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Analyzing Walksat on Random Formulas*

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Abstract

Let Φ be a uniformly distributed random k -SAT formula with n variables and m clauses. We prove that the `Walksat` algorithm from Papadimitriou (FOCS 1991)/Schöning (FOCS 1999) finds a satisfying assignment of Φ in polynomial time w.h.p. if $m/n \leq \rho \cdot 2^k/k$ for a certain constant $\rho > 0$. This is an improvement by a factor of $\Theta(k)$ over the best previous analysis of `Walksat` from Coja-Oghlan, Feige, Frieze, Krivelevich, Vilenchik (SODA 2009).

Key words: random structures, phase transitions, k -SAT, local search algorithms.

1 Introduction

Let $\Phi = \Phi_k(n, m)$ be a k -CNF on n Boolean variables x_1, \dots, x_n with m clauses chosen uniformly at random ($k \geq 3$). The interest in random k -SAT stems largely from the *experimental* observation that for certain densities r the random formula Φ is a challenging algorithmic benchmark [7, 15]. However, *analyzing* algorithms on random formulas is notoriously difficult. Indeed, the current rigorous results for random k -SAT mostly deal with algorithms that are extremely simple both to state and to analyze, or with algorithms that were specifically designed so as to allow for a rigorous analysis. More precisely, the present analysis techniques are essentially confined to simple algorithms that aim to construct a satisfying assignment by determining the value of one variable at a time *for good*, without any backtracking or reassigning variables at a later time. By contrast, most ‘real-life’ satisfiability algorithms actually rely substantially on reassigning variables.

Maybe the simplest example of a natural algorithm that eludes the standard analysis techniques is `Walksat` [17, 18]. Similar local search algorithms are quite successful in practical SAT-solving [19]. Starting from the all-true assignment, `Walksat` tries to find a satisfying assignment of its input k -CNF formula $\Phi = \Phi_1 \wedge \dots \wedge \Phi_m$ as follows. If the current assignment σ is satisfying, then clearly there is nothing to do and the algorithm terminates. Otherwise, the algorithm picks an index i such that clause Φ_i is unsatisfied uniformly at random among all such indices. Clause Φ_i is a disjunction of k literals $\Phi_{i1} \vee \dots \vee \Phi_{ik}$. `Walksat` picks an index $j \in \{1, \dots, k\}$ uniformly at random and flips the value assigned to the variable underlying the literal Φ_{ij} . Of course, this ensures that under the new assignment clause Φ_i is satisfied, but flipping Φ_{ij} may create new unsatisfied clauses. If after a certain number T_{\max} of iterations no satisfying assignment is found, `Walksat` gives up and concedes failure. The pseudocode is shown in Figure 1. In the worst case, it can be shown that $(2 - 2/k)^{(1+o(1))n}$ executions of `Walksat` with independent coins tosses will find a satisfying assignment of a satisfiable input formula Φ on n variables with probability $1 - o(1)$, for a suitable $T_{\max} = T_{\max}(k) = O(n)$ [18].

Although `Walksat` is conceptually very simple, analyzing this algorithm on random formulas is a challenge. Indeed, `Walksat` does not follow the naive template of the previously analysed algorithms that assign one variable at a time for good, because its random choices may (and will) lead `Walksat` to flipping quite a few variables several times over. This causes stochastic dependencies that seem to render the

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Algorithm 1.1 Walksat(Φ, T_{\max})

Input: A k -CNF $\Phi = \Phi_1 \wedge \dots \wedge \Phi_m$ over the variables x_1, \dots, x_n and a number $T_{\max} \geq 0$.

Output: An assignment $\sigma : V \rightarrow \{0, 1\}$.

0. Initially, let $\sigma(x_i) = 1$ for $i = 1, \dots, n$.
1. Repeat the following T_{\max} times (with independent random choices)
 2. If σ is a satisfying assignment, then halt and output σ .
 3. Otherwise, choose an index i such that clause Φ_i is unsatisfied under σ uniformly at random.
 4. Suppose that $\Phi_i = \Phi_{i1} \vee \dots \vee \Phi_{ik}$.
Choose an index $j \in \{1, \dots, k\}$ uniformly at random.
Flip the value of the variable underlying the literal Φ_{ij} in the assignment σ .
5. Return ‘failure’.

Figure 1: The Walksat algorithm.

differential equation method, the mainstay of the previous analyses of random k -SAT algorithms, useless. The goal of the present paper is to present an analysis of Walksat via a different approach that allows us to deal with the stochastic dependencies. Our main result is as follows.

Theorem 1.2 *There is a constant $k_0 > 3$ such that for any $k \geq k_0$ and*

$$0 < m/n \leq \frac{1}{25} \cdot 2^k/k,$$

Walksat($\Phi, \lceil n/k \rceil$) *outputs a satisfying assignment w.h.p.*

1.0.1 Related work.

To put Theorem 1.2 in perspective, let us compare it with other results on random k -SAT algorithms. The simplest conceivable one is presumably UnitClause. Considering all variables unassigned initially, UnitClause sets one variable at a time as follows. If there is a clause in which $k - 1$ variables have been assigned already without satisfying that clause (a ‘unit clause’), the algorithm has to assign the k th variable so as to satisfy the unit clause. If there is no unit clause, a currently unassigned variable is chosen randomly and is assigned a random truth value. As UnitClause is extremely simple and does not backtrack, it can be analyzed via the method of differential equations [1]. The result is that UnitClause finds a satisfying assignment with a non-vanishing probability so long as $m/n < (1 - o_k(1)) \frac{e}{2} \cdot 2^k/k$, where $o_k(1)$ hides a term that tends to 0 as k gets large [6]. Furthermore, ShortestClause, a natural generalization of UnitClause, succeeds for $m/n < (1 - o_k(1))e^2/8 \cdot 2^k/k$ with *high* probability [8]. Indeed, the algorithm can be modified so as to succeed with high probability even for $m/n < (1.817 - o_k(1)) \cdot 2^k/k$ by allowing a *very* limited amount of backtracking [11]. Finally, the algorithm Fix from [9], which was specifically designed for solving random k -SAT instances, succeeds up to $m/n < (1 - o_k(1))2^k \ln(k)/k$. By comparison, non-constructive arguments show that the threshold for the *existence* of a satisfying assignment is $(1 + o_k(1)) \cdot 2^k \ln 2$ [2, 3].

In summary, Theorem 1.2 shows that Walksat is broadly competitive with the other known algorithms for random k -SAT. That said, the main point of this paper is not to produce a better algorithmic bound for random k -SAT, but to address the methodological challenge of analyzing algorithms such as Walksat that may reassign variables. This difficult aspect did not occur or was sidestepped in the aforementioned previous analyses [1, 8, 9, 11]. Indeed, the lack of techniques for such analyses is arguably one of the most important shortcomings of the current theory of random discrete structures.

Theorem 1.2 improves substantially on the previous analyses of Walksat, at least for general k . The best previous result for this case showed that w.h.p. Walksat will find a satisfying assignment with $T_{\max} = n$ if $m/n < \rho' \cdot 2^k/k^2$, for a certain constant $\rho' > 0$ [10]. The proof of this result is based on a rather simple observation that allows to sidestep the analysis of the stochastic dependencies that arise in the execution of Walksat. However, it is not difficult to see that this argument is confined to clause/variable densities $m/n < 2^k/k^2$. Theorem 1.2 improves this result by a factor of $\Theta(k)$.

Furthermore, the techniques of Alekhovich and Ben-Sasson [4] show that for any k Walksat will w.h.p. find a satisfying assignment within $O(n)$ iterations if $m/n < r_{k\text{-pure}}$, where $r_{k\text{-pure}}$ is the ‘pure literal threshold’. The analysis in [4] depends heavily on the fact that the combinatorial structure of the hypergraph underlying the random k -CNF Φ is extremely simple for $m/n < r_{k\text{-pure}}$. Furthermore, because $r_{k\text{-pure}} \rightarrow 0$ in the limit of large k [16], this result is quite weak for general k . Yet [4] remains the best known result for ‘small’ k . For instance, in the case $k = 3$ the pure literal bound is $r_{3\text{-pure}} \approx 1.63$ [5].

Monasson and Semerjian [20] applied non-rigorous techniques from statistical mechanics to study the Walksat algorithm on random formulas. Their work suggests that $\text{Walksat}(\Phi, O(n))$ will find a satisfying assignment w.h.p. if $m/n < (1 - o_k(1))2^k/k$. Theorem 1.2 confirms this claim, up to the constant factor $1/25$.

In contrast to the previous ‘indirect’ attempts at analyzing Walksat on random formulas [4, 10], in the present paper we develop a technique for tracing the execution of the algorithm directly. This allows us to keep track of the arising stochastic dependencies explicitly. Before we outline our analysis, we need some notation and preliminaries.

2 Preliminaries

We let $\Omega_k(n, m)$ be the set of all k -SAT formulas with variables from $V = \{x_1, \dots, x_n\}$ that contain exactly m clauses. To be precise, we consider each formula an ordered m -tuple of clauses and each clause an ordered k -tuple of literals, allowing both literals to occur repeatedly in one clause and clauses to occur repeatedly in the formula. Thus, $|\Omega_k(n, m)| = (2n)^{km}$. Let $\Sigma_k(n, m)$ be the power set of $\Omega_k(n, m)$, and let $\mathbb{P} = \mathbb{P}_k(n, m)$ be the uniform probability measure. Throughout, we assume that $m = \lceil rn \rceil$ for a fixed number $r > 0$, the *density*.

As indicated above, we denote a uniformly random element of $\Omega_k(n, m)$ by Φ . In addition, we use the symbol Φ to denote specific (i.e., non-random) elements of $\Omega_k(n, m)$. If $\Phi \in \Omega_k(n, m)$, then Φ_i denotes the i th clause of Φ , and Φ_{ij} denotes the j th literal of Φ_i . If $Z \subset [m]$ is a set of indices, then we let $\Phi_Z = \bigwedge_{i \in Z} \Phi_i$. If $l \in \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$ is a literal, then we denote its underlying variable by $|l|$. Furthermore, we define $\text{sign}(l) = -1$ if l is a negative literal, and $\text{sign}(l) = 1$ if l is positive.

Recall that a *filtration* is a sequence $(\mathcal{F}_t)_{0 \leq t \leq \tau}$ of σ -algebras $\mathcal{F}_t \subset \Sigma_k(n, m)$ such that $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $0 \leq t < \tau$. For a random variable $X : \Omega_k(n, m) \rightarrow \mathbf{R}$ we let $\mathbb{E}[X|\mathcal{F}_t]$ denote the *conditional expectation*. Thus, $\mathbb{E}[X|\mathcal{F}_t] : \Omega_k(n, m) \rightarrow \mathbf{R}$ is a \mathcal{F}_t -measurable random variable such that for any $A \in \mathcal{F}_t$ we have

$$\sum_{\Phi \in A} \mathbb{E}[X|\mathcal{F}_t](\Phi) = \sum_{\Phi \in A} X(\Phi).$$

Also remember that $\mathbb{P}[\cdot|\mathcal{F}_t]$ assigns a probability measure $\mathbb{P}[\cdot|\mathcal{F}_t](\Phi)$ to any $\Phi \in \Omega_k(n, m)$, namely

$$\mathbb{P}[\cdot|\mathcal{F}_t](\Phi) : A \in \Sigma_k(n, m) \mapsto \mathbb{E}[\mathbf{1}_A|\mathcal{F}_t](\Phi),$$

where $\mathbf{1}_A$ is the indicator of the event A . We need the following well-known bound.

Lemma 2.1 *Let $(\mathcal{F}_t)_{0 \leq t \leq \tau}$ be a filtration and let $(X_t)_{1 \leq t \leq \tau}$ be a sequence of non-negative random variables such that each X_t is \mathcal{F}_t -measurable. Assume that there are numbers $\xi_t \geq 0$ such that $\mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq \xi_t$ for all $1 \leq t \leq \tau$. Then $\mathbb{E}[\prod_{1 \leq t \leq \tau} X_t|\mathcal{F}_0] \leq \prod_{1 \leq t \leq \tau} \xi_t$.*

Proof. For $1 \leq s \leq \tau$ we let $Y_s = \prod_{t=1}^s X_t$. Let $s > 1$. Since Y_{s-1} is \mathcal{F}_{s-1} -measurable, we obtain

$$\mathbb{E}[Y_s|\mathcal{F}_0] = \mathbb{E}[Y_{s-1}X_s|\mathcal{F}_0] = \mathbb{E}[\mathbb{E}[Y_{s-1}X_s|\mathcal{F}_{s-1}]|\mathcal{F}_0] = \mathbb{E}[Y_{s-1}\mathbb{E}[X_s|\mathcal{F}_{s-1}]|\mathcal{F}_0] \leq \xi_s \mathbb{E}[Y_{s-1}|\mathcal{F}_0],$$

whence the assertion follows by induction. \square

We also need the following tail bound (“Azuma-Hoeffding”, e.g. [13, p. 37]).

Lemma 2.2 *Let $(M_t)_{0 \leq t \leq \tau}$ be a super-martingale with respect to a filtration $(\mathcal{F}_t)_{0 \leq t \leq \tau}$ such that $M_0 = 0$. Suppose that there exist numbers c_t such that $|M_t - M_{t-1}| \leq c_t$ for all $1 \leq t \leq \tau$. Then for any $\lambda > 0$ we have $\mathbb{P}[M_\tau > \lambda] \leq \exp[-\lambda^2/(2 \sum_{t=1}^{\tau} c_t^2)]$.*

A k -CNF $\Phi = \Phi_1 \wedge \dots \wedge \Phi_m$ gives rise to a bipartite graph whose vertices are the variables V and the clauses $\{\Phi_i : i \in [m]\}$, and in which each clause is adjacent to all the variables that occur in it. This is the *factor graph* of Φ . For a vertex v of the factor graph we denote by $N(v) = N_\Phi(v)$ the neighborhood of v in the factor graph. For a set $Z \subset [m]$ we let $N(\Phi_Z) = \bigcup_{i \in Z} N(\Phi_i)$ be the set of all variables that occur in the sub-formula Φ_Z .

Let A, B be two disjoint sets of vertices of the factor graph. Recall that a l -fold matching from A to B is a set M of A - B -edges such that each $a \in A$ is incident with precisely l edges from M , while each $b \in B$ is incident with at most one edge from M . We will make use of the following simple expansion property of the factor graph of random formulas.

Lemma 2.3 *There is a constant $k_0 > 0$ such that for all $k \geq k_0$ and for $m/n \leq 2^k \ln 2$ the random formula Φ has the following property w.h.p.*

$$\text{For any set } Z \subset [m] \text{ of size } |Z| \leq n/k^2 \text{ there is a } 0.9k\text{-fold matching from } \Phi_Z \text{ to } N(\Phi_Z). \quad (1)$$

Proof. We start by proving that w.h.p. the random formula Φ has the following property.

$$\text{For any set } U \text{ of } \leq n/k \text{ variables we have } |\{i \in [m] : N(\Phi_i) \subset U\}| \leq 1.1|U|/k. \quad (2)$$

To prove (2) we use a ‘first moment’ argument. For set $U \subset V$ we let $X_U = 1$ if $|\{i \in [m] : N(\Phi_i) \subset U\}| > 1.1|U|/k$, and we set $X_U = 0$ otherwise. Then

$$\mathbb{E}[X_U] = \mathbb{P}[X_U = 1] \leq \binom{m}{1.1|U|/k} (|U|/n)^{1.1|U|}.$$

Furthermore, for any $1 \leq u \leq n/k$ we let $X_u = \sum_{U \subset V: |U|=u} X_U$. Assuming that $k \geq k_0$ is sufficiently large, we obtain

$$\begin{aligned} \mathbb{E}[X_u] &\leq \sum_{U \subset V: |U|=u} \mathbb{E}[X_U] \leq \binom{n}{u} \binom{m}{1.1u/k} \left(\frac{u}{n}\right)^{1.1u} \\ &\leq \left[\frac{en}{u} \cdot \left[\left(\frac{em}{1.1u/k}\right)^{1/k} \cdot \frac{u}{n} \right]^{1.1} \right]^u \leq \left[\frac{en}{u} \left[\left(\frac{e2^k k \ln 2}{1.1} \cdot \frac{n}{u}\right)^{1/k} \cdot \frac{u}{n} \right]^{1.1} \right]^u \\ &\leq \left[e \left(\frac{u}{n}\right)^{0.1-1/k} \left(\frac{e2^k k \ln 2}{1.1}\right)^{1.1/k} \right]^u \leq \left[e^2 \left(\frac{u}{n}\right)^{0.09} \right]^u. \end{aligned}$$

Summing the last expression over $1 \leq u \leq n/k$ and assuming that $k \geq k_0$ is large enough, we see that

$$\begin{aligned} \mathbb{E} \sum_{1 \leq u \leq n/k} X_u &\leq \sum_{1 \leq u \leq \ln^2 n} \left[e^2 \left(\frac{u}{n}\right)^{0.09} \right]^u + \sum_{\ln^2 n < u \leq n/k} [e^2 k^{-0.09}]^u \\ &\leq \ln^2 n \cdot e^2 (\ln^2 n/n)^{0.09} + \frac{n}{k} \cdot [e^2 k^{-0.09}]^{\ln^2 n} = o(1). \end{aligned}$$

Thus, $\sum_{1 \leq u \leq n/k} X_u = 0$ w.h.p. by Markov’s inequality. Hence, (2) holds true w.h.p.

