Rainbow Connection of Random Regular Graphs

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Abstract

An edge colored graph $G$ is rainbow edge connected if any two vertices are connected by a path whose edges have distinct colors. The rainbow connection of a connected graph $G$, denoted by $rc(G)$, is the smallest number of colors that are needed in order to make $G$ rainbow connected.

In this work we study the rainbow connection of the random $r$-regular graph $G = G(n, r)$ of order $n$, where $r \geq 4$ is a constant. We prove that with probability tending to one as $n$ goes to infinity the rainbow connection of $G$ satisfies $rc(G) = O(\log n)$, which is best possible up to a hidden constant.

1 Introduction

Connectivity is a fundamental graph theoretic property. Recently, the concept of rainbow connection was introduced by Chartrand, Johns, McKeon and Zhang in [7]. We say that a set of edges is rainbow colored if its every member has a distinct color. An edge colored graph $G$ is rainbow edge connected if any two vertices are connected by a rainbow colored path. Furthermore, the rainbow connection $rc(G)$ of a connected graph $G$ is the smallest number of colors that are needed in order to make $G$ rainbow edge connected.

Notice, that by definition a rainbow edge connected graph is also connected. Moreover, any connected graph has a trivial edge coloring that makes it rainbow edge connected, since one may color the edges of a given spanning tree with distinct colors.

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Other basic facts established in [7] are that \( rc(G) = 1 \) if and only if \( G \) is a clique and \( rc(G) = |V(G)| - 1 \) if and only if \( G \) is a tree. Besides its theoretical interest, rainbow connection is also of interest in applied settings, such as securing sensitive information transfer and networking (see, e.g., [5, 14]). For instance, consider the following setting in networking [5]: we want to route messages in a cellular network such that each link on the route between two vertices is assigned with a distinct channel. Then, the minimum number of channels to use is equal to the rainbow connection of the underlying network.

Caro, Lev, Roditty, Tuza and Yuster [4] prove that for a connected graph \( G \) with \( n \) vertices and minimum degree \( \delta \), the rainbow connection satisfies \( rc(G) \leq \frac{\log \delta}{\delta} n(1 + f(\delta)) \), where \( f(\delta) \) tends to zero as \( \delta \) increases. The following simpler bound was also proved in [4], \( rc(G) \leq n^{\frac{\log n + 3}{2}} \). Krivelevich and Yuster [13] removed the logarithmic factor from the upper bound in [4]. Specifically they proved that \( rc(G) \leq \frac{2n}{\frac{3}{\delta} + 1} \). Chandran, Das, Rajendraprasad and Varma [6] improved this upper bound to \( 3n \delta + 1 \), which is close to best possible.

As pointed out in [4] the random graph setting poses several intriguing questions. Specifically, let \( G = G(n, p) \) denote the binomial random graph on \( n \) vertices with edge probability \( p \). Caro, Lev, Roditty, Tuza and Yuster [4] proved that \( p = \sqrt{\log n / n} \) is the sharp threshold for the property \( rc(G) \leq 2 \). This was sharpened to a hitting time result by Heckel and Riordan [10]. He and Liang [9] studied further the rainbow connection of random graphs. Specifically, they obtain a threshold for the property \( rc(G) \leq d \) where \( d \) is constant. Frieze and Tsourakakis [8] studied the rainbow connection of \( G = G(n, p) \) at the connectivity threshold \( p = \frac{\log n + \omega}{n} \) where \( \omega \to \infty \) and \( \omega = o(\log n) \). They showed that w.h.p.\(^1 \) \( rc(G) \) is asymptotically equal to \( \max \{diam(G), Z_1(G)\} \), where \( Z_1 \) is the number of vertices of degree one.

For further results and references we refer the interested reader to the recent survey of Li, She and Sun [14].

In this paper we study the rainbow connection of the random \( r \)-regular graph \( G(n, r) \) of order \( n \), where \( r \geq 4 \) is a constant and \( n \to \infty \). It was shown in Basavaraju, Chandran, Rajendraprasad, and Ramaswamy [1] that for any bridgeless graph \( G \), \( rc(G) \leq \rho(\rho + 2) \), where \( \rho \) is the radius of \( G = (V, E) \), i.e., \( \min_{x \in V} \max_{y \in V} dist(x, y) \). Since the radius of \( G(n, r) \) is \( O(\log n) \) w.h.p., we see that [1] implies that \( rc(G(n, r)) = O(\log^2 n) \) w.h.p. The following theorem gives an improvement on this for \( r \geq 4 \).

**Theorem 1** Let \( r \geq 4 \) be a constant. Then, w.h.p. \( rc(G(n, r)) = O(\log n) \).

The rainbow connection of any graph \( G \) is at least as large as its diameter. The diameter of \( G(n, r) \) is w.h.p. asymptotically \( \log_{r-1} n \) and so the above theorem is best.

\(^1\)An event \( E_n \) occurs with high probability, or w.h.p. for brevity, if \( \lim_{n \to \infty} \Pr(E_n) = 1 \).
possible, up to a (hidden) constant factor.

We conjecture that Theorem 1 can be extended to include \( r = 3 \). Unfortunately, the approach taken in this paper does not seem to work in this case.

