Between 2- and 3-colorability

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Abstract

We consider the question of the existence of homomorphisms between $G_{n,p}$ and odd cycles when $p = c/n, 1 < c \leq 4$. We show that for any positive integer $\ell$, there exists $\varepsilon = \varepsilon(\ell)$ such that if $c = 1 + \varepsilon$ then w.h.p. $G_{n,p}$ has a homomorphism from $G_{n,p}$ to $C_{2\ell+1}$ so long as its odd-girth is at least $2\ell+1$. On the other hand, we show that if $c = 4$ then w.h.p. there is no homomorphism from $G_{n,p}$ to $C_5$. Note that in our range of interest, $\chi(G_{n,p}) = 3$ w.h.p., implying that there is a homomorphism from $G_{n,p}$ to $C_3$. These results imply the existence of random graphs with circular chromatic numbers $\chi_c$ satisfying $2 < \chi_c(G) < 2 + \delta$ for arbitrarily small $\delta$, and also that $2.5 \leq \chi_c(G_{n,\frac{4}{n}}) < 3$ w.h.p.

1 Introduction

The determination of the chromatic number of $G_{n,p}$, where $p = \frac{c}{n}$ for constant $c$, is a central topic in the theory of random graphs. For $0 < c < 1$, such graphs contain, in expectation, a bounded number of cycles, and are almost-surely 3-colorable. The chromatic number of such a graph may be 2 or 3 with positive probability, according as to whether or not any odd cycles appear.

For $c \geq 1$, we find that the chromatic number $\chi(G_{n,\frac{c}{n}}) \geq 3$ with high probability, and letting $c_k := \sup_c \chi(G_{n,\frac{c}{n}}) \leq k$, it is known for all $k$ and $c \in (c_k, c_{k+1})$ that $\chi(G_{n,\frac{c}{n}}) \in \{k, k + 1\}$, see Łuczak [7] and Achlioptas and Naor [2]; for $k > 2$, the chromatic number may well be concentrated on the single value $k$, see Friedgut [5] and Achlioptas and Friedgut [1].
In this paper, we consider finer notions of colorability for the graphs $G_{n, c}$ for $c \in (1, c_3)$, by considering homomorphisms from $G_{n, c}$ to odd cycles $C_{2\ell+1}$. A homomorphism from a graph $G$ to $C_{2\ell+1}$ implies a homomorphism to $C_{2k+1}$ for $k < \ell$. As the 3-colorability of a graph $G$ corresponds to the existence of a homomorphism from $G$ to $K_3$, the existence of a homomorphism to $C_{2\ell+1}$ implies 3-colorability. Thus considering homomorphisms to odd cycles $C_{2\ell+1}$ gives a hierarchy of 3-colorable graphs amenable to increasingly stronger constraint satisfaction problems. Note that a fixed graph having a homomorphism to all odd-cycles is bipartite.

Our main result is the following:

**Theorem 1.** For any $\ell > 1$, there is an $\varepsilon > 0$ such that with high probability, $G_{n, 1+\varepsilon/n}$ either has odd-girth $< 2\ell + 1$ or has a homomorphism to $C_{2\ell+1}$.

Conversely, we expect the following:

**Conjecture 1.** For any $c > 1$, there is an $\ell_c$ such that with high probability, there is no homomorphism from $G_{n, c}$ to $C_{2\ell+1}$ for $\ell \geq \ell_c$.

As $c_3$ is known to be at least 4.03, the following confirms Conjecture 1 for a significant portion of the interval $(1, c_3)$.

**Theorem 2.** For any $c > 2.774$, there is an $\ell_c$ such that with high probability, there is no homomorphism from $G_{n, c}$ to $C_{2\ell+1}$ for $\ell \geq \ell_c$.

We also have that $\ell_4 = 2$:

**Theorem 3.** With high probability, $G_{n, 4}$ has no homomorphism to $C_5$.

Note that as $c_3 > 4.03 > 4$, we see that there are triangle-free 3-colorable random graphs without homomorphisms to $C_5$. Our proof of Theorem 3 involves computer assisted numerical computations. The same calculations which rigorously demonstrate that $\ell_4 = 2$ suggest actually that $\ell_{3.75} = 2$ as well.

