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Packing Hamilton Cycles in Random and Pseudo-Random Hypergraphs

Alan Frieze
Carnegie Mellon University, af1p@andrew.cmu.edu

Michael Krivelevich
Tel Aviv University

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Packing Hamilton Cycles in Random and Pseudo-Random Hypergraphs

Alan Frieze∗  Michael Krivelevich†

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Abstract

We say that a $k$-uniform hypergraph $C$ is a Hamilton cycle of type $\ell$, for some $1 \leq \ell \leq k$, if there exists a cyclic ordering of the vertices of $C$ such that every edge consists of $k$ consecutive vertices and for every pair of consecutive edges $E_{i-1}, E_i$ in $C$ (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. We prove that for $k/2 < \ell \leq k$, with high probability almost all edges of the random $k$-uniform hypergraph $H(n, p, k)$ with $p(n) \gg \log^2 n/n$ can be decomposed into edge-disjoint type $\ell$ Hamilton cycles. A slightly weaker result is given for $\ell = k/2$. We also provide sufficient conditions for decomposing almost all edges of a pseudo-random $k$-uniform hypergraph into type $\ell$ Hamilton cycles, for $k/2 \leq \ell \leq k$. For the case $\ell = k$ these results show that almost all edges of corresponding random and pseudo-random hypergraphs can be packed with disjoint perfect matchings.

1 Introduction

The subject of Hamilton graphs and Hamiltonicity-related problems is undoubtedly one of the most central in Graph Theory, with a great many deep and beautiful results obtained. Hamiltonicity problems occupy a place of honor in the theory of random graphs too, the reader can consult the monographs of Bollobás [4] and of Janson, Łuczak and Ruciński [14] for an account of some of the most important results related to Hamilton cycles in random graphs. Of particular relevance to the current work is a previous result of the authors [9] who proved that for edge probability $p = p(n) \geq n^{-\epsilon}$ for some constant $\epsilon > 0$, whp1 almost all

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∗Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA15213, U.S.A. Email: alan@random.math.cmu.edu. Supported in part by NSF grant DMS-0753472.

†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by a USA-Israel BSF grant and by a grant from the Israel Science Foundation.

1An event $\mathcal{E}_n$ occurs with high probability, or whp for brevity, if $\lim_{n \to \infty} \Pr(\mathcal{E}_n) = 1$. 

edges of the random graph \( G(n, p) \) can be packed with edge-disjoint Hamilton cycles.

Quite a few results about Hamiltonicity of pseudo-random graphs are available too. Informally, a graph \( G = (V, E) \) with \(|V| = n\) vertices and \(|E| = m\) edges is pseudo-random if its edge distribution is similar, in some well-defined quantitative way, to that of a truly random graph \( G(n, p) \) with the same expected density \( p = m(n^{-1}) \). A thorough discussion about pseudo-random graphs, their alternative definitions and properties can be found in the survey [17]. It is well-known that pseudo-randomness of graphs can be guaranteed by imposing conditions on vertex degrees and co-degrees (see, e.g., [21], [5]); we will adopt a similar approach later in the paper when discussing pseudo-random hypergraphs. There are known sufficient criteria for Hamiltonicity in pseudo-random graphs. Also, the above mentioned result of [9] can be extended to the pseudo-random case as well. Since we will employ this result in our arguments, let us state it here formally. A graph \( G \) on vertex set \([n]\) is \((\alpha, \epsilon)\)-regular if

\[ Q_a: \delta(G) \geq (\alpha - \epsilon)n, \quad \text{where } \delta(G) \text{ denotes the minimum degree of } G. \]

\[ Q_b: \text{If } S, T \text{ are disjoint subsets of } [n] \text{ and } |S|, |T| \geq \epsilon n \text{ then } \left| \frac{e_G(S, T)}{|S||T|} - \alpha \right| \leq \epsilon, \quad \text{where } e_G(S, T) \text{ is the number of } S - T \text{ edges in } G. \]

The following is implied by the main theorem of [9]:

**Theorem 1** Let \( G \) be an \((\alpha, \epsilon)\)-regular graph with \( n \) vertices where

\[ \alpha \gg \epsilon \text{ and } \alpha \epsilon^3 \gg \frac{1}{(n \log n)^{1/2}}. \] (1)

Then \( G \) contains at least \((\alpha/2 - 4\epsilon)n\) edge-disjoint Hamilton cycles.

**Remark 1** Theorem 2 of [9] only claims to be true for \( \alpha \) constant. This was an unfortunate over-cautious statement. The real condition should be the one given in the above theorem. We will justify this claim in the appendix.

In contrast, much less is known about Hamiltonicity in hypergraphs in general and in random and pseudo-random hypergraphs in particular. Formally, a hypergraph \( H \) is an ordered pair \( H = (V, E) \), where \( V \) is a set of vertices, and \( E \) is a family of distinct subsets of \( V \), called edges. A hypergraph \( H \) is \( k \)-uniform if all edges of \( H \) are of size \( k \). It is generally believed that \( k \)-uniform hypergraphs for \( k \geq 3 \) are much more complicated objects of study than graphs (corresponding to \( k = 2 \)). Specifically for Hamiltonicity, even extending the definition of a Hamilton cycle in graphs to the case of (uniform) hypergraphs is not a straightforward task. In fact, several alternative definitions are possible. In this paper (in some departure from a relatively standard notation) we will use the following definition. Denote

\[ \nu_i = \frac{n}{i}, \quad 1 \leq i \leq k. \]
Suppose that $1 \leq \ell \leq k$. A type $\ell$ Hamilton cycle in a $k$-uniform hypergraph $H = (V, \mathcal{E})$ on $n$ vertices is a collection of $\nu_\ell$ edges of $H$ such that for some cyclic order of $[n]$ every edge consists of $k$ consecutive vertices and for every pair of consecutive edges $E_{i-1}, E_i$ in $C$ (in the natural ordering of the edges) we have $|E_{i-1} \setminus E_i| = \ell$. Thus, in a type $\ell$ Hamilton cycle the sets $C_i = E_i \setminus E_{i-1}$, $i = 1, 2, \ldots, \nu_\ell$, are a partition of $V$ into sets of size $\ell$. (An obvious necessary condition for the existence of a cycle of type $\ell$ in a hypergraph on $n$ vertices is that $\ell$ divides $n$. We thus always assume, when discussing Hamilton cycles of type $\ell$, that this necessary condition is fulfilled.) In the literature, when $\ell = 1$ we have a tight Hamilton cycle and when $\ell = k - 1$ we have a loose Hamilton cycle. In the extreme case $\ell = k$ the notion reduces to that of a perfect matching in a hypergraph.

Several recent papers (see, e.g., [13], [16], [18], [20]) provided sufficient conditions for the existence of a type $\ell$ Hamilton cycle in a $k$-uniform hypergraph $H$ on $n$ vertices in terms of the minimum number of edges of $H$ passing through any subset of $k - 1$ vertices, thus extending the classical Dirac sufficient condition for graph Hamiltonicity to the hypergraph case. These results however appear to be of rather limited relevance to the current paper, as here we are mostly concerned with sparse hypergraphs (with $o(|V|^k)$ edges), while the above mentioned results are for the (very) dense case.

The main goal of this paper at large is to study Hamiltonicity in random and pseudo-random hypergraphs. A random $k$-uniform hypergraph $H(n, p, k)$ is a hypergraph with vertex set $\{1, \ldots, n\} = [n]$, where each $k$-tuple of $[n]$ is an edge of the hypergraph independently with probability $p = p(n)$. For the case $k = 2$ the model $H(n, p, k)$ reduces to the classical binomial random graph $G(n, p)$. Up until recently, essentially nothing was known about Hamilton cycles in random hypergraphs. Even the most basic question of the threshold for the appearance of a cycle of type $\ell$ in $H(n, p, k)$ had not been addressed in the literature. A recent series of papers, see Frieze [8], Dudek and Frieze [6], [7], establish the thresholds for the existence of type $\ell$ Hamilton cycles, $1 \leq \ell \leq k - 1$ up to constant factors or up to arbitrarily slow growing functions of $n$. In the case $\ell = k$, i.e., the case of perfect matchings – a recent striking result of Johannson, Kahn and Vu [15], has established the order of magnitude of the threshold for the appearance of a perfect matching in a $k$-uniform random hypergraph.

In this paper, rather than studying the conditions for the existence of a single Hamilton cycle, we study the conditions for the existence of a packing of almost all edges of a random or a pseudo-random hypergraph into Hamilton cycles. For $\ell \geq k/2$ we manage to obtain non-trivial results in this direction. It appears that the cases of small $\ell$ (where adjacent edges along the Hamilton cycle have larger intersection) are harder.

Our first result is about packing Hamilton cycles in random hypergraphs.

**Theorem 2** Suppose that $k/2 < \ell \leq k$ and suppose that $np/\log^2 n \to \infty$. Then whp $H = H(n, p, k)$ contains a collection of $(1 - \epsilon)(|V|^k\ell)!)p/\nu_\ell$ edge-disjoint type $\ell$ Hamilton cycles, $\epsilon^2 = \Theta((\log n/(np)^{1/2}) = o(1)$. When $\ell = k/2$ we have the same conclusion for $\epsilon^{11} \gg \log^{7/2} n/(n^{3/2}p^4)$. 