## 2 Proof of Theorem 1

### 2.1 Outline of strategy

Let \( G = G(n, r) \), \( r \geq 4 \). Define

\[
    k_r = \log_{r-1}(K_1 \log n),
\]

where \( K_1 \) will be a sufficiently large absolute constant. Recall that the \textit{distance between two vertices} in \( G \) is the number of edges in a shortest path connecting them and the \textit{distance between two edges} in \( G \) is the number of vertices in a shortest path between them. (Hence, both adjacent vertices and incident edges have distance 1.)

For each vertex \( x \) let \( T_x \) be the subgraph of \( G \) induced by the vertices within distance \( k_r \) of \( x \). We will see (due to Lemma 5) that w.h.p., \( T_x \) is a tree for most \( x \) and that for all \( x \), \( T_x \) contains at most one cycle. We say that \( x \) is \textit{tree-like} if \( T_x \) is a tree. In which case we denote by \( L_x \) the leaves of \( T_x \). Moreover, if \( u \in L_x \), then we denote the path from \( u \) to \( x \) by \( P(u, x) \).

We will randomly color \( G \) in such a way that the edges of every path \( P(u, x) \) is rainbow colored for all \( x \). This is how we do it. We order the edges of \( G \) in some arbitrary manner as \( e_1, e_2, \ldots, e_m \), where \( m = rn/2 \). There will be a set of \( q = \lceil K_1^2 r \log n \rceil \) colors available. Then, in the order \( i = 1, 2, \ldots, m \) we randomly color \( e_i \). We choose this color uniformly from the set of colors not used by those \( e_j, j < i \) which are within distance \( k_r \) of \( e_i \). Note that the number of edges within distance \( k_r \) of \( e_i \) is at most

\[
    2 \left( (r-1) + (r-1)^2 + \cdots + (r-1)^{\lfloor k_r \rfloor - 1} \right) \leq (r-1)^{k_r} = K_1 \log n.
\]

So for \( K_1 \) sufficiently large we always have many colors that can be used for \( e_i \). Clearly, in such a coloring, the edges of a path \( P(u, x) \) are rainbow colored.

Now consider a fixed pair of tree-like vertices \( x, y \). We will show (using Corollary 4) that one can find a partial 1-1 mapping \( f = f_{x,y} \) between \( L_x \) and \( L_y \) such that if \( u \in L_x \) is in the domain \( D_{x,y} \) of \( f \) then \( P(u, x) \) and \( P(f(u), y) \) do not share any colors. The domain \( D_{x,y} \) of \( f \) is guaranteed to be of size at least \( K_2 \log n \), where \( K_2 = K_1 / 10 \).
Having identified \( f_{x,y}, D_{x,y} \) we then search for a rainbow path joining \( u \in D_{x,y} \) to \( f(u) \). To join \( u \) to \( f(u) \) we continue to grow the trees \( T_x, T_y \) until there are \( n^{1/20} \) leaves. Let the new larger trees be denoted by \( \tilde{T}_x, \tilde{T}_y \), respectively. As we grow them, we are careful to prune away edges where the edge to root path is not rainbow. We do the same with \( T_y \) and here make sure that edge to root paths are rainbow with respect to corresponding \( T_x \) paths. We then construct at least \( n^{1/21} \) vertex disjoint paths \( Q_1, Q_2, \ldots \), from the leaves of \( \tilde{T}_x \) to the leaves of \( \tilde{T}_y \). We then argue that w.h.p. one of these paths is rainbow colored and that the colors used are disjoint from the colors used on \( P(u, x) \) and \( P(f(u), y) \).

We then finish the proof by dealing with non tree-like vertices in Section 2.6.3.

2.2 Coloring lemmata

In this section we prove some auxiliary results about rainbow colorings of \( d \)-ary trees.

Recall that a complete \( d \)-ary tree \( T \) is a rooted tree in which each non-leaf vertex has exactly \( d \) children. The depth of an edge is the number of vertices in the path connecting the root to the edge. The set of all edges at a given depth is called a level of the tree. The height of a tree is the distance from the root to the deepest vertices in the tree (i.e. the leaves). Denote by \( L(T) \) the set of leaves and for \( v \in L(T) \) let \( P(v, T) \) be the path from the root of \( T \) to \( v \) in \( T \).

Lemma 2 Let \( T_1, T_2 \) be two vertex disjoint rainbow copies of the complete \( d \)-ary tree with \( \ell \) levels, where \( d \geq 2 \). Let \( T_i \) be rooted at \( x_i \), \( L_i = L(T_i) \) for \( i = 1, 2 \), and

\[
m(T_1, T_2) = |\{(v, w) \in L_1 \times L_2 : P(v, T_1) \cup P(w, T_2) \text{ is rainbow}\}|.
\]

Then,

\[
\kappa_\ell = \min_{T_1, T_2} \{ m(T_1, T_2) \} \geq \left( 1 - \sum_{i=1}^{\ell} \frac{i}{d^i} \right) d^{2\ell}.
\]

Proof. We prove this by induction on \( \ell \). If \( \ell = 1 \), then clearly

\[
\kappa_1 = d(d - 1).
\]

Suppose that (3) holds for an \( \ell \geq 2 \).