Our results can be reformulated in terms of the circular chromatic number of a random graph. Recall that the circular chromatic number $\chi_c(G)$ of $G$ is the infimum $r$ of circumferences of circles $C$ for which there is an assignment of open unit intervals of $C$ to the vertices of $G$ such that adjacent vertices are assigned disjoint intervals. (Note that if circles $C$ of circumference $r$ were replaced in this definition with line segments $S$ of length $r$, then this would give the ordinary chromatic number $\chi(G)$.) It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, that $\chi_c(G)$ is always rational, and moreover, that $\chi_c(G) \leq \frac{p}{q}$ if and only if $G$ has a homomorphism to the circulant graph $C_{p,q}$ with vertex set $\{0, 1, \ldots, q-1\}$, with $v \sim u$ whenever $\text{dist}(v, u) := \min\{|v-u|, v+q-u, u+q-v\} \geq q$. (See [9].) Since $C_{2\ell+1, \ell}$ is the odd cycle $C_{2\ell+1}$ our results can be restated as follows:
Theorem 4. In the following, inequalities for the circular chromatic number hold with high probability.

1. For any $\delta > 0$, there is an $\epsilon > 0$ such that, $G = G_n^{1+\epsilon/n}$ has $\chi_c(G) \leq 2 + \delta$ unless it has odd girth $\leq \frac{2}{\delta}$.

2. For any $c > 2.774$, there exists $r > 2$ such that $\chi_c(G_n^{c/n}) > r$.

3. $2.5 \leq \chi_c(G_n^{1/n}) < 3$.

Note that for any $c$ and $\ell > 1$, there is positive probability that $G_n^{c/n}$ has odd girth $< 2\ell + 1$, and a positive probability that it does not. In particular, as the probability that $G_n^{c/n}$ has small odd-girth can be computed precisely, Theorem 1 gives an exact probability in $(0,1)$ that $G_n^{1+\epsilon/n}$ has a homomorphism to $C_{2\ell+1}$. Indeed, Theorem 1 implies that if $c = 1 + \epsilon$ and $\epsilon$ is sufficiently small relative to $\ell$, then

$$\lim_{n \to \infty} \Pr(\chi_c(G_n^{c/n}) \in (2 + \frac{1}{\ell+1}, 2 + \frac{1}{\ell}]) = e^{-\phi_\ell(c)} - e^{-\phi_{\ell+1}(c)},$$

(1)

where

$$\phi_\ell(c) = \sum_{i=1}^{\ell-1} \frac{c^{2i+1}}{2(2i + 1)}.$$

We close with two more conjectures. The first concerns a sort of pseudo-threshold for having a homomorphism to $C_{2\ell+1}$:

**Conjecture 2.** For any $\ell$, there is a $c_\ell > 1$ such that $G_n^{c,n}$ has no homomorphism to $C_{2\ell+1}$ for $c > c_\ell$, and has either odd-girth $< 2\ell + 1$ or has a homomorphism to $C_{2\ell+1}$ for $c < c_\ell$.

The second asserts that the circular chromatic numbers of random graphs should be dense.

**Conjecture 3.** There are no real numbers $2 \leq a < b$ with the property that for any value of $c$, $\Pr(\chi_c(G_n^{c/n}) \in (a,b)) \to 0$.

Note that our Theorem 1 confirms this conjecture for the case $a = 2$.

2 Structure of the paper

We prove Theorem 1 in Section 3. We first prove some structural lemmas and then we show, given the properties in these lemmas, that we can algorithmically find a homomorphism. We prove Theorem 2 in Section 4 by the use of a simple first moment argument. We prove Theorem 3 in Section 5. This is again a first moment calculation, but it has required numerical assistance in its proof.
3 Finding homomorphisms

Lemma 1. If \( \alpha < 1/10 \) and \( c \) is a positive constant where
\[
c < c_0 = \exp \left\{ \frac{1 - 6\alpha}{3\alpha} \right\}
\]
then w.h.p. any two cycles of length less than \( \alpha \log n \) in \( G_{n,p}, p = \frac{c}{n} \), are at distance more than \( \alpha \log n \).

Proof If there are two cycles contradicting the above claim, then there exists a set \( S \) of size \( s \leq 3\alpha \log n \) that contains at least \( s + 1 \) edges. The expected number of such sets can be bounded as follows:
\[
\sum_{s=4}^{3\alpha \log n} \binom{n}{s} \left( \frac{s}{s+1} \right) \left( \frac{c}{n} \right)^{s+1} \leq \sum_{s=4}^{3\alpha \log n} \left( \frac{ne}{s} \right)^s \left( \frac{se}{2} \right)^{s+1} \left( \frac{c}{n} \right)^{s+1} \leq \frac{3\alpha \log n}{n} \sum_{s=4}^{3\alpha \log n} \left( \frac{ce^2}{2} \right)^s \leq \frac{(ce^2)^{3\alpha \log n} \log n}{n} = o(1).
\]

Our next lemma is concerned with cycles in \( K_2 \) which is the 2-core of \( G_{n,p} \). The 2-core of a graph is the graph induced by the edges that are in at least one cycle. When \( c > 1 \), the 2-core consists of a linear size sub-graph together with a few vertex disjoint cycles. By few we mean that in expectation, there are \( O(1) \) vertices on these cycles.