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Note that for the case \( \ell = k \) the above theorem provides a sufficient condition on the edge probability \( p(n) \) for being able to pack \textit{whp} almost all edges of \( H(n, p, k) \) into perfect matchings.

Other results of the paper are about packing Hamilton cycles in pseudo-random hypergraphs. For most part, we state the condition of pseudo-randomness of a hypergraph in terms of the number of edges through subsets of vertices of fixed size. These conditions are suggested by the expected numbers of such edges in truly random hypergraphs of the same edge density and are easily seen to hold \textit{whp} in random hypergraphs. Thus our results about pseudo-random hypergraphs are applicable to truly random instances as well. Naturally, the direct approach of Theorem 2 provides a better lower bound on the edge probability \( p(n) \).

In this paper we are only able to deal with the case where \( \ell \geq k/2 \). Let \( H = ([n], \mathcal{E}) \) be a \( k \)-uniform hypergraph with vertex set \([n]\) and \( m \) edges. Its density \( p = m / \binom{n}{k} \). For a set \( X \subseteq [n] \) with \( |X| = a < k \) we define its neighbourhood \( N_H(X) = \{ Y \in \binom{[n]}{k-a} : X \cup Y \in \mathcal{E} \} \) and its degree \( d_H(X) = |N_H(X)| \).

We first consider \( k/2 < \ell < k \) and list the following properties. The value \( \epsilon \) will be a parameter of \textit{regularity}. \( P_a \) says that the "degrees" of small sets are close to being regular; \( P_b \) says that the "co-degrees" of small sets are not too large.

\[
P_a(s) : \min_{S \in \binom{[n]}{s}} d_H(S) = (1 \pm \epsilon) \binom{n}{k-s} p \quad \text{and} \quad P_a = \bigcap_{1 \leq s \leq k-1} P_a(s).
\]

\[
P_b(s,t) : \max_{S_1,S_2 \in \binom{[n]}{s}} |N_H(S_1) \cap N_H(S_2)| \leq (1 + \epsilon) \binom{n}{k-s} p^2 \quad \text{and} \quad P_b = \bigcap_{1 \leq s \leq k-1} \bigcap_{0 \leq t \leq s} P_b(s,t).
\]

(the notation \( A = (1 \pm \epsilon)B \) as a shorthand for the pair of inequalities \( (1-\epsilon)B \leq A \leq (1+\epsilon)B \).)

**Theorem 3** Let \( H = ([n], \mathcal{E}) \) be a \( k \)-uniform hypergraph with \( m \) edges and let \( p = m / \binom{n}{k} \). Suppose that \( k/2 < \ell < k \) and \( 1 > \epsilon^5 \gg \log^3 n / (n^{1/2} p^2) \) and suppose that \( H \) satisfies properties \( \mathcal{P} = \{ P_a, P_b \} \). Then \( H \) contains a collection of \( (1 - 2\epsilon^{1/3})m / \nu_\ell \) edge-disjoint type \( \ell \) Hamilton cycles.

The restriction \( 1 > \epsilon \) is for relevance and the restriction \( \epsilon^5 \gg \log^3 n / (n^{1/2} p^2) \) is used in the proof (see Lemma 7). The latter condition can be relaxed a little through a more careful implementation of our argument.

We do not really need \( P_a, P_b(s,t) \) to hold for all \( s,t \) but using the above simplifies the statement of the theorem. The proof will expose the actual values of \( s,t \) for which these properties are needed. Also, one can observe that in fact, for example, the condition \( P_a \), which is a

\(^2\)We use the notation \( a_n \gg b_n \) as shorthand for \( a_n / b_n \to \infty \) as \( n \to \infty \).
conjunction of the conditions $P_a(s)$ for different $s$, is essentially implied by the condition $P_a(k - 1)$ alone. In any case any reasonable definition of a pseudo-random hypergraph is likely to yield all such conditions. This is certainly the case for random hypergraphs.

When $\ell = k/2$ we will use the result from [9] as our main technical tool, and the above stated definition of $(\alpha, \epsilon)$-regular graphs. Here the definition of a pseudo-random hypergraph is explicitly tailored to our application. Let $H = (V, E)$ be a $k$-uniform hypergraph with vertex set $V = [n]$. Let $\mathcal{P} = (X_1, X_2, \ldots, X_{\nu_{\ell}})$ be a partition of $[n]$ into $\nu_{\ell}$ parts each of size $\ell$. The graph $G_{\mathcal{P}} = G_{\mathcal{P}}(H)$ has vertex set $[\nu_{\ell}]$ and an edge $(i, j)$ whenever $E = X_i \cup X_j \in \mathcal{E}(H)$. We now say that $H$ is $(\rho, \epsilon)$-regular if for a randomly chosen $\mathcal{P}$, the graph $G_{\mathcal{P}}$ is $(\rho, \epsilon)$-regular $qs^3$.

**Theorem 4** Let $H = ([n], \mathcal{E})$ be a $k$-uniform hypergraph with $m$ edges and let $p = m/(n^k)$. Suppose that $H$ is $(\rho, \epsilon)$-regular $k$-uniform hypergraph with $\ell = k/2$ and

$$\epsilon^{11/4}p n^{1/8} \gg \log^{7/8} n.$$

Then $H$ contains a collection of $(1 - 20\epsilon)m/\nu_{\ell}$ edge-disjoint type $\ell$ Hamilton cycles.

We finally consider the case $k = \ell$. Here we will be packing perfect matchings as opposed to Hamilton cycles.

**Theorem 5** Let $H = ([n], \mathcal{E})$ be a $k$-uniform hypergraph with $m$ edges that satisfies $P_a, P_b$ and suppose that $1 \gg \epsilon \gg \log^5 n/(n^{1/2}p^2)$. Then $H$ contains a collection of $(1 - 4\epsilon^{1/2})m/\nu_k$ edge-disjoint perfect matchings.

These are the first results of any significance on packing Hamilton cycles in random and pseudo-random hypergraphs. We have no reason to believe that they are tight and it would be interesting to sharpen them.

The case $\ell < k/2$ is more difficult. Frieze, Krivelevich and Loh [10] have analysed the case $k = 3, \ell = 1$ and proved a result of the same flavour as the above. This has since been extended to more general $k, \ell$ in Bal and Frieze [2].

An interesting point of reference for our theorems is results about perfect decompositions of the edge set of a complete $k$-uniform hypergraph $K^k_n$ into Hamilton cycles of various types (assuming of course some natural divisibility conditions). These include a recent result of Bailey and Stevens [1] about packing tight Hamilton cycles and a famous result of Baranyai [3] about decomposing the edge set of $K^k_n$ into perfect matchings. While we do not – and can not for obvious reasons – achieve perfect decomposition, but rather pack almost all edges, our results apply to a wide class of hypergraphs, including relatively sparse hypergraphs.

In the next section we cite and prove several general tools applied later in the proofs. In Section 3 we focus on $H = H(n, p, k)$ and first prove Theorem 2 for $k = 3$. We then provide a

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$^3$An event $\mathcal{E}_n$ occurs quite surely, or $qs$ for brevity, if $\Pr(\mathcal{E}_n) = 1 - O(n^{-C})$ for any positive constant $C$. 


general scheme of the proof, applicable to both the random and the pseudo-random cases. In Section 5 we give a proof of the random case for general $k, \ell$. In Section 6 we prove Theorems 3, 4 and 5. The last section is devoted to concluding remarks.

2 Tools

We start with stating very standard Chernoff-type bounds for binomial random variables: for $0 \leq \epsilon \leq 1$:

\[
\Pr(\text{Bin}(m, \xi) - m\xi \leq -\epsilon m\xi) \leq e^{-\epsilon^2 m\xi/2} \quad (2)
\]

\[
\Pr(\text{Bin}(m, \xi) - m\xi \geq \epsilon m\xi) \leq e^{-\epsilon^2 m\xi/3} \quad (3)
\]

Next, we give two statements about the (high probability) existence of many disjoint perfect matchings in random and in pseudo-random bipartite graphs. For the (somewhat easier) truly random case we will need the obtained estimate on the number of matchings to hold with probability polynomially close to 1.