Let \( T_1, T_2 \) be rainbow trees of height \( \ell + 1 \). Moreover, let \( T'_1 = T_1 \setminus L(T_1) \) and \( T'_2 = T_2 \setminus L(T_2) \). We show that

\[
m(T_1, T_2) \geq d^2 \cdot m(T'_1, T'_2) - (\ell + 1) d^{\ell+1}.
\]

4
Each \((v', w') \in L'_1 \times L'_2\) gives rise to \(d^2\) pairs of leaves \((v, w) \in L_1 \times L_2\), where \(v'\) is the parent of \(v\) and \(w'\) is the parent of \(w\). Hence, the term \(d^2 \cdot m(T'_1, T'_2)\) accounts for the pairs \((v, w)\), where \(P_{v, T'_1} \cup P_{w, T'_2}\) is rainbow. We need to subtract off those pairs for which \(P_{v, T_1} \cup P_{w, T_2}\) is not rainbow. Suppose that this number is \(\nu\). Let \(v \in L(T_1)\) and let \(v'\) be its parent, and let \(c\) be the color of the edge \((v, v')\). Then \(P_{v, T_1} \cup P_{w, T_2}\) is rainbow unless \(c\) is the color of some edge of \(P_{w, T_2}\). Now let \(\nu(c)\) denote the number of root to leaf paths in \(T_2\) that contain an edge color \(c\). Thus,

\[ \nu \leq \sum_c \nu(c), \]

where the summation is taken over all colors \(c\) that appear in edges of \(T_1\) adjacent to leaves. We bound this sum trivially, by summing over all colors in \(T_2\) (i.e., over all edges in \(T_2\), since \(T_2\) is rainbow). Note that if the depth of the edge colored \(c\) in \(T_2\) is \(i\), then \(\nu(c) \leq d^{\ell+1-i}\). Thus, summing over edges of \(T_2\) gives us

\[ \sum_c \nu(c) \leq \sum_{i=1}^{\ell+1} d^{\ell+1-i} \cdot d^i = (\ell + 1)d^{\ell+1}, \]

and consequently (4) holds. Thus, by induction (applied to \(T'_1\) and \(T'_2\))

\[ m(T_1, T_2) \geq d^2 \cdot m(T'_1, T'_2) - (\ell + 1)d^{\ell+1} \]

\[ \geq d^2 \left( 1 - \sum_{i=1}^{\ell} \frac{i}{d^i} \right) d^2 - (\ell + 1)d^{\ell+1} \]

\[ \geq \left( 1 - \sum_{i=1}^{\ell+1} \frac{i}{d^i} \right) d^{2(\ell+1)}, \]

as required.

In the proof of Theorem 1 we will need a stronger version of the above lemma.

**Lemma 3** Let \(T_1, T_2\) be two vertex disjoint edge colored copies of the complete \(d\)-ary tree with \(L\) levels, where \(d \geq 2\). For \(i = 1, 2\), let \(T_i\) be rooted at \(x_i\) and suppose that edges \(e, f\) of \(T_i\) have a different color whenever the distance between \(e\) and \(f\) in \(T_i\) is at most \(L\). Let \(\kappa_L\) be as defined in Lemma 2. Then

\[ \kappa_L \geq \left( 1 - \frac{L^2}{d^{[L/2]}} - \sum_{i=1}^{[L/2]} \frac{i}{d^i} \right) d^{2L}. \]
Proof. Let \( T_\ell^i \) be the subtree of \( T_i \) spanned by the first \( \ell \) levels, where \( 1 \leq \ell \leq L \) and \( i = 1, 2 \). We show by induction on \( \ell \) that

\[
m(T_1^\ell, T_2^\ell) \geq \left( 1 - \frac{\ell^2}{d\lfloor L/2 \rfloor} - \sum_{i=1}^{\lfloor L/2 \rfloor} \frac{i}{d^i} \right) d^{2\ell}.
\]

(5)

Observe first that Lemma 2 implies (5) for \( 1 \leq \ell \leq \lfloor L/2 \rfloor - 1 \), since in this case \( T_1^\ell \) and \( T_2^\ell \) must be rainbow.

Suppose that \( \lfloor L/2 \rfloor \leq \ell < L \) and consider the case where \( T_1^\ell, T_2^\ell \) have height \( \ell + 1 \). Following the argument of Lemma 2 we observe that color \( c \) can be the color of at most \( d^\ell + 1 - \lfloor L/2 \rfloor \) leaf edges of \( T_1^\ell \). This is because for two leaf edges to have the same color, their common ancestor must be at distance (from the root) at most \( \ell - \lfloor L/2 \rfloor \).