Now, assume that Φ satisfies (2). Let $Z \subset [m]$ be a set of size $|Z| \leq n/k^2$. Let $Y \subset Z$ and let $U = N(\Phi_Y)$. Then $|U| \leq n/k$, and $N(\Phi_i) \subset U$ for any $i \in Y$. Therefore, (2) implies that $|Y| \leq 1.1|U|/k$, i.e., $|U| \geq \frac{k}{1.1}|Y| \geq 0.9k|Y|$. Hence, the assertion follows from the marriage theorem. \square

The following lemma states a second expansion-type property.

Lemma 2.4 *There exists a constant $k_0 > 0$ such that for all $k \geq k_0$ and for any $\varepsilon > 0, \lambda > 4$ satisfying $\varepsilon \leq k^{-3}$ and $\varepsilon^\lambda \leq \frac{1}{e}(2e)^{-4k}$ the random formula Φ with $m/n \leq 2^k \ln 2$ has the following property w.h.p.*

Let $Z \subset [m]$ be any set of size $|Z| \leq \varepsilon n$. If $i_1, \dots, i_l \in [m] \setminus Z$ is a sequence of pairwise distinct indices such that

$$|N(\Phi_{i_s}) \cap N(\Phi_{Z \cup \{i_j: 1 \leq j < s\}})| \geq \lambda \quad \text{for all } 1 \leq s \leq l, \quad (3)$$

then $l \leq \varepsilon n$.

Proof. It is clearly sufficient to prove that the desired property holds w.h.p. for all sets Z of size *precisely* $|Z| = \varepsilon n$. Assume that there is a set Z and a sequence $\mathbf{i} = (i_1, \dots, i_l)$ of pairwise distinct indices in $[m] \setminus Z$ of length $l = \varepsilon n$ such that $|N(\Phi_{i_s}) \cap N(\Phi_{\mathcal{I} \cup \{i_j: 1 \leq j < s\}})| \geq \lambda$ for all $1 \leq s \leq l$. Then the sets $Y = \bigcup_{j=1}^l N(\Phi_{i_j}) \setminus N(\Phi_Z) \subset V$ and Z have the following properties.

- a. $|Y| \leq \varepsilon(k - \lambda)n$.
- b. There is a set $I \subset [m] \setminus Z$ of size $|I| = \varepsilon n$ such that $N(\Phi_i) \subset N(\Phi_Z) \cup Y$ for all $i \in I$.

Property a. holds because each clause Φ_{i_j} adds no more than $k - \lambda$ ‘new’ variables to Y , and b. is true for the set $I = \{i_j : 1 \leq j \leq l\}$.

To prove that w.h.p. there do not exist Z and \mathbf{i} of length $l = \varepsilon n$ as above, we are going to show by a first moment argument that w.h.p. the random formula Φ does not feature sets Y, Z that satisfy a. and b. More precisely, for sets $Z \subset [m]$ of size $|Z| = \varepsilon n$, $Y \subset V$ of size $|Y| = \varepsilon(k - \lambda)n$, and $I \subset [m] \setminus Z$ of size $|I| = \varepsilon n$ we let $\mathcal{E}(Z, Y, I)$ be the event that $N(\Phi_i) \subset N(\Phi_Z) \cup Y$ for all $i \in I$. Then for any fixed Z, Y, I we have

$$\mathbb{P}[\mathcal{E}(Z, Y, I)] \leq \left(\frac{k|Z| + |Y|}{n} \right)^{k|I|} \leq (\varepsilon(2k - \lambda))^{k\varepsilon n},$$

because each of the $k|I|$ variable occurrences in the clauses Φ_I is uniformly distributed over V . Hence, by the union bound, for large enough k

$$\begin{aligned} \mathbb{P}[\exists Z, Y, I : \mathcal{E}(Z, Y, I)] &\leq \sum_{Z, Y, I} \mathbb{P}[\mathcal{E}(Z, Y, I)] \leq \binom{m}{\varepsilon n}^2 \binom{n}{\varepsilon n(k - \lambda)} (\varepsilon(2k - \lambda))^{k\varepsilon n} \\ &\leq \left[\left(\frac{em}{\varepsilon n} \right)^2 \left(\frac{e}{\varepsilon(k - \lambda)} \right)^{k - \lambda} (\varepsilon(2k - \lambda))^k \right]^{\varepsilon n} \\ &\leq \left[\left(\frac{e2^k}{\varepsilon} \right)^2 \left(\frac{e(2k - \lambda)}{k - \lambda} \right)^{k - \lambda} (2k\varepsilon)^\lambda \right]^{\varepsilon n} \\ &\leq \left[\left(\frac{e2^k}{\varepsilon} \right)^2 \exp(2k)(2k\varepsilon)^\lambda \right]^{\varepsilon n} \leq \left[(2e)^{2k} \varepsilon^{\lambda/2} \right]^{\varepsilon n}, \end{aligned} \quad (4)$$

where the last inequality follows from our assumption that $\varepsilon \leq k^{-3}$ with $k \geq k_0$ sufficiently large. Due to our assumption that $\varepsilon^\lambda \leq \frac{1}{e} e(2e)^{-4k}$, (4) yields $\mathbb{P}[\exists Z, Y, I : \mathcal{E}(Z, Y, I)] \leq \exp(-\varepsilon n) = o(1)$, whence the assertion follows. \square

Finally, it will be convenient to assume in our proof of Theorem 1.2 that the formula density $r = m/n$ is ‘not too small’ and that the clause length k is sufficiently large. These assumptions are justified as the case of small k or very small r is already covered by [10].

Theorem 2.5 ([10]) *There is a constant $k_0 > 3$ such that for all $k \geq k_0$ and all $r \leq \frac{1}{6} \cdot 2^k / k^2$ w.h.p. $\text{Walksat}(\Phi, n)$ will find a satisfying assignment.*

3 Outline of the analysis

Throughout this section we assume that $k \geq k_0$ for some large enough constant $k_0 > 0$, and that $r = m/n \sim \rho \cdot 2^k / k$ with $k^{-2} \leq \rho < \rho_0 = 1/25$. We can make these assumptions as otherwise the assertion of Theorem 1.2 already follows from Theorem 2.5. Furthermore, let

$$\lambda = \sqrt{k} \text{ and } \varepsilon = \exp(-k^{2/3}). \quad (5)$$

The standard approach to analyzing an algorithm on random k -SAT formulas is the *method of deferred decisions*, which often reduces the analysis to the study of a system of ordinary differential equations that

capture the dynamics of the algorithm [1]. Roughly speaking, the method of deferred decisions applies where the state of the algorithm after a given number of steps can be described by a simple probability distribution, depending only on a very few parameters determined by the past decisions of the algorithm. This is typically so in the case of simple backtrack-free algorithms such as `UnitClause`.

However, in the case of `Walksat`, this approach does not apply because the algorithm is bound to flip many variables more than once. This entails that the algorithms' future steps depend on past events in a more complicated way than the method of deferred decisions can accommodate. Hence, our approach will be to use the method of deferred decisions to trace the effect of flipping a variable *for the first time*. But we will need additional arguments to deal with the dependencies that arise out of flipping the same variable several times.

To get started, let us investigate the effect of the *first* flip that `Walksat` performs. Let $\sigma = \mathbf{1}$ be the assignment that sets every variable to true. Clearly, a clause Φ_i is unsatisfied under σ iff it consists of negative literals only. As Φ consists of m uniformly random and independent clauses, the number of unsatisfied clauses has a binomial distribution $\text{Bin}(m, 2^{-k})$, and thus there will be $(1+o(1))2^{-k}m \sim \rho n/k$ all-negative clauses w.h.p. To perform its first flip, `Walksat` chooses an index $i \in [m]$ such that Φ_i is all-negative uniformly at random, then chooses a literal index $j \in [k]$ uniformly, and sets $\sigma(\Phi_{ij})$ to false, thereby satisfying clause Φ_i .

But, of course, flipping $|\Phi_{ij}|$ may well generate new unsatisfied clauses. We need to study their number. As Φ_i is just a uniformly random all-negative clause, the random variable $|\Phi_{ij}|$ is uniformly distributed over the set of all n variables, and thus we may assume without loss that $|\Phi_{ij}| = x_1$. Furthermore, if a clause Φ_l becomes unsatisfied because variable x_1 got flipped, then x_1 must have been the only variable that appears positively in Φ_l . Now, the number of clauses whose only positive literal is x_1 has distribution $\text{Bin}(m, k/(n2^k) + O(1/n^2))$. Indeed, the probability that a random clause has precisely one positive literal is $k/2^k$, and the probability that this positive literal happens to be x_1 is $1/n$; the $O(1/n^2)$ accounts for the number of clauses in which variable x_1 occurs more than once. Hence, the *expected* number of newly created unsatisfied clauses equals $(1 + o(1))\frac{km}{2^kn} \sim \rho$.

In summary, as we are assuming that $\rho \leq \rho_0 = 1/25 < 1$, the *expected change* in the number of unsatisfied clauses as a result of the first flip is bounded from above by

$$\rho - 1 + o(1) < 0.$$

(The precise value is even smaller because x_1 may occur in further all-negative clauses.) Thus, we expect that the first flip will indeed reduce the number of unsatisfied clauses. Of course, this simple calculation does not extend to the further steps of `Walksat` because knowing the outcome of the first flip renders the various above statements about clauses/literals being uniformly distributed invalid.

To analyze the further flips, we will describe `Walksat` as a stochastic process. Our time parameter will be the number of iterations of the main loop (Steps 2–4 in Figure 1), i.e., the number of flips performed. To represent the conditioning of the random input formula imposed up to time t , we will define a sequence of random maps $(\pi_t)_{t \geq 0}$. These maps reflect for each pair $(i, j) \in [m] \times [k]$ the conditional distribution of the literals Φ_{ij} , given the information that `Walksat` has revealed after performing the first t flips. More precisely, the value of $\pi_t(i, j)$ will either be just the *sign* of the literal Φ_{ij} , or the actual literal Φ_{ij} itself. In the initial map π_0 , we have $\pi_0(i, j) = \text{sign}(\Phi_{ij})$ for all $(i, j) \in [m] \times [k]$.

At times $t \geq 1$ the map π_t will feature the occurrences of all variables that have been flipped thus far. That is, for any pair (i, j) such that `Walksat` has flipped the variable $|\Phi_{ij}|$ at least once by time t , we let $\pi_t(i, j) = \Phi_{ij}$. This information will be necessary for us to investigate the effect of flipping the same variable more than once.

In addition, we need to pay particular attention to clauses that contain many variables that have been flipped at least once. The reason is that these clauses have ‘too little randomness’ left for a direct analysis, and thus we will need to study them separately. More precisely, in our map π_t we will fully reveal all clauses Φ_i in which at least

$$k_1 = 0.57k \tag{6}$$

literals Φ_{ij} have been flipped at least once. Furthermore, we will also recursively reveal all clauses that contain at least λ variables from clauses that were fully revealed before. This recursive process ensures

- PI0.** If the assignment σ_{t-1} satisfies Φ , then the process terminates.
- PI1.** Otherwise, choose an index i_t such that Φ_{i_t} is unsatisfied under σ_{t-1} uniformly at random from the set of all such indices. In addition, choose $j_t \in [k]$ uniformly at random. Define $\sigma_t : V \rightarrow \{0, 1\}$ by letting $\sigma_t(|\Phi_{i_t, j_t}|) = 1 - \sigma_{t-1}(|\Phi_{i_t, j_t}|)$ and $\sigma_t(x) = \sigma_{t-1}(x)$ for all $x \neq |\Phi_{i_t, j_t}|$.
- PI2.** Initially, let $\mathcal{Z}_t = \mathcal{Z}_{t-1}$ and $\mathcal{N}_t = \mathcal{N}_{t-1}$.
While there is an index $i \in [m] \setminus \mathcal{Z}_t$ such that Φ_i is $(\mathcal{A}_{t-1} \cup \mathcal{N}_t \cup \{|\Phi_{i_t, j_t}|\})$ -negative and either
- there are at least k_1 indices $j \in [k]$ with $|\Phi_{ij}| \in \mathcal{A}_{t-1} \cup \{|\Phi_{i_t, j_t}|\}$, or
 - there are more than λ indices $j \in [k]$ with $|\Phi_{ij}| \in \mathcal{N}_t$,
- add the least such index i_{\min} to \mathcal{Z}_t and add the variables $\{|\Phi_{i_{\min}, j}| : j \in [k]\}$ to \mathcal{N}_t .
- PI3.** Let $\mathcal{A}_t = (\mathcal{A}_{t-1} \cup \{|\Phi_{i_t, j_t}|\}) \setminus \mathcal{N}_t$.
Define the map $\pi_t : [m] \times [k] \rightarrow \{-1, 1\} \cup L$ by letting

$$\pi_t(i, j) = \begin{cases} \Phi_{ij} & \text{if } |\Phi_{ij}| \in \mathcal{A}_t \cup \mathcal{N}_t, \\ \text{sign}(\Phi_{ij}) & \text{otherwise.} \end{cases}$$

Figure 2: the construction of the maps π_t

that we can separate the analysis of clauses that are ‘heavily conditioned’ by the past steps of `Walksat` from the bulk of the formula.

Throughout this process that mirrors the execution of `Walksat`, all variables whose occurrences have been revealed will be labeled either with an asterisk or with a zero. Those variables that got revealed because they occur either in a ‘heavily conditioned’ clause or in another clause that got revealed by the recursive process described in the previous paragraph will be labeled 0. All other variables that have been flipped by `Walksat` at least once are labeled *. We will let \mathcal{A}_t denote the set of all variables labeled *, and \mathcal{N}_t the set of all variables labeled 0.

Let us now define the maps π_t and the sets $\mathcal{A}_t, \mathcal{N}_t$ formally. Each π_t is a map $[m] \times [k] \rightarrow \{-1, 1\} \cup L$, with $L = \{x_1, \bar{x}_1, \dots, x_n, \bar{x}_n\}$ the set of literals. As mentioned above, we let $\pi_0(i, j) = \text{sign}(\Phi_{ij})$ for all $(i, j) \in [m] \times [k]$. Additionally, let $\mathcal{A}_0 = \mathcal{N}_0 = \mathcal{Z}_0 = \emptyset$, and let $\sigma_0 : V \rightarrow \{0, 1\}$, $x \mapsto 1$ be the all-true assignment. For a set $S \subset V$ we call a clause Φ_i *S-negative* if for all $j \in [k]$ with $\text{sign}(\Phi_{ij}) = 1$ we have $\Phi_{ij} \in S$. (In other words, Φ_i is *S-negative* if all of its positive literals lie in S .) For $t \geq 1$, we define the maps π_t along with the sets $\mathcal{A}_t, \mathcal{N}_t, \mathcal{Z}_t$ inductively via the process shown in Figure 2. Intuitively, the set \mathcal{Z}_t contains the clauses that are ‘heavily conditioned’ at time t , and \mathcal{N}_t is the set of variables that occur in such clauses. Moreover, \mathcal{A}_t is the set of all variables that have been flipped at least once by time t except the ones that belong to \mathcal{N}_t .

Let T be the stopping time of this process, i.e., the minimum t such that σ_t satisfies Φ (or ∞ if there is no such t). For $t > T$, we define $\pi_t = \pi_T$, $\sigma_t = \sigma_T$, $\mathcal{A}_t = \mathcal{A}_T$, $\mathcal{N}_t = \mathcal{N}_T$, and $\mathcal{Z}_t = \mathcal{Z}_T$.