Let \( 0 < x < 1 \) be such that \( xe^{-x} = ce^{-c} \). Then w.h.p. \( K_2 \) has
\[
\nu \sim (1 - x) \left( 1 - \frac{x}{c} \right) n \text{ vertices and } \mu \sim \left( 1 - \frac{x}{c} \right)^2 \frac{cn}{2} \text{ edges.}
\]
(See for example Pittel [8]).

If \( c = 1 + \varepsilon \) for \( \varepsilon \) small and positive then \( x = 1 - \eta \) where \( \eta = \varepsilon + a_1 \varepsilon^2 \), \( |a_1| \leq 2 \) for \( \varepsilon < 1/10 \).

The degree sequence of \( K_2 \) can be generated as follows, see for example Aronson, Frieze and Pittel [3]: Let \( \lambda \) be the solution to
\[
\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = \frac{2\mu}{\nu} \sim \frac{c-x}{1-x} = \frac{2 + a_1 \varepsilon}{1 + a_1 \varepsilon}.
\]
We deduce from this that
\[ \lambda \leq 4|a_1|\varepsilon \leq 8\varepsilon. \]

We generate the degrees \( d(1), d(2), \ldots, d(\nu) \) as independent copies of the random variable \( Z \) where for \( d \geq 2, \)
\[ \Pr(Z = d) = \frac{\lambda^d}{d!(e^\lambda - 1 - \lambda)}. \]

We condition that the sum \( D_1 = d(1) + d(2) + \cdots + d(n) = 2\mu. \) We let
\[ \theta_k = \frac{\Pr(d(i) = d_i, i = 1, 2, \ldots, k \mid D_1 = 2\mu)}{\Pr(d(i) = d_i, i = 1, 2, \ldots, k)} = \frac{\Pr(d(k + 1) + \cdots + d(n) = 2\mu - (d_1 + \cdots + d_k))}{\Pr(d(1) + \cdots + d(n) = 2\mu)}. \]

It is shown in [3] that if \( Z_1, Z_2, \ldots, Z_N \) are independent copies of \( Z \) then
\[ \Pr(Z_1 + \cdots + Z_N = N \mathbf{E}(Z) - t) = \frac{1}{\sigma \sqrt{2\pi N}} \left( 1 + O\left( \frac{t^2 + 1}{N\sigma^2} \right) \right) \quad (2) \]
where \( \sigma^2 = \Theta(1) \) is the variance of \( Z. \)

We observe next that the maximum degree in \( G_{n,p} \) and hence in \( K_2 \) is q.s.\textsuperscript{1} at most \( \log n. \)

It follows from this and (2) that
\[ \theta_k = 1 + o(1) \text{ for } k \leq \log^2 n \text{ and } \theta_k = O(n^{1/2}) \text{ in general.} \]

**Lemma 2.** For any \( \alpha, \beta, \) there exists \( c_0 > 1 \) such that w.h.p. any cycle \( C \) of length greater than \( \alpha \log n \) in the 2-core of \( G_{n,p}, \ p = \frac{c}{n}, 1 < c < c_0, \) has at most \( \beta|C| \) vertices of degree greater than two.

**Proof** Suppose that
\[ e^{1+8\varepsilon} \left( \frac{8\varepsilon e}{\beta} \right)^\beta < 1. \]

We will show then that w.h.p. the \( K_2 \) does not contain a cycle \( C \) where (i) \( |C| \geq \alpha \log n \) and (ii) \( C \) contains \( \beta|C| \) vertices of degree greater than two.

We can bound the probability of the existence of a “bad” cycle \( C \) as follows: In the following display we choose the vertices of our cycle in \( \binom{\nu}{k} \) ways and then arrange these vertices in a cycle \( C \) in \( (k - 1)!/2 \) ways. Then we choose \( \beta k \) vertices to have degree at least three. We then sum over possible degree sequences for the vertices in \( C. \) This explains the factor \( \theta_k \prod_{i=1}^{k} \frac{d_i(d_i - 1)}{2d_i - 2i + 1}. \) We now resort to using the configuration model of Bollobás [4]. This would explain the product \( \prod_{i=1}^{k} \frac{d_i(d_i - 1)}{2d_i - 2i + 1}. \) We use the denominator \( 2\mu - k \) to simplify the calculation. The configuration model computation will inflate our estimate by a constant