**Lemma 1** Let $k \geq 1$ be an integer. Let $G$ be a random bipartite graph with sides $A, B$ of size $|A| = |B| = n$, where each pair $(a, b), a \in A, b \in B$, is an edge of $G$ independently and with probability at least $p$. Assume $p(n) \gg \log n/n$. Then with probability $1 - o(n^{k})$, $G$ contains a family of $(1 - \delta)np$ edge disjoint perfect matchings, where

\[
\delta = \left(\frac{4(k+3)\log n}{np}\right)^{1/2} = o(1).
\]

**Proof** Due to monotonicity is is enough to prove the lemma for the case when all edge probabilities are exactly $p$. Set $n_0 = (1 - \delta)np$. The Max-Flow Min-Cut theorem tells us that the following is a necessary and sufficient condition for $G$ to have $n_0$ edge-disjoint perfect matchings: Suppose that we make up a network with source $\sigma$ and sink $\tau$ and join $\sigma$ to each vertex of $A$ by an edge of capacity $n_0$ and each vertex of $B$ to $\tau$ by an edge of capacity $n_0$. Each edge of $G$ is given capacity one. Suppose that our minimum cut is $X$ : $\bar{X}$ and $S = A \cap X$ and $T = B \cap X$ then a necessary and sufficient condition for the existence of $n_0$ disjoint perfect matchings is that

\[
(n - |S|)n_0 + |T|n_0 + e(S, B \setminus T) \geq n_0 n,
\]

which reduces to

\[
m \geq (s - t)n_0 \quad (4)
\]

for all $S \subseteq A, T \subseteq B$, if $|S| = s, |T| = t, m = e(S, B \setminus T)$.

Note that we need only verify (4) computationally for $t \leq n/2$. When $t > n/2$ we could repeat our computations to show that with the required probability $e(B \setminus T, A \setminus (A \setminus S)) \geq ((n - t) - (n - s))n_0$. 

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We have:

\[
Pr(\exists S \subseteq A, T \subseteq B, |S| > |T|, |T| \leq n/2 : e(S, B \setminus T) \leq (1 - \delta)|S|((n - |T|)p) \leq e(S, B) \leq (1 - \delta)s(n - t)p \leq \frac{n}{s} \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \binom{n}{s} \binom{n}{t} \exp \left\{-\frac{\delta^2}{2}s(n-t)p \right\} \leq \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \binom{n}{s} \binom{n}{t} \exp \left\{-\delta^2sp/4 \right\}. \tag{5}
\]

Assume first that \(\binom{n}{s} \geq \binom{n}{t}\). Then

\[
(5) \leq \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left( \frac{n^2e^2}{4s^2} \cdot e^{-\delta^2np/4} \right)^s = \sum_{s=1}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left( \frac{n^2e^2}{4s^2} \cdot n^{-k-3} \right)^t = o(n^{-k}).
\]

When \(\binom{n}{s} \leq \binom{n}{t}\) we can replace (5) by

\[
\sum_{s=n/2}^{n} \sum_{t=1}^{\min\{s-1,n/2\}} \left( \frac{n^2e^2}{4t^2} \cdot n^{-k-3} \right)^t = o(n^{-k}).
\]

Here we have used \(s \geq t\).

It follows (see (4)) that with the required probability simultaneously for all relevant sets \(S, T\),

\[
m \geq (1 - \delta)s(n - t)p \geq (1 - \delta)(s - t)np = (s - t)n_0.
\]

\[\square\]

In the next lemma we prove the existence of many disjoint perfect matchings in a bipartite pseudo-random graphs with equal sides, where pseudo-randomness is given through/controlled by vertex degrees and co-degrees.

**Lemma 2** Let \(G\) be a bipartite graph with vertex set \(A \cup B\) where \(|A| = |B| = n\). Suppose that the minimum degree in \(G\) is at least \(1 - \theta d\) and the maximum co-degree of two vertices is at most \((1 + \theta)d^2n\) for some small value \(\theta \ll 1\). Suppose further that \(\theta^{4/3}d^2n \gg 1\). Then \(G\) contains a collection of \((1 - \theta^{1/3})dn\) edge-disjoint perfect matchings.

The assumption \(\theta^{4/3}d^2n \gg 1\) in the above lemma is mostly for convenience and will be good enough for our purposes; it can be relaxed somewhat through a more careful analysis.

**Proof** Let \(d_0 = (1 - \theta)d\) and \(d_1 = (1 - \theta^{1/3})d\). Going back to (4) we see that we need to show that

\[
m \geq (s - t)d_1n \tag{6}
\]

for all \(S \subseteq A, T \subseteq B, |S| = s, |T| = t, m = e(S, B \setminus T)\). Obviously we can assume \(s > t\).
Now,

\[ m \geq s(d_0 n - t) \]

and so (6) is satisfied if

\[ s(d_0 n - t) \geq (s - t)d_1 n \]

or

\[ s(d_0 - d_1)n \geq (s - d_1 n)t. \]

In particular, (6) holds if \( t \leq (d_0 - d_1)n \). Furthermore, we also have

\[ m \geq (n - t)(d_0 n - (n - s)). \]

If \( n - t \leq (d_0 - d_1)n \) then this implies that (6) holds.

So we assume from now on that

\[ (d_0 - d_1)n < \min \{ t, n - t \}. \] (7)

We can further assume that \( t \leq n/2 \). For \( t > n/2 \) we can reverse the roles of \( A, B \) and show that \( e(B \setminus T, A \setminus (A \setminus S)) \geq ((n - t) - (n - s))d_1 n \), which is (6).

We now perform the usual double counting trick by estimating the number of paths of the form \( S, B, S \) in two ways. On one hand, each such path corresponds to a common neighbor of a pair of vertices \( a_1, a_2 \in S \). Therefore, the quantity to be estimated is at most \( (s^2)d_2^2 n \), where \( d_2 = (1 + \theta)^{1/2}d \). On the other hand, it is exactly

\[
\sum_{b \in B} \binom{d(b, S)}{2} = \sum_{b \in B \setminus L} \binom{d(b, S)}{2} + \sum_{b \in L} \binom{d(b, S)}{2}
\]

where \( d(b, S) \) is the number of neighbors of \( b \) in \( S \) in the graph \( G \). Since \( \sum_{b \in B \setminus L} d(b, S) = m \), we can estimate the first summand as follows:

\[
\sum_{b \in B \setminus T} \binom{d(b, S)}{2} \geq (n - t)\left(\frac{m}{n-t}\right) = \frac{m}{2} \left(1 - \frac{1}{n-t}\right).
\]

As for the second summand, the number of edges between \( T \) and \( S \) can be estimated from below by \( d_0 ns - m \), and therefore

\[
\sum_{b \in L} \binom{d(b, S)}{2} \geq t \left(\frac{d_0 sn - m}{2}\right) = \frac{d_0 sn - m}{2} \left(1 - \frac{1}{t}\right).
\]

It follows that

\[
m \left(\frac{m}{n-t} - 1\right) + (d_0 sn - m) \left(\frac{d_0 sn - m}{t} - 1\right) \leq s(s-1)d_2^2 n.
\]
After performing straightforward arithmetic manipulations, we get to:

\[(m - d_0 s(n - t))^2 \leq s(n - t)(std_2^2 - std_0^2 + d_0 t).\]

Recalling the definitions of \(d_0\) and \(d_2\), we see that

\[d_2^2 - d_0^2 = (1 + \theta - (1 - \theta)^2)d^2 = (3\theta + \theta^2)d^2.\]

Also, since \(s \geq t\) and \(\ell \geq (d_0 - d_1)n \geq \theta^{1/3}dn/2\) by (7), we see that \(\theta ds \geq \theta^{4/3}d^2n/2 \gg 1\) by the lemma’s assumption. Hence \(d_0 t \ll \theta d^2 st\). We thus arrive at the following inequality:

\[(m - d_0 s(n - t))^2 \leq 4\theta d^2 s^2 t(n - t).\]

Since \(t \leq n/2\), we have

\[m \geq (n - t)sd_0 \left(1 - \frac{2t^{1/2}\theta^{1/2}}{(1 - \theta)(n - t)^{1/2}}\right)\]
\[\geq (n - t)sd_0 \left(1 - \frac{2\theta^{1/2}}{1 - \theta}\right).\]

This implies (6) if

\[\frac{d_1}{d} \leq (\frac{n - t}{s - t})n \left(1 - \frac{2\theta^{1/2}}{1 - \theta}\right)(1 - \theta).\]

Since \(\frac{n(n - t)}{(s - t)n} \geq 1\), it is enough to verify that

\[\frac{d_1}{d} \leq (1 - \theta - 2\theta^{1/2}).\]

This is implied by

\[d_1 \leq d(1 - \theta^{1/3}).\]

Thus there will be \(d(1 - \theta^{1/3})\) edge-disjoint perfect matchings. \(\square\)

3 Illustrative case: random hypergraphs, \(k = 3\), \(\ell = 2\)

The proof of the general case (at least for \(\ell > k/2\)) is based on the same idea as for the case \(k = 3\), \(\ell = 2\) but is heavier on notation and will be therefore given afterwards. Hopefully, the reader will find it useful to consider the simplest case first.

