

# Vacant sets and vacant nets: Component structures induced by a random walk.

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## Abstract

Given a discrete random walk on a finite graph  $G$ , the vacant set and vacant net are, respectively, the sets of vertices and edges which remain unvisited by the walk at a given step  $t$ . Let  $\Gamma(t)$  be the subgraph of  $G$  induced by the vacant set of the walk at step  $t$ . Similarly, let  $\widehat{\Gamma}(t)$  be the subgraph of  $G$  induced by the edges of the vacant net.

For random  $r$ -regular graphs  $G_r$ , it was previously established that for a simple random walk, the graph  $\Gamma(t)$  of the vacant set undergoes a phase transition in the sense of the phase transition on Erdős-Renyi graphs  $G_{n,p}$ . Thus, for  $r \geq 3$  there is an explicit value  $t^* = t^*(r)$  of the walk, such that for  $t \leq (1-\epsilon)t^*$ ,  $\Gamma(t)$  has a unique giant component, plus components of size  $O(\log n)$ , whereas for  $t \geq (1+\epsilon)t^*$  all the components of  $\Gamma(t)$  are of size  $O(\log n)$ .

In this paper we establish the threshold value  $\widehat{t}$  for a phase transition in the graph  $\widehat{\Gamma}(t)$  of the vacant net of a simple random walk on a random  $r$ -regular graph,.

We obtain the corresponding threshold results for the vacant set and vacant net of two modified random walks. These are a non-backtracking random walk, and, for  $r$  even, a random walk which chooses unvisited edges whenever available.

This allows a direct comparison of thresholds between simple and modified walks on random  $r$ -regular graphs. The main findings are the following: As  $r$  increases the threshold for the vacant set converges to  $n \log r$  in all three walks. For the vacant net, the threshold converges to  $rn/2 \log n$  for both the simple random walk and non-backtracking random walk. When  $r \geq 4$  is even, the threshold for the vacant net of the unvisited edge process converges to  $rn/2$ , which is also the vertex cover time of the process.

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# 1 Introduction

Let  $G = (V, E)$  be a finite connected graph, with vertex set size  $|V| = n$ , and edge set size  $|E| = m$ . Let  $W$  be a simple random walk on  $G$ , with initial position  $X(0)$  at  $t = 0$ . At discrete steps  $t = 1, 2, \dots$ , the walk chooses  $X(t)$  uniformly at random (u.a.r.) from the neighbours of  $X(t - 1)$  and makes the edge transition  $(X(t - 1), X(t))$ . Let  $W(t) = (X(0), \dots, X(t))$  be the trajectory of the walk up to and including step  $t$ , and let  $\mathcal{B}(t) = \{X(s) : s \leq t\}$  be the set of vertices visited in  $W(t)$ . By analogy with site percolation, the set of unvisited vertices  $\mathcal{R}(t) = V \setminus \mathcal{B}(t)$  is referred to as the *vacant set* of the walk. The graph induced by the uncrossed edges is referred to as the *vacant net*.

In the case of random  $r$ -regular graphs, it was established independently by [6] and [12] that the graph induced by the set of unvisited vertices exhibits sharp threshold behavior. Typically, as the walk proceeds, the induced graph of the vacant set has a unique giant component, which collapses within a relatively small number of steps to leave components of at most logarithmic size. For random  $r$ -regular graphs, we establish the threshold behavior of the vacant net, i.e. the subgraph induced by the set of unvisited edges of the random walk. For comparison purposes, and ignoring terms of order  $1/r$ , the thresholds for the vacant set and vacant net occur around steps  $n \log r$  and  $(r/2)n \log r$  of the walk, respectively.

For  $v \in V$  let  $C_v$  be the expected time taken for a random walk  $W_v$  starting at vertex  $X(0) = v$ , to visit every vertex of the graph  $G$ . The *vertex cover time*  $T_{\text{COV}}^V(G)$  of a graph  $G$  is defined as  $T_{\text{COV}}^V(G) = \max_{v \in V} C_v$ . Let  $N(t) = |\mathcal{R}(t)|$  be the size of the vacant set at step  $t$  of the walk. As the walk  $W_v(t)$  proceeds, the size of the vacant set decreases from  $N(0) = n$  to  $N(t) = 0$  at expected time  $C_v$ . The change in structure of the graph  $\Gamma(t) = G[\mathcal{R}(t)]$  induced by the vacant set  $\mathcal{R}(t)$  is also of interest, insomuch as it is reasonable to ask if  $\Gamma(t)$  evolves in a typical way for most walks  $W(t)$ . Perhaps surprisingly the component structure of the vacant set can be described in detail for certain types of random graphs, and also to some extent for toroidal grids of dimension at least 5.

To motivate this description of the component structure, we recall the typical evolution of the random graph  $G_{n,p}$  as  $p$  increases from 0 to 1. Initially, at  $p = 0$ ,  $G_{n,0}$  consists of isolated vertices. As we increase  $p$ , we find that for  $p = c/n$ , when  $c < 1$  the maximum component size is logarithmic. This is followed by a phase transition around the critical value  $c = 1$ . When  $c > 1$  the maximum component size is linear in  $n$ , and all other components have logarithmic size.

In describing the evolution of the structure of the vacant set as  $t$  increases, the aim is to show that typically  $\Gamma(t)$  undergoes a reversal of the phase transition mentioned above. Thus  $\Gamma(0)$  is connected and  $\Gamma(t)$  starts to break up as  $t$  increases. There is a critical value  $t^*$  such that if  $t < t^*$  by a sufficient amount then  $\Gamma(t)$  consists of a unique giant component plus components of size  $O(\log n)$ . Once we pass through the critical value by a sufficient amount, so that  $t > t^*$ ,

then all components are of size  $O(\log n)$ . As  $t$  increases further, the maximum component size shrinks to zero. We make the following definitions. A graph with vertex set  $V_1$  is *sub-critical* if its maximum component size is  $O(\log n)$ , and *super-critical* if it has a *unique* component  $C_1(t)$  of size  $\Omega(|V_1(t)|)$ , where  $|V_1(t)| \gg \log n$ , and all other components are of size  $O(\log n)$ .

For the case of random  $r$ -regular graphs  $G_r$ , the vacant set was studied independently by Černý, Teixeira and Windisch [6] and by Cooper and Frieze [12]. Both [6] and [12] proved that w.h.p.  $\Gamma(t)$  is sub-critical for  $t \geq (1+\epsilon)t^*$  and that there is a unique linear size component for  $t \leq (1-\epsilon)t^*$ . The paper [6] conjectured that  $\Gamma(t)$  is super-critical for  $t \leq (1-\epsilon)t^*$ , and this was confirmed by [12] who also gave the detailed structure of the small ( $O(\log n)$ ) tree components as a function of  $t$ . Subsequent to this Černý and Teixeira [7] used the methods of [12] to give a sharper analysis of  $\Gamma(t)$  in the critical window around  $t^*$ . The paper [12], also established the critical value  $t^*$  for connected random graphs  $G_{n,p}$  and for strongly connected random digraphs  $D_{n,p}$ .

For the case of toroidal grids, the situation is less clear. Benjamini and Sznitman [2] and Windisch [23] investigated the structure of the vacant set of a random walk on a  $d$ -dimensional torus. The main focus of this work is to apply the method of random interlacements. For toroidal grids of dimension  $d \geq 5$ , it is shown that there is a value  $t^+(d)$ , linear in  $n$ , above which the vacant set is sub-critical, and a value of  $t^-(d)$  below which the graph is super-critical. It is believed that there is a phase transition for  $d \geq 3$ . A recent monograph by Černý and Teixeira [8] summarizes the random interlacement methodology. The monograph also gives details for the vacant set of random  $r$ -regular graphs.

Let  $\mathcal{S}(t) = \{(X(s), X(s+1)) : 0 \leq s < t\}$  be the set of *visited edges* based on transitions of the walk  $W$  up to and including step  $t$ , and let  $\mathcal{U}(t) = E(G) \setminus \mathcal{S}(t)$  be corresponding the set of *unvisited edges*. The *edge cover time*  $T_{\text{COV}}^E(G)$  of a graph  $G$  is defined in a similar way to the vertex cover time. The edge set  $\mathcal{U}(t)$  defines an edge induced subgraph  $\widehat{\Gamma}(t)$  of  $G$  whose vertices may be either visited or unvisited. By analogy with the case for vertices we will call  $\widehat{\Gamma}(t)$  the *vacant network* or *vacant net* for short. We can ask the same questions about the phase transition  $\widehat{t}$  for the vacant net, as were asked for the phase transition  $t^*$  of the vacant set.

Random walk based crawling is a simple method to search large networks, and a giant component in the vacant set can indicate the existence of a large corpus of information which has somehow been missed. Similarly, a giant component in the vacant net indicates the continuing existence of a large communications network or set of unexplored relationships. From this point of view, any way to speed up the collapse of the giant component can be seen as worthwhile. One method, which seems attractive at first sight, is to prevent the walk from backtracking over the edge it has just used. Another simple method is to walk randomly but choose unvisited edges when available.

We determine the thresholds for simple random walks and non-backtracking random walks;

and also for walks which prefer unvisited edges for the case that the vertex degree  $r$  is even. This allows a direct comparison of performance between these three types of random walk. Detailed definitions and results for simple random walks, non-backtracking walks, and walks which prefer unvisited edges are given in Sections 1.1, 1.2 and 1.3 respectively.

As an example, for random 3-regular graphs, using a non-backtracking walk reduces the threshold value by a factor of 2 for vacant sets, and by  $5/2$  for vacant nets respectively. Thus for very sparse graphs, improvements can be obtained by making the walk non-backtracking. However, the improvement gained by a non-backtracking walk is of order  $1 + O(1/r)$ , and soon becomes insignificant as  $r$  increases. In fact, for all three walks, the threshold value for the vacant set tends to  $n \log r$ . For simple and non-backtracking walks, the threshold value for the vacant net tends to  $nr/2 \log r$ . For walks which prefer unvisited edges the threshold for the vacant net tends to  $nr/2$ . This is an improvement of order  $\log r$  over the other processes, but the results only hold for  $r$  even.

As a by-product of the proofs in this paper we give an asymptotic value of  $(r/2)n$  for the vertex cover time of the unvisited edge process for  $r$  even. This confirms the order of magnitude estimate  $\Theta(n)$  and the constant  $r/2$  in the experimental results of [3]. The plot of experiments is reproduced in Section 8.1 of the Appendix. Note that the plot uses the notation  $d$  for vertex degree (rather than  $r$ ). It can be seen from the figure that the vertex cover time of the unvisited edge process exhibit a dichotomy whereby for odd vertex degree, the vertex cover time appears to be  $\Theta(n \log n)$ .

**Notation.**

Apart from  $O(\cdot), o(\cdot), \Omega(\cdot)$  as a function of  $n \rightarrow \infty$ , where  $n = |V|$ , we use the following notation. We say  $A_n \ll B_n$  or  $B_n \gg A_n$  if  $A_n/B_n \rightarrow 0$  as  $n \rightarrow \infty$ , and  $A_n \sim B_n$  if  $\lim_{n \rightarrow \infty} A_n/B_n = 1$ . The notation  $\omega(n)$  describes a function tending to infinity as  $n \rightarrow \infty$ . We measure both walk and graph probabilities in terms of  $n$ , the size of the vertex set of the graph.

We use the expression *with high probability* (w.h.p.), to mean with probability  $1 - o(1)$ , where the  $o(1)$  is a function of  $n$ , which tends to zero as  $n \rightarrow \infty$ . For the proofs in this paper, we can take  $o(1) = O(\log^{-K} n)$  for some large positive constant  $K$ . The statement of theorems in this section are w.h.p. relative to both graph sampling and walks on the sampled graph. It will be clear when we are discussing properties of the the graph, these are given in Section 2. In the case where we use deferred decisions, if  $|\mathcal{R}(t)| = N$ , the w.h.p. statements are asymptotic in  $N$ , and we assume  $N(n) \rightarrow \infty$  with  $n$ .

Let  $W$  be a random walk  $W$  on a graph  $G$ . If we need to stress the start position  $u$  of the walk  $W$ , we write  $W_u$ . The vertex occupied by  $W$  at step  $t$  is given by  $X(t)$  or  $X_u(t)$ . Generally we use  $\mathbf{Pr}(A)$  or  $\mathbf{Pr}_W(A)$  to denote the probability of event  $A = A(t)$  at some step  $t$  of the random walk  $W$ . We use  $P$  for the transition matrix of the walk, and use  $P_u^t(v)$  or  $P_u^t(v; G)$  for the  $(u, v)$ -th entry of  $P^t$ , i.e  $P_u^t(v) = \mathbf{Pr}(X_u(t) = v)$ . When using generating functions we

use simple unencumbered notation such as  $h_t, f_t, r_t$  for the probability that certain specific events occur at step  $t$ . In particular for a designated start vertex  $v$ ,  $r_t = P_v^t(v)$ . We use  $\pi_v$ , or  $\pi_G(v)$  for the stationary probability of a random walk  $W$  at vertex  $v$  of a graph  $G$ . The notation  $p_v$  has a specific meaning in the context of Lemma 5, and is reserved for that.

## 1.1 Simple random walk: Structure of vacant set and vacant net

Let  $G_r(n)$  be the space of  $r$ -regular graphs on  $n$  vertices, and let  $G$  be chosen u.a.r. from  $G_r(n)$ . The following theorem details established results for the vacant set of a simple random walk on  $G$ , as given in [6], [12].

**Theorem 1.** *Let  $W(t)$  be a simple random walk on a random  $r$ -regular graph. For  $r \geq 3$ , the following results hold w.h.p..*

- (i) *Let  $\Gamma(t)$  be the graph induced by the vacant set  $\mathcal{R}(t)$ , at step  $t$  of  $W$ , then  $G(t)$  has  $|\mathcal{R}(t)|$  vertices and  $|E(\Gamma(t))|$  edges, where*

$$|\mathcal{R}(t)| \sim n \exp\left(-\frac{r-2}{r-1} \frac{t}{n}\right), \quad |E(\Gamma(t))| \sim \frac{rn}{2} \exp\left(-\frac{2(r-2)}{r} \frac{t}{n}\right). \quad (1)$$

- (ii) *The size of the vacant net  $|\mathcal{U}(t)|$  at step  $t$  of  $W$  is*

$$|\mathcal{U}(t)| \sim \frac{rn}{2} \exp\left(-\frac{2(r-2)}{r(r-1)} \frac{t}{n}\right). \quad (2)$$

- (iii) [9] *The vertex and edge cover times of a non-backtracking walk are  $T_{\text{cov}}^V(G) \sim \frac{r-1}{r-2} n \log n$  and  $T_{\text{cov}}^E(G) \sim \frac{r(r-1)}{2(r-2)} n \log n$  respectively.*

- (iv) *The threshold for the sub-critical phase of the vacant set in  $G$  occurs at  $t^* = u^*n$  where*

$$u^* = \frac{r(r-1)}{(r-2)^2} \log(r-1). \quad (3)$$

We now come to the new results of this paper. We first consider the structure of the graph  $\widehat{\Gamma}(t)$  induced by the edges in the vacant net  $\mathcal{U}(t)$  of  $G_r$ . By using the random walk to reveal the structure of the graph, we argued in [12] that  $\Gamma(t)$  was a random graph with degree sequence  $D_s(t), s = 1, \dots, r$ . We applied the result of Molloy and Reed [20] for the existence of a giant component in fixed degree sequence graphs, to the degree sequence  $D_s(t)$  to obtain the threshold  $t^* = u^*n$  given in (3). By using a simplification of the Molloy-Reed condition in terms of moments of the degree sequence we can obtain the threshold for the vacant net  $\widehat{\Gamma}(t)$ . The proof of the next theorem is given in Section 4.

**Theorem 2.** Let  $\hat{t} = \theta^*n$ . Then w.h.p. for any  $\epsilon > 0$ , the graph  $\hat{\Gamma}(t) = (V, \mathcal{U}(t))$  induced by the unvisited edges  $\mathcal{U}(t)$  of  $G$  has the following properties:

(i) The threshold for the sub-critical phase of the vacant net in  $G$  occurs at  $\hat{t} = \theta^*n$  where

$$\theta^* = \frac{r(r^2 - 2r + 2)}{2(r - 2)^2} \log(r - 1). \quad (4)$$

(ii) For  $t \leq (1 - \epsilon)\hat{t}$ ,  $\hat{\Gamma}(t)$  is super-critical, and  $|C_1(t)| = \Omega(n)$ .

