of the large trader is nonnegative*: $V_T \ge 0$. *Then*, *in fact*, $V_T = 0$.

PROOF. Recall the notation $\lambda_0 \in \mathbf{S}^M$ for the weights and $\Sigma_0 \in \mathbf{L}^0(\mathbf{R}^M)$ for the total endowment of the initial Pareto allocation α_0 and $r = r(v, x)$ for the aggregate utility function from (3.1). Denote by α_1 the terminal wealth distribution between the market makers at maturity resulting from the strategy Q. From the characterization of Pareto allocations in Theorem 3.1 and the sub-martingale property of the process U of expected utilities, we obtain

$$\begin{aligned} \mathbb{E}\big[r(\lambda_0, \Sigma_0)\big] &= \mathbb{E}\left[\sum_{m=1}^{m} \lambda_0^m u_m\big(\alpha_0^m\big) \right] = \langle \lambda_0, U_0 \rangle \le \mathbb{E}\big[\langle \lambda_0, U_T \rangle\big] \\ &= \mathbb{E}\left[\sum_{m=1}^{m} \lambda_0^m u_m\big(\alpha_1^m\big) \right] \le \mathbb{E}\big[r(\lambda_0, \Sigma_0 - V_T)\big]. \end{aligned}$$

Since $r(\lambda_0, \cdot)$ is a strictly increasing function, the result follows. $\square$

We state now a key result of the paper where we reduce the question whether a predictable process Q is a strategy to the unique solvability of a stochastic differential equation parameterized by Q.

THEOREM 4.9. *Under Assumptions 2.1, 2.4 and 4.2, a predictable process Q with values in $\mathbf{R}^J$ is a strategy if and only if the stochastic differential equation*

$$U_t = U_0 + \int_0^t K_s(U_s, Q_s)\, dB_s, \tag{4.20}$$

has a unique strong solution U with values in $(-\infty, 0)^M$ on $[0, T]$, where

$$U_0^m \triangleq \mathbb{E}[u_m(\alpha_0^m)], \qquad m = 1, \dots, M,$$

and, for $u \in (-\infty, 0)^M$, $q \in \mathbf{R}^J$ and $t \in [0, T]$,

$$K_t(u, q) \triangleq \frac{\partial H_t}{\partial v}\left(\frac{\partial G_t}{\partial u}(u, 1, q), G_t(u, 1, q), q\right). \tag{4.21}$$

In this case, U is the process of expected utilities, and the processes of Pareto weights W and cash balance X are given by (4.17) and (4.18).

PROOF. Observe that the stochastic field $F = F_t(v, x, q)$ is positive homogeneous with respect to v:

$$F_t(cv, x, q) = cF_t(v, x, q), \qquad c > 0,$$

and that the integrand $H = H_t(v, x, q)$, clearly, shares same property. It follows that

$$\frac{\partial H_t}{\partial v}(cv, x, q) = \frac{\partial H_t}{\partial v}(v, x, q), \qquad c > 0,$$

and, therefore, that the stochastic field K from (4.21) can also be written as

$$K_t(u, q) = \frac{\partial H_t}{\partial v}\left(\frac{\partial G_t}{\partial u}(u, 1, q) \Big/ \left(\sum_{m=1}^M \frac{\partial G_t}{\partial u^m}(u, 1, q)\right), G_t(u, 1, q), q\right).$$

After this observation, the result is an immediate consequence of the definition of a strategy and the expressions (4.17) and (4.18) for the processes of Pareto weights W and cash balance X. $\square$

REMARK 4.10. In the follow-up paper [7], we provide sufficient conditions for a locally bounded predictable process Q with values in $\mathbf{R}^J$ to be a strategy, or equivalently, for (4.20) to have a unique strong solution, in terms of the "original" inputs to the model: the utility functions $(u_m)_{m=1,\dots,M}$, the initial endowment Σ_0, and the dividends ψ. In particular, these conditions also imply Assumptions 4.2 and 4.15 on $H = H_t(a)$.

As an illustration, we give an example where (4.20) is a linear equation, and, hence, can be solved explicitly.