We will construct the Hamilton cycles via the following algorithm:

\(A_1\): Choose \(r = n(np)^{1/2}\) random partitions \((X_i, Y_i), i = 1, 2, \ldots r\), of \(V\) into two sets of size \(\nu_2\).
We use the notation
\[ X_i = \{ x_{i,1} < x_{i,2} < \cdots < x_{i,\nu_2} \} \text{ and } Y_i = \{ y_{i,1} < y_{i,2} < \cdots < y_{i,\nu_2} \}. \]
For each \( i \) we choose a random permutation \( \sigma_i \) on \( X_i \) and define a sequence
\[ \Gamma_i = (x_{i,\sigma_i(1)}, x_{i,\sigma_i(2)}, \ldots, x_{i,\sigma_i(\nu_2)}, x_{i,\sigma_i(\nu_2+1)} = x_{i,\sigma_i(1)}). \]

\( A_2 \): At this point we expose the edges of \( H(n, p, 3) = ([n], \mathcal{E}_p) \).

\( A_3 \): Suppose now that for edge \( E \in \mathcal{E}_p \) there are \( f(E) \) instances \( i \) such that for some \( 1 \leq j = j(i) \leq \nu_2 \), we have \( X_i \cap E = \{ x_{i,\sigma_i(j)}, x_{i,\sigma_i(j+1)} \} \). If \( f(E) > 0 \) choose one of these \( f(E) \) instances at random and label the edge \( E \) with the chosen \( i \); if \( f(E) = 0 \), the edge \( E \) stays unlabelled. Let \( H_i \subset H \) be the subhypergraph of all edges labeled by \( i \).

\( A_4 \): Let \( G_i \) be the bipartite graph with vertex set \( A_i \cup B_i \) defined as follows: \( A_i, B_i \) are disjoint copies of \([\nu_2]\). Add edge \((a, b)\) to \( G_i \) if \( E = (x_{i,\sigma_i(a)}, y_{i,b}, x_{i,\sigma_i(a+1)}) \in H_i \) (i.e., \((a, b) \in G_i \) if the \( a \)-th pair of the sequence \( \Gamma_i \) united with the vertex \( y_{i,b} \) forms an edge \( E \) of \( H \) labeled by \( i \)).

\( A_5 \): We claim that \textbf{whp} (see Lemma 3 below) each \( G_i \) will contain at least \( n_0 = (1 - \delta)\nu_2p_0 \) edge-disjoint perfect matchings.

Here
\[ \delta = \left( \frac{72n \log n}{r} \right)^{1/2} \text{ and } f_0 = r\rho + (12r\rho \log n)^{1/2} \text{ and } p_0 = \frac{p}{f_0}. \] (8)
We set
\[ \rho = \frac{\nu_2^2}{(n^3)}. \]

**Remark 2** Note that \( \rho \) is the probability that instance \( i \) is one of the \( f(E) \) instances in \( A_3 \) for edge \( E \in \mathcal{E}_p \), as we will argue in Lemma 3. Also, we will argue that each graph \( G_i \) described above is a truly random bipartite graph with edge probability at least \( p_0 \). The parameters are set so as to ensure that \( p_0 \gg \log n/n \), thus enabling us to invoke Lemma 1.

Each such matching gives rise to a loose Hamilton cycle of \( H_i \) and these will be edge-disjoint by construction. Indeed suppose that our matching is \((e_a, \phi(e_a)), a = 1, 2, \ldots, \nu_2\), where the edges \( e_a \) are ordered according to the order of their appearance along the Hamilton cycle \( \Gamma_i \). From this we obtain the type 2 Hamilton cycle with edges \( E_a = e_a \cup \{ \phi(e_a) \} \). Since the subhypergraphs \( H_i \) are edge-disjoint and since distinct edges in the graph \( G_i \) correspond to distinct edges of \( H_i \), the so obtained Hamilton cycles in \( H \) are indeed edge-disjoint.
It follows that \textbf{whp} \( H(n, p, 3) \) contains at least \( rn_0 \) edge-disjoint Hamilton cycles. Finally,

\[
n_0 r = (1 - \delta)\nu_2 p_0 r = (1 - \delta)\frac{\nu_2 p}{f_0} r = (1 - \delta)\frac{\nu_2 pr}{r^2 p} r = (1 - \delta)\frac{\nu_2 pr}{r^2 p} r = (1 - \delta)\frac{12\log{n}}{r^2 p} \leq \frac{n^2}{r^2 p} (1 - \Theta(\log{n}/(np)^{1/2})) ,
\]

proving Theorem 2 for this case.

**Lemma 3**

\[ \Pr(G_i \text{ does not contain } n_0 \text{ edge-disjoint perfect matchings}) = o(n^{-3}). \]

**Proof**

For a triple \( E \subset [n] \), we say that \( 1 \leq i \leq r \) \textit{includes} \( E \) if the set \( E \cap X_i \) is of size 2 and is one of the edges of the cycle \( \Gamma_i \). Thus the random variable \( f(E) \) counts the number of partitions \( (X_i, Y_i) \) that include \( E \). Observe that the \( i \)-th partition includes a fixed triple \( E \) with probability

\[
\left(\begin{array}{c} 3 \\ n \end{array}\right) \left(\begin{array}{c} n - 3 \\ \nu_2 - 2 \end{array}\right) \frac{n^2}{\nu_2^2} = \frac{3n}{2(n - 1)(n - 2)} = \frac{\nu_2^2}{n^2} = \rho
\]

(first choose two elements of \( E \cap X_i \), then choose \( X_i \) to intersect \( E \) in exactly these two elements, then choose a random permutation \( \sigma_i \) \( X_i \) - due to symmetry the probability that \( E \cap X_i \) is one of the \( \nu_2 \) edges it defines is \( \nu_2^{(\nu_2)} \)). Moreover the events “\( i \) includes \( E \)” are mutually independent for different \( i \). Therefore, the random variable \( f(E) \) is distributed binomially with parameters \( r \) and \( \rho \). Now using the Chernoff bounds 2, 3, we see that with probability at least \( 1 - o(1) \) we have \( 1 \leq f(E) \leq f_0 \) for all \( \left(\begin{array}{c} n \\ 3 \end{array}\right) \) possible edges. So assume that indeed \( 1 \leq f(E) \leq f_0 \) for all \( E \). Moreover, the values of \( f(E) \) are determined by Step A1 of our construction and are thus independent of the appearance of random edges at Step A2. For \( 1 \leq a, b \leq \nu_2 \), the pair \((a, b)\) is an edge of the random auxiliary graph \( G_i \) if the corresponding triple \( E \) is an edge of the random hypergraph \( H \) and is chosen to be labeled by \( i \). Thus \((a, b) \in E(G_i)\) independently and with probability at least \( p/f_0 = p_0 \).

Therefore we can \textbf{whp} reduce our problem to showing that with probability \( 1 - o(n^{-3}) \) the random bipartite graph \( K_{\nu_2, \nu_2, p_0} \) satisfies has a family of \( n_0 \) edge disjoint perfect matchings. This is an immediate corollary of Lemma 1.
4 A general scheme

With a roadmap in mind, we proceed to the general case. There are six cases to deal with (three random and three pseudo-random) and they are all variations on the following scheme: It depends on parameters

\[ r, N_X, N_Y = n - N_X, k_X, k_Y, \nu_X = \frac{N_X}{k_X}, \nu_Y = \frac{N_Y}{k_Y}, \epsilon, f_0. \]

that vary with case.

Define

\[ \rho = \rho_k,\ell = \frac{\nu^2}{\binom{n}{k}}. \]

General Scheme:

\( S_1 \): Choose \( r \) random partitions \( (X_i, Y_i), i = 1, 2, \ldots r \), of \([n]\) into two sets of size \( N_X \) and \( N_Y \) respectively.

We use the notation

\[ X_i = \{x_{i,1} < x_{i,2} < \cdots < x_{i,N_X}\} \text{ and } Y_i = \{y_{i,1} < y_{i,2} < \cdots < y_{i,N_Y}\}. \]

\( S_2 \): For each \( i \) we let \( \sigma_i \) be a random permutation of \( X_i \) and let \( \tau_i \) be a random permutation of \( Y_i \). Form the partition \( X' = (X_{i,a}, a = 1, 2, \ldots, \nu_X) \) of \( X_i \) into sets of size \( k_X \) and the partition \( Y' = (Y_{i,b}, b = 1, 2, \ldots, \nu_Y) \) of \( Y_i \) into sets of size \( k_Y \). Here

\[ X_{i,a} = \{x_{i,\sigma_i((a-1)(k-\ell)+1)}, \ldots, x_{i,\sigma_i(a(k-\ell))}\}, \quad Y_{i,b} = \{y_{i,\tau_i((b-1)(2\ell-k)+1)}, \ldots, y_{i,\tau_i(b(2\ell-k))}\}. \]

\( S_3 \) This part of the scheme is more case specific. Some of the edges \( E \) will be represented by an edge of a graph \( G_i \).

(a) \( k/2 < \ell < k \):

\[ k_X = k - \ell, \quad k_Y = 2\ell - k, \quad N_X = \nu_k k_X, \quad N_Y = \nu_k k_Y. \]

We say that \( i \) is a candidate for \( E \in E \) if there exist \( a, b \) such that \( E = X_{i,a} \cup Y_{i,b} \cup X_{i,a+1} \).