(iii) For  $t \geq (1 + \epsilon)\hat{t}$ ,  $\hat{\Gamma}(t)$  is sub-critical, and thus  $|C_1(t)| = O(\log n)$ .

(iv) For some constant  $c > 0$  and  $t \in (\hat{t} - cn^{2/3}, \hat{t} + cn^{2/3})$ , then  $\Pr(|C_1(t)| = \Theta(n^{2/3})) \geq 1 - \epsilon$ .

## 1.2 Non-backtracking random walk: Structure of vacant set and vacant net

Speeding up random walks is a matter of both theoretical curiosity and practical interest. One plausible approach to this is to use a non-backtracking walk. A non-backtracking walk does not move back down the edge used for the previous transition unless there is no choice. Thus arguably it should be faster to cover the graph. Let  $v = X(t)$  be the vertex occupied by the walk at step  $t$ , and suppose this vertex was reached by the edge transition  $e = (X(t-1), X(t))$ . The vertex  $u = X(t+1)$  is chosen u.a.r. from  $N(v) \setminus X(t-1)$ , so that  $e \neq (X(t), X(t+1))$ . If there is no choice, i.e.  $X(t)$  is a vertex of degree 1, we can assume the walk returns along  $e$ , but as  $r \geq 3$  this case does not arise.

In the case of random  $r$ -regular graphs, a direct comparison can be made between the performance of simple and non-backtracking random walks. The details for non-backtracking walks are summarized in the following theorem, the proof of which is given in Section 5. The comparable results for simple walks are given in Section 1.1.

**Theorem 3.** Let  $W(t)$  be a non-backtracking random walk on a random  $r$ -regular graph. For  $r \geq 3$ , the following results hold w.h.p..

(i) Let  $\Gamma(t)$  be the graph induced by the vacant set  $\mathcal{R}(t)$ , at step  $t$  of  $W$ , then  $G(t)$  has  $|\mathcal{R}(t)|$  vertices and  $|E(\Gamma(t))|$  edges, where

$$|\mathcal{R}(t)| \sim n \exp(-t/n), \quad |E(\Gamma(t))| \sim \frac{rn}{2} \exp\left(-\frac{2(r-1)t}{rn}\right).$$

(ii) The size of the vacant net  $|\mathcal{U}(t)|$  at step  $t$  of  $W$  is

$$|\mathcal{U}(t)| \sim \frac{rn}{2} \exp(-2t/rn).$$

(iii) The vertex and edge cover times of a non-backtracking walk are  $T_{cov}^V(G) \sim n \log n$  and  $T_{cov}^E(G) \sim (r/2)n \log n$  respectively.

(iv) The threshold for the sub-critical phase of the vacant set occurs at  $t^* = u^*n$  where

$$u^* \sim \frac{r}{r-2} \log(r-1).$$

(v) The threshold for the sub-critical phase of the vacant net occurs at  $\hat{t} = \theta^*n$  where

$$\theta^* \sim \frac{r(r-1)}{2(r-2)} \log(r-1).$$

(vi) Let  $\tilde{t} = t^*, \tilde{\hat{t}}$ , for the vacant set and vacant net respectively. For any  $\epsilon > 0$ , some constant  $c > 0$  and  $t \in (\tilde{t} - cn^{2/3}, \tilde{\hat{t}} + cn^{2/3})$ , then  $\Pr(|C_1(t)| = \Theta(n^{2/3})) \geq 1 - \epsilon$ .

Comparing  $u^*, \theta^*$  for simple and non-backtracking walks, from (3), (4) and Theorem 3 respectively, we see that for  $r = 3$  the subcritical phases occur 2, 5/2 times earlier for vacant sets and vacant nets (resp.). This improvement decreases rapidly as  $r$  increases. A direct contrast between the densities of the vacant set for the two walks follows from the edge-vertex ratios  $|E(\Gamma(t))|/|\mathcal{R}(t)|$ . At any step  $t$  the vacant set of the simple random walk is denser w.h.p..

### 1.3 Random walks which prefer unvisited edges: Structure of vacant set and vacant net

The papers [3], [21] describe a modified random walk  $X = (X(t), t \geq 0)$  on a graph  $G$ , which uses unvisited edges when available at the currently occupied vertex. If there are *unvisited edges* incident with the current vertex, the walk picks one u.a.r. and make a transition along this edge. If there are no unvisited edges incident with the current vertex, the walk moves to a random neighbour.

In [3] this walk was called an *unvisited edge process* (or edge-process), and in [21], a *greedy random walk*. For random  $r$ -regular graphs where  $r = 2d$ , it was shown in [3] that the edge-process has vertex cover time  $\Theta(n)$ , which is best possible up to a constant. The paper also gives an upper bound of  $O(n\omega)$  for the edge cover time. The  $\omega$  term comes from the w.h.p. presence of small cycles (of length at most  $\omega$ ).

In the case of random  $r$ -regular graphs, the vacant set and vacant net of the edge-process have the following theorem which is proved in Section 6.

**Theorem 4.** *Let  $X$  be an edge-process on a random  $r$ -regular graph. For  $r \geq 4$ ,  $r = 2d$ , the following results hold w.h.p..*

- (i) *Let  $\Gamma(t)$  be the graph induced by the vacant set  $\mathcal{R}(t)$  of the edge-process at step  $t$ . Then for  $\delta > 0$  and any  $t = dt(1 - \delta)$  the vacant set has  $|\mathcal{R}(t)|$  vertices and  $|E(\Gamma(t))|$  edges, where*

$$|\mathcal{R}(t)| \sim n \left( \frac{dn - t}{dn} \right)^d, \quad |E(\Gamma(t))| \sim dn \left( \frac{dn - t}{dn} \right)^{2d-1}$$

- (ii) *The vertex cover time of the edge-process is  $T_{\text{COV}}^V(G) \sim dn$ .*

- (iii) *The threshold for the sub-critical phase of the vacant set occurs at  $t^* \sim u^*n$  where*

$$u^* \sim d \left( 1 - \left( \frac{1}{2d-1} \right)^{\frac{1}{d-1}} \right).$$

*For any  $\epsilon > 0$  and  $t = t^*(1 - \epsilon)$ , the largest component  $C_1(t)$  is of size  $\Theta(n)$ , whereas for  $t = t^*(1 + \epsilon)$ , the largest component is of size  $O(\log n)$ .*

- (iv) *For  $t = dn(1 - \delta)$ , and  $\delta \geq \sqrt{\omega \log n/n}$ , the the vacant net  $\mathcal{U}(t)$  of the edge-process is of size  $dn\delta(1 + o(1))$ .*
- (v) *The threshold for a phase transition of the vacant net occurs at  $\hat{t} \sim dn$ . For any  $\epsilon > 0$  and  $t = \hat{t}(1 - \epsilon)$ , the largest component  $C_1(t)$  is of size  $\Theta(n)$ , whereas for  $t = \hat{t}(1 + \epsilon)$ , the largest component is of size  $O(\log n)$ .*

As for the edge cover time  $T_{\text{COV}}^E(G)$  of the edge-process, trivially  $T_{\text{COV}}^E(G) \geq dn$ . It was proved in [3] that  $T_{\text{COV}}^E(G) = O(\omega n)$ . The  $\omega$  term comes from the presence of cycles size  $O(\omega)$ . We do not see any obvious reason from the proof of Theorem 4 to suppose  $T_{\text{COV}}^E(G) = \Theta(n)$ .

## 1.4 Outline of proof methodology

The proof of the vacant net threshold, Theorem 2, is given in Section 4. The proof of Theorem 3 on the properties of the vacant set and vacant net for non-backtracking random walks is given in Section 5. The technique used to analyze the structure of random walks is one the authors have developed over a sequence of papers. The results we need in the proof of this paper are given in Section 3.

The method of proof of the main theorems is similar. The main steps in the proof of (e.g.) Theorem 2 are as follows. (i) In Section 2 we state the structural graph properties we assume in order to analyse a random walk on an  $r$ -regular graph. (ii) Given these properties, in Section



4.1 we obtain the degree sequence  $\widehat{\mathbf{d}}(t)$  of the vacant net  $\widehat{\Gamma}(t)$  at step  $t$  of the walk. The degree sequence is given in an implicit form. (iii) In Section 4.2, we prove that  $\widehat{\Gamma}(t)$  is a random graph with degree sequence  $\widehat{\mathbf{d}}(t)$ . (iii) In Section 4.3 we obtain the component structure of  $\widehat{\Gamma}(t)$ . This follows from a result of Molloy and Reed [20] on the component structure of fixed degree sequence random graphs.

We next give more detail of the general method used to prove structural properties of the vacant set or vacant net. For ease of description we use the example of the vacant set of a simple random walk, and highlight any differences for the other cases as appropriate. There are two main features.

Firstly we use the random walk to generate the graph in the configuration model. If we stop the walk at any step, the un-revealed part of the graph is still random conditional on the structure of the revealed part, and the constraint that all vertices have degree  $r$ . The approach is equally valid for other Markov processes such as non-backtracking random walks. Secondly using the techniques given in Section 3.2 we can estimate the size  $N(t)$ , and degree sequence  $\mathbf{d}(t)$ , of the vacant set  $\mathcal{R}(t)$  very precisely at a given step  $t$ .

Combining these results, the graph  $\Gamma(t)$  of the vacant set is thus a random graph with  $N(t)$  vertices and degree sequence  $\mathbf{d}(t)$ . Molloy and Reed [20] derived conditions for the existence of, and size of the giant component in a random graph with a given degree sequence. We apply these conditions to  $\Gamma(t)$  to obtain the threshold etc. This is what we did in [12], and we do not reproduce in detail those aspects of (e.g.) Theorem 3 which directly repeat these methods.

## 2 Graph properties of $G_r$

Let

$$\ell_1 = \epsilon_1 \log_r n, \tag{5}$$

for some sufficiently small  $\epsilon_1$ . A cycle  $C$  is *small* if  $|C| \leq \ell_1$ . A vertex of a graph  $G$  is *nice* if it is at distance at least  $\ell_1 + 1$  from any small cycle.

Let  $D_k(v)$  be the subgraph of  $G$  induced by the vertices at distance at most  $k$  from  $v$ . A vertex  $v$  is *tree-like* to depth  $k$  if  $D_k(v)$  induces a tree, rooted at  $v$ . Thus a nice vertex is tree-like to depth  $\ell_1$ . Let  $\mathcal{N}$  denote the nice vertices of  $G$  and  $\overline{\mathcal{N}}$  denote the vertices that are not nice.

Let  $G_r$  be the space of  $r$ -regular graphs, endowed with the uniform probability measure. Let

$G$  be chosen u.a.r. from  $G_r$ . We assume the following w.h.p. properties.

$$\text{There are at most } n^{2\epsilon_1} \text{ vertices that are not nice.} \tag{6}$$

$$\text{There are no two small cycles within distance } 2\ell_1 \text{ of each other.} \tag{7}$$

$$\text{Let } \lambda = \max(\lambda_2, \lambda_n) \text{ be the second largest eigenvalue of the transition matrix } P.$$

$$\text{Then } \lambda_2 \leq (2\sqrt{r-1} + \epsilon)/r \leq 29/30, \text{ say.} \tag{8}$$

Properties (i), (ii) are straightforward to prove by first moment calculations. Property (iii) is a result of Friedman [15].

The results we prove concerning random walks on a graph  $G$  are all conditional on  $G$  having properties (6)-(8). This conditioning can only inflate the probabilities of unlikely events by  $1 + o(1)$ . This observation includes those events defined in terms of the configuration model as claimed in Lemma 10. For  $r$  constant, the underlying configuration multi-graph is simple with constant probability, and all simple  $r$ -regular graphs are equally probable. If a calculation shows that an event  $\mathcal{E}$  has probability at most  $\epsilon$  in the configuration model, then it has probability  $O(\epsilon)$  with respect to the corresponding simple graph  $G$ . We only need to multiply this bound by a further  $1 + o(1)$  in order to estimate the probability conditional on (6)-(8). We will continue using this convention without further comment.

### 3 Background material on unvisit probabilities

#### 3.1 Summary of methodology

To find the size of the vacant set or net, we estimate the probability that a given vertex or edge of the graph were not visited by the random walk during steps  $T, \dots, t$ , where  $T$  is suitably defined mixing time (see (12)). For simplicity, we refer to this quantity as an *unvisit probability*. We briefly outline of how the unvisit probability is obtained. This is given in more detail in Section 3.2.

The quantities needed to estimate the unvisit probability of a vertex  $v$  are the mixing time  $T$ , the stationary probability  $\pi_v$  of vertex  $v$  and  $R_v$ , defined below. For a simple random walk  $\pi_v = d(v)/2m$ . The mixing time  $T$  we use satisfies a convergence condition given in (12). The theorems in this paper are for random regular graphs  $G = G_r$ ,  $r \geq 3$  constant, and w.h.p.  $G$  has constant eigenvalue gap so the mixing time  $T = O(\log n)$  satisfies (12). The non-backtracking walk uses a Markov chain  $\mathcal{M}$  on directed edges. In Section 8.3 of the Appendix we prove directly that w.h.p.  $T = O(\log n)$

The unvisit probability  $\mathbf{Pr}_W(\mathcal{A}_v(t))$  is given in (23)-(24) of Corollary 6 of Lemma 5 in terms of  $p_v = (1+o(1))\pi_v/R_v$ . For regular graphs  $\pi_v = 1/n$ . The quantity  $R_v$  is defined as follows. For

a walk starting from  $v$  let  $r_0 = 1$  and let  $r_i$  be the probability the walk returns to  $v$  at step  $i$ . Then

$$R_v = \sum_{i=0}^{T-1} r_i.$$

Thus  $R_v$  is the expected number of returns to  $v$  before step  $T$ .

Because the Molloy-Reed condition is robust to small changes in degree sequence, for our proofs, we only need to find the value of  $R_v$  for nice vertices. This is obtained as follows. Let  $D_\ell(v)$  be the subgraph induced by the vertices at distance at most  $\ell$  from  $v$ . The value of  $\ell$  we use is given in (5). If  $D_\ell(v)$  is a tree, we say  $v$  is a nice vertex, and use  $\mathcal{N}$  to denote the set of nice vertices of graph  $G$ . With high probability, all but  $o(n)$  vertices of a random  $r$ -regular graph are nice. If  $v$  is nice, the subgraph  $D(v)$  is a tree with internal vertices of degree  $r$ , and we extend  $D_\ell(v)$  to an infinite  $r$ -regular tree  $\mathcal{T}$  rooted at  $v$ . The principal quantity used to calculate  $R_v$ , is  $f$ , the probability of a first return to  $v$  in  $\mathcal{T}$ . Basically, once the walk is distance  $\Theta(\log \log n)$  from  $v$  the probability of a return to  $v$  during  $T = O(\log n)$  steps is  $o(1)$ . Thus calculations for  $f$  can be made in  $\mathcal{T}$  followed by a correction of smaller order, giving  $R_v = (1 + o(1))/(1 - f)$ . This is formalized in Lemma 22 of the Appendix.

The proofs in this paper use the notion of a set  $S$  of vertices or edges not being visited by the walk during  $T, \dots, t$ . Because  $R_S$  is not well defined for general sets  $S$ , to use Corollary 6 we contract the set  $S$  to a single vertex  $\gamma(S)$ , and calculate  $R_{\gamma(S)}$  in the multi-graph  $H$  obtained from  $G$  by this contraction. Using Corollary 6 we obtain the probability that  $\gamma(S)$  is unvisited in  $H$ . Lemma 7 ensures that the probability  $\gamma(S)$  is unvisited in  $H$  is asymptotically equal to the probability the set  $S$  is unvisited in  $G$ . In the case of visits to sets of edges rather than vertices, these are subdivided by inserting a set of dummy vertices  $S$ , one in the middle of each edge in question. The set  $S$  is then contracted to a vertex  $\gamma(S)$  as before. In the case of the non-backtracking walk things get more complicated as the Markov chain  $\mathcal{M}$  of the walk is on directed edges, but the principle is the same.

The contraction operation changes the graph from  $G$  to  $H$ , which can alter the mixing time  $T$ , but does not significantly increase it for the following reasons. The effect of contracting a set of vertices increases the eigenvalue gap, (see e.g. [17] page 168) so that  $1 - \lambda_2(H) \geq 1 - \lambda_2(G)$ , and thus  $T$  can only decrease. In the case of edge subdivision, the gap could decrease. However, we only perform this operation on (at most)  $2r$  edges of an  $r$ -regular graph with constant eigenvalue gap, and with  $r$  constant. It follows that the conductance of  $H$  is still constant and thus the mixing time  $T(H)$  differs from  $T(G)$  by at most a constant multiple.