Steps **PI0–PI1** mirror the main loop of the `Walksat` algorithm; in particular, the stopping time T equals the total number of iterations of the main loop of `Walksat` before a satisfying assignment is found. The purpose of the remaining steps is to ‘update’ the sets \mathcal{A}_t and \mathcal{Z}_t and the map π_t as described above. Before we continue, it may be useful to illustrate the construction of the maps π_t with an example.

Example 3.1 *Let us go through the example of a 5-SAT formula with 6 clauses on 10 variables. For the sake of this example, we will work with $k_1 = 2$ and $\lambda = 2$. (Recall that in our proof we actually assume that $k \geq k_0$ is large enough, k_1 is as in (6) and $\lambda = \sqrt{k}$.) We will represent the maps π_t by tables whose columns correspond to the clauses Φ_i . Thus, the j th entry in column i represents the value $\pi_t(i, j)$. To improve readability, we just write + and – instead of ± 1 . Suppose that the initial map π_0 , containing the*

signs of all literals, reads

$$\pi_0 = \begin{array}{cccccc} - & - & - & + & + & + \\ - & + & - & + & - & + \\ - & - & - & - & - & + \\ - & - & - & - & + & + \\ - & - & - & - & + & + \end{array}$$

The initial assignment σ_0 is the all-true assignment, and $\mathcal{A}_0 = \mathcal{N}_0 = \mathcal{Z}_0 = \emptyset$. Throughout, we will mark the variables in \mathcal{A}_t by an asterisk $*$ and the variables in \mathcal{N}_t by a 0.

Being all-negative, clauses Φ_1 and Φ_3 are unsatisfied under σ_0 . Therefore, at time $t = 1$ step **PII** chooses $i_1 \in \{1, 3\}$ randomly; say, the outcome is $i_1 = 1$. In addition, **PII** chooses $j_1 \in [k] = \{1, 2, 3, 4, 5\}$ uniformly at random. Suppose the result is $j_1 = 5$. To carry on, we need to reveal the variable $|\Phi_{15}|$. Thus far, the process has not imposed any conditioning on $|\Phi_{15}|$, and therefore this variable is uniformly distributed over the set of all our $n = 10$ variables. Assume that indeed $|\Phi_{15}| = x_1$. Then **PII** sets $\sigma_1(x_1) = 0$ and $\sigma_1(x) = 1$ for all $x \neq x_1$.

To implement **PI2** we need to reveal all occurrences of x_1 in our random formula. As there is no previous conditioning on any of variables $|\Phi_{ij}|$ with $(i, j) \neq (1, 5)$, these variables remain independently uniformly distributed over the set of all variables, and thus the events $\{|\Phi_{ij}| = x_1\}$ occur independently with probability $1/n$. Suppose that x_1 occurs at the following positions:

$$\begin{array}{cccccc} - & - & - & \mathbf{x}_1 & + & + \\ - & \mathbf{x}_1 & - & + & - & + \\ - & - & - & - & \bar{\mathbf{x}}_1 & + \\ - & - & - & - & + & + \\ \bar{\mathbf{x}}_1 & - & - & - & + & \mathbf{x}_1 \end{array}$$

Then there is no clause with at least k_1 occurrences of a variable from $\mathcal{A}_0 \cup \mathcal{N}_0 \cup \{x_1\} = \{x_1\}$, and thus step **PI2** is void. Hence, at the end of the first iteration we have $\mathcal{A}_1 = \{x_1\}$, $\mathcal{N}_1 = \mathcal{Z}_1 = \emptyset$, and

$$\pi_1 = \begin{array}{cccccc} - & - & - & x_1^* & + & + \\ - & x_1^* & - & + & - & + \\ - & - & - & - & \bar{x}_1^* & + \\ - & - & - & - & + & + \\ \bar{x}_1^* & - & - & - & + & x_1^* \end{array}$$

At time $t = 2$ there are two unsatisfied clauses: Φ_2 , whose only positive literal got flipped to false, and Φ_3 , which was unsatisfied initially. Step **PII** chooses one of them randomly, say $i_2 = 2$, and also chooses a random position $j_2 \in [k]$, say $j_2 = 2$. As we already know from the first step, the literal in this position is $\Phi_{22} = \pi_1(2, 2) = x_1$. In effect, the second iteration reverses the flip made in the first one and thus σ_2 is the all-true assignment. Since we have revealed all the occurrences of x_1 already, step **PI2** is void and $\pi_2 = \pi_1$, $\mathcal{A}_2 = \{x_1\}$, and $\mathcal{N}_2 = \mathcal{Z}_2 = \emptyset$.

At the start of the third iteration the unsatisfied clauses are Φ_1, Φ_3 . Suppose **PII** chooses $i_3 = 1$ and $j_3 = 1$. Then we need to reveal the variable $|\Phi_{11}|$. At this point, the only conditioning imposed on this variable is that it is different from x_1 , because all occurrences of x_1 have been revealed already. Thus, $|\Phi_{11}|$ is uniformly distributed over x_2, \dots, x_{10} . Suppose that $|\Phi_{11}| = x_2$. Then $\sigma_3(x_2) = 0$ and $\sigma_3(x) = 1$ for all $x \neq x_2$. To reveal the occurrences of x_2 all over the formula, note that by the same argument we applied to $|\Phi_{11}|$ all spots marked \pm in π_2 hide variables that are uniformly distributed over x_2, \dots, x_{10} . Let us assume that x_2 occurs in the following positions.

$$\begin{array}{cccccc} \bar{\mathbf{x}}_2 & - & - & x_1^* & + & + \\ - & x_1^* & - & + & - & + \\ - & - & \bar{\mathbf{x}}_2 & - & \bar{x}_1^* & + \\ - & - & - & - & + & \mathbf{x}_2 \\ \bar{x}_1^* & - & - & - & + & x_1^* \end{array}$$

As clause Φ_1 is $\mathcal{A}_2 \cup \mathcal{N}_2 \cup \{x_2\} = \{x_1, x_2\}$ -negative and contains $k_1 = 2$ occurrences of variables from $\mathcal{A}_2 \cup \{x_2\} = \{x_1, x_2\}$, **PI2** sets $\mathcal{Z}_3 = \{1\}$, reveals the remaining three variables in Φ_1 , and adds all

variables that occur in Φ_1 to \mathcal{N}_3 . Suppose that the remaining variables in Φ_1 are $|\Phi_{12}| = x_3$, $|\Phi_{13}| = x_4$, $|\Phi_{13}| = x_5$. Then $\mathcal{N}_3 = \{x_1, x_2, x_3, x_4, x_5\}$; in particular, x_1, x_2 are now labeled 0. The new 0 label ‘overwrites’ the * because **PI3** ensures that $\mathcal{A}_3 = (\mathcal{A}_2 \cup \{x_2\}) \setminus \mathcal{N}_3 = \emptyset$. In order to carry out **PI2**, we need to reveal all occurrences of variables from \mathcal{N}_3 . Suppose this yields

$$\begin{array}{cccccc} \bar{x}_2^0 & - & - & x_1^0 & + & + \\ \bar{x}_3^0 & x_1^0 & - & \mathbf{x}_5^0 & - & + \\ \bar{x}_4^0 & - & - & - & \bar{x}_1^0 & + \\ \bar{x}_5^0 & - & - & \bar{x}_4^0 & \mathbf{x}_3^0 & x_2^0 \\ \bar{x}_1^0 & - & \bar{x}_5^0 & \bar{x}_3^0 & \mathbf{x}_4^0 & x_1^0 \end{array}$$

Then clause Φ_4 has become $\mathcal{A}_2 \cup \mathcal{N}_3 \cup \{x_2\} = \{x_1, \dots, x_5\}$ -negative (as there is no +-sign left in column four), and thus **PI2** sets $\mathcal{Z}_3 = \{1, 4\}$. To proceed, we need to reveal the remaining -sign of Φ_4 , add the underlying variable to \mathcal{N}_3 , and reveal all of its occurrences. Suppose that this yields

$$\begin{array}{cccccc} \bar{x}_2^0 & - & - & x_1^0 & + & + \\ \bar{x}_3^0 & x_1^0 & - & x_5^0 & - & + \\ \bar{x}_4^0 & - & - & \bar{x}_6^0 & \bar{x}_1^0 & \mathbf{x}_6^0 \\ \bar{x}_5^0 & - & - & \bar{x}_4^0 & x_3^0 & x_2^0 \\ \bar{x}_1^0 & - & \bar{x}_5^0 & \bar{x}_3^0 & x_4^0 & x_1^0 \end{array}$$

At this point **PI2** stops, because clauses Φ_5, Φ_6 have +-signs left and clauses Φ_2, Φ_3 contain only one variable labeled 0. Thus, at the end of the third iteration we have $\mathcal{A}_3 = \emptyset$, $\mathcal{N}_3 = \{x_1, \dots, x_6\}$, $\mathcal{Z}_3 = \{1, 4\}$, and

$$\pi_3 = \begin{array}{cccccc} \bar{x}_2^0 & - & - & x_1^0 & + & + \\ \bar{x}_3^0 & x_1^0 & - & x_5^0 & - & + \\ \bar{x}_4^0 & - & - & \bar{x}_6^0 & \bar{x}_1^0 & x_6^0 \\ \bar{x}_5^0 & - & - & \bar{x}_4^0 & x_3^0 & x_2^0 \\ \bar{x}_1^0 & - & \bar{x}_5^0 & \bar{x}_3^0 & x_4^0 & x_1^0 \end{array}$$

As the fourth iteration commences, the only unsatisfied clause left is Φ_3 , whence $i_4 = 3$. Moreover, assume that $j_4 = 1$. As we have revealed all occurrences of x_1, \dots, x_6 , at this point we know that $|\Phi_{31}|$ is uniformly distributed over $\{x_7, x_8, x_9, x_{10}\}$. Suppose that indeed $|\Phi_{31}| = x_7$. Thus, **PI1** sets $\sigma_4(x_2) = \sigma_4(x_7) = 0$ and $\sigma_4(x) = 1$ for all $x \neq x_2, x_7$. Suppose that revealing all occurrences of x_7 yields

$$\begin{array}{cccccc} \bar{x}_2^0 & - & \bar{x}_7 & x_1^0 & + & \mathbf{x}_7 \\ \bar{x}_3^0 & x_1^0 & - & x_5^0 & \bar{x}_7 & + \\ \bar{x}_4^0 & \bar{x}_7 & - & \bar{x}_6^0 & \bar{x}_1^0 & x_6^0 \\ \bar{x}_5^0 & - & - & \bar{x}_4^0 & x_3^0 & x_2^0 \\ \bar{x}_1^0 & - & \bar{x}_5^0 & \bar{x}_3^0 & x_4^0 & x_1^0 \end{array}$$

Then there are no $\mathcal{A}_3 \cup \mathcal{N}_3 \cup \{x_7\}$ -negative clauses Φ_i with $i \notin \mathcal{Z}_3$ that have at least two occurrences of a variable from $\mathcal{A}_3 \cup \{x_7\}$. Therefore, **PI2** is void, and at the end of the fourth iteration we have

$$\pi_4 = \begin{array}{cccccc} \bar{x}_2^0 & - & \bar{x}_7^* & x_1^0 & + & x_7^* \\ \bar{x}_3^0 & x_1^0 & - & x_5^0 & \bar{x}_7^* & + \\ \bar{x}_4^0 & \bar{x}_7^* & - & \bar{x}_6^0 & \bar{x}_1^0 & x_6^0 \\ \bar{x}_5^0 & - & - & \bar{x}_4^0 & x_3^0 & x_2^0 \\ \bar{x}_1^0 & - & \bar{x}_5^0 & \bar{x}_3^0 & x_4^0 & x_1^0 \end{array},$$

$\mathcal{A}_4 = \{x_7\}$, $\mathcal{N}_4 = \{x_1, \dots, x_6\}$, and $\mathcal{Z}_4 = \{1, 4\}$. As σ_4 is satisfying the process stops and $T = 4$. \square

To trace the process **PI0–PI3** over time we define a filtration $(\mathcal{F}_t)_{t \geq 0}$ by letting \mathcal{F}_t be the σ -algebra generated by the random variables i_s, j_s and $\pi_s(i, j)$ with $s \leq t$ and $(i, j) \in [m] \times [k]$. Then intuitively, a random variable X is \mathcal{F}_t -measurable if its value is determined by the first t steps of the process **PI0–PI3**. In particular, we have the following.

Fact 3.2 For any $t \geq 1$, any $x \in V$, and any $i \in [m]$ the events $\{\sigma_t(x) = 1\}$, $\{\Phi_i \text{ is satisfied under } \sigma_t\}$, $\{x \in \mathcal{A}_t\}$, $\{i \in \mathcal{Z}_t\}$, $\{x \in \mathcal{N}_t\}$, and $\{T = t\}$ are \mathcal{F}_t -measurable.

Proof. The construction in steps **PI2** and **PI3** ensures that for any $t \geq 1$ we have $\Phi_{i_t j_t} \in \mathcal{A}_t \cup \mathcal{N}_t$ and thus $\pi_t(i_t, j_t) = \Phi_{i_t j_t}$. This implies that for any variable $x \in V$ the event $\{\sigma_t(x) = 1\}$ is \mathcal{F}_t -measurable. In fact, we have $\sigma_t(x) = 1$ iff the number $|\{1 \leq s \leq t : |\pi_t(i_s, j_s)| = x\}|$ of times x has been flipped is even (because σ_0 is the all-true assignment).

This implies that for any $i \in [m]$ the event $\{\Phi_i \text{ is satisfied under } \sigma_t\}$ is \mathcal{F}_t -measurable. In fact, if there is an index $j \in [k]$ such that $\pi_t(i, j) = 1$, then Φ_{ij} is a positive literal whose underlying variable has not been flipped before, whence σ_t satisfies Φ_i . Moreover, if there is an index $j \in [k]$ such that $\Phi_{ij} \neq \pm 1$, then by the previous paragraph the event that the literal $\Phi_{ij} = \pi_t(i, j)$ is true under σ_t is \mathcal{F}_t -measurable. If there is such a satisfied literal Φ_{ij} , then Φ_i is satisfied. Conversely, if there is no $j \in [k]$ such that either $\pi_t(i, j) = 1$ or $\pi_t(i, j)$ is a literal that is satisfied under σ_t , then clause Φ_i is unsatisfied. Hence, the event $\{\sigma_t \text{ is satisfying}\}$ is \mathcal{F}_t -measurable as well, and therefore so is the event $\{T = t\}$.

Furthermore, observe that $i \in \mathcal{Z}_t$ iff for all $j \in [k]$ we have $\pi_t(i, j) \notin \{-1, 1\}$. For if $i \in \mathcal{Z}_t$, then for all $j \in [k]$ we have $|\Phi_{ij}| \in \mathcal{N}_t$ and thus $\pi_t(i, j) = \Phi_{ij} \neq \pm 1$ due to **PI3**. Conversely, if $k \geq k_0$ is large enough, any $i \in [m]$ such that $\pi_t(i, j) \notin \{-1, 1\}$ for all $j \in [k]$ must satisfy one of the two conditions that lead **PI2** to add i to \mathcal{Z}_t . Hence, for any $i \in [m]$ the event $\{i \in \mathcal{Z}_t\}$ is \mathcal{F}_t -measurable. As by construction $\mathcal{N}_t = \{\pi_t(i, j) : i \in \mathcal{Z}_t, j \in [k]\}$, we conclude that for any variable $x \in V$ the event $\{x \in \mathcal{N}_t\}$ is \mathcal{F}_t -measurable.

Finally, the construction in **PI3** ensures that $\mathcal{A}_t = \{|\pi_t(i_s, j_s)| : 1 \leq s \leq t\} \setminus \mathcal{N}_t$. As for any x the events $\{x \in \{|\pi_t(i_s, j_s)| : 1 \leq s \leq t\}\}$ and $\{x \in \mathcal{N}_t\}$ are \mathcal{F}_t -measurable, so is the event $\{x \in \mathcal{A}_t\}$. \square

If $\pi_t(i, j) = \pm 1$, then up to time t the process **PI0–PI3** has only taken the sign of the literal Φ_{ij} into account, but has been oblivious to the underlying variable. The only conditioning is that $|\Phi_{ij}| \notin \mathcal{A}_t \cup \mathcal{N}_t$ (because otherwise **PI3** would have replaced the ± 1 by the actual literal). Since the input formula Φ is random, this implies that $|\Phi_{ij}|$ is uniformly distributed over $V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)$. In fact, for all (i, j) such that $\pi_t(i, j) = \pm 1$ the underlying variables are independently uniformly distributed over $V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)$. Formally, we can state this key observation as follows.