(b) \( \ell = k/2 \):

\[ N_X = n, \quad N_Y = 0, \quad k_X = \ell, \quad k_Y = \ell. \]

We say that \( i \) is a candidate for \( E \in E \) if there exist \( a, b \) such that \( E = X_{i,a} \cup X_{i,b} \).

(c) \( \ell = k \):

\[ k_X = \lceil k/2 \rceil, \quad k_Y = \lceil k/2 \rceil, \quad N_X = \nu_k k_X, \quad N_Y = \nu_k k_Y. \]

We say that \( i \) is a candidate for \( E \in E \) if there exist \( a, b \) such that \( E = X_{i,a} \cup Y_{i,b} \).
\(S_4\): Suppose now that for \(E \in \mathcal{E}\) there are \(f(E)\) instances \(i\) that are a candidate for \(E\). If \(f(E) > 0\) then we choose one of the \(f(E)\) instances at random and label edge \(E\) with the chosen \(i = \psi(E)\). \(f(E) = 0\), the edge \(E\) stays unlabeled. Let \(H_i\) be the subhypergraph of \(H\) formed by the edges of \(H\) labeled by \(i\).

We will argue in all cases that
\[
\Pr((1 - \epsilon)f_0 \leq f(E) \leq f_0, \text{ for all } E \in \mathcal{E}) = 1 - o(1). \tag{9}
\]

\(S_5\): We now describe a graph \(G_i\) that is determined by the edges of \(H_i\). This part of the scheme is also case specific.

(a) \(k/2 < \ell < k\): \(G_i\) is the bipartite graph with vertex partition \(A_i \cup B_i\) comprising disjoint copies of \([\nu_\ell]\). For \(a \in A_i\) and \(b \in B_i\) we make \((a, b)\) an edge of \(G_i\) if \(E = X_{i,a} \cup Y_{i,b} \cup X_{i,a+1} \in \mathcal{E}\) and \(E\) is labeled with \(i\).

(b) \(\ell = k/2\): \(G_i\) is the graph with vertex set \([\nu_\ell]\), where \(a, b \in [\nu_\ell]\) are connected by an edge if \(X_{i,a} \cup X_{i,b}\) is an edge of \(H\) labeled by \(i\).

(c) \(\ell = k\): \(G_i\) is the bipartite graph with vertex partition \(A_i \cup B_i\) comprising disjoint copies of \([\nu_k]\). For \(a \in A_i\) and \(b \in B_i\) we make \((a, b)\) an edge of \(G_i\) if \(E = X_{i,a} \cup X_{i,b} \in \mathcal{E}\) and \(E\) is labeled with \(i\).

\(S_6\): Let
\[
p_0 = p/f_0.
\]

The construction succeeds:

(a) \(k/2 < \ell < k\): If for all \(i\), \(G_i\) contains \((1 - \epsilon)\nu_\ell p_0\) edge disjoint perfect matchings.

(b) \(\ell = k/2\): If for all \(i\), \(G_i\) contains \((1 - 20\epsilon)\nu_\ell r\nu_\ell\) edge-disjoint Hamilton cycles.

(c) \(\ell = k\): If for all \(i\), \(G_i\) contains \((1 - \epsilon)\nu_k p_0\) edge disjoint perfect matchings.

We argue now that each matching or cycle in \(S_6(a) - (c)\) gives rise to a type \(\ell\) Hamilton cycle in \(H\). We will argue that these Hamilton cycles are edge disjoint. Then it will be a matter of checking that the procedure succeeds \textbf{whp} once we have decided on the various parameters and that most edges of \(H\) are covered.

(a) \(k/2 < \ell < k\): A perfect matching of \(G_i\) can be expressed as a bijection \(f: A_i \rightarrow B_i\). This gives us the Hamilton cycle \(E_a = X_{i,a} \cup Y_{i,f(a)} \cup X_{i,a+1}, a = 1, 2, \ldots, \nu_\ell\) of \(H\). If \(a \neq a'\) then \(E_a \neq E_{a'}\) and an edge \(E\) can appear once only in \(G_{\psi(E)}\). Thus the Hamilton cycles produced by the algorithm are edge disjoint.

(b) \(\ell = k/2\): A Hamilton cycle of \(G_i\) can be expressed as a permutation \(\pi\) of \([\nu_\ell]\). This gives us the Hamilton cycle \(E_a = X_{i,a} \cup X_{i,\pi(a)}, a = 1, 2, \ldots, \nu_\ell\) of \(H\). The argument for Case (a) shows that the Hamilton cycles produced by the algorithm are edge disjoint.
(c) \( \ell = k \): A perfect matching of \( G_i \) gives us the Hamilton cycle \( E_a = X_{i,a} \cup Y_{i,f(a)}, a = 1, 2, \ldots, \nu_\ell \) of \( H \). The argument for Case (a) shows that the Hamilton cycles produced by the algorithm are edge disjoint.

We will now go through the six cases of the paper, one by one, showing that with carefully chosen values of \( r, N_X, k_X, k_Y, \epsilon, f_0 \), the above scheme can be shown to work whp.

We continue with the analysis of \( H(n, p, k) \). In this case we only expose the edges of \( H \) after the completion of Step \( S_2 \).

## 5 Random hypergraphs

We now provide a proof of the random case (Theorem 2). It is simpler than the proof for pseudo-random hypergraphs and hopefully will help in the understanding of the proofs of Theorems 3 and 4.

### 5.1 \( k/2 < \ell < k \)

Let

\[
r = n^{k-2}(np)^{1/2}, \quad \epsilon = \left( \frac{4(k+3)k! \log n}{\ell (np)^{1/2}} \right)^{1/2}, \quad f_0 = \rho r + (4k \rho r \log n)^{1/2}.
\]

Let also

\[
n_0 = (1 - \epsilon)\nu_ip_0.
\]

**Lemma 4**

\[
\Pr(G_i \text{ does not contain } n_0 \text{ edge-disjoint perfect matchings}) = o(n^{-k}).
\]

**Proof** The edges of \( G_i \) appear independently with probability \( p/f(E) \) where \( f(E) \) has distribution \( \text{Bin}(r, \rho) \), \( \rho = \rho_{k,\ell} = \frac{\nu_\ell^2}{\binom{k}{\ell}} \). (To see it, for a fixed partition \( (X_i, Y_i) \) and a fixed pair of permutations \( (\sigma_i, \tau_i) \) of \( X_i, Y_i \), resp., the index \( i \) includes \( \nu_\ell^2 \) \( k \)-tuples from \( [n] \). Therefore by symmetry a random \( i \) includes a fixed \( k \)-tuple \( E \) with probability \( \frac{\nu_\ell^2}{\binom{k}{\ell}} \).) Now \( \epsilon^2 \rho r \geq 4k \log n \) and so (2) and (3) imply (9).

We have thus reduced our problem to showing that with probability \( 1 - o(n^{-k}) \) the random bipartite graph \( K_{\nu_ip_0, \nu_ip_0} \) contains \( n_0 \) edge-disjoint perfect matchings. We can use \( p_0 \) as probability, because of (9). Applying Lemma 1 gives the desired result. \( \square \)
We have thus shown that \( \text{whp} \) \( H(n, p, k) \) contains at least

\[
(1 - \epsilon)r\nu_p p \geq (1 - 2\epsilon)\binom{n}{k} p/\nu_{\ell}
\]

edge-disjoint type \( \ell \) Hamilton cycles. This confirms Theorem 2 for this case.

### 5.2 \( \ell = k/2 \)

Let

\[
r = \omega n^{k-2}\epsilon^{-2}\log n, \quad \epsilon = \left(\frac{\omega^5 \log^{7/2} n}{n^{1/2}p^4}\right)^{1/11}, \quad f_0 = \rho r + (4k\rho r \log n)^{1/2}
\]

where \( \rho = \binom{n}{2}/\binom{k}{2} \).

The distribution of \( f(E) \) is \( \text{Bin}(r, \rho) \) and \( \epsilon^2 r\rho \gg \log n \) and so (9) holds by the Chernoff bounds.

**Lemma 5** Each \( G_i \) is \( \text{qs} \) \(((1 - 2\epsilon)p_0, 2\epsilon p_0)\)-regular.

**Proof** Given (9), the degree of vertex \( v \) in \( G_i \) dominates \( \text{Bin}(\nu_{\ell} - 1, p_0) \) and so Property \( Q_a \) holds from Chernoff bounds. Observe that \( \nu_{\ell}p_0 = \Omega((np)^{1/2}) \gg \log n \). Similarly the number of edges between two sets \( S, T \) dominates \( \text{Bin}(|S||T|, p_0) \) and is dominated by \( \text{Bin}(|S||T|, (1 - \epsilon)^{-1}p_0) \) and Property \( Q_b \) also holds from Chernoff bounds.

Applying Theorem 1 we see that \( \text{whp} \) \( H \) contains at least

\[
r \left( (1 - 2\epsilon)p_0/2 - 8\epsilon p_0 \right) \nu_{\ell} \geq \binom{n}{k} \frac{p}{\nu_{\ell}} (1 - 20\epsilon)
\]

edge-disjoint type \( k/2 \) Hamilton cycles, completing the proof of Theorem 2 for this case. (It is the second inequality in (1) that determines the value of \( \epsilon \) here).