## 3.2 Unvisit probabilities

Our proofs make heavy use of Lemma 5 below. Let  $P$  be the transition matrix of the walk and let  $P_u^t(v)$  be the  $(u, v)$ -th entry of  $P^t$ . Let  $W_u(t)$  be the position of the random walk  $W_u$

at step  $t$ , and let  $P_u^t(v) = \mathbf{Pr}(W_u(t) = v)$  be the  $t$ -step transition probability. We assume  $G$  is connected and aperiodic, so that random walk  $W_u$  on  $G$  has stationary distribution  $\pi$ , where  $\pi_v = d(v)/(2m)$ .

For periodic graphs, we can replace the simple random walk by a *lazy* walk, in which at each step there is a  $1/2$  probability of staying put. By ignoring the steps when the particle does not move in the lazy walk we obtain the underlying simple random walk. For large  $t$ , asymptotically half of the steps in the lazy walk will not result in a change of vertex. Therefore w.h.p. properties of the simple walk after approximately  $t$  steps can be obtained from properties of the lazy walk after  $2t$  steps. Making the walk lazy doubles the expected number of returns to a vertex and thus changes  $R_v$  (see (16)) to approximately  $2R_v$ . As we only consider the ratio  $t/R_v = 2t/2R_v$  in our proofs, our results will not alter significantly.

Suppose that the eigenvalues of the transition matrix  $P$  are  $1 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$ . Let  $\lambda = \max\{|\lambda_i| : i \geq 2\}$ . By making the chain lazy if necessary, we can always make  $\lambda_2 = \max(|\lambda_2|, |\lambda_n|)$ .

Let  $\Phi_G$  be the conductance of  $G$  i.e.

$$\Phi_G = \min_{S \subseteq V, \pi_S \leq 1/2} \frac{\sum_{x \in S} \pi_x P(x, \bar{S})}{\pi_S}, \quad (9)$$

where  $P(x, \bar{S})$  is the probability of a transition from  $x \in S$  to  $\bar{S}$ . Then,

$$1 - \Phi_G \leq \lambda_2 \leq 1 - \frac{\Phi_G^2}{2} \quad (10)$$

$$|P_u^t(x) - \pi_x| \leq (\pi_x/\pi_u)^{1/2} \lambda^t. \quad (11)$$

A proof of these can be found for example in Sinclair [22] and Lovasz [18], Theorem 5.1 respectively.

**Mixing time of  $G_r$ .** Let  $T$  be such that, for  $t \geq T$

$$\max_{u, x \in V} |P_u^t(x) - \pi_x| = \frac{\min_{x \in V} \pi_x}{n^3} = \frac{1}{n^4}. \quad (12)$$

By assumption (8) (a result of Friedman [15]) we have  $\lambda \leq (2\sqrt{r-1} + \epsilon)/r \leq 29/30$ . In which case we can take

$$T(G_r) \leq 120 \log n. \quad (13)$$

If inequality (12) holds, we say the distribution of the walk is in *near stationarity*.

**Generating function formulation.** Fix two vertices  $u, v$  of  $G$ . Let  $h_t = P_u^t(v)$  be the probability that the walk  $W_u$  visits  $v$  at step  $t$ . Let

$$H(z) = \sum_{t=T}^{\infty} h_t z^t \quad (14)$$

generate  $h_t$  for  $t \geq T$ .

We next consider the special case of returns to vertex  $v$  made by a walk  $W_v$ , starting at  $v$ . Let  $r_t = P_v^t(v)$  be the probability that the walk returns to  $v$  at step  $t = 0, 1, \dots$ . In particular note that  $r_0 = 1$ , as the walk starts at  $v$ . Let

$$R(z) = \sum_{t=0}^{\infty} r_t z^t$$

generate  $r_t$ , and let

$$R_T(z) = \sum_{j=0}^{T-1} r_j z^j. \quad (15)$$

Thus, evaluating  $R_T(z)$  at  $z = 1$ , we have  $R_T(1) \geq r_0 = 1$ . Let

$$R_v = R_T(1) = \sum_{i=0}^{T-1} r_i. \quad (16)$$

The quantity  $R_v$ , the expected number of returns to  $v$  during the mixing time, has a particular importance in our proofs.

For  $t \geq T$  let  $f_t = f_t(u \rightarrow v)$  be the probability that the first visit made to  $v$  by the walk  $W_u$  to  $v$  in the period  $[T, T+1, \dots]$  occurs at step  $t$ . Let

$$F(z) = \sum_{t=T}^{\infty} f_t z^t$$

generate  $f_t$ . The relationship between  $h_j$  and  $f_j, r_j$  is given by

$$h_t = \sum_{k=1}^t f_k r_{t-k}. \quad (17)$$

In terms of generating functions, this becomes

$$H(z) = F(z)R(z). \quad (18)$$

The following lemma gives the probability that a walk, starting from near stationarity makes a first visit to vertex  $v$  at a given step. The content of the lemma is to extend  $F(z) = H(z)/R(z)$

analytically beyond  $|z| = 1$  and extract the asymptotic coefficients. For the proof of Lemma 5 and Corollary 6, see Lemma 6 and Corollary 7 of [10]. We use the lemma to estimate  $\mathbf{E}|\mathcal{R}_T(t)|$ , the expected number of vertices unvisited after  $T$ . The value of  $\mathbf{E}|\mathcal{R}_T(t)|$  differs from  $\mathbf{E}|\mathcal{R}(t)|$  by at most  $T$  vertices, so as  $T = O(\log n)$  and  $\mathbf{E}|\mathcal{R}_T(t)| = \Theta(n)$  this simplification will not affect our results.

**Lemma 5.** *For some sufficiently large constant  $K$ , let*

$$\lambda = \frac{1}{KT}, \quad (19)$$

where  $T$  satisfies (12). Suppose that

(i) For some constant  $\theta > 0$ , we have

$$\min_{|z| \leq 1+\lambda} |R_T(z)| \geq \theta.$$

(ii)  $T\pi_v = o(1)$  and  $T\pi_v = \Omega(n^{-2})$ .

There exists

$$p_v = \frac{\pi_v}{R_v(1 + O(T\pi_v))}, \quad (20)$$

such that for all  $t \geq T$ ,

$$f_t(u \rightarrow v) = (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^{t+1}} + O(T\pi_v e^{-\lambda t/2}). \quad (21)$$

$$= (1 + O(T\pi_v)) \frac{p_v}{(1 + p_v)^t} \quad \text{for } t \geq \log^3 n. \quad (22)$$

Lemma 5 depends on two conditions (i), (ii). For nice  $G_r$ , as as  $T\pi_v = O(\log n/n) = o(1)$ , condition (ii) holds. For the case where  $R_v \geq 1$  constant, it was shown in [11] Lemma 18 that condition (i) always holds. The following corollary follows directly by adding up  $f_s(u \rightarrow v)$  for  $s \geq t$ .

**Corollary 6.** *For  $t \geq T$  let  $\mathcal{A}_v(t)$  be the event that  $W_u$  does not visit  $v$  at steps  $T, T+1, \dots, t$ . Then, under the assumptions of Lemma 5,*

$$\mathbf{Pr}_W(\mathcal{A}_v(t)) = \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} + O(T^2\pi_v e^{-\lambda t/2}) \quad (23)$$

$$= \frac{(1 + O(T\pi_v))}{(1 + p_v)^t} \quad \text{for } t \geq \log^3 n. \quad (24)$$

We use the notation  $\mathbf{Pr}_W$  here to emphasize that we are dealing with the probability space of walks on a fixed  $G$ .

Corollary 6 gives the probability of not visiting a single vertex in time  $[T, t]$ . We need to extend this result to certain small sets of vertices. In particular we need to consider sets consisting of  $v$  and a subset of its neighbours  $N(v)$ . Let  $S$  be such a subset.

Suppose now that  $S$  is a subset of  $V$  with  $|S| = o(n)$ . By contracting  $S$  to single vertex  $\gamma = \gamma(S)$ , we form a graph  $H = H(S)$  in which the set  $S$  is replaced by  $\gamma$  and the edges that were contained in  $S$  are contracted to loops. The probability of no visit to  $S$  in  $G$  can be found (up to a multiplicative error of  $1 + O(1/n^3)$ ) from the probability of a first visit to  $\gamma$  in  $H$ . This is the content of Lemma 7 below.

We can estimate the mixing time of a random walk on  $H$  as from the conductance of  $G$  as follows. Note that the conductance of  $H$  is at least that of  $G$ . As some subsets of vertices of  $V$  have been removed by the contraction of  $S$ , the set of values that we minimise over, to calculate the conductance of  $H$ , (see (9)), is a subset of the set of values that we minimise over for  $G$ . It follows that the conductance of  $H$  is bounded below by the conductance of  $G$ . Assuming that the conductance of  $G$  is constant, which is the case in this paper, then using (10), (11), we see that the mixing time for  $W$  in  $H$  is  $O(\log n)$ .

Say that the stationary distribution  $\pi_G$  of the walk in  $G$  and  $\pi_H$  of the walk in  $H$  are *compatible* if  $\pi_H(\gamma(S)) = \sum_{v \in S} \pi_G(v)$  and for  $w \notin S$ ,  $\pi_G(w) = \pi_H(w)$ . For example, if  $G$  is an undirected graph then the stationary distributions are always compatible, because the stationary distribution of  $\gamma(S)$  is given by  $\pi_H(\gamma(S)) = d(S)/2m = \sum_{v \in S} \pi_G(v)$ . If  $G$  is directed, compatibility does not follow automatically, and needs to be checked.

**Lemma 7.** [10] *Let  $W_u$  be a random walk in  $G$  starting at  $u \notin S$ , and let  $\mathcal{X}_u$  be a random walk in  $H$  starting at  $u \neq \gamma$ . Let  $T$  be a mixing time satisfying (12) in both  $G$  and  $H$ . Then provided  $\pi_G$  and  $\pi_H$  are compatible,*

$$\Pr(\mathcal{A}_\gamma(t); H) = \Pr(\bigwedge_{v \in S} \mathcal{A}_v(t); G) \left( 1 + O\left(\frac{1}{n^3}\right) \right),$$

where the probabilities are those derived from the walk in the given graph.

**Proof** Let  $W_x(j)$  (resp.  $X_x(j)$ ) be the position of walk  $W_x$  (resp.  $\mathcal{X}_x(j)$ ) at step  $j$ . Let  $\Gamma = G, H$  and let  $P_u^s(x; \Gamma)$  be the transition probability in  $\Gamma$ , for the walk to go from  $u$  to  $x$

in  $s$  steps.

$$\begin{aligned} \Pr(\mathcal{A}_\gamma(t); H) &= \sum_{x \neq \gamma} P_u^T(x; H) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; H) \\ &= \sum_{x \neq \gamma} (\pi_H(x)(1 + O(n^{-3}))) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; H) \end{aligned} \quad (25)$$

$$= \sum_{x \neq \gamma} (\pi_G(x)(1 + O(n^{-3}))) \Pr(X_x(s-T) \neq \gamma, T \leq s \leq t; H) \quad (26)$$

$$= \sum_{x \notin S} (P_u^T(x; G)(1 + O(n^{-3}))) \Pr(W_x(s-T) \notin S, T \leq s \leq t; G) \quad (27)$$

$$= \Pr(\bigwedge_{v \in S} \mathcal{A}_v(t); G)(1 + O(1/n^3)).$$

Equation (25) follows from (12). Equation (26) from compatibility of  $\pi_H$  and  $\pi_G$ . Equation (27) follows because there is a natural measure preserving map  $\phi$  between walks in  $G$  that start at  $x \notin S$  and avoid  $S$  and walks in  $H$  that start at  $x \neq \gamma$  and avoid  $\gamma$ .  $\square$

## 4 Simple random walk. Proof of Theorem 2.

### 4.1 Degree sequence of the vacant net

We need some definitions. For any edge  $e$  of  $G$ , we say  $e$  is *red at  $t$*  if the walk made no transition along  $e$  during  $[T, t]$ . If  $e$  is a red edge, we say  $e$  is *unvisited at  $t$* , (i.e. unvisited between  $T$  and  $t$ ). For any vertex  $v$ , we assume there is a labeling  $e_1(v), \dots, e_r(v)$  of the edges incident with vertex  $v$ . Sometimes we write  $e(v)$  for a particular edge incident with  $v$ . If  $v$  has exactly  $s$  red edges at  $t$ , we say the *red degree* of  $v$  is  $s$ , and write  $d_R(v, t) = s$ . Recall that if a vertex  $v$  is *nice* ( $v \in \mathcal{N}$ ), then it is tree-like to depth least  $\ell_1 = \epsilon_1 \log_r n$ .

**Lemma 8.** For  $\ell = 1, \dots, r$ , let

$$\alpha_\ell = \frac{\ell}{r} \left( 2 - \left( \frac{1}{r-1} + \frac{\ell(r-1)}{r(r-2) + \ell} \right) \right). \quad (28)$$

For  $u \in \mathcal{N}$ , let  $e_1, \dots, e_\ell$  be a set of edges incident with  $u$ . Let

$$\bar{P}(u, \ell, t) = \Pr(\text{edges } e_1, \dots, e_\ell \text{ are red at } t), \quad (29)$$

then

$$\bar{P}(u, \ell, t) = \exp \left( -\alpha_\ell \frac{t}{n} (1 + o(1)) \right). \quad (30)$$



*Proof.* Let  $S = \{e_1, \dots, e_\ell\}$  be a set of edges incident with a nice vertex  $u$  of the graph  $G$ . To prove (29) we need to apply the results of Lemma 5 and Corollary 6 to the set  $S$ . As  $S$  is not a vertex the results of Corollary 6 do not apply directly, but we can get round this. We define a graph  $H$  with distinguished vertex  $\gamma(\ell)$ , obtained by modifying the structure of  $S$  in  $G$  in way detailed below, which we call *subdivide-contract*. The graph  $H$  is obtained as follows:

- (i) Subdivide the edges  $e_i = (u, v_i)$ ,  $i = 1, \dots, \ell$  incident with vertex  $u$  into  $(u, w_i)$ ,  $(w_i, v_i)$  by inserting a vertex  $w_i$ .
- (ii) Contract  $\{w_1, \dots, w_\ell\}$  to a vertex  $\gamma(\ell)$ , keeping the parallel edges that are created, and let  $H$  be the resulting multigraph obtained from  $G$  by this process.

We apply Corollary 6 to  $H$  with  $v = \gamma(\ell)$ . Let  $W_x$  be a walk in  $H$  starting from vertex  $x$ . Let  $p_{\gamma(\ell)} \sim \pi_{\gamma(\ell)}/R_{\gamma(\ell)}$  as given in (20). Here  $\pi_{\gamma(\ell)}$  is the stationary probability of  $\gamma(\ell)$  and  $R_{\gamma(\ell)}$  is given by (16). For  $t \geq \log^3 n$  let  $\mathcal{A}_{\gamma(\ell)}(t)$  be the event that  $W_x$  does not visit  $\gamma(\ell)$  at steps  $T, T+1, \dots, t$ . Then from (24)

$$\Pr_W(\mathcal{A}_{\gamma(\ell)}(t)) = \frac{(1 + O(T\pi_\gamma))}{(1 + p_{\gamma(\ell)})^t}. \quad (31)$$

We next prove that

$$p_{\gamma(\ell)} = (1 + o(1)) \frac{\alpha_\ell}{n}, \quad (32)$$

where  $\alpha_\ell$  is given by (28). The first step is to obtain  $\pi_{\gamma(\ell)}$  and  $R_{\gamma(\ell)}$ . By direct calculation

$$\pi_\gamma(\ell) = \frac{2\ell}{rn + 2\ell}. \quad (33)$$

We next prove that  $R_{\gamma(\ell)} = (1 + o(1))1/(1 - f_\gamma)$ , where  $\gamma = \gamma(\ell)$ , and

$$f_\gamma = \frac{1}{2} \left( \frac{1}{r-1} + \frac{\ell(r-1)}{r(r-2) + \ell} \right). \quad (34)$$

Before we inserted  $w_1, \dots, w_\ell$  into  $S$  and contracted them to  $\gamma$ , the vertex  $u$  was tree-like to depth  $\ell_1$ . Let  $D(u) = D_{\ell_1}(u)$  be the subgraph of  $G$  induced by the vertices at distance at most  $\ell_1$  from  $u$ . Let  $\mathcal{T}_u$  be an infinite  $r$ -regular tree rooted at  $u$ . Thus  $D(u)$  can be regarded as the subgraph of  $\mathcal{T}_u$  induced by the vertices at distance at most  $\ell_1$  from  $u$ . In this way we extend  $D(u)$  to an infinite  $r$ -regular tree. Let  $D'$  be the corresponding subgraph in  $H$ , and let  $\mathcal{T}'_u$  be the corresponding infinite graph. Apart from  $\gamma(\ell)$  which has degree  $2\ell$  and  $\ell$  parallel edges between  $\gamma(\ell)$  and  $u$ , the graph  $\mathcal{T}'_u$  has the same  $r$ -regular structure as  $\mathcal{T}_u$ .