EXAMPLE 4.11 (Bachelier model with price impact). Consider an economy with a single market maker and one stock. The market maker's utility function is exponential:

$$u(x) = -\frac{1}{\gamma} e^{-\gamma x}, \qquad x \in \mathbf{R},$$

where the constant $\gamma > 0$ is the absolute risk-aversion coefficient. The initial endowment of the market maker and the payoff of the stock are given by

$$\Sigma_0 = \alpha_0 = b + \frac{\mu}{\gamma\sigma} B_T,$$
$$\psi = s + \mu T + \sigma B_T,$$

where the constants $b, \mu, s \in \mathbf{R}$ and $\sigma > 0$. Note that the initial Pareto pricing measure $\mathbb{Q} = \mathbb{Q}_0$ and the stock price S have the expressions

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \operatorname{const} u'(\Sigma_0) = e^{-(\mu/\sigma)B_T - (\mu^2/(2\sigma^2))T},$$
$$S_t \triangleq \mathbb{E}_{\mathbb{Q}_0}[\psi|\mathcal{F}_t] = s + \mu t + \sigma B_t, \qquad t \in [0, T],$$

and coincide with the martingale measure and the stock price in the classical Bachelier model for a "small" investor.

Direct computations show that, for $a = (v, x, q) \in \mathbf{A}$,

$$F_t(a) = v e^{-\gamma x} N_t(q),$$

where the martingale $N(q)$ evolves as

$$dN_t(q) = -\left(\frac{\mu}{\sigma} + \gamma\sigma q\right) N_t(q)\, dB_t. \tag{4.22}$$

For the integrand $H = H_t(a)$ in (4.10) and the stochastic field $G = G_t(b)$, we obtain

$$\frac{\partial H_t}{\partial v}(a) = -\left(\frac{\mu}{\sigma} + \gamma\sigma q\right) e^{-\gamma x} N_t(q),$$
$$u = e^{-\gamma G_t(u,1,q)} N_t(q), \qquad u \in (-\infty, 0),$$

where the second equality follows from (3.18). The stochastic field $K = K_t(u, q)$ in (4.21) is then given by

$$K_t(u, q) = -\left(\frac{\mu}{\sigma} + \gamma\sigma q\right) u, \qquad u \in (-\infty, 0).$$

From Theorem 4.9, we obtain that a predictable process Q is a strategy if and only if

$$\int_0^T Q_t^2\, dt < \infty,$$

and that, in this case, the expected utility process U for the market maker evolves as

$$dU_t = -\left(\frac{\mu}{\sigma} + \gamma\sigma Q_t\right) U_t \, dB_t. \tag{4.23}$$

Observe now that, by (4.19), the cumulative gain V_t of the large trader satisfies

$$U_t = e^{\gamma V_t} N_t(0).$$

From (4.22) and (4.23) and the fact that $V_0 = 0$, we deduce

$$\begin{aligned} V_t &= \int_0^t \left[(-Q_r)(\mu \, dr + \sigma \, dB_r) - \frac{\gamma\sigma^2}{2} Q_r^2 \, dr\right] \\ &= \int_0^t \left[(-Q_r) \, dS_r - \frac{\gamma\sigma^2}{2} Q_r^2 \, dr\right]. \end{aligned}$$

Recall that $-Q$ denotes the number of shares owned by the large investor and then observe that the first, linear with respect to Q, term yields the wealth evolution in the classical Bachelier model. The second, quadratic, term thus describes the feedback effect of the large trader's actions on stock prices, with the risk-aversion coefficient $\gamma > 0$ playing the role of a *price impact coefficient*.

4.3. *Maximal local strategies*. For a stochastic process X and a stopping time σ with values in $[0, T]$, recall the notation $X^\sigma \triangleq (X_{t\wedge\sigma})_{0\le t\le T}$ for X "stopped" at σ. The following localization fact for strategies will be used later on several occasions.