### 5.3 \( \ell = k \)

Let

\[
r = n^{k-2}(np)^{1/2}, \quad \epsilon = 10k! \left(\frac{4(k + 3)k^2 \log n}{k!(np)^{1/2}}\right)^{1/2}, \quad f_0 = \rho r + (4k\rho r \log n)^{1/2}
\]

where \( \rho = \rho_{k,k} \).

It is left to argue that \( \text{whp} \) each \( G_i \) contains at least \( (1 - \epsilon)\nu_k p_0 \) edge-disjoint perfect matchings.
Lemma 6

\[ \Pr(G_i \text{ does not contain } (1 - \epsilon)\nu_k p_0 \text{ edge-disjoint perfect matchings}) = o(n^{-k}). \]

Proof \quad Equation (9) holds, as in the case \( k/2 < \ell < k \). The claim follows again from Lemma 1.

We have thus shown that \( \text{whp } H(n, p, k) \) contains at least

\[ (1 - \epsilon)\nu_k p_0 = \frac{(1 - \epsilon)\nu_k p}{pr + (4kpr \log n)^{1/2}} \geq (1 - 2\epsilon) \binom{n}{k} p / \nu_k \]

edge-disjoint type \( k \) Hamilton cycles. This confirms Theorem 2 for this case.

6 Pseudo-random hypergraphs

In this section we prove Theorems 3, 4 and 5. We follow the same strategy as described in Section 4. There are complications caused by the notation that we have to add and also by the fact that \( H \) is not random.

6.1 \( k/2 < \ell < k \) (Theorem 3).

We first choose \( f_0 \) such that

\[ \frac{\log^2 n}{\epsilon^4} \ll f_0^2 \ll \frac{\epsilon n^{1/2} p^2}{\log n}. \] (12)

We then choose

\[ r = \left(1 - \frac{\epsilon}{2}\right) \binom{n}{k} f_0 / \nu_\ell^2. \]

\( f(E) \) is distributed as \( \text{Bin}(r, \rho_{k, \ell}) \) for each edge of \( H \) and the lower bound on \( f_0 \) together with the definition of \( r \) will then imply that (9) holds.

(The two inequalities in (12) determine our lower bound on \( \epsilon \).)

We know from Lemma 2 that if we can prove that the degrees and co-degrees of our bipartite graphs \( G_i \) “behave”, then we can deduce the existence of many disjoint perfect matchings and so get our packing of Hamilton cycles. Given Lemma 2, all we need to do is to estimate the degrees and co-degrees of vertices in a fixed \( G_i \).

Lemma 7 Whp, over our random choices of \( X_i, Y_i, \sigma_i, \tau_i \), each \( G_i \) has minimum degree at least \( (1 - 2\epsilon)\nu_i p_0 \) and maximum co-degree of at most \( (1 + 5\epsilon)\nu_i p_0^2 \).
Proof We fix $i$ and focus on $G_i$. We first show that the minimum degree in $G_i$ is large. We first fix $a \in A_i$. The vertex $a$ corresponds to the block $X_{i,a}$ of $\sigma_i$. Condition on $X_{i,a} \cup X_{i,a+1} = S$ for some $S \subseteq [n], \left| S \right| = 2(k - \ell)$. We expose a random subset $Y_i$ first. Let $Z_a^*$ be the number of edges $E \in \mathcal{E}$ such that $S \subseteq E$ and $E \cap Y_i = E - S$. For each edge $E \in N_H(S)$

\[
\Pr(E \cap Y_i = E - S) = \frac{(N_y - (2k - k))}{(n - 2(k - \ell))} = \left(1 + O\left(\frac{1}{n}\right)\right) \left(\frac{2k - k}{\ell}\right)^{2\ell - k}.
\]

Therefore by assumption $P_a$,

\[
\mathcal{E}(Z_a^*) \geq (1 - \epsilon) \left(1 + O\left(\frac{1}{n}\right)\right) \left(\frac{2k - k}{\ell}\right)^{2\ell - k} \left(n - 2(k - \ell)\right) p.
\]

Since changing the fate of one vertex with respect to the choice of $Y_i$ changes the value of $Z_a^*$ by at most

\[
\Delta_a = \max_{S' \subseteq ([n] \setminus S)} d_H(S'),
\]

and the latter quantity is bounded by $(1 + \epsilon)(2n - 1)p$ by assumption $P_a$, we get by the Azuma-Hoeffding inequality that for any $t > 0$

\[
\Pr(Z_a^* \leq \mathcal{E}(Z_a^*) - t) \leq \exp\left\{-\frac{2t^2}{n\Delta_a^2}\right\}.
\]

(13)

Here we are using the following inequality: Let $S_n$ denote the set of permutations of $[n]$ and let $f : S_n \to \mathbb{R}$ be such that $|f(\pi) - f(\pi')| \leq u$ whenever $\pi'$ is obtained from $\pi$ by transposing two elements. Then if $\pi$ is chosen randomly from $S_n$ then

\[
\Pr(f(\pi) - f(\pi) \leq - t) \leq \exp\left\{-\frac{2t^2}{nu^2}\right\}.
\]

(14)

For a proof see e.g., Section 3.2 of [19] or Lemma 11 of [12].

In this context, think of choosing a random $m$-subset of $[n]$ as choosing a random $\pi$ and then taking the first $m$ elements as your subset.

Plugging in the estimates on $\mathcal{E}(Z_a^*)$ and $\Delta_a$ stated above in (14), we get that $qs$ for every $a \in A_i$,

\[
Z_a^* \geq (1 - \epsilon) \left(1 + O\left(\frac{1}{n}\right)\right) \left(\frac{2k - k}{\ell}\right)^{2\ell - k} \left(n - 2(k - \ell)\right) p - n^{2\ell - k - 1/2} p \log n
\]

\[\geq (1 - \frac{3}{2}\epsilon) \left(\frac{2k - k}{\ell}\right)^{2\ell - k} \left(\frac{n}{2\ell - k}\right) p.
\]

(15)

So assume that $Y_i$ is chosen so that (15) holds. Now we expose the random permutation $\tau_i$ of $Y_i$. Let $Z_a$ be the degree of $a$ in $G_i$, which is the number of edges $E \in \mathcal{E}$ such that
1. \( E \cap Y_i = E - S \) (the number of such edges is \( Z^*_a \));
2. \( E \cap Y_i \) forms a block \( Y_{i,b} \) under \( \tau_i \);
3. \( E \) is labeled by \( i \) (this happens independently and with probability \( 1/f(E) \geq 1/f_0 \)).

Hence,
\[
E(Z_a) \geq Z^*_a \frac{\nu \ell}{(N_{\ell-k})} \frac{1}{f_0}.
\]

Observe that changing \( \tau_i \) by a single transposition changes the value of \( Z_a \) by at most 2 (at most two blocks \( Y_{i,b} \) are affected by such a change). Therefore, applying concentration results for permutation graphs we get that for any \( t > 0 \)
\[
\Pr(Z_a \leq E(Z_a) - t) \leq \exp \left\{ -\frac{t^2}{2N_Y} \right\}.
\]

Thus \( qs \) for every partition \( i \) and for every \( a \in A_i \), its degree is \( G_i \) is at least
\[
(1 - \frac{3}{2}\ell) \left( \frac{2\ell - k}{\ell} \right)^{2\ell-k} \left( \frac{n}{2\ell - k} \right)p \frac{\nu \ell}{(N_{\ell-k})} \frac{1}{f_0} - n^{1/2} \log n \geq (1 - 2\ell)\nu \ell p_0,
\]
due to our assumption on \( \epsilon \).

The argument for the degrees of the vertices of \( B_i \) is quite similar. Fix \( b \in B_i \). The vertex \( b \) corresponds to the block \( Y_{i,b} \) of \( \tau_i \). Condition on \( Y_{i,b} = S \) for some \( S \subset [n], |S| = 2\ell - k \). We expose a random subset \( X_i \) first. Let \( Z^*_b \) be the number of edges \( E \in \mathcal{E} \) such that \( S \subset E \) and \( E \cap X_i = E - S \). For each edge \( E \in N_H(S) \)
\[
\Pr(E \cap X_i = E - S) = \frac{(N_X - k n - (2\ell-k))}{(n-k)} = \frac{(N_X)2k-2\ell}{(n-(2\ell-k))2k-2\ell} = \left( 1 + O \left( \frac{1}{n} \right) \right) \left( \frac{k - \ell}{\ell} \right)^{2k-2\ell}.
\]

Therefore by assumption \( P_a \),
\[
E(Z^*_b) \geq (1 - \epsilon) \left( 1 + O \left( \frac{1}{n} \right) \right) \left( \frac{k - \ell}{\ell} \right)^{2k-2\ell} \left( \frac{n - 2\ell + k}{2(k - \ell)} \right)p.
\]