Let  $\mathcal{T}$  be an infinite  $r$ -regular tree rooted at a fixed vertex  $v$  of arbitrary positive degree  $d(v)$ . Lemma 22 proves that the probability  $\phi$  of a first return to  $v$  in  $\mathcal{T}$  is given by  $\phi = 1/(r-1)$ . Let  $f_\gamma$  be the probability of a first return to  $\gamma$  in  $\mathcal{T}'_u$ . With probability  $1/2$  a walk starting at  $\gamma$  passes to one of  $v_1, \dots, v_\ell$  in which case the probability of a return to  $\gamma$  is  $\phi = 1/(r-1)$ . With probability  $1/2$  the walk passes from  $\gamma$  to  $u$  from whence it returns to  $\gamma$  with probability

$\ell/r$  at each visit to  $u$ . If the walk exits to a neighbour of  $u$  other than  $\gamma$  the probability of a return to  $u$  is  $\phi = 1/(r-1)$ . Thus in  $\mathcal{T}'_u$ , a first return to  $\gamma$  has probability

$$\begin{aligned} f_\gamma &= \frac{1}{2} \left( \phi + \frac{\ell}{r} \sum_{k \geq 0} \left( \left( \frac{r-\ell}{r} \right) \phi \right)^k \right) \\ &= \frac{1}{2} \left( \phi + \frac{\ell}{r - (r-\ell)\phi} \right). \end{aligned}$$

This establishes (34). It follows from Lemma 22 that the value of  $R_{\gamma(\ell)} = (1+o(1))1/(1-f_\gamma)$ . Combining (33) and (34) gives the value of  $p_{\gamma(\ell)}$  in (32) where  $\alpha_\ell$  is (28).

The last step is to get back from the walk in  $H$  to the walk in  $G$ . By Lemma 7, the event that  $\gamma(\ell)$  is unvisited at steps  $T, \dots, t$  of a random walk in  $H$ , has the same asymptotic probability as the event (29) in  $G$  that there is no transition along the edge set  $\{e_i = (u, v_i), i = 1, \dots, \ell\}$  during steps  $T, \dots, t$  of a random walk in  $G$ . This, and (31) gives

$$\bar{P}(u, \ell, t) = (1+o(1)) \Pr_W(\mathcal{A}_{\gamma(\ell)}(t)) = \frac{(1+o(1))}{(1+p_{\gamma(\ell)})^t} = (1+o(1))e^{-tp_{\gamma(\ell)}(1+O(p_{\gamma(\ell)}))}.$$

This, along with (32) completes the proof of the lemma.  $\square$

Let  $d_R(v, t)$  be the red degree of vertex  $v$  at step  $t$  and let  $S(v, s, t) = \binom{d_R(v, t)}{s}$  be the number of  $s$ -subsets of red edges incident with vertex  $v$  at step  $t$ . Let  $M(s, t)$  be given by

$$M(s, t) = \sum_{v \in \mathcal{N}} S(v, s, t).$$

Thus  $M(s, t)$  enumerates sets of incident red edges of size  $s$  over nice vertices.

Recall that we have defined an edge to be red if it is unvisited in  $T, \dots, t$ . By definition, all edges start red at step  $T$ . For  $t \geq T$ , the random variable  $M(s, t)$  is monotone non-increasing in  $t$ . For any  $s \geq 1$  there will be some step  $t(s)$  at which  $M(s, t(s)) = 0$ .

**Lemma 9.** *Let  $\alpha_s$  be given by (28). The following results hold w.h.p.,*

(i)

$$\mathbf{EM}(s, t) = (1+o(1))n \binom{r}{s} \exp(-p_{\gamma(\ell)}) = (1+o(1))n \binom{r}{s} \exp\left(- (1+o(1))\alpha_s \frac{t}{n}\right). \quad (35)$$

(ii) For  $s \geq 1$  let  $t_s = (n \log n)/\alpha_s$ . The values  $t_s$  satisfy  $t_r < t_{r-1} < \dots < t_1$ .

Let  $\omega = \epsilon \log n$ . For  $t < t_s - \omega n$ ,  $\mathbf{EM}(s, t) \rightarrow \infty$  whereas for  $t > t_s + \omega n$ ,  $\mathbf{EM}(s, t) = o(1)$ .

For  $t = O(n)$ ,  $|\bar{\mathcal{N}}| = o(\mathbf{EM}(s, t))$ .

(iii) For all  $0 \leq t \leq t_s - \omega n$ , the value of  $M(s, t)$  is concentrated within  $(1 + o(1))\mathbf{E}M(s, t)$ .

*Proof.* (i), (ii). The value of  $\mathbf{E}M(s, t)$  follows from (30) by linearity of expectation, and the fact that  $|\mathcal{N}| = (1 - o(1))n$ . Thus

$$\mathbf{E}M(s, t) = \sum_{u \in \mathcal{N}} \binom{r}{s} \bar{P}(u, s, t) = (1 + o(1))n \binom{r}{s} e^{-\alpha_s \frac{t}{n}(1+o(1))}. \quad (36)$$

For  $t \leq t_s - \omega n$ ,  $\mathbf{E}M(s, t) = \Omega(n^\epsilon)$ .

The function  $\alpha_s$  is strictly monotone increasing in  $s$ . For  $r \geq 3$ , the derivative  $d\alpha(x)/dx$  is positive for  $x \in [0, r)$ , and zero at  $x = r$ . Thus the values  $t_s$  satisfy  $t_i < t_j$  if  $i > j$ .

*Proof of (iii).* Fix  $s, t$  where  $s = 1, \dots, r$ , and  $t \leq t_s - \omega n$ . We use the Chebyshev inequality to prove concentration of  $Z = M(s, t)$ . Suppose that  $\delta \ll \epsilon$ , and  $\omega'(n) = \delta \log n$ , then

$$\log \log n \ll \omega' = \omega'(n) = \delta \log n \ll \omega = \epsilon \log n. \quad (37)$$

We first show that

$$\mathbf{Var}(Z) = \mathbf{E}Z + O(r^{\omega'} \mathbf{E}Z) + e^{-a\omega'} (\mathbf{E}Z)^2, \quad (38)$$

for some constant  $a > 0$ .

Let  $v, w \in \mathcal{N}$ . Let  $Q_s(v) = \{e_1(v), \dots, e_s(v)\}$  be a set of edges incident with  $v$ , and let  $Q_s(w) = \{f_1(w), \dots, f_s(w)\}$  be a set of edges incident with  $w$ . Let  $\mathcal{E}_v = \mathcal{E}(Q_s(v))$  be the event that the edges in  $Q_s(v)$  are red at  $t$ . Similarly, let  $\mathcal{E}_w = \mathcal{E}(Q_s(w))$  be the event that the  $Q_s(w)$  edges are red at  $t$ .

Let  $v, w$  be at distance at least  $\omega'$  apart then we claim that

$$\mathbf{Pr}(\mathcal{E}_v \cap \mathcal{E}_w) = (1 + e^{-\Omega(\omega')}) \mathbf{Pr}(\mathcal{E}_v) \mathbf{Pr}(\mathcal{E}_w). \quad (39)$$

To prove this we use the same method as Lemma 8. That is to say, we use Corollary 6 to find the unvisit probability of a vertex  $\gamma$  that we construct from  $Q_s(v) \cup Q_s(w)$  using subdivide-contract. We carry out the subdivide-contract process on the edges of  $Q_s(v), Q_s(w)$  by inserting an extra vertex  $x_i$  into  $e_i$  and an extra vertex  $y_i$  into  $f_i$ , and contracting  $S = \{x_1, \dots, x_s, y_1, \dots, y_s\}$  to  $\gamma(S)$ .

For the random walk on the associated graph  $H = H(\gamma(S))$  we have that  $p_{\gamma(S)}$  in (20) is given by  $p_{\gamma(S)} \sim \pi_{\gamma(S)}/R_{\gamma(S)}$ , where

$$\pi_{\gamma(S)} = \frac{4s}{rn + 4s}.$$

By Lemma 16 we can write  $1/R_{\gamma(S)} = (1 + o(1))(1 - f_{\gamma(S)})$ . We next prove that the value of  $f_{\gamma(S)}$  is given by

$$f_{\gamma(S)} = \frac{1}{2} (f_{\gamma(S_x)} + f_{\gamma(S_y)} + O(f^*)).$$

In this expression,  $f^*$  is an error term defined below, and  $\gamma(S_x), \gamma(S_y)$  are the contractions of  $S_x = \{x_1, \dots, x_s\}$ , and  $S_y = \{y_1, \dots, y_s\}$  respectively, as obtained in Lemma 8 and (e.g.)  $f_{\gamma(S_x)}$  is evaluated in  $H(\gamma(S_x))$ . Indeed, with probability  $1/2$ , the first move from  $\gamma(S)$  will be to a vertex  $u$  which is a neighbour of one of  $S_x = \{x_1, \dots, x_s\}$  on the subdivided edges  $e_1, \dots, e_s$ . Assume it is to a neighbour of  $S_x$ . The probability of a first return directly to  $\gamma(S_x)$  will be  $f_{\gamma(S_x)} = (1 + o(1))f$  as given by Lemma 8.

The  $O(f^*)$  term is a correction for the probability that a walk starting from  $\gamma(S_x)$  makes a transition across any of the edges in  $Q_s(w)$  during the mixing time. This event is not counted as a return in walks on  $H(\gamma(S_x))$  but would be in  $H(\gamma(S))$ . However, because  $v$  and  $w$  are at distance at least  $\omega'$ , using (80), the probability  $f^*$  of a visit to  $Q_s(w)$  during  $T$  can be bounded by  $T(n^{-1} + \lambda_{\max}^{\omega'})$ . Thus

$$p_{\gamma(S)} = (1 + O(1/n) + O(Te^{-\Omega(\omega')})) (p_{\gamma(S_x)} + p_{\gamma(S_y)}). \quad (40)$$

Equation (39) follows on using equation (40), Corollary 6 with  $p_{\gamma(S)}, p_{\gamma(S_x)}$  and  $p_{\gamma(S_y)}$  followed by Lemma 7. This confirms (39) and gives

$$\Pr(\mathcal{E}_v \cap \mathcal{E}_w) = (1 + e^{-\Omega(\omega')}) \Pr(\mathcal{E}_v) \Pr(\mathcal{E}_w) = (1 + o(1)) \bar{P}(v, s, t) \bar{P}(w, s, t),$$

where  $\bar{P}(v, s, t)$  is given by (30) in Lemma 8.

Summing over  $v, w \in \mathcal{N}$  and edge sets  $Q_s(v), Q_s(w)$  incident with  $v, w$  respectively,

$$\begin{aligned} \mathbf{E}(Z^2(t)) &= \mathbf{E}Z + \sum_{\substack{v, w \\ Q_s(v), Q_s(w) \\ \text{dist}(v, w) \geq \omega'}} \Pr(\mathcal{E}_v \cap \mathcal{E}_w) + \sum_{\substack{v, w \\ Q_s(v), Q_s(w) \\ \text{dist}(v, w) < \omega'}} \Pr(\mathcal{E}_v \cap \mathcal{E}_w) \\ &\leq \mathbf{E}Z + (1 + e^{-a\omega'}) (\mathbf{E}Z)^2 + r^{\omega'} \mathbf{E}(Z) \end{aligned}$$

and (38) follows. Applying the Chebyshev inequality we see that

$$\Pr\left(|Z - \mathbf{E}Z| \geq \mathbf{E}Z e^{-a\omega'/3}\right) \leq \frac{2r^{\omega'} e^{a\omega'}}{\mathbf{E}Z} + e^{-a\omega'/3}. \quad (41)$$

When  $t \leq t_s - \omega n$ ,  $\mathbf{E}Z \geq e^{\omega\alpha_s}/2 = \Omega(n^\epsilon) \gg n^\delta$  and our choice of  $\omega'$  in (37) implies that we can find a  $\delta_1$  such that the RHS of (41) is  $O(n^{-\delta_1}) = o(1)$  for such  $t$ .

The result (41) from the Chebyshev inequality is too weak to prove concentration of  $M(s, t)$  directly for all of  $t_s$  steps. We copy the approach used in [12], Theorem 4(a). Interpolate the interval  $[0, t_s]$  at  $A = n^{\delta_1/2}$  integer points  $s_1, \dots, s_A$  at distance  $\sigma = t_s n^{-\delta_1/2}$  apart (ignoring rounding), for some small constant  $\delta_1 > 0$  determined by (41). The concentration at the interpolation points follows from (41). We use the monotone non-increasing property of  $M(s, t)$  to bound the value of  $M(s, t)$  between  $s_i$  and  $s_{i+1}$ . The proof of this is identical to the one in [12] and is not given in further detail here.  $\square$

## 4.2 Uniformity: Using random walks in the configuration model

We use the random walk to generate the graph  $G$  in question. The main idea is to realize that as  $G$  is a random graph, the graph  $\Gamma(t)$  of the vacant set or vacant net has a simple description. Intuitively, if we condition on  $\mathcal{R}(t)$  and the history of the process, (the walk trajectory up to step  $t$ ), and if  $G_1, G_2$  are graphs with vertex set  $\mathcal{R}(t)$  and the same degree sequence, then substituting  $G_2$  for  $G_1$  will not conflict with the history. Every extension of  $G_1$  is an extension of  $G_2$  and vice-versa.

We briefly and informally explain what we do. By working in the configuration model, we can use the random walk to generate a random  $r$ -regular multigraph. Because the configuration points (half edges) at any vertex have labels, we can sample u.a.r. from these points to determine the next edge transition of the walk without exposing all the edges at the vertex in the underlying multigraph. In this way the walk discovers the edges of the multigraph as it proceeds. If we stop the walk at some step  $t$ , the undiscovered part of the multigraph is random, conditional on the subgraph exposed by the walk so far, and the constraint that all vertices have degree  $r$ .

We use the configuration or pairing model of Bollobás [4], derived from a counting formula of Canfield [5]. We start with  $n$  disjoint sets of  $S_1, S_2, \dots, S_n$  each of size  $r$ . The elements of  $S_v = \{v(1), \dots, v(r)\}$  correspond to the labeled endpoints of the half edges incident with vertex  $v$ . We refer to these elements as (configuration) points.

Let  $S = \bigcup_{i=1}^n S_i$ . A *configuration* or *pairing*  $F$  is a partition of  $S$  into  $rn/2$  pairs. Let  $\Omega$  be the set of configurations. Any  $F \in \Omega$  defines an  $r$ -regular multi-graph  $G_F = ([n], E_F)$  where  $E_F = \{(i, j) : \exists \{x, y\} \in F : x \in S_i, y \in S_j\}$ , i.e. we contract  $S_i$  to a vertex  $i$  for  $i \in [n]$ .

Let  $U_0 = S, F_0 = \emptyset$ . Given  $U_{i-1}, F_{i-1}$  we construct  $F_i$  as follows. Choose  $x_i$  arbitrarily from  $U_{i-1}$ . Choose  $y_i$  u.a.r. from  $U_{i-1} \setminus \{x_i\}$ . Set  $F_i = F_{i-1} \cup \{\{x_i, y_i\}\}, U_i = U_{i-1} \setminus \{x_i, y_i\}$ . If we stop at step  $i$ , the points in  $U_i$  are unpaired, and can be paired u.a.r. The underlying multigraph of this pairing of  $U_i$  is a random multigraph in which the degree of vertex  $v$  is  $d(v) = |S_v \cap U_i|$ .

It is known that: (i) Each simple graph arises the same number of times as  $G_F$ . i.e. if  $G, G'$  are simple, then  $|\{F : G_F = G\}| = |\{F' : G'_F = G'\}|$ . (ii) Provided  $r$  is constant, the probability  $G_F$  is simple is bounded below by a constant. Thus if  $F$  is chosen uniformly at random from  $\Omega$  then any event that occurs w.h.p. for  $F$ , occurs w.h.p. for  $G_F$ , and hence w.h.p. for  $G_r$ .