\textsuperscript{1}A sequence of events \( \mathcal{E}_n \) is said to occur quite surely q.s. if \( \Pr(\neg \mathcal{E}_n) = O(n^{-C}) \) for any constant \( C > 0. \)
factor that we hide with the notation \( \leq b \). We write \( A \leq b B \) for \( A = O(B) \) when \( O(B) \) is “ugly looking”.

\[
\Pr(\exists C) \leq_b \sum_{k=\alpha \log n}^{\nu} \frac{\nu}{k} \left( \frac{k-1}{2} \right)! \binom{k}{\beta k} \theta_k \sum_{d_1,\ldots,d_{\beta k} \geq 3} \prod_{i=1}^{k} \frac{x_{d_i}}{d_i! (e^{x_{d_i}} - 1 - \lambda)} \cdot \frac{d_i (d_i - 1)}{2 \mu - 2k}
\]

\[
\leq \sum_{k=\alpha \log n}^{\nu} \frac{e^{k/\mu}}{2k} \left( \frac{\nu}{2\mu (e^{\lambda} - 1 - \lambda)} \right)^k \lambda^{2k} \binom{k}{\beta k} \theta_k (e^{\lambda} - 1)^{\beta k} e^{(1-\beta)k\lambda}
\]

\[
= \sum_{k=\alpha \log n}^{\nu} \frac{\theta_k}{2k} \left( e^{k/\mu} \cdot \frac{\lambda}{(e^{\lambda} - 1)^{1-\beta}} \cdot \left( \frac{e}{\beta} \right)^{\beta} \cdot e^{(1-\beta)\lambda} \right)^k
\]

\[
= o(1).
\]

**Lemma 3.** For any \( \alpha \) and any \( k \in \mathbb{N} \), there exists \( \varepsilon_0 > 0 \) such that w.h.p. we can decompose the edges of the \( G = G_{n,p} \), \( p = \frac{1+\varepsilon}{n} \), \( 0 < \varepsilon < \varepsilon_0 \), as \( F \cup M \), where \( F \) is a forest, and where the distance in \( F \) between any two edges in \( M \) is at least \( k \).

**Proof**

By choosing \( \beta < \frac{1}{2k} \) in Lemma 2 we can find, in every cycle of length \( > \alpha \log n \) of the 2-core \( K_2 \) of \( G \) (which includes all cycles of \( G \)), a path of length at least \( 2k + 1 \) whose interior vertices are all of degree 2. We can thus choose in each cycle of \( K_2 \) of length \( > \alpha \log n \) such a path of maximum length, and let \( \mathcal{P} \) denote the set of such paths. (Note that, in general, there will be fewer paths in \( \mathcal{P} \) than long cycles in \( K_2 \) due to duplicates, but that the elements of \( \mathcal{P} \) are nevertheless disjoint paths in \( K_2 \).) We now choose from each path in \( \mathcal{P} \) an edge from the center of the path to give a set \( M_1 \). Note that the set of cycles in \( G \setminus M_1 \) is the same as the set of cycles in \( G \setminus \bigcup_{P \in \mathcal{P}} P \). (In particular, the only cycles which remain have length \( \leq \alpha \log n \) and are at distance \( \geq k \) from \( M \).) Thus, letting \( M_2 \) consist of one edge from each cycle of \( G \setminus M_1 \), Lemma 1 implies that \( M = M_1 \cup M_2 \) is as desired. \( \square \)
**Proof of Theorem 1.** Our goal in this section is to give a $C_{2\ell+1}$-coloring of $G = G_{n, \frac{1+\varepsilon}{2}}$ for $\varepsilon > 0$ sufficiently small. By this we will mean an assignment $c : V(G) \to \{0, 1, \ldots, 2\ell\}$ such that $x \sim y$ in $G$ implies that $c(x) \sim c(y)$ as vertices of $C_{2\ell+1}$; that is, that $x = y \pm 1$ (mod $2\ell+1$).

Consider a decomposition of $G$ as $F \cup M$ as given by Lemma 3, with $k = 4\ell - 2$.

We begin by 2-coloring $F$. Let $c_F : V \to \{0, 1\}$ be such a coloring. Our goal will be to modify this coloring to give a good $C_{2\ell+1}$ coloring of $S$.

Let $B$ be the set of edges $xy \in M$ for which $c_F(x) = c_F(y)$, and let $B_i$ be a set of distinct representatives for $B$, and for $i = 0, 1$, let $B^i = \{v \in B \mid c_F(v) = i\}$.