LEMMA 4.12. *Let Assumptions 2.1, 2.4 and 4.2 hold, σ be a stopping time with values in $[0, T]$, Q be a strategy and W, X, V and U be its processes of Pareto weights, cash balance, cumulative gain and expected utilities. Then Q^σ is also a strategy and W^σ and X^σ are its processes of Pareto weights and cash balance. The processes of cumulative gain, $V(Q^\sigma)$, and of expected utilities, $U(Q^\sigma)$, for the strategy Q^σ coincide with V and U on $[0, \sigma]$, while on $(\sigma, T]$ they are given by*

$$\begin{aligned} U(Q^\sigma)_t &= \frac{\partial F_t}{\partial v}(W_\sigma, X_\sigma, Q_\sigma), \\ V(Q^\sigma)_t &= -G_t(U(Q^\sigma)_t, 1, 0). \end{aligned}$$

PROOF. The proof follows directly from Definition 4.6 and the construction of U and V in (4.16) and (4.19). $\square$

Let τ be a stopping time with values in $(0, T] \cup \{\infty\}$ and U be a process with values in $(-\infty, 0)^M$ defined on $[0, \tau) \cap [0, T]$. Recall that, for the equation (4.20), τ and U are called the *explosion* time and the *maximal local solution* if for every stopping time σ with values in $[0,\tau)\cap[0,T]$ the process U^σ is the unique solution to (4.20) on $[0,\sigma]$ and

$$\limsup_{t\uparrow\tau}\left|\log(-U_t)\right| = \infty \qquad \text{on } \{\tau<\infty\}. \tag{4.24}$$

Observe that, for $m=1,\dots,M$, the sub-martingale property of $U^m<0$ insures the existence of the limit: $\lim_{t\uparrow\tau} U_t^m$ and prevents it from being $-\infty$. Hence, (4.24) is equivalent to

$$\lim_{t\uparrow\tau}\max_{m=1,\dots,M} U_t^m = 0 \qquad \text{on } \{\tau<\infty\}.$$

For convenience of future references, we introduce a similar localized concept for strategies.

DEFINITION 4.13. A predictable process Q with values in $\mathbf{R}^J$ is called a *maximal local strategy* if there are a stopping time τ with values in $(0,T]\cup\{\infty\}$ and processes V, W and X on $[0,\tau)\cap[0,T]$ with values in $\mathbf{R}$, $\mathbf{S}^M$ and $\mathbf{R}$, respectively, such that

$$\lim_{t\uparrow\tau} V_t = -\infty \qquad \text{on } \{\tau<\infty\} \tag{4.25}$$

and for every stopping time σ with values in $[0,\tau)\cap[0,T]$ the process Q^σ is a strategy with Pareto weights W^σ and cash balance X^σ whose cumulative gain equals V on $[0,\sigma]$.

Similar to the "global" case we call V, W and X from Definition 4.13 the processes of cumulative gain, Pareto weights and cash balance, respectively; the process U of expected utilities is defined on $[0,\tau)\cap[0,T]$ as in (4.16). In view of (4.25), we call τ the *explosion* time for V. Note that, by Lemma 4.12, the class of maximal local strategies contains the class of (global) strategies.

THEOREM 4.14. *Let Assumptions* 2.1, 2.4 *and* 4.2 *hold and* τ *be a stopping time with values in* $(0,T]\cup\{\infty\}$. *A predictable process* Q *with values in* $\mathbf{R}^J$ *is a maximal local strategy and* τ *is the explosion time for its cumulative gain process* V *if and only if the stochastic differential equation* (4.20) *admits the unique maximal local solution* U *with the explosion time* τ.

If, in addition, Q *is locally bounded, then* τ *is also the explosion time for its cash balance process*:

$$\lim_{t\uparrow\tau} X_t = \infty \qquad \textit{on } \{\tau<\infty\}.$$

PROOF. By Theorem 4.1 in [6], for every $t\in[0,T]$ the random field $G_t(\cdot)$ has sample paths in a certain space $\widetilde{\mathbf{G}}^1$ of continuously differentiable saddle functions on $\mathbf{B}$. Among other properties, a function $g = g(b) = g(u, y, q)$ in $\widetilde{\mathbf{G}}^1$ is convex with respect to q, strictly increasing with respect to u, and

$$\lim_{n\to\infty} g(u_n, 1, q) = \infty \tag{4.26}$$

for every sequence $(u_n)_{n\geq 1}$ in $(-\infty, 0)^M$ converging to a boundary point of $(-\infty, 0)^M$; see the properties (G2), (G3) and (G6) of the elements of $\widetilde{\mathbf{G}}^1$ in [6].