We next explain how to use a random walk on  $[n]$  to generate a random  $F$ , and hence a random multigraph  $G$ . To do this, we begin with a starting vertex  $u = i_0$ . Suppose that at the  $t$ -th step we are at some vertex  $i_t$ , and have a partition of  $S$  into red and blue points,  $R_t, B_t$  respectively. Initially,  $R_0 = S$  and  $B_0 = \emptyset$ . In addition we have a collection  $F_t$  of

disjoint pairs from  $S$  where  $F_0 = \emptyset$ .

At step  $t + 1$  we choose a random edge incident with  $i_t$ . Obviously  $i_t \in \mathcal{B}(t)$ , as it is visited by the walk, but we treat the configuration points in  $S_{i_t}$  as blue or red, depending on whether the corresponding edge is previously traversed (blue) or not (red). Let  $x$  be chosen randomly from  $S_{i_t}$ . There are two cases of how  $i_{t+1}$  is chosen.

If  $x \in B_t$  then it was previously paired with a  $y \in S_j \cap B_t$ , and thus  $j \in \mathcal{B}(t)$ . The walk moves from  $i_t$  to  $i_{t+1} = j$  along an existing edge corresponding to some  $\{x, y\} \in F$ . We let  $R_{t+1} = R_t, B_{t+1} = B_t$  and we let  $F_{t+1} = F_t$ .

If  $x \in R_t$ , then the edge is unvisited, so we choose  $y$  randomly from  $R_t \setminus \{x\}$ . Suppose that  $y \in S_j$ . This is equivalent to moving from  $i_t \in \mathcal{B}(t)$  to  $i_{t+1} = j$ . We now check vertex  $j$  to see if it was previously visited. If  $j \in \mathcal{B}(t)$  this is equivalent to moving between blue vertices on a previously unvisited edge. If  $j \in \mathcal{R}(t)$ , this is equivalent to moving to a previously unvisited vertex. In either case we update as follows.  $R_{t+1} = R_t \setminus \{x, y\}$  and  $B_{t+1} = B_t \cup \{x, y\}$ , and  $F_{t+1} = F_t \cup \{\{x, y\}\}$ .

After  $t$  steps we have a random pairing  $F_t$  of at most  $t$  disjoint pairs from  $S$ . The entries in  $F_t$  consist of a known pairing of  $B_t$ , and constitute the revealed edges of the random graph. The points in  $R_t$  are still unpaired. In principle we can extend  $F_t$  to a random configuration  $F$  by adding a random pairing of  $R_t$  to it. The vacant net,  $\widehat{\Gamma}(t)$  is the subgraph of  $V$  induced by the edges unvisited during steps  $1, \dots, t$ , and is the underlying multigraph of a u.a.r. pairing of  $R_t$ . To generate  $\Gamma(t)$ , the subgraph induced by the vacant set  $\mathcal{R}(t)$ , we extend the pairing  $F_t$  to a pairing  $F_{t'}$  by method **Extend- $\mathcal{B}(t)$**  defined as follows.

**Extend- $\mathcal{B}(t)$ .** Let  $S_B = \cup_{v \in \mathcal{B}(t)} S_v$ . Let  $K = S_B \cap R_\tau$ . For  $\tau \geq t$ , and while  $K \neq \emptyset$  choose an arbitrary point  $x$  of  $K$ . Pair  $x$  with a u.a.r. point  $y$  of  $R_\tau - \{x\}$ . Let  $R_\tau = R_\tau \setminus \{x, y\}$ . If  $y \in K$  let  $K = K \setminus \{x, y\}$  else let  $K = K \setminus \{x\}$ . Set  $\tau = \tau + 1$ . Let  $t' = \tau$  be the first step at which  $K = \emptyset$ . Pair  $R_{t'}$  u.a.r. to generate the multigraph  $\Gamma(t)$ .

The next lemma summarizes this discussion.

**Lemma 10.**

- i) The pairing  $F_t$  can be generated in the configuration model by a random walk  $W_u(t)$  without exposing any pairings not in  $F_t$ . The underlying multigraph of  $F_t$  gives the edges covered by the walk  $W_u(t)$ .
- ii) The pairing  $F_t$  plus a u.a.r. pairing of  $R_t$  is a uniform random member of  $\Omega$ .
- iii) The vertex  $v \in V$  is in  $\mathcal{R}(t)$  if and only if  $S_v \subseteq R_t$ .
- iv) **Vacant net.** The u.a.r. pairing of  $R_t$  gives the vacant net,  $\widehat{\Gamma}(t)$  as a random multigraph with degree sequence determined by  $\widehat{d}(v) = |S_v \cap R_t|$  for  $v \in V$ . Let  $\widehat{\mathbf{d}}(t)$  be the degree

sequence of  $\widehat{\Gamma}(t)$ . Conditional on  $\widehat{\Gamma}(t)$  being simple,  $\widehat{\Gamma}(t)$  is a u.a.r. graph with degree sequence  $\widehat{\mathbf{d}}(t)$ .

- v) **Vacant set.** Extend  $F_t$  to  $F_{t'}$  using method **Extend- $\mathcal{B}(t)$**  described above. The u.a.r. pairing of  $R_{t'}$  gives  $\Gamma(t)$ , the induced subgraph of the vacant set, as a random multigraph with degree sequence determined by  $d(v) = |S_v \cap R_{t'}|$  for  $v \in \mathcal{R}(t)$ . Let  $\mathbf{d}(t)$  be the degree sequence of  $\Gamma(t)$ . Conditional on  $\Gamma(t)$  being simple,  $\Gamma(t)$  is a u.a.r. graph with degree sequence  $\mathbf{d}(t)$ .

### 4.3 Applying the Molloy-Reed Condition

The Molloy-Reed condition for bounded degree graphs can be stated as follows.

**Theorem 11.** Let  $G_{N,\mathbf{d}}$  be the graphs with vertex set  $[N]$  and degree sequence  $\mathbf{d} = (d_1, d_2, \dots, d_N)$ , and endowed with the uniform measure. Let  $D(s) = |\{j : d_j = s\}|$ , be the number of vertices of degree  $s = 0, 1, \dots, r$ , where  $D(s) = (1 + o(1))\lambda_s N$  for  $s = 0, 1, \dots, r$ , and  $\lambda_0, \lambda_1, \dots, \lambda_r \in [0, 1]$  are such that  $\lambda_0 + \lambda_1 + \dots + \lambda_r = 1$ . Let

$$L(\mathbf{d}) = \sum_{s=0}^r s(s-2)\lambda_s. \quad (42)$$

- (a) If  $L(\mathbf{d}) < 0$  then w.h.p.  $G_{n,\mathbf{d}}$  is sub-critical.  
(b) If  $L(\mathbf{d}) > 0$  then w.h.p.  $G_{n,\mathbf{d}}$  is super-critical.

The following theorem on the scaling window is adapted from Theorem 1.1 of Hatami and Molloy [16], with the observation (after Theorem 3.2) from Černý and Teixeira [7] that including a constant proportion of vertices of degree zero does not modify the validity of the result.

**Theorem 12.** [16] Let  $G_{N,\mathbf{d}}$  be the graphs with vertex set  $[N]$  and degree sequence  $\mathbf{d} = (d_1, d_2, \dots, d_N)$ , and endowed with the uniform measure. Let  $R = \sum_{u \in V} d_u(d_u - 2)^2 / 2|E(G)|$ . Assume that  $R > 0$  constant, and  $D(2) < N(1 - \delta)$  for some  $\delta > 0$ . For any  $c > 0, \epsilon > 0$ , and  $-cN^{2/3} \leq NL(\mathbf{d}) \leq cN^{2/3}$ ,

$$\Pr(|C_1| = \Theta(N^{2/3})) \geq 1 - \epsilon.$$

To complete the proof of Theorem 2 we need to evaluate  $L(\mathbf{d})$  for  $\widehat{\Gamma}(t)$  to obtain  $\widehat{t}$ . It is convenient for us to express  $L(\mathbf{d}) = \sum_{s=0}^r s(s-2)\lambda_s$  in a form which uses the results of Lemma 8 and Lemma 9 of Section 4.1.

**Lemma 13.** Let  $G = (V, E, \mathbf{d})$  be a graph with degree sequence  $\mathbf{d}$  of maximum degree  $r$ . Let  $D(s)$ ,  $s = 0, \dots, r$ , be the number of vertices of degree  $s$ . Let  $U \subseteq V$  be a set of vertices, and  $\bar{U} = V \setminus U$ . Let  $M_U(s) = \sum_{u \in U} \binom{d(u)}{s}$ , and let  $R = \sum_{u \in V} d(u)(d(u) - 2)^2 / 2|E(G)|$ . Then  $L(\mathbf{d})$  can be written as

$$L(\mathbf{d}) \cdot N = (1 + o(1)) (2M_U(2) - M_U(1) + O(r^2|\bar{U}|)), \quad (43)$$

and  $R$  can be written as

$$R \cdot (M_U(1) + O(r|\bar{U}|)) = (1 + o(1)) (6M_U(3) - 2M_U(2) + M_U(1) + O(r^3|\bar{U}|)). \quad (44)$$

*Proof.* Let

$$Q = \sum_{s=0}^r s(s-2)D(s), \quad (45)$$

then  $Q$  can be written as

$$\begin{aligned} Q &= \sum_{s=0}^r s(s-1)D(s) - \sum_{s=0}^r sD(s) \\ &= \sum_{v \in V} d(v)(d(v)-1) - \sum_{v \in V} d(v) \\ &= \sum_{v \in U} d(v)(d(v)-1) - \sum_{v \in U} d(v) + \left( \sum_{v \notin U} d(v)(d(v)-1) - \sum_{v \notin U} d(v) \right) \\ &= 2M_U(2) - M_U(1) + O(r^2|\bar{U}|). \end{aligned} \quad (46)$$

The case for  $R$  is similar.  $\square$

In our proofs, we choose  $U = \mathcal{N}$ , the set of nice vertices. It follows from Lemma 9 that  $r^2\bar{U} = o(M_U(1) + M_U(2))$ . The next lemma proves the Molloy-Reed threshold condition is equivalent to  $M_{\mathcal{N}}(1) \sim 2M_{\mathcal{N}}(2)$ .

**Lemma 14.** (i) The asymptotic solution to  $L(\mathbf{d}) = 0$  in (42) obtained at  $\hat{t} = (1 + o(1))\theta^*n$  where

$$\theta^* = \frac{r(r^2 - 2r + 2)}{2(r-2)^2} \log(r-1). \quad (47)$$

(ii) The assumptions of Theorem 12 are valid and the scaling window is of order  $\Theta(n^{2/3})$ .

*Proof.* Let  $\mathbf{d}$  be the degree sequence of  $\hat{\Gamma}(t)$ , let  $\mathbf{D}$  be the degree sequence of nice vertices  $\mathcal{N}$ , and  $\bar{\mathbf{D}}$  the degree sequence of  $\bar{\mathcal{N}}$ . For nice vertices and any  $0 \leq s \leq r$  we use the notation  $M(s, t) = M_{\mathcal{N}}(s, t)$ . Thus using (43) with  $U = \mathcal{N}$ ,

$$nL(\mathbf{d}) \sim Q(\mathbf{d}) = 2M(2, t) - M(1, t) + O(r^2|\bar{\mathcal{N}}|) - O(T). \quad (48)$$



Thus the condition  $L(\mathbf{d}) \sim 0$  is equivalent to  $Q(\mathbf{D})/n \rightarrow 0$ . The term  $O(T)$  removes any vertices/edges visited during the mixing time  $T$ , but unvisited during  $T, \dots, t$  and hence marked red. From (6),  $|\overline{\mathcal{N}}| = O(n^\epsilon)$ . For nice vertices, and  $t = cn$  for any  $c \geq 0$  constant gives  $M(2, t) = \Theta(n)$ ,  $M(1, t) = \Theta(n)$ . Thus when  $M(1, t) \sim 2M(2, t)$  then  $L(\mathbf{d}) \sim 0$ . By Lemma 9,  $M(s, t)$  is asymptotic to (35), which is

$$M(1, t) = (1 + o(1))r \exp\left(-\frac{t}{n} \frac{2(r-2)}{r(r-1)}\right)$$

$$M(2, t) = (1 + o(1))r(r-1)/2 \exp\left(-\frac{t}{n} \frac{2}{r} \left(2 - \left(\frac{1}{r-1} + \frac{2(r-1)}{r(r-2)+2}\right)\right)\right).$$

Thus  $L(\mathbf{d}) \rightarrow 0$  when  $t \sim \tilde{t} = \theta^* n$  where  $\theta^*$  is given by (47).

Regarding the expression for  $R = R(t)$  in (44), with  $U = \mathcal{N}$ .

$$(M(1, t) + O(r\overline{\mathcal{N}}))R(t) = (1 + o(1))(6M(3, t) - 2M(2, t) + M(1, t)) + O(r^3\overline{\mathcal{N}}).$$

By Lemma 9, for  $t = cn$ , any  $c \geq 0$  constant, and  $s = 1, 2, 3$  we have that w.h.p.  $M(s, t) = \Theta(n)$ . On the other hand from (48), and the assumption of the scaling window

$$\frac{2M(2, t) - M(1, t) + O(r^3\overline{\mathcal{N}})}{M(1, t)} = O(L(\mathbf{d})) = o(1).$$

Thus  $R(t) > 0$  constant. □

## 5 Non-backtracking random walk. Proof of Theorem 3

Note that, as in the case of a simple random walk, we can use a non-backtracking random walk to generate the underlying graph in the configuration model. The only change to the sampling procedure given in Section 4.2, is as follows. Suppose the walk arrives at vertex  $v$  by a transition  $(u, v)$ . In the configuration model, this is equivalent to a pairing  $\{x(u), y(v)\}$  where  $x(u) \in S_u$ ,  $y(v) \in S_v$ . To make the walk non-backtracking, we sample the configuration point of  $v$  used for the next transition u.a.r from  $S_v \setminus \{y(v)\}$ .

For a connected graph  $G = (V, E)$  of minimum degree 2, the state space of a non-backtracking walk  $W$  on  $G$  can be described by a digraph  $M = (U, D)$  with vertex set  $U$  and directed edges  $D$ . To avoid any confusion with the vertex set  $V$  of  $G$ , we refer to the elements  $\sigma$  of  $U$  as states, rather than vertices. The states  $\sigma \in U$  are orientations  $(u, v)$  of edges  $\{u, v\} \in E(G)$ . The state  $\sigma = (u, v)$  is read as ‘the walk  $W$  arrived at  $v$  by a transition along  $(u, v)$ ’. Let  $N(u) = N_G(u)$  denote the neighbours of  $u$  in  $G$ . The in-neighbours of  $(u, v)$  in  $M$  are states  $\{(x, u), x \in N(u), x \neq v\}$ . Hence the state  $(u, v)$  has in-degree  $(r-1)$  in  $M$ . Similarly  $(u, v)$  has out-degree  $(r-1)$  and out-neighbours  $\{(v, w), w \in N(v), w \neq u\}$ .

Let  $\mathcal{M}$  be a simple random walk on  $M$ . The walk  $\mathcal{M}$  on  $M$  is a Markov process which corresponds directly to the non-backtracking walk on  $G$ . For states  $\sigma = (u, v)$ ,  $\sigma' = (v, w)$ , the transition matrix  $P = P(\mathcal{M})$  has entries  $P(\sigma, \sigma') = 1/(d(v) - 1)$  if  $w \neq u$  and  $P(\sigma, \sigma') = 0$  otherwise. The total number of states  $|U| = 2|E(G)| = 2m$ . Using  $\pi = \pi P$ ,

$$\pi(u, v) = \sum_{x \in N(u), x \neq v} \frac{\pi(x, u)}{d(u) - 1},$$

which has solution  $\pi(\sigma) = 1/2m$ .

For random  $r$ -regular graphs, Alon et al. [1] established that a non-backtracking walk on  $G$  has mixing time  $T_G = O(\log n)$  w.h.p. The analysis in [1] was made on the graph  $G$  whereas, to apply Corollary 6, we need the mixing time  $T_{\mathcal{M}}$  of the Markov chain  $\mathcal{M}$ . The proof of Lemma 15 below is given in Section 8.3 of the Appendix.