Fact 3.3 Let $t \geq 0$. Let \mathcal{E}_t be the set of all pairs (i, j) such that $\pi_t(i, j) \in \{-1, 1\}$. The conditional joint distribution of the variables $(|\Phi_{ij}|)_{(i,j) \in \mathcal{E}_t}$ given \mathcal{F}_t is uniform over $(V \setminus (\mathcal{A}_t \cup \mathcal{N}_t))^{\mathcal{E}_t}$. That is, for any map $f : \mathcal{E}_t \rightarrow V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)$ we have

$$\mathbb{P}[\forall (i, j) \in \mathcal{E}_t : |\Phi_{ij}| = f(i, j) | \mathcal{F}_t] = |V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)|^{-|\mathcal{E}_t|}.$$

Let

$$T^* = \theta n \quad \text{with } \theta = 0.38/k.$$

Our overall goal is to prove that the stopping time of the process **PI0–PI3** satisfies $T \leq T^*$ w.h.p. (The number θ is chosen somewhat arbitrarily; for the analysis to work it seems to be essential that $\theta = c/k$ for some $c > 0$ that is neither “too small” nor “too large”. The concrete constant above happens to work.) To prove this, we will define non-negative random variables S_t, H_t such that $S_t + H_t = 0$ implies that σ_t is a satisfying assignment. We will then trace S_t, H_t for $1 \leq t \leq T^*$.

For any $t \geq 1$ let

$$\mathcal{D}_t = \{i \in [m] : \Phi_i \text{ is } \mathcal{A}_t \cup \mathcal{N}_t\text{-negative}\}.$$

As **PI3** ensures that Φ_i is $\mathcal{A}_t \cup \mathcal{N}_t$ -negative iff $\pi_t(i, j) \neq 1$ for all $j \in [k]$, the event $\{i \in \mathcal{D}_t\}$ is \mathcal{F}_t -measurable for any $i \in [m]$. We define

$$S_0 = |\mathcal{D}_0| \quad \text{and} \quad S_t = |\mathcal{D}_t| - |\mathcal{A}_t| \quad \text{for } t \geq 1. \quad (7)$$

Any clause Φ_i with $i \notin \mathcal{D}_t$ is satisfied under σ_t . For if $j \in [k]$ is such that $\pi_t(i, j) = 1$, then Φ_{ij} is a positive literal and $\sigma_t(\Phi_{ij}) = 1$, because WalkSAT starts with the all-true assignment σ_0 and the variable Φ_{ij} has not been flipped up to time t . Clearly, in order to study the random variable S_t it is crucial to estimate $|\mathcal{D}_t|$. This is the purpose of the following proposition, whose proof we defer to Section 4.

Proposition 3.4 *Wh.p. we have $|\mathcal{D}_t| \leq 2^{2-k}m$ for all $t \leq T^*$.*

To define the random variables H_t , let us call an assignment $\tau : \mathcal{N}_t \rightarrow \{0, 1\}$ *rich* for \mathcal{Z}_t if in each clause Φ_i with $i \in \mathcal{Z}_t$ at least $0.8k$ literals Φ_{ij} are satisfied under τ .

Proposition 3.5 *Wh.p. there is a sequence $(\tau_t)_{1 \leq t \leq T^*}$ with the following properties.*

1. For any $1 \leq t \leq T^*$, τ_t is a rich assignment for \mathcal{Z}_t .
2. For any $1 < t \leq T^*$ and any $x \in \mathcal{N}_{t-1}$ we have $\tau_t(x) = \tau_{t-1}(x)$.

Moreover, τ_t is \mathcal{F}_t -measurable for all t .

Assuming that there is a sequence $(\tau_t)_{1 \leq t \leq T^*}$ as in Proposition 3.5, we define $H_0 = 0$ and

$$H_t = |\{x \in \mathcal{N}_t : \sigma_t(x) \neq \tau_t(x)\}| \text{ for } 1 \leq t \leq T^*,$$

and $H_t = |\mathcal{N}_t|$ for $t > T^*$. For the sake of completeness, we also let $H_t = |\mathcal{N}_t|$ if there is no such sequence $(\tau_t)_{1 \leq t \leq T^*}$. The proof of Proposition 3.5 hinges upon the following fact.

Proposition 3.6 *Wh.p. we have $|\mathcal{Z}_t| \leq \varepsilon n$ for all $t \leq T^*$.*

We defer the proof of Proposition 3.6 to Section 5. Assuming Proposition 3.6, we can derive Proposition 3.5 rather easily.

Proof of Proposition 3.5 (assuming Proposition 3.6). By Lemma 2.3, we may assume that Φ has the expansion property (1). Furthermore, by Proposition 3.6 we may assume that $|\mathcal{Z}_t| \leq \varepsilon n$ for all $t \leq T^*$. Under these assumptions we will construct the sequence $(\tau_t)_{1 \leq t \leq T^*}$ by induction on $t \geq 1$. Thus, suppose that $1 \leq t \leq T^*$ and that we have already got assignments τ_s with $1 \leq s < t$ that satisfy 1.–2.

The set $Z = \mathcal{Z}_t \setminus \mathcal{Z}_{t-1}$ of indices that \mathcal{Z}_t gained at time t has size $|Z| \leq |\mathcal{Z}_t| \leq \varepsilon n$. Therefore, (1) ensures that there is a $0.9k$ -fold matching M from Z to the set

$$N = N(\Phi_Z) = \{|\Phi_{ij}| : (i, j) \in Z \times [k]\} \subset \mathcal{N}_t$$

of variables that occur in the clauses Φ_i with $i \in Z$. The construction in **PI2** ensures that none of these clauses Φ_i has more than λ occurrences of a variable from \mathcal{N}_{t-1} (as otherwise $i \in \mathcal{Z}_{t-1}$). Therefore, in the matching M' obtained from M by omitting all edges $e = \{i, x\}$ with $i \in Z$ and $x \in \mathcal{N}_{t-1}$ each clause Φ_i with $i \in Z$ is incident with at least $0.9k - \lambda \geq 0.8k$ edges. Now, for each edge $e = \{i, x\} \in M'$ let $\tau_t(x)$ be the truth value that makes the corresponding literal in Φ_i evaluate to true. Furthermore, for all $y \in \mathcal{N}_{t-1}$ let $\tau_t(y) = \tau_{t-1}(y)$, and for all other variables $x' \in \mathcal{N}_t$ let $\tau_t(x') = 1$. This ensures that τ_t satisfies the conditions in Proposition 3.5. \square

Having defined the random variables S_t, H_t , we are now going to verify that they suit their intended purpose, i.e., that $S_t + H_t = 0$ implies that σ_t is satisfying.

Proposition 3.7 *Let $1 \leq t \leq T^*$. If $S_t + H_t = 0$, then σ_t is a satisfying assignment.*

Proof. Let U_t be the number of clause indices $i \in [m] \setminus \mathcal{Z}_t$ such that Φ_i is unsatisfied under σ_t . We claim that

$$U_t \leq S_t = |\mathcal{D}_t| - |\mathcal{A}_t|. \tag{8}$$

To see this, recall that any index $i \in [m]$ such that Φ_i is unsatisfied under σ_t belongs to \mathcal{D}_t . Therefore, to prove (8) it suffices to construct injective maps $s_t : \mathcal{A}_t \rightarrow \mathcal{D}_t$ such that for any $x \in \mathcal{A}_t$ the clause $\Phi_{s_t(x)}$ is satisfied under σ_t . In fact, the map s_t will have the property that for each $x \in \mathcal{A}_t$ there is an index $j \in [k]$ such that $x = |\Phi_{s_t(x)j}|$ and such that the literal $\Phi_{s_t(x)j}$ is true under σ_t .

The construction of the maps s_t is inductive. For $t = 0$ we have $\mathcal{A}_0 = \emptyset$ and thus there is nothing to do. Thus, suppose that $1 \leq t \leq T$ and that we have defined s_{t-1} already. Let $y = |\Phi_{i_t j_t}|$ be the variable flipped at time t . If $i_t \notin \mathcal{Z}_t$, then $y \in \mathcal{A}_t$ and we define $s_t(y) = i_t$. Moreover, we let $s_t(x) = s_{t-1}(x)$ for

all $x \in \mathcal{A}_t \setminus \{y\} \subset \mathcal{A}_{t-1}$. (Note that it is possible that $y \in \mathcal{A}_{t-1}$ as y may have been flipped before.) For $t > T$ we set $s_t = s_{t-1}$.

To verify that s_t has the desired properties, assume that $T \geq t$ and observe that **PII** ensures that Φ_{i_t} was unsatisfied under σ_{t-1} . Thus, $i_t \in \mathcal{D}_{t-1} \subset \mathcal{D}_t$. But as **PII** sets $\sigma_t(y) = 1 - \sigma_{t-1}(y)$, Φ_{i_t} is satisfied under σ_t . Furthermore, for all $x \in \mathcal{A}_t \setminus \{y\}$ we have $\sigma_t(x) = \sigma_{t-1}(x)$, and thus each of these variables contributes a true literal to its clause $\Phi_{s_t(x)} = \Phi_{s_{t-1}(x)}$ by induction. Since s_{t-1} is injective but Φ_{i_t} was unsatisfied under σ_{t-1} , we have $i_t \notin \text{Im}(s_{t-1})$, whence s_t is injective. This establishes (8).

As (8) shows, $S_t = 0$ implies $U_t = 0$, i.e., σ_t satisfies all clauses Φ_i with $i \notin \mathcal{Z}_t$. To complete the proof, we need to show that if $H_t = 0$, then σ_t also satisfies all clauses Φ_i with $i \in \mathcal{Z}_t$. But if $H_t = 0$, then $\sigma_t(x) = \tau_t(x)$ for all $x \in \mathcal{N}_t$, and τ_t is a satisfying assignment of $\Phi_{\mathcal{Z}_t}$. \square

Finally, we have all the pieces in place to prove Theorem 1.2.

Proof of Theorem 1.2 (assuming Propositions 3.4 and 3.6). Proposition 3.7 shows that

$$\mathbb{P}[T \geq T^*] = \mathbb{P}[T \geq T^* \wedge \forall 1 \leq t \leq T^* : S_t + H_t > 0].$$

We are going to bound the probability on the r.h.s. To this end, we work with two random variables S'_t, H'_t that are easier to analyze than the original S_t, H_t . Namely, we let $S'_0 = H'_0 = 0$, and

$$S'_t = S'_{t-1} - \begin{cases} 1 & \text{if } \pi_{t-1}(i_t, j_t) = -1, \\ 0 & \text{otherwise} \end{cases} \quad (t \geq 1).$$

In other words, we let $S'_t = S'_{t-1} - 1$ if the variable flipped at time t had not been flipped before and does not occur in any of the ‘exceptional’ clauses $\Phi_{\mathcal{Z}_{t-1}}$. Otherwise, $S'_t = S'_{t-1}$.

We claim that

$$S_t \leq |\mathcal{D}_t| + k|\mathcal{Z}_t| + S'_t \quad \text{for any } t \geq 0. \quad (9)$$

To see this, recall from (7) that $S_t = |\mathcal{D}_t| - |\mathcal{A}_t|$. By **PI3**, the set \mathcal{A}_t contains all variables $|\Phi_{i_s j_s}|$ such that $\pi_{s-1}(i_s, j_s) = -1$ with $s \leq t$, except the ones that belong to \mathcal{N}_t . Since $|\mathcal{N}_t| \leq k|\mathcal{Z}_t|$, we obtain (9).

Furthermore, we let $H'_0 = 0$ and

$$H'_t = H'_{t-1} + \begin{cases} -1 & \text{if } |\Phi_{i_t j_t}| \in \mathcal{N}_{t-1} \text{ and } \sigma_t(|\Phi_{i_t j_t}|) = \tau_t(|\Phi_{i_t j_t}|), \\ 1 & \text{if } |\Phi_{i_t j_t}| \in \mathcal{N}_{t-1} \text{ and } \sigma_t(|\Phi_{i_t j_t}|) \neq \tau_t(|\Phi_{i_t j_t}|), \\ 0 & \text{otherwise} \end{cases} \quad (t \geq 1).$$

Thus, starting at 0, we decrease the value of H'_t by one if the variable flipped at time t lies in \mathcal{N}_{t-1} and its new value coincides with the ‘ideal’ assignment τ_t , while we increase by one if these values differ.

We claim that

$$H_t \leq k|\mathcal{Z}_t| + H'_t \quad \text{for any } t \geq 0. \quad (10)$$

For $H_0 = H'_0$ and

$$\begin{aligned} H_t - H_{t-1} &= |\{x \in \mathcal{N}_t : \sigma_t(x) \neq \tau_t(x)\}| - |\{x \in \mathcal{N}_{t-1} : \sigma_{t-1}(x) \neq \tau_{t-1}(x)\}| \\ &\leq |\mathcal{N}_t \setminus \mathcal{N}_{t-1}| + H'_t - H'_{t-1} \leq k|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| + H'_t - H'_{t-1} \end{aligned} \quad \text{for any } t \geq 1.$$

Combining (9) and (10) with Propositions 3.4 and 3.6, we see that w.h.p.

$$\begin{aligned} S_t + H_t &\leq |\mathcal{D}_t| + 2k|\mathcal{Z}_t| + S'_t + H'_t \\ &\leq 2^{2-k}m + 2k|\mathcal{Z}_t| + S'_t + H'_t \leq \frac{4\rho n}{k} + 2k\varepsilon n + S'_t + H'_t \end{aligned} \quad \text{for any } t \leq T^*. \quad (11)$$

Hence, we are left to analyze $S'_t + H'_t$.

The sequence $(S'_t + H'_t)_t$ is a super-martingale. More precisely, we claim that with $\gamma = 0.429$ we have

$$\mathbb{E}[S'_t + H'_t | \mathcal{F}_{t-1}] < S'_{t-1} + H'_{t-1} - \gamma \quad \text{for all } t \leq \min\{T, T^*\}. \quad (12)$$

There are two cases to consider.

Case 1: $i_t \notin \mathcal{Z}_{t-1}$. The construction in step **PI2** ensures that there are fewer than λ indices j such that $|\Phi_{i_t j}| \in \mathcal{N}_{t-1}$. Furthermore, **PI2** ensures that there are less than k_1 indices j such that $|\Phi_{i_t j}| \in \mathcal{A}_{t-1}$. Moreover, there is no index j such that $\pi_{t-1}(i_t, j) = 1$, because otherwise clause Φ_{i_t} would have been satisfied under σ_{t-1} . This means that for at least $k - k_1 - \lambda$ indices $j \in [k]$ we have $\pi_{t-1}(i_t, j) = -1$. Therefore, as $j_t \in [k]$ is chosen uniformly at random, with probability at least $1 - (k_1 + \lambda)/k \geq 0.43 - \lambda/k$ we have $S'_t = S'_{t-1} - 1$. In addition, as Φ_{i_t} contains at most λ variables from \mathcal{N}_{t-1} , the probability that $H'_t = H'_{t-1} + 1$ is bounded from above by $\lambda/k < 0.00001$. Thus, (12) holds.

Case 2: $i_t \in \mathcal{Z}_{t-1}$. As the assignment τ_{t-1} is rich, there are at least $0.8k$ indices j such that $\tau_t(\Phi_{i_t j}) = \tau_{t-1}(\Phi_{i_t j}) = 1$. However, for all of these indices j we have $\sigma_{t-1}(\Phi_{i_t j}) = 0$, because Φ_{i_t} is unsatisfied under σ_{t-1} . Hence, the probability that $\tau_t(\Phi_{i_t j_t}) = 1$ and $\sigma_{t-1}(\Phi_{i_t j_t}) = 0$ is at least 0.8 , and if this event indeed occurs then $\sigma_t(\Phi_{i_t j_t}) = \tau_t(\Phi_{i_t j_t}) = 1$. Therefore, $H'_t - H'_{t-1}$ has expectation $\leq -0.8 + 0.2 \leq -0.6$. Moreover, $S'_t \leq S'_{t-1}$ with certainty. This implies (12).