Therefore,

\[
m(T_1^{\ell+1}, T_2^{\ell+1}) \geq d^2 \cdot m(T_1^\ell, T_2^\ell) - d^{\ell+1-\lfloor L/2 \rfloor} \sum_c \nu(c)
\]

\[
\geq d^2 \cdot m(T_1^\ell, T_2^\ell) - d^{\ell+1-\lfloor L/2 \rfloor} (\ell + 1)d^{\ell+1}
\]

\[
= d^2 \cdot m(T_1^\ell, T_2^\ell) - (\ell + 1)d^{2(\ell+1)-\lfloor L/2 \rfloor}.
\]

Thus, by induction

\[
m(T_1^{\ell+1}, T_2^{\ell+1}) \geq d^2 \left( 1 - \frac{\ell^2}{d\lfloor L/2 \rfloor} - \sum_{i=1}^{\lfloor L/2 \rfloor} \frac{i}{d^i} \right) d^{2\ell} - (\ell + 1)d^{2(\ell+1)-\lfloor L/2 \rfloor}
\]

\[
= \left( 1 - \frac{\ell^2 + \ell + 1}{d\lfloor L/2 \rfloor} - \sum_{i=1}^{\lfloor L/2 \rfloor} \frac{i}{d^i} \right) d^{2(\ell+1)}
\]

\[
\geq \left( 1 - \frac{(\ell + 1)^2}{d\lfloor L/2 \rfloor} - \sum_{i=1}^{\lfloor L/2 \rfloor} \frac{i}{d^i} \right) d^{2(\ell+1)}
\]

yielding (5) and consequently the statement of the lemma.

\[\square\]

Corollary 4 Let \( T_1, T_2 \) be as in Lemma 3, except that the root degrees are \( d + 1 \) instead of \( d \). If \( d \geq 3 \) and \( L \) is sufficiently large, then there exist \( S_i \subseteq L_i, i = 1, 2 \) and \( f : S_1 \rightarrow S_2 \) such that

(a) \( |S_i| \geq d^L/10 \), and

(b) \( x \in S_1 \) implies that \( P_{x,T_1} \cup P_{f(x),T_2} \) is rainbow.
Proof. To deal with the root degrees being $d + 1$ we simply ignore one of the subtrees of each of the roots. Then note that if $d \geq 3$ then

$$1 - \frac{L^2}{dL/2} - \sum_{i=1}^{\lfloor L/2 \rfloor} \frac{i}{d} \geq 1 - \frac{L^2}{dL/2} - \sum_{i=1}^{\infty} \frac{i}{d^i} = 1 - \frac{L^2}{dL/2} - \frac{d}{(d-1)^2} \geq \frac{1}{5}$$

for $L$ sufficiently large. Now we choose $S_1, S_2$ in a greedy manner. Having chosen a matching $(x_i, y_i = f(x_i)) \in L_1 \times L_2$, $i = 1, 2, \ldots, p$, and $p < dL/10$, there will still be at least $d^{2L/5} - 2pd^L > 0$ pairs in $m(T_1, T_2)$ that can be added to the matching. \qed

2.3 Configuration model

We will use the configuration model of Bollobás [2] in our proofs (see, e.g., [3, 11, 15] for details). Let $W = [2m = rn]$ be our set of configuration points and let $W_i = [(i-1)r + 1, ir]$, $i \in [n]$, partition $W$. The function $\phi : W \to [n]$ is defined by $w \in W_{\phi(w)}$. Given a pairing $F$ (i.e., a partition of $W$ into $m$ pairs) we obtain a (multi-)graph $G_F$ with vertex set $[n]$ and an edge $(\phi(u), \phi(v))$ for each $\{u, v\} \in F$. Choosing a pairing $F$ uniformly at random from among all possible pairings $\Omega_W$ of the points of $W$ produces a random (multi-)graph $G_F$. Each $r$-regular simple graph $G$ on vertex set $[n]$ is equally likely to be generated as $G_F$. Here simple means without loops or multiple edges. Furthermore, if $r$ is a constant, then $G_F$ is simple with a probability bounded below by a positive value independent of $n$. Therefore, any event that occurs w.h.p. in $G_F$ will also occur w.h.p. in $G(n, r)$.

2.4 Density of small sets

Here we show that w.h.p. almost every subgraph of a random regular graph induced by the vertices within a certain small distance is a tree. Let

$$t_0 = \frac{1}{10} \log_{r-1} n. \quad (6)$$

Lemma 5 Let $k_r$ and $t_0$ be defined in (1) and (6). Then, w.h.p. in $G(n, r)$

(a) no set of $s \leq t_0$ vertices contains more than $s$ edges, and

(b) there are at most $\log^{O(1)} n$ vertices that are within distance $k_r$ of a cycle of length at most $k_r$. 

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Proof. We use the configuration model described in Section 2.3. It follows directly from the definition of this model that the probability that a given set of \( k \) disjoint pairs in \( W \) is contained in a random configuration is given by

\[
p_k = \frac{1}{(rn - 1)(rn - 3) \ldots (rn - 2k + 1)} \leq \frac{1}{(rn - 2k)^k} \leq \frac{1}{r^k(n - k)^k}.
\]

Thus, in order to prove (a) we bound:

\[
\Pr(\exists S \subseteq [n], |S| \leq t_0, e[S] \geq |S| + 1) \leq \sum_{s=3}^{t_0} \binom{n}{s} \left( \frac{s}{2} \right) r^{2(s+1)} p_{s+1} \leq \sum_{s=3}^{t_0} \left( \frac{es}{2} \right)^s \frac{r^{s+1}}{s} \frac{r}{n - (s + 1)} \leq \frac{et_0}{2} \cdot \frac{r}{n - (t_0 + 1)} \sum_{s=3}^{t_0} \left( e^2 r \right)^s \leq \frac{et_0}{2} \cdot \frac{r}{n - (t_0 + 1)} \cdot t_0 \cdot (e^2 r)^{t_0} \leq \frac{e^2 r}{2(n - (t_0 + 1))} \cdot n \frac{\log_{e^2 r}^{-1} n}{10} = o(1),
\]

as required.