**Lemma 15.** *For  $G \in G_r$ ,  $r \geq 3$  constant, w.h.p.  $T_{\mathcal{M}} = O(\log n)$ .*

In Section 4 we described a technique called subdivide-contract which we used to obtain first returns to a suitably constructed set  $S$  which was contracted to a vertex  $\gamma(S)$ . It remains to establish the value of  $R_{\gamma(S)}$  obtained by applying the subdivide-contract method to the various sets  $S$  of vertices and edges used in our proof. In each case we outline the construction of the set  $S$  and state the relevant value of  $p_{\gamma(S)}$  as given by (20) which we use in Corollary 6. Because the walk cannot backtrack, the calculation of  $R_{\gamma}$  for sets  $S$  of tree-like (i.e. nice) vertices is greatly simplified. Let  $\mathcal{T}_{\gamma}$  be an infinite  $r$ -regular tree rooted at a vertex  $\gamma$  of arbitrary degree. For a non-backtracking walk starting from  $\gamma$ , a first return to  $\gamma$  after moving to an adjacent vertex, is impossible.

## 5.1 Properties of the vacant set

**Size of vacant set.** Let  $v$  be a nice vertex of  $G$ , and let  $S = [v]$  be a set of states of  $M$ , where  $[v] = \{(u, v), u \in N(v)\}$ . A visit to  $[v]$  in  $M$  is equivalent to a visit to  $v$  in  $G$ . If  $v$  is nice then, (i) states  $(u, v), (x, v) \in [v]$  are directed distance at least  $2\ell = \epsilon \log_r n$  apart in  $M$ ; (ii) the state  $(u, v)$  induces an  $(r - 1)$ -regular in-arborescence and out-arborescence in  $M$ .

Contract the set  $[v]$  of states of  $M$  to a single state  $\gamma([v])$  retaining all edges incident with  $[v]$ . This gives a multi-digraph  $H$  with states  $\tilde{U} = (U \setminus [v]) \cup \{\gamma([v])\}$ . We only apply this construction to nice vertices  $v$ , in which case the digraph rooted at  $\gamma([v])$  is an arborescence to depth  $\ell$ . To simplify notation, if we contract a set  $S$  of states of  $M$  to  $\gamma(S)$ , and  $f$  is any state of  $M$  not in  $S$ , we use the indexing  $f \notin S$ , both for  $M$  and  $H$ , i.e. as shorthand for  $f \neq \gamma([v])$ .

The set  $[v]$  consists of  $r$  states of  $U$  each of in-degree and out-degree  $(r - 1)$ . As we contracted without removing edges, the vertex  $\gamma([v])$  has in-degree and out-degree  $r(r - 1)$ . For any state  $(v, w)$  of  $H$  there are  $r - 1$  parallel edges directed from  $\gamma([v])$  to  $(v, w)$  and no others. For a state  $\sigma$  of  $H$ , let  $N^-(\sigma)$  be the in-neighbours of  $\sigma$ , and let  $d^+(\sigma)$  be the out-degree of  $\sigma$ .

Let  $\mathcal{H}$  be a simple random walk on  $H$ . Apart from transitions to and from  $[v]$  (resp.  $\gamma([v])$ ), the transition matrices of the walks  $\mathcal{M}$  and  $\mathcal{H}$  are identical. Let  $\pi$  be the stationary distribution of  $P(\mathcal{M})$  in  $M$  and  $\tilde{\pi}$  the stationary distribution of  $\tilde{P}(\mathcal{H})$  in  $H$ .

For irreducible aperiodic Markov chain with transition matrix  $P$ , the stationary distribution is the unique vector of probabilities  $\pi$  which satisfies the equations  $\pi = \pi P$ . Given  $\pi$  we only have to check this condition.

We claim that  $\tilde{\pi}(\gamma([v])) = 1/n$ . For any state  $f$  of  $H$  other than  $\gamma([v])$ , we claim  $\tilde{\pi}(f) = 1/rn$ , and thus  $\tilde{\pi}(f) = \pi(f)$  for such states. This includes out-neighbours  $(v, w)$  of  $\gamma([v])$ . Considering  $\tilde{\pi} = \tilde{\pi}\tilde{P}$ , we have

$$\tilde{\pi}(\gamma([v])) = \sum_{f \in N^-(\gamma([v]))} \frac{\tilde{\pi}(f)}{d^+(f)} \quad (49)$$

$$\tilde{\pi}(v, w) = \tilde{\pi}(\gamma) \frac{(r - 1)}{d^+(\gamma)} = \frac{\tilde{\pi}(\gamma([v]))}{r}. \quad (50)$$

For (49), as  $d^-(\gamma([v])) = r(r - 1)$  and  $d^+(f) = (r - 1)$ , this confirms  $\tilde{\pi}(\gamma) = 1/n$ . For (50), the  $(r - 1)$  comes from the parallel edges from  $\gamma([v])$  to  $(v, w)$ , and confirms  $\tilde{\pi}(v, w) = 1/rn$ . For any other state  $f$ , the relevant rows of  $\tilde{P}$  are identical with those of  $P$  confirming  $\tilde{\pi}(f) = \pi(f) = 1/rn$ .

We use Lemma 7 to apply results obtained for  $\mathcal{H}$  to the walk  $\mathcal{M}$ . The lemma needs the stationary distributions  $\pi = \pi_{\mathcal{M}}$  and  $\tilde{\pi} = \pi_{\mathcal{H}}$  to be compatible i.e.  $\pi(f) = \tilde{\pi}(f)$  for  $f \notin [v]$  (resp.  $f \neq \gamma([v])$ ). This follows immediately from the values of  $\pi, \tilde{\pi}$  obtained above.

Finally we calculate  $R_{\gamma([v])}$ . We first give a general explanation of the method. Let  $\mathcal{T}_\gamma$  be an infinite arborescence with root vertex  $\gamma$  of out-degree  $r(r - 1)$  and all other vertices of out-degree  $(r - 1)$ . Similar to Lemma 22, we relate first returns to  $\gamma([v])$  in  $H$  to first returns to  $\gamma$  in  $\mathcal{T}_\gamma$ , to obtain a value of  $R_{\gamma([v])}$  given by

$$R_\gamma = (1 + o(1))/(1 - f), \quad (51)$$

where  $f$  is a first return probability to  $\gamma = \gamma([v])$  in the arborescence  $\mathcal{T}_\gamma$ . Let  $v$  be a nice vertex of  $G$ , i.e.  $v$  is tree-like to distance  $\ell = \epsilon \log_r n$ . Thus any cycle containing  $v$  has girth at least  $2\ell$ . Because the walk is non-backtracking, once it leaves  $v$  it cannot begin to return to  $v$ , until it has traveled far enough to change its direction, i.e. after at least  $\ell$  steps. A direct return to  $v$  from a vertex  $u$  at distance  $\ell$ , can be modeled as a biased random walk, in which the walk succeeds only if it moves closer to  $v$  at every step, with probability  $1/(r - 1)$ . If this

fails, the walk moves away from  $v$  once more to distance  $\ell$ . Thus the probability of any return to  $v$ , and hence  $\gamma([v])$  from distance  $\ell$  during  $T$  steps is given by  $O(T/(r-1)^\ell) = o(1)$ .

In the case of  $\gamma([v])$ ,  $\gamma$  has no loops, so the first return probability in  $\mathcal{T}_\gamma$  is  $f = 0$ . This gives

$$p_\gamma = (1+o(1))(1-f)\tilde{\pi}_\gamma = (1+o(1))\frac{1}{n}.$$

Applying Corollary 6 to  $\gamma([v])$  in  $H$  we have

$$\Pr_{\mathcal{H}}(\mathcal{A}_{\gamma([v])}(t)) = (1+o(1))\exp(-(1+o(1))t/n). \quad (52)$$

To estimate the probability  $\Pr_W(\mathcal{A}_v(t))$ , that  $v$  is unvisited during  $T, \dots, t$ , we use the equivalent walk  $\mathcal{M}$  in the digraph  $M$ , and contract  $[v]$  to a vertex  $\gamma = \gamma([v])$  to give a walk  $\mathcal{H}$  in  $H$ . Using Lemma 7 with (52) establishes the result that

$$\Pr_W(\mathcal{A}_v(t)) = \Pr_{\mathcal{M}}(\mathcal{A}_{[v]}(t)) = (1+o(1))\Pr_{\mathcal{H}}(\mathcal{A}_{\gamma([v])}(t)) = (1+o(1))\exp(-(1+o(1))t/n).$$

It follows that at step  $t$  of  $W$ , the vacant set  $\mathcal{R}(t)$  is of expected size

$$\mathbf{E}|\mathcal{R}(t)| = \sum_{v \in V} \Pr_W(\mathcal{A}_v(t)) = |\mathcal{N}|e^{-(1+o(1))t/n} + |\overline{\mathcal{N}}| \sim ne^{-(1+o(1))t/n}.$$

The concentration of  $|\mathcal{R}(t)|$  follows from the methods of Lemma 9. Theorem 3(i) for  $|\mathcal{R}(t)|$  follows from  $\mathbf{E}|\mathcal{R}(t)|$ , the concentration of  $|\mathcal{R}(t)|$  and the fact that  $o(n)$  vertices are not nice. Theorem 3(iii), for vertex cover time follows from equating  $\mathbf{E}|\mathcal{R}(t)| = o(1)$  and applying the techniques used in [9] to obtain a lower bound.

**Number of edges in the vacant set.** The vertices  $u, v$  are unvisited in  $G$  if and only if the corresponding set of states  $S = [u] \cup [v]$  is unvisited in  $M$ . Let  $u, v \in \mathcal{R}(t)$  and let  $\{u, v\}$  be an edge of  $G$  and hence of  $\Gamma(t)$ . In this case, for nice  $u, v$ , the corresponding set of states  $S$  of  $M$  induces into two disjoint components given by

$$\begin{aligned} S_u &= \{(u, v)\} \cup \{(x, u), x \in N(u), x \neq v\} \\ S_v &= \{(v, u)\} \cup \{(x, v), x \in N(v), x \neq u\}. \end{aligned}$$

The total in-degree and out-degree of  $S_u$  is  $r(r-1)$ . The details of the edges incident with e.g.  $S_u$  are as follows. The set  $S_u$  induces  $(r-1)$  internal edges in  $M$  of the form  $((x, u), (u, v))$ . For a state  $e = (x, u) \in S_u$  there are  $(r-1)$  states  $f$  of  $M$ ,  $f = (a, x), x \neq u$  which point to  $e$ , a total in-degree from  $U \setminus S$  to  $S_u$  of  $(r-1)^2$ . Similarly,  $S_u$  points to  $(r-1) + (r-1)(r-2)$  distinct states of  $U$  not in  $S$ . In total, the in-degree and out-degree of  $\gamma(S)$  is  $2r(r-1)$  of which  $2(r-1)$  edges are loops at  $\gamma(S)$ . This means  $2(r-1)^2$  states (other than  $\gamma(S)$ ) point to  $\gamma(S)$ .

We claim  $\tilde{\pi}(\gamma(S)) = 2/n$ , and that for  $f \notin S$ , we have  $\tilde{\pi}(f) = 1/rn = \pi(f)$ . We use  $\tilde{\pi} = \tilde{\pi}\tilde{P}$  to confirm this. For  $\gamma(S)$  we have

$$\begin{aligned}\tilde{\pi}(\gamma(S)) &= \sum_{f \in N^-(\gamma(S))} \frac{\tilde{\pi}(f)}{r-1} + \tilde{\pi}(\gamma(S)) \frac{2(r-1)}{2r(r-1)} \\ &= 2(r-1)^2 \frac{1}{rn} \frac{1}{r-1} + \frac{2}{n} \frac{1}{r} = \frac{2}{n}.\end{aligned}$$

If  $f \neq \gamma(S)$ , but  $f \in N^+(\gamma(S))$  then

$$\begin{aligned}\tilde{\pi}(f) &= \frac{\tilde{\pi}(\gamma(S))}{2r(r-1)} + \sum_{\substack{e \in N^-(f) \\ e \neq \gamma(S)}} \frac{\tilde{\pi}(e)}{r-1} \\ &= \frac{2}{n} \frac{1}{2r(r-1)} + \frac{1}{rn} \frac{r-2}{r-1} = \frac{1}{rn}.\end{aligned}$$

For any other state  $f$ , the relevant rows of  $\tilde{P}$  are identical with  $P$  confirming  $\tilde{\pi}(f) = \pi(f) = 1/rn$ . Hence for  $f \neq \gamma(S)$ ,  $\tilde{\pi}(f) = \pi(f)$  so  $\tilde{\pi}_{\mathcal{H}}$  is compatible with  $\pi_{\mathcal{M}}$  in Lemma 7.

Consider next  $R_{\gamma(S)}$ . In the infinite arborescence  $\mathcal{T}_{\gamma}$  there are  $(r-1)$  loops at  $\gamma$  so  $f = (r-1)/r(r-1) = 1/r$ . From (51) we obtain

$$p_{\gamma(S)} \sim \tilde{\pi}(\gamma(S))(1-f) = 2(r-1)/rn. \quad (53)$$

Using the observation that at most  $o(rn)$  edges of  $\Gamma(t)$  are incident with vertices which are not nice ( $v \in \bar{\mathcal{N}}$ ), the expected size of the edge set  $E(\Gamma(t))$  of the graph  $\Gamma(t)$  induced by the vacant set is

$$\mathbf{E}(|E(\Gamma(t))|) \sim \frac{rn}{2} e^{-\frac{2(r-1)t}{rn}(1+o(1))}.$$

This plus a concentration argument similar to Lemma 9, completes the proof of Theorem 3(i).

**Number of paths length two in the vacant set.** Let  $u, v, w \in \mathcal{R}(t)$  be such that  $u, w \in N(v)$ . Thus  $uvw$  is a path of length two in  $G$  and hence  $\Gamma(t)$ . The assumption that  $u, v, w$  are unvisited in  $G$  is equivalent to  $[u] \cup [v] \cup [w]$  unvisited in  $M$ . Let  $S = [u] \cup [v] \cup [w]$ . The set  $S$  can be written as

$$\begin{aligned}S &= \{(u, v), (v, w), (w, v), (v, u)\} \cup \{(x, u), x \in N(u), x \neq v\} \\ &\cup \{(y, v), y \in N(v), y \neq w, u\} \cup \{(z, w), z \in N(w), z \neq v\}.\end{aligned}$$

Thus  $S$  induces a single component in the underlying graph of  $M$ . Counting the elements of the sets  $S$  in the order above we see that  $S$  has size  $4 + (r-1) + (r-2) + (r-1) = 3r$ , and hence a total in-degree (resp. out-degree) of  $3r(r-1)$ . Of these edges,  $2 + (r-1) + 2(r-2) + (r-1) = 4(r-1)$  are internal.

We claim that  $\tilde{\pi}(\gamma(S)) = 3/n$ , and for  $f \neq \gamma(S)$ ,  $\tilde{\pi}(f) = 1/rn$ . We use  $\tilde{\pi} = \tilde{\pi}\tilde{P}$  to confirm this. For  $\gamma(S)$ ,

$$\begin{aligned}\tilde{\pi}(\gamma(S)) &= \sum_{f \in N^-(\gamma(S))} \frac{\tilde{\pi}(f)}{r-1} + \tilde{\pi}(\gamma(S)) \frac{4(r-1)}{3r(r-1)} \\ &= \frac{1}{rn} \frac{3r(r-1) - 4(r-1)}{r-1} + \frac{3}{n} \frac{4(r-1)}{3r(r-1)} = \frac{3}{n}.\end{aligned}$$

For any state  $f$  which is an out-neighbour of  $\gamma(S)$ , there are  $(r-1)$  parallel edges from  $\gamma(S)$  to  $f$ . For example let  $f = (w, x)$ ,  $x \neq v$ , then states  $(z, w)$ ,  $z \neq x$  of  $S$  point to  $(x, w)$ . Thus

$$\tilde{\pi}(f) = \tilde{\pi}(\gamma(S)) \frac{r-1}{3r(r-1)} = \frac{1}{rn}.$$

We obtain that  $\tilde{\pi}_{\mathcal{H}}$  is compatible with  $\pi_{\mathcal{M}}$  in Lemma 7.