To complete the proof, we are going to apply Azuma's inequality (Lemma 2.2 in Section 2) to the random variable $S'_{T^*} + H'_{T^*}$. The inequality applies because (12) shows that $(S'_t + H'_t)_{t \geq 0}$ is a supermartingale. However, there is a minor technical intricacy: to use the inequality, we need an upper bound on the expectation $\mathbb{E}[S'_{T^*} + H'_{T^*}]$. But as (12) only holds for $t \leq \min\{T, T^*\}$, this would require knowledge of the probability that $T \geq T^*$, the very quantity that we want to estimate.

To circumvent this problem, we define further random variables R_t by letting $R_t = S'_t + H'_t$ for $t \leq \min\{T^*, T\}$ and $R_t = R_{t-1} - \gamma$ for $t > \min\{T^*, T\}$. Then $R_0 = 0$ and $\mathbb{E}[R_t | \mathcal{F}_{t-1}] \leq R_{t-1} - \gamma$ for all $t \geq 0$. Thus, $\mathbb{E}[R_{T^*}] \leq -\gamma T^*$. Recalling the definition (5) of ε , we obtain for $k \geq k_0$ sufficiently large and $\rho \leq \rho_0 = 1/25$ the bound

$$\mathbb{E}[R_{T^*}] \leq -\gamma \cdot T^* \leq -4\rho n/k - 10k\varepsilon n. \quad (13)$$

Furthermore, $|R_t - R_{t-1}| \leq 2$ for all $t \geq 0$ by the definitions of S'_t, H'_t . Therefore, Azuma's inequality and (13) yield

$$\mathbb{P}[R_{T^*} > -4\rho n/k - 2k\varepsilon n] \leq \mathbb{P}\left[R_{T^*} > \mathbb{E}[R_{T^*}] + n^{2/3}\right] \leq \exp\left[-\frac{n^{4/3}}{8T^*}\right] = o(1). \quad (14)$$

Finally, we obtain from (9), (10), and Proposition 3.7

$$\begin{aligned} \mathbb{P}[T > T^*] &\leq \mathbb{P}[\forall t \leq T^* : |\mathcal{D}_t| + 2k|\mathcal{Z}_t| + R_t > 0] \leq \mathbb{P}[|\mathcal{D}_{T^*}| + 2k|\mathcal{Z}_{T^*}| + R_{T^*} > 0] \\ &\leq \mathbb{P}[|\mathcal{D}_{T^*}| + 2k|\mathcal{Z}_{T^*}| > 4\rho n/k + 2k\varepsilon n] + \mathbb{P}[R_{T^*} > -4\rho n/k - 2k\varepsilon n] \stackrel{(11), (14)}{=} o(1), \end{aligned}$$

thereby completing the proof. \square

Our remaining task is to establish Propositions 3.4 and 3.6. From a formal point of view, we should start with Proposition 3.6 because the proof of Proposition 3.4 depends on it. However, the argument that is used in the proof of Proposition 3.4 is conceptually similar to but technically far simpler than the one that we use to prove Proposition 3.6. Hence, for didactical reasons we will start with the proof of Proposition 3.4 in Section 4 and postpone the proof of Proposition 3.6 to Section 5.

4 Proof of Proposition 3.4

In this section we keep the notation and the assumptions from Proposition 3.4.

Our goal is to bound the number $|\mathcal{D}_{T^*}|$ of $\mathcal{A}_{T^*} \cup \mathcal{N}_{T^*}$ -negative clauses Φ_i , i.e., clauses whose positive literals all belong to $\mathcal{A}_{T^*} \cup \mathcal{N}_{T^*}$. Thus, we need to study how the process **PI0-PI3** 'hits' the positions $(i, j) \in [m] \times [k]$ that represent positive literals by adding their underlying variable to $\mathcal{A}_{T^*} \cup \mathcal{N}_{T^*}$. To this end, we consider the two random variables

$$K_t^*(i, j) = \begin{cases} 1 & \text{if } \pi_{t-1}(i, j) = 1 \text{ and } \Phi_{ij} \in \mathcal{A}_t, \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

$$K_t^0(i, j) = \begin{cases} 1 & \text{if } \pi_{t-1}(i, j) = 1 \text{ and } \Phi_{ij} \in \mathcal{N}_t, \\ 0 & \text{otherwise,} \end{cases} \quad (16)$$

for any $(i, j) \in [m] \times [k]$ and $t \geq 1$. Recall that $\pi_{t-1}(i, j) = \text{sign}(\Phi_{ij})$ iff Φ_{ij} is a literal such that $|\Phi_{ij}| \notin \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ (cf. **PI3**). To simplify the notation, we define for a set $\mathcal{I} \subset [m] \times [k]$

$$K_t^*(\mathcal{I}) = \prod_{(i,j) \in \mathcal{I}} K_t^*(i, j), \quad K_t^0(\mathcal{I}) = \prod_{(i,j) \in \mathcal{I}} K_t^0(i, j).$$

If $\mathcal{I}^*, \mathcal{I}^0 \subset [m] \times [k]$ are both non-empty, then

$$K_t^*(\mathcal{I}^*) \cdot K_t^0(\mathcal{I}^0) = 0. \quad (17)$$

Indeed, suppose that $K_t^0(\mathcal{I}^0) \neq 0$. Then **PI2** must have added at least one clause to \mathcal{Z}_t . But the construction in **PI2** ensures that the first clause that gets added to \mathcal{Z}_t contains the variable $|\Phi_{i_t j_t}|$ flipped at time t . Thus, $\mathcal{A}_t \subset \mathcal{A}_{t-1}$ by **PI3**, and thus there cannot be a pair (i, j) with $K_t^*(i, j) = 1$. In effect, $K_t^*(\mathcal{I}^*) = 0$.

Lemma 4.1 *Let $t \geq 1$ and $\emptyset \neq \mathcal{I}^* \subset [m] \times [k]$. Let $\mathcal{E}_t^*(\mathcal{I}^*)$ be the event that $|\Phi_{ij}| = |\Phi_{i_t j_t}| \notin \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ for all $(i, j) \in \mathcal{I}^*$, and that $(i_t, j_t) \notin \mathcal{I}^*$. Then*

$$\mathbb{P}[\mathcal{E}_t^*(\mathcal{I}^*) | \mathcal{F}_{t-1}] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|}. \quad (18)$$

Proof. Since clause Φ_{i_t} is unsatisfied under σ_{t-1} , Φ_{i_t} is $\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ -negative and thus $\pi_{t-1}(i_t, j_t) \neq 1$. Hence, **PI3** ensures that either $|\Phi_{i_t j_t}| \in \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ or $\pi_{t-1}(i_t, j_t) = -1$. If $\mathcal{E}_t^*(\mathcal{I}^*)$ occurs, then $|\Phi_{i_t j_t}| \notin \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ and thus $\pi_{t-1}(i_t, j_t) = -1$. Furthermore, if \mathcal{I}^* occurs, then $|\Phi_{ij}| \notin \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ for all $(i, j) \in \mathcal{I}^*$, and thus $\pi_{t-1}(i, j) \in \{-1, 1\}$ by **PI3**. Thus, by Fact 3.3 $|\Phi_{i_t j_t}|$ and $|\Phi_{ij}|$ with $(i, j) \in \mathcal{I}^*$ are independently uniformly distributed over $V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})$. Therefore,

$$\mathbb{P}[\mathcal{E}_t^*(\mathcal{I}^*) | \mathcal{F}_{t-1}] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|},$$

as claimed. \square

Corollary 4.2 *For any $t \geq 1$, $\mathcal{I}^* \subset [m] \times [k]$ we have*

$$\mathbb{E}[K_t^*(\mathcal{I}^*) | \mathcal{F}_{t-1}] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|}.$$

Proof. If $\prod_{(i,j) \in \mathcal{I}^*} K_t^*(i, j) = 1$, then the event $\mathcal{E}_t^*(\mathcal{I}^*)$ occurs. Hence, Lemma 4.1 implies that

$$\mathbb{E} \left[\prod_{(i,j) \in \mathcal{I}^*} K_t^*(i, j) | \mathcal{F}_{t-1} \right] \leq \mathbb{P}[\mathcal{E}_t^*(\mathcal{I}^*) | \mathcal{F}_{t-1}] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|}, \quad (19)$$

as claimed. \square

Lemma 4.3 *For any $t \geq 1$, $\delta_t \geq 0$ and $\mathcal{I}^0 \subset [m] \times [k]$ we have*

$$\mathbb{E}[K_t^0(\mathcal{I}^0) \cdot \mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\} | \mathcal{F}_{t-1}] \leq \left(\frac{k\delta_t}{\max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t\}} \right)^{|\mathcal{I}^0|}.$$

Proof. We may assume that $\mathcal{I}^0 \neq \emptyset$. We may also assume that $\pi_{t-1}(i, j) = 1$ for all $(i, j) \in \mathcal{I}^0$ as otherwise $K_t^0(\mathcal{I}^0) = 0$. We are going to work with the conditional distribution

$$p[\cdot] = \mathbb{P}[\cdot | \mathcal{F}_{t-1}].$$

Let \mathcal{E}^0 be the event that $K_t^0(\mathcal{I}^0) = 1$ and $|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t$. Then our goal is to estimate $p[\mathcal{E}^0]$.

If the event \mathcal{E}^0 occurs, then $\pi_{t-1}(i_t, j_t) = -1$ and $|\Phi_{i_t j_t}| \in \mathcal{N}_t$. Indeed, being unsatisfied under the assignment σ_{t-1} , clause Φ_{i_t} is $\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ -negative, and thus $\pi_{t-1}(i_t, j_t) \neq 1$. Furthermore, if $\pi_{t-1}(i_t, j_t) = \Phi_{i_t j_t}$, then $|\Phi_{i_t j_t}| \in \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ by **PI3**, and thus $\mathcal{Z}_t = \mathcal{Z}_{t-1}$ and $\mathcal{N}_t = \mathcal{N}_{t-1}$ by the construction in step **PI2**. But if $\mathcal{N}_t = \mathcal{N}_{t-1}$, then $K_t^0(\mathcal{I}^0) = 0$ by definition.

Thus, assume that $\pi_{t-1}(i_t, j_t) = -1$ and $|\Phi_{i_t j_t}| \in \mathcal{N}_t$. We need to trace the process described in **PI2** that enhances the sets \mathcal{N}_t and \mathcal{Z}_t . This process may add a sequence of clause indices to the set \mathcal{Z}_t and the variables that occur in these clauses to \mathcal{N}_t . As these variables get added to the set \mathcal{N}_t one by one, we will study the probability that they occur in one of the positions $(i, j) \in \mathcal{I}^0$. The first clause that **PI2** adds to \mathcal{Z}_t necessarily contains the newly flipped variable $|\Phi_{i_t j_t}|$, and thus we may assume that this is the first variable that gets added to \mathcal{N}_t . In addition, if $|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t$, **PI2** may add up to $k\delta_t - 1$ further variables to \mathcal{N}_t . To track this process, we need a bit of notation.

Let s_1, \dots, s_y be the clause indices that **PI2** adds to \mathcal{Z}_t , in the order in which they get added by the process. Let $y^* = \min\{y, \delta_t\}$. For each $1 \leq i \leq y^*$ let $1 \leq j_{i,1} < \dots < j_{i,l_i} \leq k$ be the unique sequence of indices such that $\pi_{t-1}(s_i, j_{i,q}) = -1$ and

$$|\Phi_{s_i j_{i,q}}| \notin \{|\Phi_{i_t j_t}|\} \cup \mathcal{N}_{t-1} \cup \bigcup_{h=1}^{i-1} N(\Phi_{s_h}) \cup \{|\Phi_{s_i j_{i,u}}| : u < q\} \text{ for all } q \leq l_i.$$

This means that $\{|\Phi_{s_i j_{i,q}}| : 1 \leq q \leq l_i\}$ are the new variables that Φ_{s_i} contributes to \mathcal{N}_t and that did not belong to \mathcal{A}_{t-1} already. Let $\xi_0 = |\Phi_{i_t j_t}|$ and let ξ_1, \dots, ξ_L be the sequence of variables $|\Phi_{s_i j_{i,q}}|$ with $q = 1, \dots, l_i$ and $i = 1, \dots, y^*$. Hence, ξ_0, \dots, ξ_L is the sequence of variables not in \mathcal{A}_{t-1} that **PI2** adds to \mathcal{N}_t , in the order in which the process adds these variables to \mathcal{N}_t . By our choice of y^* , the total number of these variables satisfies

$$L + 1 \leq ky^* \leq k\delta_t.$$

Of course, L and ξ_0, \dots, ξ_L are random variables.

If \mathcal{E}^0 occurs, then each of the variables Φ_{ij} with $(i, j) \in \mathcal{I}^0$ occurs in the sequence ξ_0, \dots, ξ_L . Hence, there exists a map $f : \mathcal{I}^0 \rightarrow \{0, 1, \dots, k\delta_t - 1\}$ such that $f(i, j) \leq L$ and $\Phi_{ij} = \xi_{f(i,j)}$ for all $(i, j) \in \mathcal{I}^0$. For a given f let $\mathcal{E}^0(f)$ denote this event. Then by the union bound,

$$p[\mathcal{E}^0] \leq \sum_{f: \mathcal{I}^0 \rightarrow \{0, 1, \dots, k\delta_t - 1\}} p[\mathcal{E}^0(f)] \leq (k\delta_t)^{|\mathcal{I}^0|} \max_{f: \mathcal{I}^0 \rightarrow \{0, 1, \dots, k\delta_t - 1\}} p[\mathcal{E}^0(f)]. \quad (20)$$

We claim that

$$p[\mathcal{E}^0(f)] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t\}^{-|\mathcal{I}^0|} \quad (21)$$

for any f . To prove (21), let $\mathcal{I}_l^0 = f^{-1}(l)$ be the set of positions $(i, j) \in \mathcal{I}^0$ where the variable ξ_l occurs ($0 \leq l \leq L$). Moreover, let $\mathcal{E}_l^0(f)$ be the event that

- a. $\Phi_{ij} = \xi_l$ for all $(i, j) \in \mathcal{I}_l^0$, and
- b. $\Phi_{ij} \neq \xi_l$ for all $(i, j) \in \mathcal{I}^0 \setminus \mathcal{I}_l^0$.

As $\pi_{t-1}(i, j) = 1$ for all $(i, j) \in \mathcal{I}_l^0$, given \mathcal{F}_{t-1} the variables Φ_{ij} with $(i, j) \in \mathcal{I}_l^0$ are independently uniformly distributed over $V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})$ by Fact 3.3. Hence, given the event $\bigcap_{\nu < l} \mathcal{E}_\nu^0(f)$, the variables $|\Phi_{ij}|$ with $(i, j) \in \mathcal{I}_l^0$ are uniformly distributed over the set $V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1} \cup \{\xi_0, \dots, \xi_{l-1}\})$ (for if $\mathcal{E}_\nu^0(f)$ occurs for some $\nu < l$, then $\Phi_{ij} \neq \xi_\nu$ for all $(i, j) \in \mathcal{I}_l^0$). Therefore, we obtain

$$p\left[\mathcal{E}_l^0(f) \mid \bigcap_{\nu < l} \mathcal{E}_\nu^0(f)\right] \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - l + 1\}^{-|\mathcal{I}_l^0|} \quad \text{for any } 0 \leq l \leq L.$$

Multiplying these conditional probabilities up for $0 \leq l \leq L < k\delta_t$, we obtain (21). Finally, combining (18), (20), and (21) completes the proof. \square

Corollary 4.4 *For any $t \geq 1$, $\delta_t \geq 0$ and $\mathcal{I}^*, \mathcal{I}^0 \subset [m] \times [k]$ we have*

$$\begin{aligned} & \mathbb{E}[K_t^*(\mathcal{I}^*)K_t^0(\mathcal{I}^0)\mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\} | \mathcal{F}_{t-1}] \\ & \leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|} \cdot \left(\frac{k\delta_t}{\max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t\}}\right)^{|\mathcal{I}^0|}. \end{aligned}$$

Proof. This is immediate from (17) and Corollary 4.2 and Lemma 4.3. \square

Why does the bound provided by Corollary 4.4 “make sense”? First, observe that the only reason we need to take the max of the respective expression and one is because a priori it could happen that, e.g., $V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}) = \emptyset$. Apart from this issue, the first factor basically comes from the fact that for each pair (i, j) with $\pi_{t-1}(i, j) = 1$ the variable Φ_{ij} is uniformly distributed over $V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})$. Hence, it seems reasonable that the probability that one such Φ_{ij} equals the variable flipped at time t is $1/|V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|$, and that these events occur independently. With respect to the second factor, a similar intuition applies. Due to the $\mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\}$ factor on the left hand side, at most $k\delta_t$ variables are added to \mathcal{N}_t that were not already in \mathcal{N}_{t-1} . Hence, for each Φ_{ij} with $\pi_{t-1}(i, j) = 1$ there are now $k\delta_t$ “good” cases that would make $K_t^0(i, j) = 1$. Moreover, as we reveal the $k\delta_t$ variables, there remain at least $|V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t$ “possible” cases. We will now establish the following.