These properties readily imply that if $(g_n)_{n\geq 1}$ is a sequence in $\widetilde{\mathbf{G}}^1$ which converges to $g \in \widetilde{\mathbf{G}}^1$ in $\mathbf{C}^1(\mathbf{B})$, then

$$\lim_{n\to\infty} \inf_{q\in C} g_n(u_n, 1, q) = \infty \tag{4.27}$$

for every compact set $C \subset \mathbf{R}^J$ and every sequence $(u_n)_{n\geq 1}$ in $(-\infty, 0)^M$ converging to a boundary point of $(-\infty, 0)^M$. Indeed, because of the q-convexity and the u-monotonicity, it is sufficient to consider the case when C is a singleton and the sequence $(u_n)_{n\geq 1}$ is increasing. Then, for $q \in \mathbf{R}^J$,

$$\liminf_{n\to\infty} g_n(u_n, 1, q) \geq \lim_{k\to\infty} \liminf_{n\to\infty} g_n(u_k, 1, q) = \lim_{k\to\infty} g(u_k, 1, q) = \infty,$$

where the last equality follows from (4.26).

Since, by Lemma 4.3, the stochastic field $G = G_t(b)$ has sample paths in $\mathbf{C}([0, T], \mathbf{C}^1(\mathbf{B}))$, the property (4.27) readily yields the result as soon as we recall the constructions of V and X in (4.19) and (4.18). Observe that in the argument concerning X we can assume, by localization, that Q is (globally) bounded and, hence, takes values in some compact set $C \subset \mathbf{R}^J$. $\square$

To establish the existence of a maximal local strategy or, equivalently, the existence and uniqueness of a maximal local solution to (4.20) we shall also require Assumption 2.2 and a stronger version of Assumption 4.2.

ASSUMPTION 4.15. For every $t \in [0, T]$, the random field $H_t(\cdot)$ from Assumption 4.2 has sample paths in $\mathbf{C}^2(\mathbf{A}, \mathbf{R}^d)$ and, for every compact set $C \subset \mathbf{A}$,

$$\int_0^T \|H_t\|_{2,C}^2\, dt < \infty.$$

The role of these additional assumptions is to guarantee the local Lipschitz property with respect to u for the stochastic field K in (4.21).

LEMMA 4.16. *Let Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15 hold and K be the stochastic field defined in (4.21). Then for every $t \in [0, T]$ the random field $K_t(\cdot)$ has sample paths in $\mathbf{C}^1((-\infty, 0)^M \times \mathbf{R}^J, \mathbf{R}^{M\times d})$ and, for every compact set $C \subset (-\infty, 0)^M \times \mathbf{R}^J$,*

$$\int_0^T \|K_t\|_{1,C}^2\, dt < \infty.$$

PROOF. This follows from Assumption 4.15 and the fact that by Lemma 4.3, the stochastic field $G = G_t(b)$ has sample paths in $\mathbf{C}([0, T], \mathbf{C}^2(\mathbf{B}))$. □

THEOREM 4.17. *Let Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15 hold and Q be a predictable process with values in $\mathbf{R}^J$ such that, for every compact set $C \subset (-\infty, 0)^M$,*

$$\int_0^T \|K_t(\cdot, Q_t)\|_{1,C}^2 \, dt < \infty. \tag{4.28}$$

Then Q is a maximal local strategy.

PROOF. It is well known (see, e.g., Theorem 3.4.5 in [19]) that (4.28) implies the existence of a unique *maximal local solution* to (4.20). The result now follows from Theorem 4.14. □

THEOREM 4.18. *Under Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15 every locally bounded predictable process Q is a maximal local strategy.*

PROOF. This follows from Theorem 4.17 if we observe that, by Lemma 4.16, a locally bounded Q satisfies (4.28). □

The preceding result allows us to finally reconcile Definition 4.6 with the construction of simple strategies in Theorems 2.7 and 4.1 since it resolves the uniqueness issue raised in Remark 4.7.