To estimate  $R_{\gamma(S)}$  consider  $\mathcal{T}_{\gamma(S)}$ . The vertex  $\gamma(S)$  has  $4(r-1)$  loops and total out-degree  $3r(r-1)$  giving a value for  $f$  in (51) of  $f = 4/3r$ . Thus

$$p_{\gamma(S)} \sim (3r-4)/rn. \tag{54}$$

**Threshold for the vacant set.** Theorem 3(iv) follows from using  $Q = 2M(2) - M(1)$  (see (46)), and equating  $Q = 0$  in Lemma 14 with the appropriate values of  $\mathbf{E}M(1, t)$ ,  $\mathbf{E}M(2, t)$  as in (36). From (53), (54) we have  $\alpha_1 = 2(r-1)/r$ ,  $\alpha_2 = (3r-4)/r$ . Equating  $M(1, t) = 2M(2, t)$  and setting  $t = u^*n$  gives

$$u^* = \frac{r-2}{r} \log(r-1).$$

## 5.2 Properties of the vacant net

**Size of the vacant net.** The calculations for the vacant net are much simpler than for the vacant set. For the case of an unvisited edge  $\{u, v\}$  of  $E(G)$ , where  $u, v$  are nice, the corresponding unvisited states of  $U$  in  $M$  are  $S = \{(u, v), (v, u)\}$ . Contract  $S$  to a vertex  $\gamma(S)$ . The equations  $\tilde{\pi} = \tilde{\pi}\tilde{P}$  for the walk  $\mathcal{H}$  in  $H$  are solved by  $\tilde{\pi}(\gamma(S)) = 2/rn$  for  $\gamma(S)$ , and  $\tilde{\pi}(\sigma) = 1/rn$  for any other state  $\sigma$  of  $H$ . Thus  $\tilde{\pi}_{\mathcal{H}}$  is compatible with  $\pi_{\mathcal{M}}$  in Lemma 7. No first return to  $\gamma(S)$  is possible in the arborescence  $\mathcal{T}_{\gamma(S)}$ , and so  $f = 0$  in (51). Thus

$$p_{\gamma} \sim 2/rn, \tag{55}$$

and the vacant net is of expected size  $\mathbf{E}|\mathcal{U}(t)| \sim (rn/2)e^{-2t/rn}$ . The concentration of the random variable  $|\mathcal{U}(t)|$  follows from the methods of Lemma 9. Theorem 3(ii) follows from this. The edge cover time in Theorem 3(iii) is obtained by equating  $\mathbf{E}|\mathcal{U}(t)| = o(1)$  and applying the techniques used in [9] to obtain a lower bound.

**The number of paths length two in the vacant net.** Let  $\{v, u\}, \{u, w\}$  be adjacent unvisited edges of  $E(G)$ . The corresponding states of  $U$  in  $M$  are  $S = \{(u, v), (v, w), (w, v), (v, u)\}$  which contacts to a vertex  $\gamma(S)$  with total in-degree and out-degree  $4(r - 1)$ . At  $\gamma(S)$  there are two loops and  $4r - 6$  in-neighbours other than  $\gamma(S)$ . We obtain a stationary probability  $\tilde{\pi}(\gamma(S)) = 4/rn$  and  $\tilde{\pi}(f) = \pi(f)$  for  $f \notin S$  which confirms  $\tilde{\pi}$  and  $\pi$  are compatible.

The total out-degree of  $\gamma(S)$  is  $4(r - 1)$ , but there are two loops at  $\gamma(S)$  which can be chosen for a first return in  $\mathcal{T}_{\gamma(S)}$ , with probability  $2/4(r - 1)$ . If the walk moves away from  $\gamma(S)$ , no first return is possible in  $\mathcal{T}_{\gamma(S)}$ . This gives  $f = 1/2(r - 1)$  in (51). Thus

$$p_\gamma \sim 2(2r - 3)/r(r - 1)n. \quad (56)$$

**Threshold for the vacant net.** Theorem 3(v) follows from using  $Q = 2M(2) - M(1)$  (see (46)), and equating  $Q = 0$  in Lemma 14 with the appropriate values of  $\mathbf{EM}(1, t)$ ,  $\mathbf{EM}(2, t)$  from (36). From (55), (56) we have  $\alpha_1 = 2/r$ ,  $\alpha_2 = 2(2r - 3)/r(r - 1)$ . Equating  $M(1, t) = 2M(2, t)$  and setting  $t = \theta^*n$  gives

$$\theta^* = \frac{r(r - 1)}{2(r - 2)} \log(r - 1).$$

## 6 Random walks which prefer unvisited edges

The unvisited edge process is a modified random walk  $X = (X(t), t \geq 0)$  on a graph  $G = (V, E)$ , which uses unvisited edges when available at the currently occupied vertex. If there are *unvisited edges* incident with the current vertex, the walk picks one u.a.r. and makes a transition along this edge. If there are no unvisited edges incident with the current vertex, the walk moves to a random neighbour.

**Partitioning the edge-process into red and blue walks.** At any step  $t$  of the walk, we partition the edges of  $G$  into red (unvisited) edges and blue (visited) edges. Thus  $t = t_R + t_B$  where  $t_R$  is the number of transitions along red edges up to step  $t$ , hence recoloring those edges blue, and  $t_B$  the number of transitions along blue edges. Note that in [3] the unvisited edges were designated blue and the visited edges red, the opposite of the terminology in this paper.

At each step  $t$  the next transition is either along a red or blue edge. We speak of the sequence of these edge transitions as the red (sub)-walk and the blue (sub)-walk. The walk thus consists of red and blue phases which are maximal sequences of edge transitions of the given edge type (unvisited or visited). For any vertex  $v$ , and step  $t$ , let  $d_B(v, t)$  the blue degree of  $v$ , be the number of blue edges incident with  $v$  at the start of step  $t$ . Similarly define  $d_R(v, t)$ .

For graphs of even degree, each red phase starts at some step  $s$  at a vertex  $u$  of positive even red degree  $d_R(u, s) \geq 2$ , and ends at some step  $t$  when the walk returns to  $u$  along the last red edge incident with  $u$ . Thus  $d_R(u, t) = 0$  and a blue phase begins at step  $t + 1$ . Thus for  $r$ -regular graphs  $r = 2d$ , if we ignore the red phases of the edge-process  $X$ , then the resulting blue phases describe a simple random walk  $W$  on the graph  $G$ . To illustrate this, suppose the edge-process  $X$  starts at  $X(0) = u$ , then  $W$  also starts at vertex  $u$  after the completion of the first red phase at  $t_R$ . After some number of steps  $t_B$ , the blue walk  $W$  arrives at a vertex  $u'$  with unvisited edges, and a red phase starts from  $u'$ , at step  $t_R + 1$ , as counted in the red walk. This is followed by a blue phase starting from  $u'$  at step  $t_B + 1$  of the blue walk. Thus the walks interlace seamlessly, and at step  $t$  of the edge-process, we have  $t = t_R + t_B$ , where  $t_R, t_B$  are the number of red and blue edge transitions.

In summary the red walk is a walk with jumps which consists of a sequence of closed tours each with a distinct start vertex. The blue walk is a simple random walk. Given a step  $s = t_R + t_B$  of the edge-process, we extend the notation  $d_R(u, s)$  for the red degree of vertex  $u$  at step  $s$  of the edge-process to  $d_R(u, t_R)$  the red degree of vertex  $u$  at step  $t_R$  of the red walk.

## 6.1 Thresholds measured in the red walk

To make our analysis, we first consider only the red walk steps  $t = t_R$ . Let  $r = 2d$  and let  $R_j(t)$  be the number of vertices of red degree  $j$  for  $j = 0, 1, \dots, 2d$  at step  $t$  of the red walk. Unless the walk is at the vertex  $u$  which starts the red phase, (in which case all vertices have even red degree), then with the exception of  $u$  and the current position  $v$  of the walk, all other vertices have even red degree at any step of a red phase.

We generate the red walk in the configuration model, and derive its approximate degree sequence. The intuition is as follows. Suppose the red walk arrives at vertex  $v$  at the end of step  $t$ , and leaves  $v$  at the start of step  $t + 1$ . To simplify things we could agree to say the degree of  $v$  changes by 2 at the start of step  $t + 1$ . Thus we consider the following process which samples u.a.r. without replacement from the sets  $S_v, v \in V$  of configuration points of a graph with  $m = dn$  edges.

### **Pairs-process.**

At each step  $t = 1, \dots, m$ :

Pick an unused configuration point  $\alpha$  u.a.r., remove  $\alpha$  from the set of available points. Pick another unused point u.a.r.  $\beta$  from the same vertex as  $\alpha$ , remove  $\beta$  from the set of available points.

Add  $Y_t = (\alpha, \beta)$  to the ordered list of samples  $Y_1, \dots, Y_{t-1}$ .

Let the random variables  $\mathbf{N}_k(t)$ ,  $k = 0, 1, \dots, d$ , be the number of vertices of degree  $2k$



generated by the Pairs-process, and let  $N_k(t) = \mathbf{E}\mathbf{N}_k(t)$ . Here the degree of a vertex is the number of *unpaired* points associated with that vertex.

We condition on the pairings in our process and the ordering  $\alpha, \beta$  within pairs. After this, we have a permutation of  $dn$  objects, where each object is a pair. The probability  $p(k)$  that a vertex contributes to  $\mathbf{N}_k(t)$  is the probability that exactly  $d - k$  out of a fixed set of  $d$  objects appear before the  $t$ -th element in our permutation. Thus  $p(k)$  has a hypergeometric distribution, and

$$N_k(t) = np(k) = n \frac{\binom{t}{d-k} \binom{dn-t}{k}}{\binom{dn}{d}}.$$

Thus  $N_k(t)$  can be written as,

$$N_k(t) = \left(1 + O\left(\frac{1}{t} + \frac{1}{dn-t}\right)\right) n \binom{d}{k} \left(\frac{dn-t}{dn}\right)^k \left(\frac{t}{dn}\right)^{d-k}, \quad i = 0, 1, \dots, d. \quad (57)$$

By a martingale argument on the configuration sequence of pairs of points of length  $dn$ , the random variables  $\mathbf{N}_k(t)$  are concentrated within  $O(\sqrt{dn \log n})$  of  $N_k(t)$  for any  $0 \leq t \leq dn$ . Suppose the first difference between a pair of sequences  $Y, Y'$  occurs at vertices  $v, v'$  with  $Y_i = (\alpha_i, \beta_i)$  and  $Y'_i = (\alpha'_i, \beta'_i)$ . Let  $Y_j$  be the first occurrence of  $v'$  after step  $i$  in  $Y$ . Map this to the first occurrence  $Y'_j$  of  $v$  in  $Y'$ . For  $u \neq v, v'$  let all other entries of the sequence be the same. Map subsequent pairings  $Y_\ell, Y'_\ell$  between (possibly) different configuration points of  $v$  as appropriate; and similarly for  $v'$ . The maximum difference between  $\mathbf{N}_k(t)$  in the mapped sequences is 2. Thus

$$\mathbf{Pr}(|\mathbf{N}_k(t) - N_k(t)| \geq \sqrt{An \log n}) = O(n^{-A/8}). \quad (58)$$

We next explain how  $N_k(t)$  can be used to approximate  $R_{2k}(t)$ , the number of vertices of red degree  $j = 2k$ . Let  $Y$  be a Pairs-process and  $W$  a red walk generated in the configuration model. Let the vertices which start the red phases of  $W$  be  $\mathbf{u} = (u_1, u_2, \dots, u_J)$ . There is an isomorphism between  $(W, \mathbf{u})$  and  $(Y, \mathbf{u})$ . Let  $Y_i = (\alpha_i, \beta_i)$  be the pair generated at step  $i$  of Pairs. Let  $Y_\ell$  be the last occurrence in  $Y$ , of configuration points from vertex  $u = u_1$  (i.e. from  $S_u$ ). The subsequence  $P = (Y_1, \dots, Y_\ell)$  of Pairs is isomorphic to the first red phase  $Q$  of  $W$  by the following mapping which moves  $\beta_\ell$  to the front of  $P$  to form  $Q$ .

$$\begin{aligned} P &= (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_{i-1}, \beta_{i-1}), (\alpha_i, \beta_i), \dots, (\alpha_\ell, \beta_\ell), \\ Q &= (\beta_\ell, \alpha_1, (\beta_1, \alpha_2), \dots, (\beta_{i-1}, \alpha_i), (\beta_i, \alpha_{i+1}), \dots, (\beta_{\ell-1}, \alpha_\ell). \end{aligned}$$

Given  $W$  and sequence  $\mathbf{u} = (u_1, \dots, u_J)$ , the last occurrence of  $u_i$  is before the last occurrence of  $u_{i+1}$ . Thus there is always a unique  $Y$  to match this  $W$ .

We next relate the probability of a given  $(W, \mathbf{u})$  to that of the corresponding  $(Y, \mathbf{u})$ . Let  $v$  be the vertex chosen to pair at step  $i$  of  $Y$ . In the Pairs-process, let  $d(v, i)$  be the number of

remaining unmatched configuration points of  $v$  at the start of step  $i$ . The total degree of the underlying graph is  $2m$ . Thus

$$\begin{aligned}\Pr(Y_1) &= \frac{1}{2m} \frac{1}{d(v, 1) - 1} \\ \Pr(Y_i | Y_1, \dots, Y_{i-1}) &= \frac{1}{2m - (2i - 2)} \frac{1}{d(v, i) - 1}.\end{aligned}$$

When  $v \neq u$ , and the transition is  $W_{i+1} = (\beta_i, \alpha_{i+1})$ , the vertex  $v$  which corresponds to  $\alpha_i$  in  $Y_i = (\alpha_i, \beta_i)$  had degree  $d(v, i)$  when  $\alpha_i$  was chosen u.a.r., and so  $\beta_i$  was chosen from a set of size  $d(v, i) - 1$  and so

$$\Pr(W_{i+1} | W_1, \dots, W_i) = \frac{1}{d(v, i) - 1} \frac{1}{2m - (2i + 1)}.$$

However if  $v = u$ , because we moved  $\beta_\ell$  to the front of  $W$  the red degree of  $u$  at step  $i$  is less by one than it was in Pairs. Thus

$$\begin{aligned}\Pr(W_1) &= \frac{1}{d(u)} \frac{1}{2m - 1} \\ \Pr(W_{i+1} | W_1, \dots, W_i) &= \frac{1}{d(u, i) - 2} \frac{1}{2m - (2i + 1)}.\end{aligned}$$

This means that, at step  $\ell$  when the red degree of  $u = u_1$  becomes zero,

$$\Pr_W(Q) = \Pr_Y(P) \frac{(d(u) - 1)(d(u) - 3) \dots 1}{d(u)(d(u) - 2) \dots 2} \prod_{i=0}^{\ell-1} \frac{2m - 2i}{2m - 2i - 1}.$$

We repeat this analysis starting with  $u = u_2$  etc. Thus with  $\Pr(Z)$  being the probability of process  $Z$ ,

$$\Pr(W) \leq \Pr(Y) \prod_{i=0}^{m-1} \left( \frac{2m - 2i}{2m - 2i - 1} \right) = \Pr(Y) \frac{(2^m m!)^2}{(2m)!} = O(\sqrt{m}) \Pr(Y).$$

Recall that  $R_{2k}(t)$  is the number of vertices at step  $t$  of the red walk. Suppose we generate a red walk starting from  $u$  in the configuration model, stopping at step  $j$  to give  $Q = (a_1, b_1, a_2, b_2, \dots, a_j, b_j)$ . Then  $P = (b_1, a_2), \dots, (b_{j-1}, a_j)$  is a Pairs sequence, and for any  $j$

$$R_{2k}(j) = N_k(j - 1) + C, \quad |C| \leq 2.$$

Using (58) with  $A = 24$ , we have,

$$\Pr(\exists t, R_{2k}(t) \geq |N_k(t) + O(\sqrt{n \log n})|) \leq O(n\sqrt{dn} n^{-A/8}) = O(n^{-1}). \quad (59)$$

Using  $t_R$  to denote red steps we can obtain the size of the vacant set  $R_d(t_R)$ . We do this next. Theorem 4 is expressed in terms of step  $t$  of the Edge-process, where  $t = t_R + t_B$  and  $t_B$  are blue steps. Thus, to prove Theorem 4 we need to add back the number of blue steps  $t_B$ . We do this in Section 6.3.