Proposition 4.5 *W.h.p. we have either $\mathcal{Z}_{T^*} > \varepsilon n$ or $|\mathcal{D}_{T^*}| \leq 2^{2-k}m$.*

Proof. Let \mathcal{E} be the event that $|\mathcal{Z}_{T^*}| \leq \varepsilon n$ but $|\mathcal{D}_{T^*}| > 2^{2-k}m$. Our goal is to show that $\mathbb{P}[\mathcal{E}] = o(1)$. To this end, we will decompose \mathcal{E} into various ‘sub-events’ that are sufficiently detailed for us to bound their probabilities via Corollary 4.4. In order to bound the probability of \mathcal{E} we will then use the union bound.

As a first step, we need to decompose \mathcal{E} according to the sequence $(|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}|)_{t \geq 1}$ of increments of the sets \mathcal{Z}_t . More precisely, let Δ be the set of all sequences $\delta = (\delta_t)_{1 \leq t \leq T^*}$ of non-negative integers with $\sum_{t=1}^{T^*} \delta_t \leq \varepsilon n$. Let $\mathcal{E}(\delta)$ be the event that $|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t$ for all $1 \leq t \leq T^*$ and $|\mathcal{D}_{T^*}| > 2^{2-k}m$. If the event \mathcal{E} occurs, then there is a sequence δ such that the event $\mathcal{E}(\delta)$ occurs. Hence, by the union bound

$$\mathbb{P}[\mathcal{E}] \leq \sum_{\delta \in \Delta} \mathbb{P}[\mathcal{E}(\delta)] \leq |\Delta| \cdot \max_{\delta \in \Delta} \mathbb{P}[\mathcal{E}(\delta)].$$

As it is well known that $|\Delta| = \binom{\varepsilon n + T^* - 1}{T^* - 1} \leq \binom{\varepsilon n + T^*}{\varepsilon n}$, we obtain

$$\mathbb{P}[\mathcal{E}] \leq \binom{\varepsilon n + T^*}{\varepsilon n} \max_{\delta \in \Delta} \mathbb{P}[\mathcal{E}(\delta)]. \quad (22)$$

Fixing any sequence $\delta \in \Delta$, we now decompose the event $\mathcal{E}(\delta)$ further according to the precise set M of clauses that end up in \mathcal{D}_{T^*} , and according to the precise ‘reason’ why each clause $i \in M$ belongs to \mathcal{D}_{T^*} . More precisely, let $M \subset [m]$ be a set of size $\mu = 2^{2-k}m$. Moreover, for disjoint $Q^*, Q^0 \subset M \times [k]$ let $\mathcal{E}_0(Q^*, Q^0)$ be the event that

$$\pi_0(i, j) = 1 \text{ for } (i, j) \in Q^* \cup Q^0, \text{ while } \pi_0(i, j) = -1 \text{ for } (i, j) \in M \times [k] \setminus (Q^* \cup Q^0).$$

Furthermore, for maps $\tau^* : Q^* \rightarrow [T^*]$, $\tau^0 : Q^0 \rightarrow [T^*]$ let $\mathcal{E}(\delta, \tau^*, \tau^0)$ be the event that $|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t$ for all $1 \leq t \leq T^*$ and

$$\begin{aligned} \pi_{\tau^*(i,j)-1}(i, j) &= 1 \quad \text{while} \quad \Phi_{ij} \in \mathcal{A}_{\tau^*(i,j)} \text{ for all } (i, j) \in Q^*, \\ \pi_{\tau^0(i,j)-1}(i, j) &= 1 \quad \text{while} \quad \Phi_{ij} \in \mathcal{N}_{\tau^0(i,j)} \text{ for all } (i, j) \in Q^0. \end{aligned}$$

If the event $\mathcal{E}(\delta)$ occurs, then there exist Q^*, Q^0 and τ^*, τ^0 such that the events $\mathcal{E}_0(Q^*, Q^0)$ and $\mathcal{E}(\delta, \tau^*, \tau^0)$ occur. In fact, if $\mathcal{E}(\delta)$ occurs, then $|\mathcal{D}_{T^*}| \geq \mu$. Thus, select a subset $M \subset \mathcal{D}_{T^*}$ of size μ . By the definition of \mathcal{D}_{T^*} , each $i \in M$ is $\mathcal{A}_{T^*} \cup \mathcal{N}_{T^*}$ -negative. Thus, for any $j \in [k]$ such that Φ_{ij} is a positive literal there is a time $1 \leq t = t(i, j) \leq T^*$ such that $\pi_{t-1}(i, j) = 1$ but $\pi_t(i, j) \in \mathcal{A}_t \cup \mathcal{N}_t$. If $\pi_t(i, j) \in \mathcal{A}_{t(i,j)}$, then include (i, j) in Q^* and set $\tau^*(i, j) = t$. Otherwise, add (i, j) to Q^0 and let $\tau^0(i, j) = t$. Then indeed both $\mathcal{E}_0(Q^*, Q^0)$ and $\mathcal{E}(\delta, \tau^*, \tau^0)$ occur. Thus, by the union bound,

$$\mathbb{P}[\mathcal{E}(\delta)] \leq \sum_{Q^*, Q^0, \tau^*, \tau^0} \mathbb{P}[\mathcal{E}_0(Q^*, Q^0) \cap \mathcal{E}(\delta, \tau^*, \tau^0)]. \quad (23)$$

The event $\mathcal{E}_0(Q^*, Q^0)$ depends only on the signs of the literals and is therefore \mathcal{F}_0 -measurable. Furthermore, as signs of the literals Φ_{ij} are mutually independent, we get

$$\mathbb{P}[\mathcal{E}_0(Q^*, Q^0)] = 2^{-k\mu}.$$

Therefore, (23) yields

$$\mathbb{P}[\mathcal{E}(\delta)] \leq 2^{-k\mu} \sum_{Q^*, Q^0, \tau^*, \tau^0} \mathbb{P}[\mathcal{E}(\delta, \tau^*, \tau^0) | \mathcal{F}_0]. \quad (24)$$

Thus, we are left to estimate $\mathbb{P}[\mathcal{E}(\delta, \tau^*, \tau^0) | \mathcal{F}_0]$.

We defined the random variables $K_t^*(\cdot, \cdot)$, $K_t^0(\cdot, \cdot)$ so that if the event $\mathcal{E}(\delta, \tau^*, \tau^0)$ occurs, then

$$\prod_{(i,j) \in Q^*} K_{\tau^*(i,j)}(i, j) \cdot \prod_{(i,j) \in Q^0} K_{\tau^0(i,j)}(i, j) \cdot \prod_{t=1}^{T^*} \mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\} = 1.$$

In order to apply Corollary 4.4 to the above expression, we are going to reorder the product according to the time parameter. More precisely, let $Q_t^* = \tau^{*-1}(t)$ and $Q_t^0 = \tau^{0-1}(t)$. Then

$$\begin{aligned} \mathbb{P}[\mathcal{E}(\delta, \tau^*, \tau^0) | \mathcal{F}_0] &\leq \mathbb{E} \left[\prod_{(i,j) \in Q^*} K_{\tau^*(i,j)}(i, j) \prod_{(i,j) \in Q^0} K_{\tau^0(i,j)}(i, j) \prod_{t=1}^{T^*} \mathbf{1}\{|\mathcal{Z}_t - \mathcal{Z}_{t-1}| \leq \delta_t\} = 1 | \mathcal{F}_0 \right] \\ &= \mathbb{E} \left[\prod_{t=1}^{T^*} K_t(Q_t^*) K_t(Q_t^0) \cdot \mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\} = 1 | \mathcal{F}_0 \right]. \end{aligned}$$

If $|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t$ for all $t \leq T^*$, then $|\mathcal{N}_{t-1}| + k\delta_t \leq k \sum_{s \leq t} \delta_s \leq k\varepsilon n$ for all $t \leq T^*$. Furthermore, $|\mathcal{A}_t| \leq t \leq T^* = \frac{n}{k}$ for all $t \geq 0$. Hence, $|V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t \geq n(1 - k\varepsilon - 1/k) \geq n/1.01$ for all $t \leq T^*$, provided that $k \geq k_0$ is large enough. Thus, Corollary 4.4 entails in combination with Lemma 2.1

$$\mathbb{P}[\mathcal{E}(\delta, \tau^*, \tau^0) | \mathcal{F}_0] = \left(\frac{1.01}{n}\right)^{|Q^*|} \cdot \prod_{(i,j) \in Q^0} \frac{1.01k\delta_{\tau^0(i,j)}}{n}. \quad (25)$$

For any $M \subset [m]$ of size μ and any two disjoint $Q^*, Q^0 \subset M \times [k]$ let

$$S(M, Q^*, Q^0) = \sum_{\tau^*, \tau^0} \left(\frac{1.01}{n}\right)^{|Q^*|} \cdot \prod_{(i,j) \in Q^0} \frac{1.01k\delta_{\tau^0(i,j)}}{n},$$

with the sum ranging over all maps $\tau^* : Q^* \rightarrow [T^*]$, $\tau^0 : Q^0 \rightarrow [T^*]$. Recall that $\theta = T^*/n$. As $\sum_{t \leq T^*} \delta_t \leq \varepsilon n$, we obtain

$$\begin{aligned} S(M, Q^*, Q^0) &\leq \left(\frac{1.01T^*}{n}\right)^{|Q^*|} \left(\frac{1.01k}{n}\right)^{|Q^0|} \sum_{\tau^0} \prod_{(i,j) \in Q^0} \delta_{\tau^0(i,j)} \\ &= \left(\frac{1.01T^*}{n}\right)^{|Q^*|} \left(\frac{1.01k}{n}\right)^{|Q^0|} \left(\sum_{t=1}^{T^*} \delta_t\right)^{|Q^0|} \leq (1.01\theta)^{|Q^*|} (1.01\varepsilon k)^{|Q^0|}. \end{aligned} \quad (26)$$

Combining (24), (25), and (26), we thus get for any $\delta \in \Delta$

$$\begin{aligned} \mathbb{P}[\mathcal{E}(\delta)] &\leq 2^{-k\mu} \sum_{M \subset [m]: |M|=\mu} \sum_{Q^*, Q^0 \subset M \times [k]: Q^* \cap Q^0 = \emptyset} S(M, Q^*, Q^0) \\ &\leq 2^{-k\mu} \binom{m}{\mu} \sum_{q^*, q^0: q^* + q^0 \leq k\mu} \sum_{Q^*, Q^0: |Q^*|=q^*, |Q^0|=q^0} (1.01\theta)^{q^*} (1.01\varepsilon k)^{q^0} \\ &\leq 2^{-k\mu} \binom{m}{\mu} \sum_{q^*, q^0: q^* + q^0 \leq k\mu} \binom{k\mu}{q^*, q^0, k\mu - q^* - q^0} (1.01\theta)^{q^*} (1.01k\varepsilon)^{q^0} \\ &\leq \binom{m}{\mu} \left(\frac{1 + 1.01(\theta + k\varepsilon)}{2}\right)^{k\mu} \leq \left[\frac{em}{\mu} \cdot \left(\frac{1 + 1.01(\theta + k\varepsilon)}{2}\right)^k\right]^\mu \\ &\leq \left[e2^{k-2} \cdot \left(\frac{1 + 1.01(\theta + k\varepsilon)}{2}\right)^k\right]^\mu \leq 0.999^\mu, \end{aligned} \quad (27)$$

provided that $k \geq k_0$ is sufficiently big. Finally, combining (22) and (27), we obtain

$$\mathbb{P}[\mathcal{E}] \leq \binom{\varepsilon n + T^*}{\varepsilon n} 0.999^\mu \leq \left(\frac{e(\varepsilon n + \theta n)}{\varepsilon n} \right)^{\varepsilon n} 0.999^\mu \leq (e(1 + \theta/\varepsilon))^{\varepsilon n} 0.999^\mu. \quad (28)$$

By our assumption that $\rho \geq k^{-3}$ (cf. the first paragraph in Section 3), we have $\mu = 2^{2-k}m \geq \rho n/k \geq k^{-4}n$. Hence, recalling that $\theta \leq 1/k$ and $\varepsilon = \exp(-k^{2/3})$ (cf. (5)), we obtain from (28)

$$\mathbb{P}[\mathcal{E}] \leq \exp[n(\varepsilon \ln(2e/\varepsilon) - k^{-4})] \leq \exp\left[n\left(k \exp(-k^{2/3}) + k^{-4} \ln 0.999\right)\right] = \exp(-\Omega(n)) = o(1),$$

provided that $k \geq k_0$ is sufficiently large. \square

Finally, Proposition 3.4 is immediate from Propositions 3.6 and 4.5.

5 Proof of Proposition 3.6

Throughout this section we keep the notation and the assumptions of Proposition 3.6.

5.1 Outline

The goal in this section is to bound the size of the set \mathcal{Z}_{T^*} . There are two reasons why step **PI2** may add a clause index $i \in [m]$ to the set \mathcal{Z}_t for some $1 \leq t \leq T^*$. First, the clause Φ_i may feature at least k_1 variables from the set $\mathcal{A}_{t-1} \cup \{|\Phi_{i_t j_t}|\}$, i.e., variables that have been flipped at least once. Second, Φ_i may contain at least λ variables that also occur in clauses that were added to \mathcal{Z}_t previously. The key issue is to deal with the first case. Once that is done, we can bound the number of clauses that get included for the second reason via Lemma 2.4, i.e., via the expansion properties of the random formula.

Thus, we need to investigate how a clause Φ_i comes to contain a lot of variables from $\mathcal{A}_{t-1} \cup \{|\Phi_{i_t j_t}|\}$ for some $t \leq T^*$. There are two ways in which this may occur. First, `Walksat` may have tried to satisfy Φ_i ‘actively’ several times, i.e., $i_s = i$ for several $s \leq t$. Second, Φ_i may contain several of the variables $|\Phi_{i_s j_s}|$ flipped at times $s < t$ ‘accidentally’, i.e., without `Walksat` trying to actively satisfy i . More precisely, for any $t \geq 0$ we call a pair $(i, j) \in [m] \times [k]$

- *t-active* if there is $1 \leq s \leq t$ such that $(i, j) = (i_s, j_s)$ and $\pi_{s-1}(i, j) = -1$.
- *t-passive* if there is $1 \leq s \leq t$ such that $(i, j) \neq (i_s, j_s)$ but $|\Phi_{ij}| = |\Phi_{i_s j_s}|$ and $\pi_{s-1}(i, j) \in \{-1, 1\}$.

Furthermore, we say that $i \in [m]$ is *t-active* if there are $k_2 = k_1 - 10^{-6}k$ indices j such that (i, j) is *t-active*. Similarly, we say that i is *t-passive* if there are $k_3 = 10^{-6}k$ indices j such that (i, j) is *t-passive*. These definitions ensure that any $i \in [m]$ for which there are at least k_1 indices $j \in [k]$ such that $|\Phi_{ij}| \in \mathcal{A}_{t-1} \cup \{|\Phi_{i_t j_t}|\}$ is either *t-active* or *t-passive*.

To prove Proposition 3.6, we will deal separately with *t-active* and *t-passive* clauses. Let A_t be the number of *t-active* clauses, and let P_t be the number of *t-passive* clauses.

Lemma 5.1 *For any $1 \leq t \leq T^*$ we have $\mathbb{P}[A_t < \varepsilon n/4 \vee |\mathcal{Z}_t| > \varepsilon n] \geq 1 - 1/n^2$.*

We defer the proof of Lemma 5.1 to Section 5.2.