We now define a new $C_{2\ell+1}$ coloring $c : V \to \{0, 1, \ldots, 2\ell\}$, by

$$c(v) = \begin{cases} c_F(v) & \text{if } \text{dist}_F(v, B) \geq 2\ell - 1 \\ c_F(x) - (-1)^i(\text{dist}_F(x, v) + 1) & \text{if } \exists x \in B^i \text{ s.t. dist}(x, v)_F < 2\ell - 1. \end{cases}$$

(Color addition and subtraction are computed modulo $2 \ell + 1$.)

Since edges in $M$ are separated by distances $\geq 4\ell - 2$, this coloring is well-defined (i.e., there is at most one choice for $x$). Moreover, $c$ is certainly a good $C_{2\ell+1}$-coloring of $F$. Thus if $c$ is not a good $C_{2\ell+1}$-coloring of $S$, it is bad along some edge $xy \in M$. But if such an edge was already properly colored in the 2-coloring $c_F$, it is still properly colored by $c$, since it has distance $\geq 4\ell - 2 \geq 2\ell - 1$ from other edges in $M$. On the other hand, if previously we had $c_F(x) = c_F(y) = i$, and WLOG $x \in B^i$, then the definition of $c(v)$ gives that we now have that $c(x) \in \{i - 1, i + 1\}$ (modulo $2\ell - 1$). Thus if $c$ is not a good $C_{2\ell+1}$-coloring of $S$, then there is an edge $xy \in M$ such that $x \in B^i$ and $y$’s color also changes in the coloring $c$; but by the distance between edges in $M$, this can only happen if $x$ and $y$ are at $F$-distance $< 2\ell - 1$. Note also that $c_F(x) = c_F(y)$ implies that $\text{dist}_F(x, y)$ is even. Thus in this case, $F \cup \{xy\}$ contains an odd cycle of length $\leq 2\ell - 1$, and so $G$ has odd girth $< 2\ell + 1$, as desired.

\[\square\]

### 4 Avoiding homomorphisms to long odd cycles

For large $\ell$, one can prove the non-existence of homomorphisms to $C_{2\ell+1}$ using the following simple observation:

**Observation 4.** If $G$ has a homomorphism to $C_{2\ell+1}$, then $G$ has an induced bipartite subgraph with at least $\frac{2\ell}{2\ell+1}|V(G)|$ vertices.

**Proof.** Delete the smallest color class. \[\square\]
Proof of Theorem 2. The probability that $G_{n,c}$ has an induced bipartite subgraph on $\beta n$ vertices is at most 
\[
\left( \frac{n}{\beta n} \right)^{2\beta n} \left( 1 - \frac{c}{n} \right)^{\beta^2 n^2 / 4} < \left( \frac{2^\beta e^{-c\beta^2 / 4}}{\beta^2 (1 - \beta)^{1 - \beta}} \right)^n \tag{4}
\]

The expression inside the parentheses is unimodal in $\beta$ for fixed $c$, and, for $c > 2.774$, is less than 1 for $\beta > 0.999971$. In particular, for $c > 2.774$, $G_{n,c}$ has no homomorphism to $C_{2\ell+1}$ for $2\ell + 1 \geq 1,427,583$.

5 Avoiding homomorphisms to $C_5$

A homomorphism of $G = G_{n,p}, p = \frac{c}{n}$ into $C_5$ induces a partition of $[n]$ into sets $V_i, i = 0, 1, \ldots, 4$. This partition can be assumed to have the following properties:

P1 The sets $V_i, i = 0, 1, \ldots, 4$ are all independent sets.

P2 There are no edges between $V_i$ and $V_{i+2} \cup V_{i-2}$. Here addition and subtraction in an index are taken to be modulo 5.

P3 Every $v \in V_i, i = 1, 2, 3, 4$ has a neighbor in $V_{i-1}$.

P4 Every $v \in V_2$ has a neighbor in $V_3$.

Hatami [6], Lemma 2.1 shows that we can assume P1, P2, P3. Given P1, P2, P3, if $v \in V_2$ has no neighbors in $V_3$ then we can move $v$ from from $V_2$ to $V_0$ and still have a homomorphism. Furthermore, this move does not upset P1, P2, P3.