We prove (b) in a similar manner. The expected number of vertices within \( k_r \) of a cycle of length at most \( k_r \) can be bounded from above by

\[
\sum_{\ell=0}^{k_r} \binom{n}{\ell} \sum_{k=3}^{k_r} \binom{n}{k} \frac{(k - 1)!}{2} r^{2(k+\ell)} p_k p_{k+\ell} \leq \sum_{\ell=0}^{k_r} \sum_{k=3}^{k_r} n^{k+\ell} \frac{r}{n - (k + \ell)} \leq \sum_{\ell=0}^{k_r} \sum_{k=3}^{k_r} (2r)^{k+\ell} \leq k_r^2 (2r)^{2k_r} = \log^O(1) n.
\]

Now (b) follows from the Markov inequality. \( \square \)
2.5 Chernoff bounds

In the next section we will use the following bounds on the tails of the binomial distribution $\text{Bin}(n, p)$ (for details, see, e.g., [11]):

$$\Pr(\text{Bin}(n, p) \leq \alpha np) \leq e^{-(1-\alpha)^2np/2}, \quad 0 \leq \varepsilon \leq 1,$$  
(7)

$$\Pr(\text{Bin}(n, p) \geq \alpha np) \leq (e^{\alpha})^{-\alpha np}, \quad \alpha \geq 1.$$  
(8)

2.6 Coloring the edges

We now consider the problem of coloring the edges of $G = G(n, r)$. Let $H$ denote the line graph of $G$ and let $\Gamma = H^{k_r}$ denote the graph with the same vertex set as $H$ and an edge between vertices $e, f$ of $\Gamma$ if there is a path of length at most $k_r$ between $e$ and $f$ in $H$. Due to (2) the maximum degree $\Delta(\Gamma)$ satisfies

$$\Delta(\Gamma) \leq K_1 \log n.$$  
(9)

We will construct a proper coloring of $\Gamma$ using

$$q = \lceil K_2^2 r \log n \rceil$$  
(10)

colors. Let $e_1, e_2, \ldots, e_m$ with $m = rn/2$ be an arbitrary ordering of the vertices of $\Gamma$. For $i = 1, 2, \ldots, m$, color $e_i$ with a random color, chosen uniformly from the set of colors not currently appearing on any neighbor in $\Gamma$. At this point only $e_1, e_2, \ldots, e_{i-1}$ will have been colored.

Suppose then that we color the edges of $G$ using the above method. Fix a pair of vertices $x, y$ of $G$.

2.6.1 Tree-like and disjoint

Assume first that $T_x, T_y$ are vertex disjoint and that $x, y$ are both tree-like. We see immediately, that $T_x, T_y$ fit the conditions of Corollary 4 with $d = r - 1$ and $L = k_r$. Let $S_x \subseteq L(T_x), \; S_y \subseteq L(T_y), \; f : S_x \to S_y$ be the sets and function promised by Corollary 4. Note that $|S_x|, |S_y| \geq K_2 \log n$, where $K_2 = K_1/10$.

In the analysis below we will expose the pairings in the configuration as we need to. Thus an unpaired point of $W$ will always be paired to a random unpaired point in $W$.

We now define a sequence $A_0 = S_x, A_1, \ldots, A_{t_0}$, where $t_0$ defined as in (6). They are defined so that $T_x \cup A_{\leq t}$ spans a tree $T_{x,t}$ where $A_{\leq t} = \bigcup_{j \leq t} A_j$. Given
$A_1, A_2, \ldots, A_i = \{v_1, v_2, \ldots, v_p\}$ we go through $A_i$ in the order $v_1, v_2, \ldots, v_p$ and construct $A_{i+1}$. Initially, $A_{i+1} = \emptyset$. When dealing with $v_j$ we add $w$ to $A_{i+1}$ if:

(a) $w$ is a neighbor of $v_j$;

(b) $w \notin T_x \cup T_y \cup A_{\leq i+1}$ (we include $A_{i+1}$ in the union because we do not want to add $w$ to $A_{i+1}$ twice);

(c) If the path $P(v_j, x)$ from $v_j$ to $x$ in $T_{x,i}$ goes through $v \in S_x$ then the set of edges $E(w)$ is rainbow colored, where $E(w)$ comprises the edges in $P(v_j, x) + (v_j, w)$ and the edges in the path $P(f(v), y)$ in $T_y$ from $y$ to $f(v)$.