Lemma 5.2 *For any $1 \leq t \leq T^*$ we have $\mathbb{P}[P_t < \varepsilon n/4 \vee |\mathcal{Z}_{t-1}| > \varepsilon n] \geq 1 - 1/n^2$.*

Proof. As in the proof of Proposition 4.5, we are going to break the event of interest, i.e.,

$$\mathcal{E} = \{P_t \geq \varepsilon n/4 \wedge |\mathcal{Z}_{t-1}| \leq \varepsilon n\},$$

down into sub-events whose probabilities can be estimated via Lemma 4.1. Then we will use the union bound to estimate the probability of \mathcal{E} .

For a set $M \subset [m]$ of $\mu = \varepsilon n/4$ clause indices let $\mathcal{E}(M)$ be the event that $|\mathcal{Z}_{t-1}| \leq \varepsilon n$ and all $i \in M$ are t -passive. If \mathcal{E} occurs, then there is a set M such that the event $\mathcal{E}(M)$ occurs. Hence, by the union bound

$$\mathrm{P}[\mathcal{E}] \leq \sum_{M \subset [m]: |M|=\mu} \mathrm{P}[\mathcal{E}(M)] \leq \binom{m}{\mu} \max_M \mathrm{P}[\mathcal{E}(M)]. \quad (29)$$

Thus, fix a set $M \subset [m]$ of size μ . Let $Q \subset M \times [k]$ be a set such that for each $i \in M$ there are precisely k_3 indices $j \in [k]$ such that $(i, j) \in Q$. Let $\mathcal{E}(M, Q)$ be the event that $|\mathcal{Z}_{t-1}| \leq \varepsilon n$ and all pairs $(i, j) \in Q$ are t -passive. If the event $\mathcal{E}(M)$ occurs, then there exists a set Q such that $\mathcal{E}(M, Q)$ occurs. Therefore, again by the union bound

$$\mathrm{P}[\mathcal{E}(M)] \leq \sum_Q \mathrm{P}[\mathcal{E}(M, Q)] \leq \binom{k}{k_3}^\mu \max_Q \mathrm{P}[\mathcal{E}(M, Q)]. \quad (30)$$

For a map $\tau : Q \rightarrow [t]$ let $\mathcal{E}(M, Q, \tau)$ be the event that $|\mathcal{Z}_{t-1}| \leq \varepsilon n$ and

$$\tau(i, j) = \min \{s \in [t] : (i, j) \text{ is } s\text{-passive}\} \text{ for all } (i, j) \in Q.$$

If the event $\mathcal{E}(M, Q)$ occurs, then there is a map τ such that the event $\mathcal{E}(M, Q, \tau)$ occurs. Consequently, for any M, Q we have

$$\mathrm{P}[\mathcal{E}(M, Q)] \leq \sum_\tau \mathrm{P}[\mathcal{E}(M, Q, \tau)] \leq t^{|Q|} \max_\tau \mathrm{P}[\mathcal{E}(M, Q, \tau)]. \quad (31)$$

Combining (29), (30), and (31), we see that

$$\mathrm{P}[\mathcal{E}] \leq \binom{m}{\mu} \binom{k}{k_3}^\mu t^{k_3 \mu} \max_{M, Q, \tau} \mathrm{P}[\mathcal{E}(M, Q, \tau)]. \quad (32)$$

Hence, fix any M, Q, τ . Let $Q_s = \tau^{-1}(s)$ for any $1 \leq s \leq t$, and let $\mathcal{E}_s^*(Q_s)$ be the event that $|\Phi_{ij}| = |\Phi_{i_t j_t}| \notin \mathcal{A}_{t-1} \cup \mathcal{N}_{t-1}$ for all $(i, j) \in Q_s$, and $(i_t, j_t) \notin Q_s$. If $\mathcal{E}(M, Q, \tau)$ occurs, then the events $\mathcal{E}_s^*(Q_s)$ occur for all $1 \leq s \leq t$. Moreover, the construction **PI0–PI3** ensures that $|\mathcal{A}_s| \leq s$, and that $|\mathcal{N}_{s-1}| \leq k|\mathcal{Z}_{s-1}| \leq k\varepsilon n$ for all $1 \leq s \leq t$. Therefore, Lemma 4.1 implies

$$\mathrm{P}[\mathcal{E}(M, Q, \tau)] \leq \mathrm{P}\left[\bigcap_{s=1}^t \mathcal{E}_s^*(Q_s) \cap \{|\mathcal{N}_{s-1}| \leq k\varepsilon n\}\right] \leq \prod_{s=1}^t \max\{1, n - s + 1 - k\varepsilon n\}^{-|Q_s|} \quad (33)$$

As $s \leq t \leq T^* \leq n/k$, $\varepsilon = \exp(-k^2/3)$, and because we are assuming that $k \geq k_0$ is sufficiently large, we have $n - s + 1 - k\varepsilon n \geq n/1.001$. Hence, (33) yields

$$\mathrm{P}[\mathcal{E}(M, Q, \tau)] \leq \prod_{s=1}^t \max\{1, n - s + 1 - k\varepsilon n\}^{-|Q_s|} \leq (1.001/n)^{\mu k_3}. \quad (34)$$

Finally, combining (32) and (34) and recalling that $\theta = T^*/n$, we get

$$\mathrm{P}[\mathcal{E}] \leq \binom{m}{\mu} \binom{k}{k_3}^\mu t^{k_3 \mu} (1.001/n)^{\mu k_3} \leq \left[\frac{em}{\mu} \cdot \left(\frac{1.001ek\theta}{k_3} \right)^{k_3} \right]^\mu \leq \left[\frac{4e2^k \rho}{\varepsilon k} \left(\frac{1.001ek\theta}{k_3} \right)^{k_3} \right]^\mu.$$

By our choice of θ we have $1.001ek\theta \leq 10$. Hence, we obtain for $k \geq k_0$ large enough

$$\mathrm{P}[\mathcal{E}] \leq \left[\frac{4e2^k \rho}{\varepsilon k} k_3^{-k_3/2} \right]^\mu \leq \exp(-\mu) = o(1),$$

thereby completing the proof. \square

Proof of Proposition 3.6. In order to bound $|\mathcal{Z}_t|$ for $0 \leq t \leq T^*$, we are going to consider a superset $\mathcal{Y}_t \supset \mathcal{Z}_t$ whose size is easier to estimate. To define \mathcal{Y}_t , we let \mathcal{Y}_t^* be the set of all i that are either t -active or t -passive. Now, \mathcal{Y}_t is the outcome of the following process.

Initially, let $\mathcal{Y}_t = \mathcal{Y}_t^*$.

While there is a clause $i \in [m] \setminus \mathcal{Y}_t$ such that $|\{j \in [k] : |\Phi_{ij}| \in N(\Phi_{\mathcal{Y}_t})\}| \geq \lambda$, add i to \mathcal{Y}_t .

Comparing the above process with the construction in **PI2**, we see that indeed

$$\mathcal{Y}_t \supset \mathcal{Z}_t. \quad (35)$$

Also note that $\mathcal{Y}_t \supset \mathcal{Y}_{t-1}$ for all $t \geq 1$.

To bound $|\mathcal{Y}_t|$, we proceed by induction on t . Let Y_t be the event that either the random formula Φ violates the property (3), or $|\mathcal{Y}_t| > \varepsilon n$. We claim that $\mathbb{P}[Y_0] = o(1)$ and that

$$\mathbb{P}[Y_t] \leq \mathbb{P}[Y_{t-1}] + 2n^{-2} \quad \text{for all } 1 \leq t \leq T^*. \quad (36)$$

Since trivially $\mathcal{Y}_0 = \emptyset$, Y_0 is simply the event that Φ violates (3). Hence, Lemma 2.4 shows directly that

$$\mathbb{P}[Y_0] = o(1). \quad (37)$$

Now, consider some $1 \leq t \leq T^*$. Lemmas 5.1 (applied to $t-1$) and Lemma 5.2 (applied to t) show that

$$\mathbb{P}[A_t + P_t \leq \varepsilon n/2 \vee |\mathcal{Z}_{t-1}| > \varepsilon n] \geq 1 - 2/n^2.$$

Furthermore, if Y_{t-1} does not occur, then we know that $|\mathcal{Z}_{t-1}| \leq |\mathcal{Y}_{t-1}| \leq \varepsilon n$ and that (3) is satisfied. If in addition $A_t + P_t \leq \varepsilon n/2$, then (3) ensures that $|\mathcal{Y}_t| \leq \varepsilon n$, and thus Y_t does not occur. Therefore,

$$\mathbb{P}[Y_t] = \mathbb{P}[Y_{t-1}] + \mathbb{P}[Y_t \setminus Y_{t-1}] \leq \mathbb{P}[Y_{t-1}] + \mathbb{P}\left[A_t + P_t > \frac{\varepsilon n}{2} \wedge |\mathcal{Z}_{t-1}| \leq \varepsilon n\right] \leq \mathbb{P}[Y_{t-1}] + 2/n^2.$$

Finally, (36) and (37) yield

$$\mathbb{P}[|\mathcal{Y}_{T^*}| > \varepsilon n] \leq \mathbb{P}[Y_{T^*}] \leq \mathbb{P}[Y_0] + \sum_{t=1}^{T^*} 2/n^2 = o(1) + 2T^*/n^2 = o(1).$$

In combination with (35), this implies the assertion. \square

5.2 Proof of Lemma 5.1

How can a clause Φ_i become t -active? If this occurs, then `Walksat` must have tried ‘actively’ to satisfy Φ_i at least k_2 times by flipping one of its variables. But each time, the variable that `Walksat` flipped to satisfy Φ_i got flipped again because flipping it rendered another clause unsatisfied.

More precisely, if Φ_i is t -active, then there exist distinct ‘slots’ $j_1, \dots, j_{k_2} \in [k]$ and times $s_1, \dots, s_{k_2} \in [t]$ such that (i, j_l) is s_l -active for $l = 1, \dots, k_2$. This means that at the times s_l , `Walksat` actively tried to satisfy Φ_i by flipping $|\Phi_{ij_l}|$ ($l = 1, \dots, k_2$). However, as `Walksat` had to make k_2 attempts, each of the variables $|\Phi_{ij_l}|$ with $l < k_2$ must have been flipped once more by time s_{l+1} . Hence, $|\Phi_{ij_l}|$ occurs positively in a clause Φ_{h_l} that is unsatisfied at some time $s_l < q_l < s_{l+1}$. In particular, $h_l \in \mathcal{D}_{q_l} \subset \mathcal{D}_t$.

Thus, in order to prove Lemma 5.1 we are going to bound the probability that there are at least $\varepsilon n/4$ clauses Φ_i that admit $j_1, \dots, j_{k_2} \in [k]$ such that for each $1 \leq l < k_2$ there is another clause Φ_{h_l} with the following properties.

A1. We have $\text{sign}(\Phi_{ij_l}) = -1$, and there is an index $j \in [k]$ such that $\text{sign}(\Phi_{hj}) = 1$ and $\Phi_{h_l} = |\Phi_{ij_l}|$.

A2. $h_l \in \mathcal{D}_t$, i.e., Φ_{h_l} is $\mathcal{A}_t \cup \mathcal{N}_t$ -negative.

In order to deal with **A1** we will need to refine our filtration. Given a subset $Q \subset [m] \times [k]$ and a map $g : Q \rightarrow [m] \times [k]$, we let Ω_g be the event that

$$\text{sign}(\Phi_{ij}) = -1, \text{sign}(\Phi_{g(i,j)}) = 1 \text{ and } |\Phi_{ij}| = |\Phi_{g(i,j)}| \text{ for all } (i, j) \in Q.$$

Since the literals of the random formula Φ are independently uniformly distributed, we see that

$$\mathbb{P}[\Omega_g] \leq 2^{-|Q \cup g(Q)|} n^{-|Q|}. \quad (38)$$

We consider Ω_g as a probability space equipped with the uniform distribution (in other words, we are going to condition on Ω_g). Further, we define a filtration $(\mathcal{F}_{g,t})_{t \geq 0}$ on Ω_g by letting $\mathcal{F}_{g,t} = \{\mathcal{E} \cap \Omega_g : \mathcal{E} \in \mathcal{F}_t\}$. In other words, $\mathcal{F}_{g,t}$ is the projection of \mathcal{F}_t onto Ω_g . Hence, Fact 3.2 directly implies the following.

Fact 5.3 *For any $t \geq 0$, any $x \in V$, and any $i \in [m]$ the events $\{\sigma_t(x) = 1\}$, $\{\Phi_i$ is satisfied under $\sigma_t\}$, $\{x \in \mathcal{A}_t\}$, $\{i \in \mathcal{Z}_t\}$, $\{x \in \mathcal{N}_t\}$, and $\{T = t\}$ are $\mathcal{F}_{g,t}$ -measurable.*

Moreover, since the only conditioning we impose in Ω_g concerns the literals Φ_{ij} with $(i, j) \in Q \cup g(Q)$, Fact 3.3 yields the following.

Fact 5.4 *Let $t \geq 0$. Let \mathcal{E}_t be the set of all pairs $(i, j) \in [m] \times [k] \setminus (Q \cup g(Q))$ such that $\pi_t(i, j) \in \{-1, 1\}$. The conditional joint distribution of the variables $(|\Phi_{ij}|)_{(i,j) \in \mathcal{E}_t}$ given $\mathcal{F}_{t,g}$ is uniform over $(V \setminus (\mathcal{A}_t \cup \mathcal{N}_t))^{\mathcal{E}_t}$. That is, for any map $f : \mathcal{E}_t \rightarrow V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)$ we have*

$$\mathbb{P}[\forall (i, j) \in \mathcal{E}_t : |\Phi_{ij}| = f(i, j) | \mathcal{F}_{t,g}] = |V \setminus (\mathcal{A}_t \cup \mathcal{N}_t)|^{-|\mathcal{E}_t|}.$$

Similarly, with respect to the random variables $K_t^*(\cdot, \cdot)$ and $K_t^0(\cdot, \cdot)$ defined in (15) and (16), Corollary 4.4 implies the following.

Corollary 5.5 *For any $t \geq 1$, $\delta_t \geq 0$ and $\mathcal{I}^*, \mathcal{I}^0 \subset [m] \times [k] \setminus (Q \cup g(Q))$ we have*

$$\begin{aligned} \mathbb{E}[K_t^*(\mathcal{I}^*)K_t^0(\mathcal{I}^0)\mathbf{1}\{|\mathcal{Z}_t \setminus \mathcal{Z}_{t-1}| \leq \delta_t\} | \mathcal{F}_{g,t-1}] &\leq \max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})|\}^{-|\mathcal{I}^*|} \\ &\cdot \left(\frac{k\delta_t}{\max\{1, |V \setminus (\mathcal{A}_{t-1} \cup \mathcal{N}_{t-1})| - k\delta_t\}} \right)^{|\mathcal{I}^0|}. \end{aligned}$$

As a further preparation, we need the following lemma.

Lemma 5.6 *Let $1 \leq t \leq T^*$. Set $\mu = \varepsilon n/4$ and let $M \subset [m]$ be a set of size $|M| = \mu$. Furthermore, let $Q \subset M \times [k]$, let $I \subset [m]$ be a set of size $|I| \leq |Q|$, and let $g : Q \rightarrow I \times [k]$. Let $\mathcal{E}(M, Q, I, g)$ denote the event that $|\mathcal{Z}_t| \leq \varepsilon n$ and the following three statements hold.*

- a. *For all $(i, j) \in Q$ we have $\text{sign}(\Phi_{ij}) = -1$, $\text{sign}(\Phi_{g(i,j)}) = 1$, and $\Phi_{g(i,j)} = |\Phi_{ij}|$.*
- b. *$I \subset \mathcal{D}_t$.*
- c. *For each $i \in I$ there is $j \in [k]$ such that $(i, j) \in g(Q)$.*

Then $\mathbb{P}[\mathcal{E}(M, Q, I, g)] \leq 2^{\binom{T^* + \varepsilon n}{\varepsilon n}} (2n)^{-|Q|} 2^{-|I \times [k]|} \exp(1.011k\theta|I|)$.