We let $|V_i| = n_i$ for $i = 0, 1, \ldots, 4$. For a fixed partition we then have

\[ \Pr(P1 \land P2) = (1 - p)^S \text{ where } S = \binom{n}{2} - \sum_{i=0}^{4} n_in_{i+1}. \tag{5} \]

\[ \Pr(P3 \mid P1 \land P2) = \prod_{i=1}^{4} (1 - (1 - p)^{n_{i-1}})^{n_i}. \tag{6} \]

\[ \Pr(P4 \mid P1 \land P2 \land P3) \leq \left( 1 - \left( 1 - \frac{1}{n_2} \right)^{n_3} (1 - p)^{n_3} \right)^{n_2}. \tag{7} \]

Equations (5) and (6) are self evident, but we need to justify (7). Consider the bipartite subgraph $\Gamma'$ of $G_{n,p}$ induced by $V_2 \cup V_3$. P3 tells us that each $v \in V_3$ has a neighbor in $V_2$. Denote this event by $\mathcal{A}$. Suppose now that we choose a random mapping $\phi$ from $V_3$ to $V_2$. We then create a bipartite graph $\Gamma'$ with edge set $E_1 \cup E_2$. Here $E_1 = \{ xy : x \in V_3, y = \phi(x) \}$ and $E_2$ is obtained by independently including each of the $n_2n_3$ possible edges between $V_2$ and $V_3$ with probability $p$. We now claim that we can couple $\Gamma, \Gamma'$ so that $\Gamma \subseteq \Gamma'$.
Event $A$ can be construed as follows: A vertex in $v \in V_3$ chooses $B_v$ neighbors in $V_2$ where $B_v$ is distributed as a binomial $\text{Bin}(n_2, p)$, conditioned to be at least one. The neighbors of $v$ in $V_2$ will then be a random $B_v$ subset of $V_2$. We only have to prove then that if $v$ chooses $B_v'$ random neighbors in $\Gamma'$ then $B_v'$ stochastically dominates $B_v$. But $B_v'$ is one plus $\text{Bin}(n_2 - 1, p)$ and domination is easy to confirm. We have $n_2 - 1$ instead of $n_2$, since we do not wish to count the edge $v$ to $\phi(v)$ twice.

We now write $n_i = \alpha_i n$ for $i = 0, \ldots, 4$. We are particularly interested in the case where $c = 4$. Now (4) implies that $G_{n_4}^{\frac{1}{4}}$ has no induced bipartite subgraph of size $\beta n$ for $\beta > 0.94$. Thus we may assume that $\alpha_i \geq 0.06$ for $i = 0, \ldots, 4$. In which case we can write

$$
\Pr(P_1 \wedge P_2 \wedge P_3 \wedge P_4) \leq e^{o(n)} \times \exp \left\{ -c \left( \frac{1}{2} - \sum_{i=0}^{4} \alpha_i \alpha_{i+1} \right) n \right\} \times \left( \prod_{i=1}^{4} (1 - e^{-\alpha_{i-1}} \alpha_i) \right)^n \times (1 - e^{-\alpha_3/\alpha_2} e^{-\alpha_3})^n.
$$

The number of choices for $V_0, \ldots, V_4$ with these sizes is

$$
\binom{n}{n_0, n_1, n_2, n_3, n_4} = e^{o(n)} \times \left( \frac{1}{\prod_{i=0}^{4} \alpha_i^{\alpha_i}} \right)^n \leq 5^n.
$$

Putting $\alpha_4 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$ and

$$
b = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) = \frac{1}{\alpha_0^{\alpha_0} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} \alpha_4^{\alpha_4}} e^{(\alpha_0 \alpha_4 - \frac{1}{2})(e^{\alpha_0} - 1)^{\alpha_1} (e^{\alpha_1} - 1)^{\alpha_2} (e^{\alpha_2} - 1)^{\alpha_3} (e^{\alpha_3} - 1)^{\alpha_4} (1 - e^{-\alpha_3/\alpha_2} e^{-\alpha_3})^{\alpha_2}},
$$

we see that since there are $O(n^4)$ choices for $n_0, \ldots, n_4$ we have

$$
\Pr(\exists \text{ a homomorphism from } G_{n_4}^{\frac{1}{4}} \text{ to } C_5) \leq e^{o(n)} \left( \max_{\alpha_0 + \ldots + \alpha_4 \leq 0.94 \atop \alpha_0, \ldots, \alpha_4 \geq 0.06} b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \right)^n. \quad (8)
$$

In the next section, we describe a numerical procedure for verifying that the maximum in (8) is less than 1. This will complete the proof of Theorem 3.

### 6 Bounding the function.

Our aim now is to bound the partial derivatives of $b(4.0, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$, to translate numerical computations of the function on a grid to a rigorous upper bound.