THEOREM 4.19. *Under Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15 every simple predictable process Q with values in $\mathbf{R}^J$ is a strategy and its processes of Pareto weights W and cash balance X are simple and given by (4.6)–(4.7) and (4.3)–(4.4).*

PROOF. The fact, that, for W and X given by (4.6)–(4.7) and (4.3)–(4.4), the process $A \triangleq (W, X, Q)$ satisfies (4.15) and (4.9) has been already established in our discussion following Theorem 4.1. The uniqueness follows from Theorem 4.18. □

5. Approximation by simple strategies. In this final section, we provide a justification for the construction of the general strategies in Definition 4.6 by discussing approximations based on simple strategies. To simplify the presentation, we restrict ourselves to the case of locally bounded processes.

For measurable stochastic processes, in addition to the ucp convergence defined by the metric

$$d_{\mathrm{ucp}}(X, Y) \triangleq \mathbb{E}\Big[\sup_{t \in [0,T]} |X_t - Y_t| \wedge 1\Big],$$

we also consider the convergence in the space $\mathbf{L}^0(d\mathbb{P} \times dt)$ with the metric

$$d_{\mathbf{L}^0}(X, Y) \triangleq \mathbb{E}\left[\int_0^T \big(|X_t - Y_t| \wedge 1\big)\, dt\right].$$

We call a sequence of stochastic processes $(X^n)_{n\geq 1}$ *uniformly locally bounded from above* if there is an increasing sequence of stopping times $(\sigma_n)_{n\geq 1}$ such that $\mathbb{P}[\sigma_n < T] \to 0$, $n \to \infty$ and $X_t^k \leq n$ on $[0, \sigma_n]$ for $k \geq 1$. The sequence $(X^n)_{n\geq 1}$ is called *uniformly locally bounded* if the sequence of its absolute values $(|X^n|)_{n\geq 1}$ is uniformly locally bounded from above.

We begin with a general convergence result:

THEOREM 5.1. *Let Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15 hold and consider a sequence of strategies $(Q^n)_{n\geq 1}$ which is uniformly locally bounded and converges to a strategy Q in $\mathbf{L}^0(d\mathbb{P} \times dt)$.*

Then the processes $(U^n, V^n)_{n\geq 1}$, of expected utilities and cumulative gains, converge to (U, V) in ucp, *the processes $(W^n, X^n)_{n\geq 1}$, of Pareto weights and cash balance, converge to (W, X) in $\mathbf{L}^0(d\mathbb{P} \times dt)$, and the sequence $(X^n)_{n\geq 1}$ is uniformly locally bounded. If, in addition, the sequence $(Q^n)_{n\geq 1}$ converges to Q in* ucp, *then the sequence $(W^n, X^n)_{n\geq 1}$ also converges to (W, X) in* ucp.

PROOF. By standard localization arguments, we can assume the existence of constants $a > 0$ and $b > 0$ such that

$$\max\Big(\big|\ln(-U)\big|, |Q|, \sup_{n\geq 1}\big|Q^n\big|\Big) \leq a,$$

and, in view of Lemma 4.16, such that

$$\int_0^T \|K_s(\cdot)\|_{1,C(a)}^2\, ds \leq b, \tag{5.1}$$

where

$$C(a) \triangleq \big\{(u, q) \in (-\infty, 0)^M \times \mathbf{R}^J : \max\big(\big|\ln(-u)\big|, |q|\big) \leq 2a\big\}.$$