Since changing the fate of one vertex with respect to the choice of \( X_i \) changes the value of \( Z^*_b \) by at most
\[
\Delta_b = \max_{S' \in ([n] \setminus \ell-k+1)} d_H(S'),
\]
and the latter quantity is bounded by \( (1 + \epsilon)\left( \frac{n}{2k-2\ell-1} \right)p \) by assumption \( P_a \), we get by (14) that for any \( t > 0 \)
\[
\Pr(Z^*_b \leq E(Z^*_b) - t) \leq \exp \left\{ -\frac{2t^2}{n\Delta_b^2} \right\}.
\]
Plugging in the estimates on $\mathbf{E}(Z^*_b)$ and $\Delta_b$ stated above, we get that $qs$ for every $b \in B_i$,
\[
Z^*_b \geq (1 - \epsilon) \left( 1 + O \left( \frac{1}{n} \right) \right) \left( \frac{k - \ell}{\ell} \right)^{2k - 2\ell} \left( \frac{n - 2\ell + k}{2(k - \ell)} \right) p - n^{2k - 2\ell - 1/2} p \log n \\
\geq (1 - \frac{3}{2} \epsilon) \left( \frac{k - \ell}{\ell} \right)^{2k - 2\ell} \left( \frac{n}{2k - 2\ell} \right) p.
\]
So assume that $X_i$ is chosen so that (16) holds. Now we expose the random permutation $\sigma_i$ of $X_i$. Let $Z_b$ be the degree of $b$ in $G_i$, which is the number of edges $E \in \mathcal{E}$ such that
\[1. \ E \cap X_i = E \setminus S \ (\text{the number of such edges is } Z^*_b); \]
\[2. \ E \cap X_i \text{ forms two consecutive blocks } X_{i,a}, X_{i,a+1} \text{ under } \sigma_i; \]
\[3. \ E \text{ is labeled by } i \ (\text{this happens independently and with probability } 1 / f(E) \geq 1 / f_0). \]
Hence,
\[
\mathbf{E}(Z_b) \geq Z^*_b \frac{\nu_k}{\binom{n}{2k - 2\ell}} \frac{1}{f_0}.
\]
Observe that changing $\sigma_i$ by a single transposition changes the value of $Z_b$ by at most $4$.
Therefore, applying again concentration results for permutation graphs we get that for any $t > 0$
\[
\Pr(Z_b \leq \mathbf{E}(Z_b) - t) \leq \exp \left\{ - \frac{t^2}{8N_X} \right\}.
\]
Thus $qs$ for every partition $i$ and for every $b \in B_i$, its degree is $G_i$ is at least
\[
(1 - \frac{3}{2} \epsilon) \left( \frac{k - \ell}{\ell} \right)^{2k - 2\ell} \left( \frac{n}{2k - 2\ell} \right) p \frac{\nu_k}{\binom{n}{2k - 2\ell}} \frac{1}{f_0} - n^{1/2} \log n \geq (1 - 2\epsilon) \nu_k p_0,
\]
due to our assumption on $\epsilon$.
Now we treat typical co-degrees in the graph $G_i$. First fix $b_1, b_2 \in B_i$ and $Y_{i,b_1}, Y_{i,b_2}$ and expose a random set $X_i$. Let $Z^{*}_{b_1,b_2}$ be the number of subsets $S_1 \subset [n]$ of cardinality $|S_1| = 2(k - \ell)$ such that $S_1 \subset X_i$ and both $S_1 \cup Y_{i,b_1}$ and $S_1 \cup Y_{i,b_2}$ form an edge in $\mathcal{E}$. By our assumption $P_b$,
\[
\mathbf{E}(Z^{*}_{b_1,b_2}) \leq (1 + \epsilon) \left( \frac{k - \ell}{\ell} \right)^{2k - 2\ell} \left( \frac{n - 2(2\ell - k)}{2(k - \ell)} \right) p^2.
\]
Using assumption $P_a$ we see that changing $X_i$ by one element changes $Z^{*}_{b_1,b_2}$ by at most
\[
\Delta_{b_1,b_2} = \max_{S \in \binom{[n]}{2\ell - k + 1}} |N_H(S)| \leq (1 + \epsilon) \binom{n}{2(k - \ell) - 1} p.
\]
Applying (14) we see that $q_s$ for every $b_1, b_2 \in B_i$,

$$Z^*_{b_1, b_2} \leq \left(1 + \frac{3}{2} \epsilon\right) \left(\frac{k - \ell}{\ell}\right)^{2(k - \ell)} \left(\frac{n}{2(k - \ell)}\right)^2 p^2. \quad (17)$$

Assume $X_i$ is chosen so that (17) holds. Expose the random permutation $\sigma_i$ of $X_i$. Let $Z_{b_1, b_2}$ be the co-degree of $b_1, b_2$ in $G_i$, which is the number of blocks $X_{i,a}$ of $X_i$ under $\sigma_i$ such that $E_1 = X_{i,a} \cup X_{i,a+1} \cup Y_{i,b_1}$, $E_2 = X_{i,a} \cup X_{i,a+1} \cup Y_{b_2} \in E$, and both edges $E_1, E_2$ are labeled by $i$. Then, recalling that an edge $E \in E$ is labeled by $i$ with probability $\frac{1}{f(E)} \leq \frac{1}{(1 - \frac{2}{3}) f_0}$, we get

$$\mathbb{E}(Z_{b_1, b_2}) \leq Z^*_{b_1, b_2} \frac{\nu_\ell}{(2k - 2\ell)} \frac{1}{(1 - \frac{3}{2} \epsilon)^2 f_0^2}.$$ 

Transposing one pair of elements of $\sigma_i$ changes $Z_{b_1, b_2}$ by at most 4. Using (14) again, we obtain that $q_s$ for every partition $i$ and every pair $b_1, b_2 \in B_i$, the co-degree $Z_{b_1, b_2}$ of $b_1, b_2$ in $G_i$ satisfies:

$$Z_{b_1, b_2} \leq \left(1 + \frac{3}{2} \epsilon\right) \left(\frac{k - \ell}{\ell}\right)^{2k - 2\ell} \left(\frac{n}{2k - 2\ell}\right)^2 p^2 \frac{\nu_\ell}{(2k - 2\ell)} \frac{1}{(1 - \frac{3}{2} \epsilon)^2 f_0^2} - n^{1/2} \log n \leq (1 + 5\epsilon) \nu_\ell p_0^2.$$ 

Now consider $a_1, a_2 \in A_i$ and $X_{i,a_1}, X_{i,a_1+1}, X_{i,b_1}, X_{i,b_1+1}$ and expose a random set $Y_i$. Let $Z*_{a_1, a_2}$ be the number of subsets $S_i \subseteq [n]$ of cardinality $|S_i| = 2\ell - k$ such that $S_i \in Y_i$ and both $S_i \cup X_{i,a_1} \cup X_{i,a_1+1}$ and $S_i \cup X_{i,a_2} \cup X_{i,a_2+1}$ form an edge in $E$. By our assumption $P_b$, 

$$\mathbb{E}(Z^*_{a_1, a_2}) \leq (1 + \epsilon) \left(\frac{2\ell - k}{\ell}\right)^{2\ell - k} \left(\frac{n}{2\ell - k}\right)^2 p^2.$$ 

Using assumption $P_a$ we see that changing $Y_i$ by one element changes $Z^*_{a_1, a_2}$ by at most

$$\Delta_{a_1, a_2} = \max_{S \in [n]} |N_H(S)| \leq (1 + \epsilon) \left(\frac{n}{2\ell - k - 1}\right) p.$$ 

Applying (14) we see that $q_s$ for every $b_1, b_2 \in B_i$,

$$Z^*_{a_1, a_2} \leq \left(1 + \frac{3}{2} \epsilon\right) \left(\frac{k - \ell}{\ell}\right)^{2k - 2\ell} \left(\frac{n}{2k - 2\ell}\right)^2 p^2. \quad (18)$$

Assume $Y_i$ is chosen so that (18) holds. Expose the random permutation $\tau_i$ of $Y_i$. Let $Z_{a_1, a_2}$ be the co-degree of $a_1, a_2$ in $G_i$, which is the number of blocks $Y_{i,b}$ of $Y_i$ under $\tau_i$ such that
\[ E_1 = X_{i,a_1} \cup X_{i,a_1+1} \cup Y_{i,b}, \quad E_2 = X_{i,a_2} \cup X_{i,a_2+1} \cup Y_{i,b} \in \mathcal{E}, \text{ and both edges } E_1, E_2 \text{ are labeled by } i. \] Then, recalling that an edge \( E \in \mathcal{E} \) is labeled by \( i \) with probability \( \frac{1}{f(E)} \leq \frac{1}{(1-\frac{3}{2} \epsilon)} f_0 \), we get

\[ \mathbf{E}(Z_{a_1,a_2}) \leq Z_{a_1,a_2}^* \frac{\nu_t}{(\frac{N}{2\ell-k})} \frac{1}{(1-\frac{3}{2} \epsilon)^2 f_0^2}. \]

Transposing one pair of elements of \( \tau_i \) changes \( Z_{a_1,a_2} \) by at most 4. Using (14) again, we obtain that \( q_s \) for every partition \( i \) and every pair \( a_1, a_2 \in B_i \), the co-degree \( Z_{a_1,a_2} \) of \( a_1, a_2 \) in \( G_i \) satisfies:

\[ Z_{a_1,a_2} \leq \left(1 + \frac{3}{2} \epsilon \right) \left(\frac{2\ell - k}{\ell}\right)^{2\ell-k} \left(\frac{n}{2\ell - k}\right)^p \frac{\nu_t}{(\frac{N}{2\ell-k})} \frac{1}{(1-\frac{3}{2} \epsilon)^2 f_0^2} - n^{1/2} \log n \]

\[ \leq (1 + 5\epsilon) \nu \ell p_0^2. \]

We can now apply Lemma 2 below with \( n = \nu_t, d = p_0, \theta = 5\epsilon \) to show that each \( G_i \) contains at least \( (1- (5\epsilon)^{1/3}) \nu q p_0 \geq (1-2\epsilon^{1/3})m/rn \) edge-disjoint perfect matchings. This completes the proof of Theorem 3 for the case \( k/2 < \ell < k \).