We do not add neighbors of $v_j$ to $A_{i+1}$ if ever one of (b) or (c) fails. We prove next that

$$\Pr \left( |A_{i+1}| \leq (r - 1.1)|A_i| \mid K_2 \log n \leq |A_i| \leq n^{2/3} \right) = o(n^{-3}). \quad (11)$$

Let $X_b$ and $X_c$ be the number of vertices lost because of case (b) and (c), respectively. Observe that

$$(r - 1)|A_i| - X_b - X_c \leq |A_{i+1}| \leq (r - 1)|A_i| \quad (12)$$

First we show that $X_b$ is dominated by the binomial random variable

$$Y_b \sim (r - 1)\text{Bin} \left( (r - 1)|A_i|, \frac{r|A_i|}{rn/2 - rn^{2/3}} \right)$$

conditioning on $K_2 \log n \leq |A_i| \leq n^{2/3}$. This is because we have to pair up $(r - 1)|A_i|$ points and each point has a probability less than $\frac{r|A_i|}{rn/2 - rn^{2/3}}$ of being paired with a point in $A_i$. (It cannot be paired with a point in $A_{\leq i-1}$ because these points are already paired up at this time). We multiply by $(r - 1)$ because one “bad” point “spoils” the vertex. Thus, (8) implies that

$$\Pr(X_b \geq |A_i|/20) \leq \Pr(Y_b \geq |A_i|/20) \leq \left( \frac{40er(r - 1)^2|A_i|}{n} \right)^{|A_i|/20} = o(n^{-3}).$$

We next observe that $X_c$ is dominated by

$$Y_c \sim (r - 1)\text{Bin} \left( r|A_i|, \frac{4\log r - 1}{q} \right).$$

To see this we first observe that $|E(w)| \leq 2\log_{r-1} n$, with room to spare. Consider an edge $e = (v_j, w)$ and condition on the colors of every edge other than $e$. We examine the effect of this conditioning, which we refer to as $C$. 
We let \( c(e) \) denote the color of edge \( e \) in a given coloring. To prove our assertion about binomial domination, we prove that for any color \( x \),

\[
\Pr(c(e) = x \mid C) \leq \frac{2}{q}. 
\]

(13)

We observe first that for a particular coloring \( c_1, c_2, \ldots, c_m \) of the edges \( e_1, e_2, \ldots, e_m \) we have

\[
\Pr(c(e_i) = c_i, i = 1, 2, \ldots, m) = \prod_{i=1}^{m} \frac{1}{a_i}
\]

where \( q - \Delta \leq a_i \leq q \) is the number of colors available for the color of the edge \( e_i \) given the coloring so far i.e. the number of colors unused by the neighbors of \( e_i \) in \( \Gamma \) when it is about to be colored.

Now fix an edge \( e = e_i \) and the colors \( c_j, j \neq i \). Let \( C \) be the set of colors not used by the neighbors of \( e_i \) in \( \Gamma \). The choice by \( e_i \) of its color under this conditioning is not quite random, but close. Indeed, we claim that for \( c, c' \in C \)

\[
\frac{\Pr(c(e) = c \mid c(e_j) = c_j, j \neq i)}{\Pr(c(e) = c' \mid c(e_j) = c_j, j \neq i)} \leq \left( \frac{q - \Delta}{q - \Delta - 1} \right)^{\Delta}. 
\]

(14)

This is because, changing the color of \( e \) only affects the number of colors available to neighbors of \( e_i \), and only by at most one. Thus, for \( c \in C \), we have

\[
\Pr(c(e) = c \mid c(e_j) = c_j, j \neq i) \leq \frac{1}{q - \Delta} \left( \frac{q - \Delta}{q - \Delta - 1} \right)^{\Delta}. 
\]

(14)

Now from (9) and (10) we see that \( \Delta \leq \frac{q}{K q r} \) and so (14) implies (13).

Applying (8) we now see that

\[
\Pr(X_c \geq |A_i|/20) \leq \Pr(Y_c \geq |A_i|/20) \leq \left( \frac{80e(r - 1)}{K_1^2} \right)^{|A_i|/20} = o(n^{-3}).
\]

This completes the proof of (11). Thus, (11) and (12) implies that w.h.p.

\[
|A_{t_0}| \geq (r - 1.1)^{t_0} \geq (r - 1)^{\frac{1}{2}t_0} = n^{1/20}
\]

and

\[
|A_{t_0}| \leq (r - 1)^{t_0} |A_0| \leq K_1 n^{1/10} \log n,
\]

since trivially \( |A_0| \leq K_1 \log n \).
In a similar way, we define a sequence of sets $B_0 = S_y, B_1, \ldots, B_{t_0}$ disjoint from $A_{\leq t_0}$. Here $T_y \cup B_{\leq t_0}$ spans a tree $T_{y,t_0}$. As we go along we keep an injection $f_i : B_i \rightarrow A_i$ for $0 \leq i \leq t_0$. Suppose that $v \in B_i$. If $f_i(v)$ has no neighbors in $A_{i+1}$ because (b) or (c) failed then we do not try to add its neighbors to $B_{i+1}$. Otherwise, we pair up its ($r-1$) neighbors $b_1, b_2, \ldots, b_{r-1}$ outside $A_{\leq i}$ in an arbitrary manner with the ($r-1$) neighbors $a_1, a_2, \ldots, a_{r-1}$. We will add $b_1, b_2, \ldots, b_{r-1}$ to $B_{i+1}$ and define $f_{i+1}(b_j) = a_j$, $j = 1, 2, \ldots, r-1$ if for each $1 \leq j \leq r-1$ we have $b_j \notin A_{\leq t_0} \cup T_x \cup T_y \cup B_{\leq i+1}$ and the unique path $P(b_j, y)$ of length $i + k_r$ from $b_i$ to $y$ in $T_{y,i}$ is rainbow colored and furthermore, its colors are disjoint from the colors in the path $P(a_j, x)$ in $T_{x,i}$. Otherwise, we do not grow from $v$. The argument that we used for (11) will show that

$$\Pr (|B_{j+1}| \leq (r-1)|B_j| \mid K_2 \log n \leq |B_j| \leq n^{2/3}) = o(n^{-3}).$$  \hspace{1cm} (15)$$

The upshot is that w.h.p. we have $B_{t_0}$ and $A'_{t_0} = f_{t_0}(B_{t_0})$ of size at least $n^{1/20}$.