Define the stopping times

$$\sigma_n \triangleq \inf\big\{t \in [0, T] : \big|\ln\big(-U_t^n\big)\big| \geq 2a\big\}, \qquad n \geq 1,$$

where we follow the convention that $\inf \varnothing \triangleq \infty$. Observe that the ucp convergence of $(U^n)_{n\geq 1}$ to U holds if

$$\big(U - U^n\big)_{T\wedge\sigma_n}^* \to 0, \qquad n \to \infty. \tag{5.2}$$

To prove (5.2), note first that for every two stopping times $0 \le \tau_* \le \tau^* \le \sigma_n$ we have using Doob's inequality

$$
\begin{aligned}
\mathbb{E}\,\Big[\sup_{\tau_* \le t \le \tau^*} |U_t - U_t^n|^2\Big]
&\le \mathbb{E}\bigg[2\big|U_{\tau_*} - U_{\tau_*}^n\big|^2 + 2 \sup_{\tau_* \le t \le \tau^*} \bigg|\int_{\tau_*}^t \big(K_s(U_s, Q_s) - K_s\big(U_s^n, Q_s^n\big)\big)\, dB_s\bigg|^2\bigg] \\
&\le 2\mathbb{E}\big|U_{\tau_*} - U_{\tau_*}^n\big|^2 + 8\mathbb{E}\bigg[\int_{\tau_*}^{\tau^*} \big|K_s(U_s, Q_s) - K_s\big(U_s^n, Q_s^n\big)\big|^2\, ds\bigg] \\
&\le 2\mathbb{E}\big|U_{\tau_*} - U_{\tau_*}^n\big|^2 \\
&\quad + 8\mathbb{E}\bigg[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|_{1,C(a)}^2 \big(|U_s - U_s^n|^2 + |Q_s - Q_s^n|^2\big)\, ds\bigg] \\
&\le 2\mathbb{E}\big|U_{\tau_*} - U_{\tau_*}^n\big|^2 + 8\mathbb{E}\bigg[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|_{1,C(a)}^2\, ds \sup_{\tau_* \le t \le \tau^*} |U_t - U_t^n|^2\bigg] \\
&\quad + 8\mathbb{E}\bigg[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|_{1,C(a)}^2 |Q_s - Q_s^n|^2\, ds\bigg].
\end{aligned}
$$

Rearranging terms, we thus obtain

$$
\begin{aligned}
&\mathbb{E}\bigg[\bigg(1 - 8\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|_{1,C(a)}^2\, ds\bigg) \sup_{\tau_* \le t \le \tau^*} |U_t - U_t^n|^2\bigg] \\
&\qquad \le 2\mathbb{E}\big|U_{\tau_*} - U_{\tau_*}^n\big|^2 + 8\mathbb{E}\bigg[\int_{\tau_*}^{\tau^*} \|K_s(\cdot)\|_{1,C(a)}^2 |Q_s - Q_s^n|^2\, ds\bigg].
\end{aligned}
\tag{5.3}
$$

Now choose $\tau_0 \triangleq 0$ and, for $i = 1, 2, \dots,$ let

$$
\tau_i \triangleq \inf\bigg\{t \ge \tau_{i-1} : 8\int_{\tau_{i-1}}^t \|K_s(\cdot)\|_{1,C(a)}^2\, ds \ge \frac{1}{2}\bigg\} \wedge T.
$$

Note that because of (5.1) we have $\tau_i = T$ for $i \ge i_0$, where i_0 is the smallest integer greater than $16b$. Hence, to establish (5.2), it suffices to prove

$$
\mathbb{E}\Big[\sup_{\tau_{i-1} \wedge \sigma^n \le s \le \tau_i \wedge \sigma^n} |U_s - U_s^n|^2\Big] \to 0, \qquad n \to \infty \text{ for } i = 1, \dots, i_0.
$$

For $i = 1$, this follows from estimate (5.3) with $\tau_* \triangleq \tau_0 = 0$ and $\tau^* \triangleq \tau_1 \wedge \sigma^n$ because $U_0 = U_0^n$ and because of our assumption on the sequence $(Q^n)_{n \ge 1}$. For $i = 2, 3, \dots$ this convergence holds by induction, since with $\tau_* \triangleq \tau_{i-1} \wedge \sigma^n$ and $\tau^* \triangleq \tau_i \wedge \sigma^n$ the first term on the right-hand side of (5.3) vanishes for $n \to \infty$ because of the validity of our claim for $i - 1$ and the second term disappears again by assumption on $(Q^n)_{n \ge 1}$. This completes the proof of the ucp convergence of $(U^n)_{n \ge 1}$ to U.