Before doing this we verify that w.h.p. $G_{n,p=\frac{4}{n}}$ has no independent set $S$ of size $s = 3n/5$ or more. Indeed,

$$
\Pr(\exists S) \leq 2^n (1 - p)^{\binom{s}{2}} \leq 2^n e^{-18n/25} e^{12/5} = o(1).
$$
In the calculations below we will make use of the following bounds: They assume that $0.06 \leq \alpha_i \leq 0.6$ for $i \geq 0$.

\[
\log(\alpha_i) > -2.82; \quad -1.31 < \log(e^{4\alpha_i} - 1) < 2.31; \quad \frac{e^{4\alpha_i}}{e^{4\alpha_i} - 1} < 4.69
\]
\[
\frac{1}{e^{4\alpha_1} - 1} < 3.69; \quad \log(e^{\alpha_2 \alpha_3 + 4\alpha_3} - 1) > -0.91; \quad \frac{1 + 4\alpha_2}{e^{\alpha_3 \alpha_2} e^{4\alpha_3} - 1} < 8.40.
\]

We now use these estimates to bound the absolute values of the $\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}$. Our target value for these is 30. We will be well within these bounds except for $i = 2$

Taking logarithms to differentiate with respect to $\alpha_0$, we find

\[
\frac{\partial b}{\partial \alpha_0} = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times
\left( c \left( -\alpha_0 + \alpha_1 + \frac{\alpha_1}{e^{\alpha_0 c} - 1} + \alpha_4 \right) - \log(\alpha_0) + \log(\alpha_4) - \log(e^{\alpha_3 c} - 1) \right), \tag{9}
\]

In particular, for $c = 4$,

\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_0} \geq -4\alpha_0 + \log(\alpha_4) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 2.31,
\]
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_0} \leq 4 \left( \alpha_1 + \frac{\alpha_1}{e^{\alpha_0 c} - 1} + \alpha_4 \right) - \log(\alpha_0) - \log(e^{4\alpha_3} - 1) < 4 \times 4.69 + 2.82 + 1.31.
\]

Similarly, we find

\[
\frac{\partial b}{\partial \alpha_1} = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times
\left( c \left( -\alpha_0 + \alpha_2 + \frac{\alpha_2}{e^{\alpha_1 c} - 1} \right) - \log(\alpha_1) + \log(\alpha_4) + \log \left( \frac{e^{\alpha_0 c} - 1}{e^{\alpha_3 c} - 1} \right) \right), \tag{10}
\]

and so for $c = 4$,

\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_1} \geq -4\alpha_0 + \log(\alpha_4) + \log(e^{4\alpha_0} - 1) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 3.62,
\]
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_1} \leq 4 \left( \alpha_2 + \frac{\alpha_2}{e^{\alpha_1 c} - 1} \right) - \log(\alpha_1) - \log(e^{4\alpha_3} - 1) < 2.4 \times 4.69 + 2.82 + 1.31.
\]

We next find that

\[
\frac{\partial b}{\partial \alpha_2} = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times
\left( c \left( -\alpha_0 + \alpha_3 + \frac{\alpha_3}{e^{\alpha_2 c} - 1} \right) - \frac{\alpha_3 / \alpha_2}{e^{\alpha_3 \alpha_2 + \alpha_3} - 1} - \log(\alpha_4) + \log(\alpha_2 + \log(e^{\alpha_1 c} - 1) - \frac{\alpha_3}{\alpha_2} - \alpha_3 - \log(e^{\alpha_3 / \alpha_2 + \alpha_3} - 1) \right); \tag{11}
\]
and so for $c = 4$,
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} \geq -4 \alpha_0 - \frac{\alpha_3}{\alpha_2} e^{\alpha_3/\alpha_2+\alpha_3} - \log(e^{\alpha_3/\alpha_2+\alpha_3} - 1) + \log(\alpha_4) + \log \left( \frac{e^{4\alpha_1} - 1}{e^{4\alpha_3} - 1} \right)
\]

We need to be a little careful here. Now $\alpha_3/\alpha_2 \leq 10$ and if $\alpha_3/\alpha_2 \geq 9$ then $\alpha_3 \geq 0.54$ and then $\alpha_1 \leq 0.46 - 3 \times 0.06 = 0.28$ for $i \neq 3$. We bound $-\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}$ for both possibilities.