Our aim now is to show that w.h.p. one can find vertex disjoint paths of length $O(\log_{r-1} n)$ joining $u \in B_{t_0}$ to $f_{t_0}(u) \in A_{t_0}$ for at least half of the choices for $u$.

Suppose then that $B_{t_0} = \{u_1, u_2, \ldots, u_p\}$ and we have found vertex disjoint paths $Q_j$ joining $u_j$ and $v_j = f_{t_0}(j)$ for $1 \leq j < i$. Then we will try to grow breadth first trees $T_i, T'_i$ from $u_i$ and $v_i$ until we can be almost sure of finding an edge joining their leaves. We will consider the colors of edges once we have found enough paths.

Let $R = A_{\leq t_0} \cup B_{\leq t_0} \cup T_x \cup T_y$. Then fix $i$ and define a sequence of sets $S_0 = \{u_i\}, S_1, S_2, \ldots, S_t$ where we stop when either $S_t = \emptyset$ or $|S_t|$ first reaches size $n^{3/5}$. Here $S_{j+1} = N(S_j) \setminus (R \cup S_{\leq j})$. ($N(S)$ will be the set of neighbors of $S$ that are not in $S$). The number of vertices excluded from $S_{j+1}$ is less than $O(n^{1/10} \log n)$ (for $R$) plus $O(n^{1/10} \log n \cdot n^{3/5})$ for $S_{\leq j}$. Since

$$\frac{O(n^{1/10} \log n \cdot n^{3/5})}{n} = O(n^{-3/10} \log n) = O(n^{-3/11}),$$

$|S_{j+1}|$ dominates the binomial random variable

$$Z \sim Bin \left( (r-1)|S_j|, 1 - O(n^{-3/11}) \right).$$

Thus, by (7)

$$\Pr (|S_{j+1}| \leq (r-1)|S_j| \mid 100 < |S_j| \leq n^{3/5}) \leq \Pr (Z \leq (r-1)|S_j| \mid 100 < |S_j| \leq n^{3/5}) = o(n^{-3}).$$

Therefore w.h.p., $|S_j|$ will grow at a rate $(r-1.1)$ once it reaches a size exceeding 100. We must therefore estimate the number of times that this size is not reached. We
can bound this as follows. If $S_j$ never reaches 100 in size then some time in the construction of the first $\log_{r-1} 100$ $S_j$'s there will be an edge discovered between an $S_j$ and an excluded vertex. The probability of this can be bounded by $100 \cdot O(n^{-3/11}) = O(n^{-3/11})$. So, if $\beta$ denotes the number of $i$ that fail to produce $S_t$ of size $n^{3/5}$ then

$$\Pr(\beta \geq 20) \leq o(n^{-3}) + \left(\frac{n^{1/10} \log n}{20}\right) \cdot O(n^{-3/11})^{20} = o(n^{-3}).$$

Thus w.h.p. there will be at least $n^{1/20} - 20 > n^{1/21}$ of the $u_i$ from which we can grow a tree with $n^{3/5}$ leaves $L_{i,y}$ such that all these trees are vertex disjoint from each other and $R$.

By the same argument we can find at least $n^{1/21}$ of the $v_i$ from which we can grow a tree $L_{i,x}$ with $n^{3/5}$ leaves such that all these trees are vertex disjoint from each other and $R$ and the trees grown from the $u_i$. We then observe that if $e(L_{i,x}, L_{i,y})$ denotes the edges from $L_{i,x}$ to $L_{i,y}$ then

$$\Pr(\exists i : e(L_{i,x}, L_{i,y}) = \emptyset) \leq n^{1/20} \left(1 - \frac{(r - 1)n^{3/5}}{\log r - 1} \right)^{2 \log_{r-1} n} = o(n^{-3}).$$

We can therefore w.h.p. choose an edge $f_i \in e(L_{i,x}, L_{i,y})$ for $1 \leq i \leq n^{1/21}$. Each edge $f_i$ defines a path $Q_i$ from $x$ to $y$ of length at most $2 \log_{r-1} n$. Let $Q'_i$ denote that part of $Q_i$ that goes from $u_i \in A_{t_0}$ to $v_i \in B_{t_0}$. The path $Q_i$ will be rainbow colored if the edges of $Q'_i$ are rainbow colored and distinct from the colors in the path from $x$ to $u_i$ in $T_{x,t_0}$ and the colors in the path from $y$ to $v_i$ in $T_{y,t_0}$. The probability that $Q'_i$ satisfies this condition is at least $\left(1 - \frac{2 \log_{r-1} n}{q}\right)^{2 \log_{r-1} n}$. Here we have used (13). In fact, using (13) we see that

$$\Pr(\exists i : Q_i \text{ is rainbow colored}) \leq \left(1 - \left(1 - \frac{2 \log_{r-1} n}{q}\right)^{2 \log_{r-1} n}\right)^{n^{1/21}} \leq \left(1 - \frac{1}{n^{4/(3K_r^2)}}\right)^{n^{1/21}} = o(n^{-3}).$$

This completes the case where $x, y$ are both tree-like and $T_x \cap T_y = \emptyset$.