The rest of the assertions follows from the representations (4.17), (4.18) and (4.19) for Pareto weights, cash balances and cumulative gains in terms of the stochastic field $G = G_t(b)$ and the fact that, by Lemma 4.3, G has sample paths in $\mathbf{C}(\mathbf{C}^1(\mathbf{B}), [0, T])$. $\square$

THEOREM 5.2. *Under Assumptions 2.1, 2.2, 2.4, 4.2 and 4.15, a predictable locally bounded process Q with values in $\mathbf{R}^J$ is a strategy if and only if there is a sequence $(Q^n)_{n\geq 1}$ of simple strategies, which is uniformly locally bounded, converges to Q in $\mathbf{L}^0(d\mathbb{P} \times dt)$, and for which the sequence of associated cash balances $(X^n)_{n\geq 1}$ is uniformly locally bounded from above.*

For the proof, we need a lemma.

LEMMA 5.3. *Under Assumptions 2.1, 2.2, 2.4 and 4.2, for every strategy Q and every $t \in [0, T]$*

$$\begin{aligned}
&\sum_{m=1}^{M} \left(\frac{1}{c} \log\bigl((-U_t^m) \vee 1\bigr) + c \log\bigl((-U_t^m) \wedge 1\bigr) \right) \\
&\quad \leq G_t(-\mathbf{1}, 1, Q_t) - X_t \\
&\quad \leq \sum_{m=1}^{M} \left(\frac{1}{c} \log\bigl((-U_t^m) \wedge 1\bigr) + c \log\bigl((-U_t^m) \vee 1\bigr) \right),
\end{aligned} \tag{5.4}$$

where $c > 0$ is taken from Assumption 2.2, $\mathbf{1} \triangleq (1, \ldots, 1) \in \mathbf{R}^M$, and X and U are the processes of cash balance and expected utilities for Q.

PROOF. Theorem 4.2 in [6] implies that under Assumptions 2.1, 2.2 and 2.4, for every $t \in [0, T]$ the random field $G_t(\cdot)$ has sample paths in a certain space $\widetilde{\mathbf{G}}^2(c)$ of twice-differentiable saddle functions on $\mathbf{B}$. The property (G7) of the elements of $\widetilde{\mathbf{G}}^2(c)$ states that

$$\frac{1}{c} \leq -u^m \frac{\partial G_t}{\partial u^m}(u, 1, q) \leq c, \qquad m = 1, \ldots, M.$$

This yields the result if we account for the representation (4.18) for X. $\square$

PROOF OF THEOREM 5.2. The "only if" part follows from Theorem 5.1 and the fact that every locally bounded predictable process Q can be approximated in $\mathbf{L}^0(d\mathbb{P} \times dt)$ by a sequence of simple predictable processes $(Q^n)_{n\geq 1}$ which is uniformly locally bounded. Hereafter, we shall focus on sufficiency.

By Theorem 4.18, Q is a maximal local strategy. Denote by U and X its processes of expected utilities and cash balance and by τ the explosion time of X; see Theorem 4.14. We have to show that $\tau = \infty$.

For $a > 0$ and $b > a$, define the stopping times

$$\tau(a) \triangleq \inf\Big\{t \in [0, T]: \max_{m=1,\dots,M} U_t^m > -a\Big\},$$

$$\tau_n(a) \triangleq \inf\Big\{t \in [0, T]: \sup_{k \geq n} \max_{m=1,\dots,M} U_t^{k,m} > -a\Big\}, \qquad n \geq 1,$$

$$\sigma(b) \triangleq \inf\Big\{t \in [0, T]: \min_{m=1,\dots,M} U_t^m < -b\Big\},$$

$$\sigma_n(b) \triangleq \inf\Big\{t \in [0, T]: \inf_{k \geq n} \min_{m=1,\dots,M} U_t^{k,m} < -b\Big\}, \qquad n \geq 1,$$

where U^n is the process of expected utilities for Q^n and where we let $\inf \varnothing \triangleq \infty$. Note that, by Theorem 4.14, $\tau(a) \to \tau$, $a \to 0$, and hence, $\tau = \infty$ if and only if