### 6.2 \( \ell = k/2 \) (Theorem 4).

We first choose \( f_0 \) such that

\[ \frac{\log n}{\epsilon^2} \ll f_0 \ll \epsilon^{3/4} p n^{1/8} \log^{1/8} n. \]

We then choose

\[ r = \frac{(2 - \epsilon) \binom{n}{k} f_0}{\nu_t^2}. \]

The random variable \( f(E) \) is distributed as Bin\((r, \rho)\) where \( \rho = \frac{\binom{\nu_t}{k}}{\binom{n}{k}} \) and so with the above parameter definitions, (9) holds whp.

**Lemma 8** Let \( G \) be a \( \nu \) vertex, \((p, \epsilon)\)-regular graph. Suppose that \( G_0 \) is the random subgraph of \( G \) where each edge \( e \) of \( G \) is included independently with probability \( p_e \), where \( (1-\epsilon)p^* \leq p_e \leq p^* \). Suppose that

\[ \epsilon^2 \nu p^* \gg \log \nu \text{ and } \epsilon^3 \nu p^* \gg \log 1/\epsilon. \]

Then \( G_0 \) is \(((1-2\epsilon)p^*, 2\epsilon p^*)\)-regular, qs.

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The degree of vertex $v$ in $G_0$ dominates $\text{Bin}(p\nu, (1-\epsilon)p^*)$ and so Property $Q_a$ holds from Chernoff bounds. Similarly the number of edges between two sets $S, T$ dominates $\text{Bin}((p-\epsilon)|S||T|, (1-\epsilon)p^*)$ and is dominated by $\text{Bin}((p-\epsilon)|S||T|, p^*)$ and Property $Q_b$ also holds from Chernoff bounds.

We apply Lemma 8 to each $G_{\mathcal{P}_i}$, where $\mathcal{P}_i$ is the partition induced by $\sigma_i$. Here $p^* = (1-\epsilon)^{-1}f_0$. Now by assumption $G_{\mathcal{P}_i}$ is $(p, \epsilon)$ regular. It follows that $\text{qs}$, $G_i$ is $((1-3\epsilon)p/f_0, 2\epsilon p/f_0)$ regular. Applying Theorem 1 and using the upper constraint on $f_0$ we see that $\text{qs}$ each $G_i$ contains $p(1-3\epsilon)/2f_0 - 8\epsilon p/f_0)\nu_i$ edge disjoint type $\ell$ Hamilton cycles. Thus $H$ contains $r$ times as many edge disjoint Hamilton cycles, verifying this case.

6.3 $\ell = k$ (Theorem 5).

Here the aim is to find many edge-disjoint perfect matchings. We first choose $f_0$ such that (12) holds and then choose

$$r = (1-\epsilon)\left(\frac{n}{k}\right)\frac{f_0}{\nu_k^2}.$$

We claim (see Lemmas 2 and 9) that $\text{whp}$ each $G_i$ will contain at least

$$n_0 = (1 - 2\epsilon^{1/3})\nu_k p_0$$

edge-disjoint perfect matchings. So $H$ will contain at least $rn_0$ edge-disjoint perfect matchings, completing the proof of Theorem 5.

**Lemma 9** $\text{Whp}$, over our random choices of $X_i, Y_i, \sigma_i, \tau_i$, each $G_i$ has minimum degree at least $2\epsilon (1 - 2\epsilon) \nu_k p_0$ and maximum co-degree at most $(1 + 5\epsilon)\nu_k p_0^2$.

**Proof** The arguments here are very similar to those in Lemma 7, so we will be rather brief. We fix $i$ and focus on $G_i$. We first show that the minimum degree in $G_i$ is large. For $a \in A_i$, denote by $Z_a$ its degree in $G_i$. Then, using assumption $P_a$ and martingale-type arguments, we can show that

$$\mathbb{E}(Z_a) \geq \left(1 - \frac{3}{2}\epsilon\right)\left(\frac{k_Y}{k}\right)^{k_Y} \left(\frac{n}{k_Y}\right)^p \frac{\nu_k}{\left(\frac{k_Y}{k}\right) f_0} \frac{1}{f_0}.$$

Using concentration results for permutation graphs again, we derive that $\text{qs}$ for every partition $i$ and every $a \in A$, the degree of $a$ in $G_i$ is at least

$$\left(1 - \frac{3}{2}\epsilon\right)\left(\frac{k_Y}{k}\right)^{k_Y} \left(\frac{n}{k_Y}\right)^p \frac{\nu_k}{\left(\frac{k_Y}{k}\right) f_0} \frac{1}{f_0} - n^{1/2} \log n \geq (1 - 2\epsilon)\nu_k p_0,$$

due to our assumption on $\epsilon$. 

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Let now $Z_b$ denote the degree of vertex $b \in B_i$ in $G_i$. We can argue similarly, while invoking assumption $P_a$ to show that $qs$ for every partition $i$ and every $b \in B_i$,

$$Z_b \geq (1 - 2\epsilon) \nu_k p_0.$$ 

Finally, we treat the co-degrees of pairs of vertices in $G_i$. Let $b_1, b_2 \in B_i$. Let $Z_{b_1,b_2}$ be their co-degree in $G_i$. Then using assumption $R_e$ and martingale-type concentration arguments, we can prove that $qs$ for every partition $i$ and every pair of vertices $b_1, b_2 \in B_i$

$$Z_{b_1,b_2} \leq \left(1 + \frac{3}{2} \epsilon\right) \left(\frac{k_X}{k}\right)^{k_X} \binom{n}{k_X} p^2 \frac{\nu_k}{\binom{n_X}{k_X}} \frac{1}{1 - \frac{3}{2} \epsilon} f_0^2 - n^{1/2} \log n$$

$$\leq (1 + 5\epsilon) \nu_k p_0^2.$$ 

Similarly, if $a_1, a_2 \in A_i$, let $Z_{a_1,a_2}$ be their co-degree in $G_i$. Then using assumption $P_b$ and martingale-type concentration arguments, we can prove that $qs$ for every partition $i$ and every pair of vertices $a_1, a_2 \in A_i$

$$Z_{a_1,a_2} \leq \left(1 + \frac{3}{2} \epsilon\right) \left(\frac{k_Y}{k}\right)^{k_Y} \binom{n}{k_Y} p^2 \frac{\nu_k}{\binom{n_Y}{k_Y}} \frac{1}{1 - \frac{3}{2} \epsilon} f_0^2 - n^{1/2} \log n$$

$$\leq (1 + 5\epsilon) \nu_k p_0^2.$$ 

\hfill \Box

7 Concluding remarks

In this paper we have derived sufficient conditions for packing almost edges of $k$-uniform random and pseudo-random hypergraphs into disjoint type $\ell$ Hamilton cycles. This appears to be a first result of this kind for the problem of packing Hamilton cycles in this setting. There is no reason to believe our assumptions on the edge probability $p(n)$ or the density of a pseudo-random hypergraph are tight, and it would be quite natural to try and extend them and to obtain tight(er) bounds.

In the paper [11] we were able to show how to use the results of [9] in a game theoretic setting. More precisely, we showed how to play a Maker-Breaker type of game on the complete graph where Maker is able to construct an $(1/2 - \epsilon, \epsilon)$-regular graph, $\epsilon = o(1)$. We could then use the results of [9] to show that Maker could construct approximately $n/4$ edge-disjoint Hamilton cycles when alternately choosing edges against an adversary. The techniques of that paper can be extended to the hypergraph setting in a straightforward manner.

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References


A Claimed improvement for Theorem 1

The proof in [9] requires

1. \( \epsilon^2 n/\alpha \gg \log n \) and \( \epsilon^3 n/\alpha \gg \log(1/\epsilon) \) for the proof of Lemma 1.

2. \( \alpha \epsilon^3 \gg \frac{1}{(n \log n)^{1/2}} \) is needed to verify condition Q2.

With these inequalities, the proof of Theorem 1 of [9] goes through.