Continuing we get
\[
\frac{\alpha_3}{\alpha_2} \geq 9 : \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} > -1.12 - 10.01 - 12.4 - 2.82 - 3.62 = -29.97,
\]
\[
\frac{\alpha_3}{\alpha_2} \leq 9 : \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} > -2.4 - 9.01 - 11.4 - 2.82 - 3.62,
\]
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} \leq 4 \left( \alpha_3 + \frac{\alpha_3}{e^{4\alpha_2} - 1} \right) - \log(\alpha_2) + \log \left( \frac{e^{4\alpha_1} - 1}{e^{4\alpha_3} - 1} \right) - \log(e^{\alpha_3/\alpha_2+\alpha_3} - 1)
\]
\[
< 2.4 \times 3.69 + 2.82 + 3.62 + 0.91.
\]

Finally, we find that
\[
\frac{\partial b}{\partial \alpha_3} = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times c \left( -\alpha_0 + \frac{\alpha_4}{e^{\alpha_3} - 1} \right) + \frac{1 + 4\alpha_2}{\alpha_3} + \log(\alpha_4) - \log(\alpha_3) + \log \left( \frac{e^{\alpha_2} - 1}{e^{\alpha_3} - 1} \right) \tag{12}
\]

and so for $c = 4$
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_3} \geq -4 \alpha_0 + \log(\alpha_4) + \log(e^{4\alpha_2} - 1) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 3.62,
\]
\[
\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_3} \leq 4 \alpha_4 \frac{e^{4\alpha_3} - 1}{e^{\alpha_3} - 1} + \frac{1 + 4\alpha_2}{\alpha_3 e^{4\alpha_3} - 1} - \log(\alpha_3) + \log \left( \frac{e^{4\alpha_2} - 1}{e^{4\alpha_3} - 1} \right)
\]
\[
< 2.4 \times 4.69 + 8.40 + 2.82 + 3.62.
\]

We see that $|\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}| < 30$ for all $0 \leq i \leq 3$. Thus, if we know that $b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \leq B$ for some $B$, this means that we can bound $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < \rho$ by checking that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < \rho - \epsilon$ on a grid with step-size $\delta = \epsilon/(2 \cdot B \cdot 30)$.

The C++ program in Appendix A checks that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < .949$ on a grid with step-size $\delta = .0008$ (it completes in around an hour or less on a standard desktop computer, and is available for download from the authors’ websites). Suppose now that $B \geq 1$ is the supremum of $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$ in the region of interest. For $\epsilon = 60\delta B = 0.048B$, we must have at some $\delta$-grid point that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \geq B - \epsilon = .962B \geq .962$. This contradicts the computer-assisted bound of $< .949$ on the grid, completing the proof of Theorem 3. \qed

References


A  C++ code to check function bound

```cpp
#include <iostream>
#include <math.h>
#include <stdlib.h>
using namespace std;

int main(int argc, char* argv[]){
    double delta=.0008; //step size
    double maxIndSet=.6; //no independent sets larger than this fraction
    double minClass=.06; //all color classes larger than this fraction
    double val=0;
    double maxval=0;
    double maxa0,maxa1,maxa2,maxa3; //to record the coordinates of max value
    maxa0=maxa1=maxa2=maxa3=0;
    double A23,A,B,C; //For precomputing parts of the function
    double c=4;
    for (double a3=minClass; a3 + 4*minClass<1; a3+=delta){
        B=exp(c*a3)-1;
        for (double a2=minClass; a3 + a2 + 3*minClass<1; a2+=delta){
            A23=1/(pow(a2,a2)*pow(a3,a3)) * exp(-c/2)
                * pow(exp(c*a2)-1,a3) * pow(1-exp(-a3/a2)*exp(-c*a3),a2);
            for (double a1=minClass;
                a3+a1<maxIndSet && a3 + a2 + a1 + 2*minClass<1;
                a1+=delta){
                A=A23/pow(a1,a1)* pow(exp(c*a1)-1,a2);
            for (double a0=max(max(minClass,.4-a2-a3),.4-a1-a3);
                a2+a0<maxIndSet && a3+a0<maxIndSet
                && a3 + a2 + a1 + a0 + minClass<1;
                a0+=delta){
                double a4=1-a0-a1-a2-a3;
                C=exp(c*a0);
                val=1/pow(a0,a0) * A * pow(B*C/a4,a4) * pow(C-1,a1);
                if (val>maxval){
                    maxval=val;
                    maxa0=a0; maxa1=a1; maxa2=a2; maxa3=a3;
                }
            }
        }
    }
    cout << "Max is "<<maxval<<", obtained at (" <<maxa0<<","<<maxa1<<","<<maxa2<<","<<maxa3<<","<<1-maxa0-maxa1-maxa2-maxa3<<")"<<endl;
}
```

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program output:

$./bound
Max is 0.948754, obtained at (0.2904,0.2568,0.1704,0.1632,0.1192)