Proof. To estimate $\mathbb{P}[\mathcal{E}(M, Q, I, g)]$, we need to decompose the event $\mathcal{E}(M, Q, I, g)$ into ‘more detailed’ sub-events whose probabilities can be bounded directly via Corollary 5.5. To this end, let $\mathcal{I}^*, \mathcal{I}^0$ be two disjoint subsets of $I \times [k] \setminus g(Q)$, and let $t^* : \mathcal{I}^* \rightarrow [T^*]$, $t^0 : \mathcal{I}^0 \rightarrow [T^*]$ be two maps. Let $\mathcal{E}(M, Q, I, g, t^*, t^0)$ be the event that $|\mathcal{Z}_t| \leq \varepsilon n$ and that the following statements are true.

- a. *For all $(i, j) \in Q$ we have $\text{sign}(\Phi_{ij}) = -1$, $\text{sign}(\Phi_{g(i,j)}) = 1$, and $\Phi_{g(i,j)} = |\Phi_{ij}|$.*
- b.
 - i. *If $(i, j) \in I \times [k] \setminus (g(Q) \cup \mathcal{I}^* \cup \mathcal{I}^0)$, then $\text{sign}(\Phi_{ij}) = -1$.*
 - ii. *If $(i, j) \in \mathcal{I}^*$, then $\text{sign}(\Phi_{ij}) = \pi_{t^*(i,j)-1}(i, j) = 1$ and $\Phi_{ij} \in \mathcal{A}_{t^*(i,j)}$.*
 - iii. *If $(i, j) \in \mathcal{I}^0$, then $\text{sign}(\Phi_{ij}) = \pi_{t^0(i,j)-1}(i, j) = 1$ and $\Phi_{ij} \in \mathcal{N}_{t^0(i,j)}$.*
- c. *For each $i \in I$ there is $j \in [k]$ such that $(i, j) \in g(Q)$.*

If the event $\mathcal{E}(M, Q, I, g)$ occurs, then there exist $\mathcal{I}^*, \mathcal{I}^0, t^*, t^0$ such that the event $\mathcal{E}(M, Q, I, g, t^*, t^0)$ occurs. Indeed, the definition of the set \mathcal{D}_{T^*} is such that if $i \in \mathcal{D}_{T^*}$, then for any $(i, j) \in I \times [k]$ such that $\text{sign}(\Phi_{ij}) = 1$ we have $\Phi_{ij} \in \mathcal{A}_{T^*} \cup \mathcal{N}_{T^*}$. Thus, by the union bound,

$$\mathbb{P}[\mathcal{E}(M, Q, I, g)] \leq \sum_{\mathcal{I}^*, \mathcal{I}^0} \sum_{t^*, t^0} \mathbb{P}[\mathcal{E}(M, Q, I, g, t^*, t^0)]. \quad (39)$$

Furthermore, let $\delta = (\delta_1, \dots, \delta_t)$ be a sequence such that $\sum_{s=1}^t \delta_s \leq \varepsilon n$. Let $\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)$ be the event that $|\mathcal{Z}_s \setminus \mathcal{Z}_{s-1}| \leq \delta_s$ for all $1 \leq s < t$ and that $\mathcal{E}(M, Q, I, g, t^*, t^0)$ occurs. Then by the union bound,

$$\begin{aligned} \mathbb{P}[\mathcal{E}(M, Q, I, g, t^*, t^0)] &\leq \sum_{\delta} \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)] \\ &\leq \binom{T^* + \varepsilon n}{\varepsilon n} \max_{\delta} \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)]. \end{aligned} \quad (40)$$

The event $\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)$ is sufficiently specific so that we can estimate its probability easily. Namely, if $\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)$ occurs, then Ω_g occurs and

$$\prod_{(i,j) \in \mathcal{I}^*} K_{t^*}^*(i,j)(i,j) \prod_{(i,j) \in \mathcal{I}^0} K_{t^*}^0(i,j)(i,j) \prod_{s=1}^t \mathbf{1}\{|\mathcal{Z}_s \setminus \mathcal{Z}_{s-1}| \leq \delta_s\} = 1. \quad (41)$$

To bound the probability that (41) occurs, we reorder the product by the time parameter. That is, letting $\mathcal{I}_s^* = t^{*-1}(s)$, $\mathcal{I}_s^0 = t^{0-1}(s)$, we get

$$\begin{aligned} &\mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0) | \mathcal{F}_{g,0}] \\ &\leq \mathbb{E} \left[\prod_{(i,j) \in \mathcal{I}^*} K_{t^*}^*(i,j)(i,j) \prod_{(i,j) \in \mathcal{I}^0} K_{t^*}^0(i,j)(i,j) \prod_{s=1}^t \mathbf{1}\{|\mathcal{Z}_s \setminus \mathcal{Z}_{s-1}| \leq \delta_s\} = 1 | \mathcal{F}_{g,0} \right] \\ &\leq \mathbb{E} \left[\prod_{s=1}^t K_s^*(\mathcal{I}_s^*) K_s^0(\mathcal{I}_s^0) \mathbf{1}\{|\mathcal{Z}_s \setminus \mathcal{Z}_{s-1}| \leq \delta_s\} | \mathcal{F}_{g,0} \right]. \end{aligned} \quad (42)$$

Since for any $s \leq t \leq T^*$ we have $|\mathcal{A}_s| \leq s \leq T^* \leq \frac{n}{k}$, and as $|\mathcal{N}_s| \leq k \sum_{q=1}^s \delta_q \leq k\varepsilon n$, we see that $|\mathcal{A}_s \cup \mathcal{N}_s| + k\delta_s \leq 0.001n$ for all $s \leq t$. Hence, (42) and Corollary 5.5 yield

$$\begin{aligned} \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0) | \mathcal{F}_{g,0}] &\leq \prod_{s=1}^t \left(\frac{1.01}{n} \right)^{|\mathcal{I}_s^*|} \left(\frac{1.01k\delta_s}{n} \right)^{|\mathcal{I}_s^0|} \\ &\leq \left(\frac{1.01}{n} \right)^{|\mathcal{I}^*| + |\mathcal{I}^0|} \prod_{(i,j) \in \mathcal{I}^0} k\delta_{t^0(i,j)}. \end{aligned} \quad (43)$$

Furthermore, if the event $\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)$ occurs, then for all $(i, j) \in I \times [k] \setminus (g(Q) \cup \mathcal{I}^* \cup \mathcal{I}^0)$ we have $\text{sign}(\Phi_{ij}) = -1$, while $\text{sign}(\Phi_{ij}) = 1$ for all $(i, j) \in \mathcal{I}^* \cup \mathcal{I}^0$. This event is $\mathcal{F}_{0,g}$ -measurable. Hence, as the signs of the literals Φ_{ij} are independently uniformly distributed, we obtain from (43)

$$\mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0) | \Omega_g] \leq 2^{-|I \times [k] \setminus g(Q)|} \left(\frac{1.01}{n} \right)^{|\mathcal{I}^* \cup \mathcal{I}^0|} \prod_{(i,j) \in \mathcal{I}^0} k\delta_{t^0(i,j)}. \quad (44)$$

Combining (38) and (44), we get

$$\begin{aligned} \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)] &= \mathbb{P}[\Omega_g] \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0) | \Omega_g] \\ &\leq (2n)^{-|Q|} 2^{-|I \times [k]|} \left(\frac{1.01}{n} \right)^{|\mathcal{I}^* \cup \mathcal{I}^0|} \prod_{(i,j) \in \mathcal{I}^0} k\delta_{t^0(i,j)}. \end{aligned} \quad (45)$$

As (39) and (40) show, in order to obtain $\mathbb{P}[\mathcal{E}(M, Q, I, g)]$, we need to sum (45) over all possible choices of $\delta, \mathcal{I}^*, \mathcal{I}^0, t^*, t^0$:

$$\begin{aligned}
\mathbb{P}[\mathcal{E}(M, Q, I, g)] &\leq \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} \sum_{\mathcal{I}^*, \mathcal{I}^0} \sum_{t^*: \mathcal{I}^* \rightarrow [t]} \sum_{t^0: \mathcal{I}^0 \rightarrow [t]} \mathbb{P}[\mathcal{E}(\delta, M, Q, I, g, t^*, t^0)] \\
&\leq \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} \sum_{\mathcal{I}^*, \mathcal{I}^0} \left(\frac{1.01}{n} \right)^{|\mathcal{I}^* \cup \mathcal{I}^0|} t^{|\mathcal{I}^*|} \sum_{t^0: \mathcal{I}^0 \rightarrow [t]} \prod_{(i,j) \in \mathcal{I}^0} k \delta_{t^0(i,j)} \\
&\leq \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} \sum_{\mathcal{I}^*, \mathcal{I}^0} \left(\frac{1.01}{n} \right)^{|\mathcal{I}^* \cup \mathcal{I}^0|} t^{|\mathcal{I}^*|} \left(\sum_{s=1}^t k \delta_s \right)^{|\mathcal{I}^0|} \\
&\leq \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} \sum_{\mathcal{I}^*, \mathcal{I}^0} (1.01t/n)^{|\mathcal{I}^*|} (1.01k\varepsilon)^{|\mathcal{I}^0|} \quad [\text{as } \sum_{s=1}^t \delta_s \leq \varepsilon n] \\
&\leq \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} (1 + 1.01(\theta + k\varepsilon))^{k|I|}
\end{aligned}$$

Hence,

$$\mathbb{P}[\mathcal{E}(M, Q, I, g)] \leq 2 \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} \exp(1.011k|I|\theta).$$

as desired. \square

Proof of Lemma 5.1. Let $\mu = \varepsilon n/4$ and fix some $1 \leq t \leq T^*$. Let \mathcal{E} be the event that $|\mathcal{Z}_t| \leq \varepsilon n$ and $A_t \geq \mu$. For a set $M \subset [m]$ of size $|M| = \mu$ we let $\mathcal{E}(M)$ signify the event that all clauses $i \in M$ are t -active. If \mathcal{E} occurs, then there is a set M of size μ such that $\mathcal{E}(M)$ occurs. Hence, by the union bound

$$\mathbb{P}[\mathcal{E}] \leq \sum_{M \subset [m]: |M|=\mu} \mathbb{P}[\mathcal{E}(M)] \leq \binom{m}{\mu} \max_M \mathbb{P}[\mathcal{E}(M)]. \quad (46)$$

To bound the expression on the r.h.s., fix some set $M \subset [m]$ of size μ . Let $\mathcal{Q}(M)$ be the set of all $Q \subset M \times [k]$ such that for each $i \in M$ we have $|\{j \in [k] : (i, j) \in Q\}| = k_2 - 1$. For a set $Q \in \mathcal{Q}(M)$ let $\mathcal{E}(M, Q)$ be the event that $|\mathcal{Z}_t| \leq \varepsilon n$ and

- all pairs $(i, j) \in Q$ are $s(i, j)$ -active for some $s(i, j) \leq t$, and
- for each $i \in M$ there is $j' \in [k]$ such that (i, j') is s -active at some time s satisfying

$$\max_{j: (i,j) \in Q} s(i, j) < s \leq t.$$

If the event $\mathcal{E}(M)$ occurs, then there exists $Q \in \mathcal{Q}(M)$ such that $\mathcal{E}(M, Q)$ occurs. (In fact, if $\mathcal{E}(M)$ occurs, then by the definition of t -active, for any $i \in M$ there are at least k_2 indices j such that (i, j) is s -active for some $s \leq t$. We can thus let Q contain the pairs (i, j) for the ‘earliest’ $k_2 - 1$ such indices j .) Hence, by the union bound

$$\mathbb{P}[\mathcal{E}(M)] \leq \sum_{Q \in \mathcal{Q}} \mathbb{P}[\mathcal{E}(M, Q)] \leq \binom{k}{k_2 - 1}^\mu \max_{Q \in \mathcal{Q}} \mathbb{P}[\mathcal{E}(M, Q)]. \quad (47)$$

Now, fix a set $M \subset [m]$, $|M| = \mu$, and a set $Q \in \mathcal{Q}(M)$. If the event $\mathcal{E}(M, Q)$ occurs, then there exist I, g such that the event $\mathcal{E}(M, Q, I, g)$ as in Lemma 5.6 occurs. Indeed, this is precisely what we pointed out in **A1, A2** above. Thus, by the union bound

$$\begin{aligned}
\mathbb{P}[\mathcal{E}(M, Q)] &\leq \sum_{I, g} \mathbb{P}[\mathcal{E}(M, Q, I, g)] \leq \sum_{\nu=1}^{(k_2-1)\mu} \sum_{I \subset [m]: |I|=\nu} \sum_{g: Q \rightarrow I \times [k]} \mathbb{P}[\mathcal{E}(M, Q, I, g)] \\
&\leq \sum_{\nu=1}^{(k_2-1)\mu} \binom{m}{\nu} (k\nu)^{(k_2-1)\mu} \max_{I, g: |I|=\nu, g: Q \rightarrow I \times [k]} \mathbb{P}[\mathcal{E}(M, Q, I, g)]. \quad (48)
\end{aligned}$$

According to Lemma 5.6,

$$P[\mathcal{E}(M, Q, I, g)] \leq 2 \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} 2^{-|I \times [k]|} \exp(1.011k\theta\nu). \quad (49)$$

Combining (48) and (49), we obtain

$$\begin{aligned} P[\mathcal{E}(M, Q)] &\leq 2 \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} \sum_{\nu=1}^{(k_2-1)\mu} \binom{m}{\nu} (k\nu)^{(k_2-1)\mu} 2^{-k\nu} \exp(1.011k\theta\nu) \\ &\leq 2 \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} \sum_{\nu=1}^{(k_2-1)\mu} \left(\frac{em}{\nu 2^k}\right)^\nu (k\nu)^{(k_2-1)\mu} \exp(1.011k\theta\nu) \\ &\leq 2 \binom{T^* + \varepsilon n}{\varepsilon n} (2n)^{-|Q|} \sum_{\nu=1}^{(k_2-1)\mu} \left(\frac{e\rho n}{k\nu}\right)^\nu (k\nu)^{(k_2-1)\mu} \exp(1.011k\theta\nu). \end{aligned}$$

Since the largest summand is the one with $\nu = (k_2 - 1)\mu$ and as $|Q| = (k_2 - 1)\mu$, we obtain

$$P[\mathcal{E}(M, Q)] \leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \left(\frac{\exp(1 + 1.011k\theta)\rho}{2}\right)^{(k_2-1)\mu} \quad (50)$$

Let $\xi > 0$ be such that $\binom{k}{k_2-1} = (2\xi)^{k_2-1}$ and let $\zeta = \exp(1 + 1.011k\theta)$. Plugging (50) into (47), we get

$$P[\mathcal{E}(M)] \leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \binom{k}{k_2-1}^\mu \left(\frac{\zeta\rho}{2}\right)^{(k_2-1)\mu} \leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} (\xi\zeta\rho)^{(k_2-1)\mu}. \quad (51)$$

Finally, (46) and (51) yield

$$\begin{aligned} P[\mathcal{E}] &\leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \binom{m}{\mu} (\zeta\xi\rho)^{(k_2-1)\mu} \leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \left[\frac{em}{\mu} (\zeta\xi\rho)^{(k_2-1)}\right]^\mu \\ &\leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \left[\frac{4e2^k\rho}{k\varepsilon} (\zeta\xi\rho)^{(k_2-1)}\right]^\mu. \end{aligned} \quad (52)$$

If $\rho \leq \rho_0 = 1/25$, then

$$\frac{4e2^k\rho}{k\varepsilon} (\zeta\xi\rho)^{(k_2-1)} < \exp(-k_2/100) \quad (53)$$

for $k \geq k_0$ large enough. Hence, (52) and (53) yield for $k \geq k_0$ large enough

$$\begin{aligned} P[\mathcal{E}] &\leq 2k\mu \binom{T^* + \varepsilon n}{\varepsilon n} \exp(-k_2\mu/100) \leq 2k\mu \left(\frac{e(T^* + \varepsilon n)}{\varepsilon n}\right)^{\varepsilon n+1} \exp(-k_2\mu/100) \\ &\leq 2k\mu \left(\frac{e(1/k + \varepsilon)}{\varepsilon}\right)^{\varepsilon n+1} \exp(-k_2\mu/100) \\ &\leq \exp[2\varepsilon n - \varepsilon n \ln \varepsilon - k_2\mu/100 + o(n)] \\ &\leq \exp[n(2\varepsilon - \varepsilon \ln \varepsilon - k_2\varepsilon/400 + o(1))] && \text{[by our choice of } \mu \text{]} \\ &\leq \exp[-nk_2\varepsilon/401] = o(1), && \text{[by our choice of } \varepsilon, \text{ cf. (5)],} \end{aligned}$$

as desired. \square

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