### 2.6.2 Tree-like but not disjoint

Suppose now that $x, y$ are both tree-like and $T_x \cap T_y \neq \emptyset$. If $x \in T_y$ or $y \in T_x$ then there is nothing more to do as each root to leaf path of $T_x$ or $T_y$ is rainbow.
Let \( a \in T_y \cap T_x \) be such that its parent in \( T_x \) is not in \( T_y \). Then \( a \) must be a leaf of \( T_y \). We now bound the number of leaves \( \lambda_a \) in \( T_y \) that are descendants of \( a \) in \( T_x \). For this we need the distance of \( y \) from \( T_x \). Suppose that this is \( h \). Then
\[
\lambda_a = 1+(r-2)+(r-1)(r-2)+(r-1)^2(r-2)+\cdots+(r-1)^{kr-h-1}(r-2) = (r-1)^{kr-h}+1.
\]
Now from Lemma 5 we see that there will be at most two choices for \( a \). Otherwise, \( T_x \cup T_y \) will contain at least two cycles of length less than \( 2k_r \). It follows that w.h.p.
\[
\lambda_0 = 2((r-1)^{kr}+1) \leq |S_y|/10.
\]
Similarly, if \( (r-1)^h \geq 201 \) then at most \( |S_x|/10 \) leaves of \( T_x \) will be in \( T_y \). In which case we can use the proof for \( T_x \cap T_y = \emptyset \) with \( S_x, S_y \) cut down by a factor of at most \( 4/5 \).

If \( (r-1)^h \leq 200 \), implying that \( h \leq 5 \) then we proceed as follows: We just replace \( k_r \) by \( k_r + 5 \) in our definition of \( T_x, T_y \), for these pairs. Nothing much will change. We will need to make \( q \) bigger by a constant factor, but now we will have \( y \in T_x \) and we are done.

### 2.6.3 Non tree-like

We can assume that if \( x \) is non tree-like then \( T_x \) contains exactly one cycle \( C \). We first consider the case where \( C \) contains an edge \( e \) that is more than distance 5 away from \( x \). Let \( e = (u,v) \) where \( u \) is the parent of \( v \) and \( u \) is at distance 5 from \( x \). Let \( \tilde{T}_x \) be obtained from \( T_x \) by deleting the edge \( e \) and adding two trees \( H_u, H_v \), one rooted at \( u \) and one rooted at \( v \) so that \( \tilde{T}_x \) is a complete \( (r-1) \)-ary tree of height \( k_r \). Now color \( H_u, H_v \) so that Lemma 3 can be applied. We create \( \tilde{T}_y \) from \( T_y \) in the same way, if necessary. We obtain at least \( (r-1)^{2k_r}/5 \) pairs. But now we must subtract pairs that correspond to leaves of \( H_u, H_v \). By construction there are at most \( 4(r-1)^{2k_r-5} \leq (r-1)^{2k_r}/10 \). So, at least \( (r-1)^{2k_r}/10 \) pairs can be used to complete the rest of the proof as before.

We finally deal with those \( T_x \) containing a cycle of length 10 or less, no edge of which is further than distance 10 from \( x \). Now the expected number of vertices on cycles of length \( k \leq 10 \) is given by
\[
k\left(n\right)\begin{pmatrix}k-1\end{pmatrix}\frac{(n-1)!}{2}2^k\frac{\Psi(rn-2k)}{\Psi(rn)}\sim\frac{(r-1)^k}{2k},
\]
where \( \Psi(m) = m!/(2^{m/2}(m/2)!). \)

It follows that the expected number of edges \( \mu \) that are within 10 or less from a cycle of length 10 or less is bounded by a constant. Hence \( \mu = o(\log n) \) w.h.p. and
we can give each of these edges a distinct new color after the first round of coloring. Any rainbow colored set of edges will remain rainbow colored after this change.

Then to find a rainbow path beginning at $x$ we first take a rainbow path to some $x'$ that is distance 10 from $x$ and then seek a rainbow path from $x'$. The path from $x$ to $x'$ will not cause a problem as the edges on this path are unique to it.

3 Conclusion

We have shown that w.h.p. $r_c(G(n, r)) = O(\log n)$ for $r \geq 4$ and $r = O(1)$. We have conjectured that this remains true for the case $r = 3$. We know there are examples of coloring $T_1, T_2$ in Lemma 2 where $\kappa_\ell = 2^\ell$ when $d = 2$. So more has to be done on this part of the proof. At a more technical level, we should also consider the case where $r \to \infty$ with $n$. Part of this can be handled by the sandwiching results of Kim and Vu [12].

References