$$\lim_{a \to 0} \mathbb{P}\big[\tau(a) \leq T\big] = 0. \tag{5.5}$$

From Theorem 4.14 and Lemma 4.12, we deduce that $Q^{\tau(a) \wedge T}$ is a strategy whose expected utility process coincides with U on $[0, \tau(a) \wedge T]$. Hence, by Theorem 5.1,

$$\big(U^n - U\big)^*_{\tau(a) \wedge T} \to 0, \qquad n \to \infty. \tag{5.6}$$

Hereafter, we shall assume that a is rational and that, for every such a, the convergence above takes place almost surely. This can always be arranged by passing to a subsequence.

Since

$$\big\{\tau(a) < \tau_n(2a)\big\} \subset \bigcap_{k \geq n} \big\{\big(U^k - U\big)^*_{\tau(a) \wedge T} \geq a\big\},$$

we obtain

$$\lim_{n \to \infty} \mathbb{P}\big[\tau(a) < \tau_n(2a)\big] = 0. \tag{5.7}$$

Similarly, as

$$\big\{\sigma_n(2b) \wedge \tau(a) < \sigma(b) \wedge \tau(a)\big\} \subset \bigcup_{k \geq n} \big\{\big(U^k - U\big)^*_{\tau(a) \wedge T} \geq b\big\},$$

and since the convergence in (5.6) takes place almost surely, we deduce

$$\lim_{n \to \infty} \mathbb{P}\big[\sigma_n(2b) \wedge \tau(a) < \sigma(b) \wedge \tau(a)\big] = 0.$$

The latter convergence implies that

$$\limsup_{n \to \infty} \mathbb{P}\big[\sigma_n(2b) < \tau(a)\big] \leq \mathbb{P}\big[\sigma(b) < \tau(a)\big] \leq \mathbb{P}\big[\sigma(b) < \tau\big]. \tag{5.8}$$

From (5.7) and (5.8), we deduce

$$\mathbb{P}\big[\tau(a) \leq T\big] \leq \mathbb{P}\big[\sigma(b) < \tau\big] + \limsup_{n \to \infty} \mathbb{P}\big[\tau_n(2a) \leq \sigma_n(2b) \wedge T\big].$$

Therefore, (5.5) holds if

$$\lim_{b\to\infty} \mathbb{P}[\sigma(b) < \tau] = 0, \tag{5.9}$$

and, for every $b > 0$,

$$\lim_{a\to 0} \limsup_{n\to\infty} \mathbb{P}[\tau_n(a) \le \sigma_n(b) \wedge T] = 0. \tag{5.10}$$

The verification of (5.9) is straightforward due to the sub-martingale property of U. The uniform local boundedness conditions on $(Q^n)_{n\ge 1}$ and $(X^n)_{n\ge 1}$ (from above) and the fact that G has trajectories in $\mathbf{C}(\mathbf{C}(\mathbf{B}), [0, T])$ imply that the process

$$Y_t \triangleq \inf_{n\ge 1}\big(G(-\mathbf{1}, 1, Q^n_t, t) - X^n_t\big), \qquad 0 \le t \le T,$$

is locally bounded from below. The convergence (5.10) follows now from the second inequality in (5.4) of Lemma 5.3. $\square$

We conclude this section with affirmative answers to our Questions 2.9 and 2.10 from Section 2.2. Recall that the acronym LCRL means left-continuous with right limits.

THEOREM 5.4. *Under Assumptions* 2.1, 2.2, 2.4, 4.2 *and* 4.15, *a predictable process Q with values in $\mathbf{R}^J$ and LCRL trajectories is a strategy if and only if there is a predictable process X with values in $\mathbf{R}$ and a sequence of simple strategies $(Q^n)_{n\ge 1}$ converging to Q in* ucp *such that the sequence of its cash balances $(X^n)_{n\ge 1}$ converges to X in* ucp. *In this case*, *X is the cash balance process for Q*.

PROOF. This follows from Theorems 5.1 and 5.2 and the fact that every predictable process with LCRL trajectories is a limit in ucp of a sequence of simple processes which then necessarily is also uniformly locally bounded. $\square$

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