**Vacant set properties at any step of the red walk.** Let  $t = dn(1 - \delta)$  where  $\delta > 0$  constant. Then  $N_d(t) = \Theta(n)$ , and w.h.p. the size of the vacant set at red step  $t$  is

$$|\mathcal{R}(t)| = R_{2d}(t) = (1 + o(1))N_d(t) = (1 + o(1))n \left( \frac{dn - t}{dn} \right)^d. \quad (60)$$

Let  $M(1, t)$  denote the expected number of edges (resp.  $M(2, t)$  denote twice the expected number of pairs of edges) induced by the vacant set at each vertex of the vacant set. Working in the configuration model, with  $r = 2d$ ,

$$M(1, t) \sim N_d(t)r \frac{2d}{2dn - 2t} N_d(t) \sim rn \left( \frac{dn - t}{dn} \right)^{2d-1} \quad (61)$$

$$M(2, t) \sim N_d(t) \binom{r}{2} \left( \frac{2d}{2dn - 2t} N_d(t) \right)^2 \sim \binom{r}{2} n \left( \frac{dn - t}{dn} \right)^{3d-2}. \quad (62)$$

The expected number of edges induced by the vacant set is

$$\mathbf{E}|E(\Gamma(t))| \sim M(1, t)/2 \sim dn \left( \frac{dn - t}{dn} \right)^{2d-1}. \quad (63)$$

The concentration of  $M(1, t)$ ,  $M(2, t)$  follow from the methods of Lemma 9 (the Chebychev inequality and the interpolation).

The threshold  $t = t^*$  for the subcritical phase comes from applying the Molloy-Reed condition given by  $L(\mathbf{d}) = 0$ . In (43) we examine  $2M(2, t) - M(1, t) - O(|\overline{\mathcal{N}}|)$ , where  $M(1, t), M(2, t)$  are given by (61)-(62), and  $|\overline{\mathcal{N}}| = O(n^\epsilon)$  is the number of non-nice vertices (see (6)). Let  $t^* = u^*n$ , where

$$u^* = d \left( 1 - \left( \frac{1}{2d-1} \right)^{\frac{1}{d-1}} \right). \quad (64)$$

Note that  $M(1, t^*) = \Theta(n), M(2, t^*) = \Theta(n)$ . At  $t^*$ ,  $2M(2, t^*) \sim M(1, t^*)$ . If we choose  $t = u^*n(1 + \varepsilon)$ , where  $|\varepsilon| > 0$  constant, then using (61)-(62) gives

$$2M(2, t) - M(1, t) = |\Theta(n)| \left( (1 - \varepsilon((r-1)^{1/(d-1)} - 1))^{d-1} - 1 \right). \quad (65)$$

Thus for  $t = t^*(1 - \varepsilon)$  this difference is positive. As  $|\mathcal{R}(t)| = \Theta(n)$ , w.h.p. this confirms the w.h.p. existence of a giant component linear in the graph size. At  $t = t^*(1 + \varepsilon)$  the difference in (65) is negative, and the maximum component size is  $O(\log n)$ . It remains to find out how many blue steps have elapsed by red step  $t^*(1 + \varepsilon)$ . We defer this until Section 6.3.

**Vacant net properties at any step of the red walk.** The vacant net has exactly  $U(t) = dn - t$  edges. Thus, similarly to the vacant set,

$$M(1, t) \sim 2dn - 2t \quad (66)$$

$$M(2, t) \sim \sum_{k=1}^d \binom{2k}{2} N_k(t) \sim \frac{dn - t}{dn} (dn + 2(d-1)(dn - t)). \quad (67)$$

In (43) for the Molloy-Reed condition we require  $2M(2, t) - M(1, t) - O(|\overline{\mathcal{N}}|) > 0$ , where  $M(1, t), M(2, t)$  are given by (66)-(67), and  $|\overline{\mathcal{N}}| = O(n^\epsilon)$  is the number of non-nice vertices (see (6)).

The solution to  $M(1, t) = 2M(2, t)$  obtained by using the right hand side values of (66)-(67) is at the end of the red walk, i.e. red step  $\hat{t} = \theta^* n$  where

$$\theta^* = d.$$

For any red step  $t(\delta) = (1 - \delta)dn$  where  $\delta > 0$ ,  $M(1, t) = \Theta(n)$ ,  $M(2, t) = \Theta(n)$ , and

$$2M(2, t) - M(1, t) \sim 4(d - 1) \frac{(dn - t)^2}{dn}.$$

Thus at red step  $t(\delta) = (1 - \delta)\hat{n}$ , for any  $\delta > 0$ , w.h.p. the vacant net has a giant component linear in the graph size. It remains to find out how many blue steps  $t_B$  have elapsed before this value of  $t = t_R$ , and also to analyse the sub-critical case  $dn - t = o(n)$ . We defer this until Section 6.3.

## 6.2 Number of blue steps before a given red step

Suppose a red phase starts at red step  $s$  from vertex  $v$  of red degree  $2k$ . At step  $s$  the walk leaves  $v$  along a red edge, and returns to  $v$  at some step  $t' \geq s$ . We have  $d_R(v, \tau) = 2k - 1$  for  $s \leq \tau < t'$  and  $d_R(v, t') = 2k - 2$ . Thus a red phase at  $v$  consists of  $k$  *excursion rounds* with starts  $s_1, \dots, s_k$  and ends  $t_1, \dots, t_k$ , where  $s_1 = s, t_k = t$  and  $s_i = t_{i-1} + 1$ . At the final return,  $d_R(v, t) = 0$  and a blue phase begins.

**Lemma 16.** *Let  $L(s, k)$  be the finish time of a red phase starting at red step  $s$  from a vertex  $v$  of red degree  $2k$ . Let  $r = 2d$ . Then for  $t \leq dn(1 - \delta)$ , and  $\delta \geq \omega/\sqrt{n}$*

$$\Pr(L(s, k) = t) \leq (1 + O(k/\omega)) \frac{k}{2^{2k}} \binom{2k}{k} \frac{(t - s)^{k-1}}{(dn - s)^{k-1/2} (dn - t)^{1/2}}. \quad (68)$$

*Proof.* Let  $\rho = 2k - 1$ . For a walk starting from  $v$  at  $s$ , let  $T_v^+$  be the first return time to  $v$ . Then working in the configuration model,

$$\begin{aligned} \Pr(T_v^+ = t \mid s, 2k) &= \prod_{\sigma=s}^{t-1} \left( 1 - \frac{\rho}{2dn - 2\sigma - 1} \right) \frac{\rho}{2dn - 2t - 1} \\ &= \frac{\rho}{2dn - 2t - 1} \exp \left( -\frac{\rho}{2} \left( \frac{1}{dn - s} + \dots + \frac{1}{dn - t} \right) + O \left( \frac{t - s}{(dn - t)^2} \right) \right) \\ &= \frac{\rho}{2dn - 2t - 1} \exp \left( \frac{\rho}{2} \left( \log \frac{dn - t}{dn - s} + O \left( \frac{1}{n\delta^2} \right) \right) \right) \\ &= (1 + O(1/\omega)) \frac{\rho}{2(dn - t)} \left( \frac{dn - t}{dn - s} \right)^{\rho/2}. \end{aligned}$$

Thus

$$\begin{aligned}
\Pr(L(s, k) = t) &= \sum_{s < t_1 < \dots < t_{k-1} < t} \prod_{i=1}^k \Pr(T_v^+ = t_i \mid s_i, \rho_i = 2k - 2(i-1) - 1) \\
&= (1 + O(1/\omega))^k ((2k-1)(2k-3)\dots 1) \frac{1}{2^k} \sum_{s < t_1 < \dots < t_{k-1} < t} \prod_{i=1}^k \frac{1}{dn - t_i} \left( \frac{dn - t_i}{dn - s_i} \right)^{\rho_i/2} \\
&= (1 + O(k/\omega)) \frac{(2k)!}{k! 2^{2k}} \frac{1}{(dn - s)^{k-1/2}} \frac{1}{(dn - t)^{1/2}} \sum_{s < t_1 < \dots < t_{k-1} < t} 1 \\
&= (1 + O(k/\omega)) \binom{t-s}{k-1} \frac{(2k)!}{k! 2^{2k}} \frac{1}{(dn - s)^{k-1/2}} \frac{1}{(dn - t)^{1/2}}.
\end{aligned}$$

□

We use Lemma 16 to upper bound the number of red phases before a given red step  $t$ .

**Lemma 17.** *Let  $J(t)$  be the number of red phases completed at or before step  $t = t_R$  of the red walk. For  $\delta > 0$  and any  $t \leq dn(1 - \delta)$ , and  $\delta \geq \omega/\sqrt{n}$*

$$\Pr\left(J(t) \geq \frac{de^3}{\delta}\right) = O\left(\frac{1}{n}\right).$$

*Proof.* The  $J$  red phases start at  $s_1, \dots, s_J$  and end at  $t_1, \dots, t_J$ , where  $s_1 = 0$ ,  $t_J = t$ , and  $s_i = t_{i-1} + 1$ . The total number of excursion rounds is  $K = k_1 + \dots + k_J$ , where  $J < K \leq dJ$ . Let  $\tau = (t_1, \dots, t_J)$  and  $\kappa = (k_1, \dots, k_J)$ . Let  $\mathcal{E}(\tau, \kappa)$  be the event that  $L(s_i, k_i) = t_i$ ,  $i = 1, \dots, J$ . Then

$$\Pr(\mathcal{E}(\tau, \kappa)) = \prod_{i=1}^J \Pr(L(s_i, k_i) = t_i).$$

To simplify (68), as  $\binom{2k}{k} \leq 2^{2k}$  and  $k \leq d$ , then

$$\frac{k}{2^{2k}} \binom{2k}{k} \leq k \leq d.$$

Let  $s = adn$ ,  $t = bdn$  where  $0 \leq a \leq b \leq (1 - \delta)$ . Then, as  $(t - s) \leq (dn - s)$ ,

$$\frac{(t - s)^{k-1}}{(dn - s)^{k-1/2} (dn - t)^{1/2}} = \left( \frac{t - s}{dn - s} \right)^{k-1} \frac{1}{(dn - s)^{1/2} (dn - t)^{1/2}} \leq \frac{1}{dn\delta}.$$

Thus

$$\Pr(\mathcal{E}(\tau, \kappa)) \leq (1 + O(K/\omega)) \frac{d^J}{(dn)^J \delta^J}. \quad (69)$$

For a given realization of the edge-process, the sequence  $\kappa$  is a fixed input determined by the blue walk, and the right hand side of (69) is independent of the value of this input. For any given red step  $t_J = t$ , let  $J(t) = J(t, \kappa)$  be the number of red phases completed at or before step  $t$ . Then,

$$\begin{aligned} \Pr(J(t), \kappa) &= \sum_{\tau} \Pr(\mathcal{E}(\tau, \kappa)) \leq (1 + O(K/\omega)) \binom{t}{J-1} \frac{1}{n^J \delta^J} \\ &= O(1 + O(K/\omega)) \frac{t^{J-1} e^J}{n^J \delta^J J^J} = O\left(\frac{1}{n}\right) \left(\frac{de^{1+O(d/\omega)}}{J\delta}\right)^J. \end{aligned}$$

The last line follows from  $K \leq dJ$  and  $t \leq dn$ . Finally, we put  $J = de^3/\delta$ .  $\square$

### 6.3 Proof of Theorem 4

Let  $T_G$  be a mixing time for a random walk on  $G$ , such that, for  $t \geq T_G$ ,

$$\max_{u, x \in V} |P_u^{(t)}(x) - \pi_x| \leq \frac{1}{n^3}. \quad (70)$$

We use the following results from [3].

**Lemma 18.** *Let  $T_G$  be a mixing time of a random walk  $W_u$  on  $G$  satisfying (70). For any start vertex  $u$  let  $\mathbf{A}_{t,u}(v)$  be the event that  $W_u$  has not visited vertex  $v$  at or before step  $t$ . Then*

$$\Pr(\mathbf{A}_{t,u}(v)) \leq e^{-\lfloor t/(T_G + 3\mathbf{E}_\pi H_v) \rfloor},$$

where  $\mathbf{E}_\pi H_v$  is the hitting time of  $v$  starting from stationarity.

**Lemma 19.** *Let  $G = (V, E)$ , let  $|E| = m$ . Let  $S \subseteq V$ , and let  $d(S)$  be the degree of  $S$ . Then  $\mathbf{E}_\pi H_S$ , the expected hitting time of  $S$  from stationarity satisfies*

$$\mathbf{E}_\pi H_S \leq \frac{2m}{d(S)(1 - \lambda(G))}.$$

In (8) we used the crude bound  $\lambda_2 \leq 29/30$ , in which case Lemma 19 gives an upper bound  $\mathbf{E}_\pi H_S \leq 30n/|S|$ . From (13) we can take  $T_G = 120 \log n$ . Lemma 18 then implies that

$$\Pr \mathbf{A}_{t,u}(v) \leq \exp\left(-\frac{t}{120(n/|S| + \log n)}\right). \quad (71)$$

## Various useful properties.

**Lemma 20.** (i) Let  $S(t)$  be the vertex set of the vacant net at red step  $t$ . Then for any  $t = dn(1 - \delta)$ ,  $|S(t)| \geq \delta n/2$ .

(ii) Let  $t_R = dn(1 - \delta)$  where  $\sqrt{(\omega \log n)/n} \leq \delta = o(1)$ . Then w.h.p. by red step  $t_R$  at most  $t(\delta) = dn(1 - \delta + O(1/\omega))$  steps of the edge-process have elapsed.

(iii) There exists red step  $t_1 = dn(1 - \delta)$  with  $\delta = \Omega(\sqrt{(\omega \log n)/n})$ , such that w.h.p. by red step  $t_1$ , for  $k \geq 3$ ,  $R_{2k}(t_1) = 0$ , and  $R_4(t_1) = O(\omega \log n)$ . The corresponding step of the edge-process is  $t = dn(1 + O(1/\omega))$ .

*Proof.* Part (i). At red step  $t = dn(1 - \delta)$  there are, by definition,  $\delta dn$  red edges. Let  $S(t) = \{v \in V : d_R(v, t) > 0\}$  denote the vertex set of the vacant net at red step  $t$ . Then deterministically

$$|S(t)| \geq \delta dn/r = \delta n/2.$$

Part (ii). At  $t = dn(1 - \delta)$  the vertex set of the vacant net is of size  $|S(t)| \geq \delta n/2$ . Let  $S = S(t)$  in Lemma 19. Contract  $S(t)$  to a vertex  $v(S)$ . Let  $\tau$  denote a step of the blue walk, and apply Lemma 18 at  $\tau = A \log n/\delta$  for some large  $A$ . Using (71), and choosing  $A = 500$ , the probability  $P_B(\tau)$  that a blue phase lasts more than  $T_G + \tau \leq 2\tau$  steps is upper bounded by

$$P_B(\tau) \leq \exp \left\{ -\frac{A \log n/\delta}{120(\log n + 2/\delta)} \right\} = O(1/n^2).$$

Let  $t_R$  be the end of the first red phase at which  $|S(t)| \leq n\delta$ . Let  $t_B$  be the number of blue steps before  $t_R$ , then

$$t_B = \sum_{i \leq J} t_{B,i} \leq 2\tau J(t_R) = O\left(\frac{\log n}{\delta^2}\right).$$

Thus provided  $\delta \geq \sqrt{\omega \log n/n}$ ,  $t_B = O(n/\omega)$  and

$$t(\delta) = t_R + t_B = dn(1 - \delta) + O(n/\omega) = dn(1 - \delta + O(1/\omega)).$$

Part (iii). Let  $t = dn(1 - \delta_0(k))$  where  $\delta_0(k) = (\log n/n)^{1/2k}$ . At  $t_R = t$ , (57) and (58) imply that w.h.p.

$$R_{2k}(t) = O(n\delta_0^k) + O(\sqrt{n \log n}) = O(\sqrt{n \log n}).$$

Next choose  $\delta_1 = \sqrt{\omega \log n/n}$ . Let  $B_{2k}(t)$  be the set of vertices of red degree  $2k$  at red step  $t$ . Let  $t' = t + dn\delta_1$  and let  $P_B(v)$  be the probability the red walk did not visit  $v$  during  $dn\delta_1$  steps. Thus

$$\begin{aligned} P_B(v) &= \prod_{s=t}^{t'} \left(1 - \frac{k}{dn - s}\right) = O(1) \exp \left(-k \log \frac{dn - t}{dn - t'}\right) \\ &= O(1) \exp(-k \log \delta_0/\delta_1) = O(\omega^{1/2} (\log n/n)^{(k-1)/2}). \end{aligned}$$

Thus for  $k = d, d - 1, \dots, 3$ ,

$$\Pr(R_{2k}(t') \neq 0) = O\left(\sqrt{\omega n \log n} (\log n/n)^{(k-1)/2}\right) = o(1).$$

For  $k = 2$

$$\mathbf{E}R_4(t') = O(\sqrt{\omega} \log n),$$

and thus

$$\Pr(R_4(t') \geq \omega \log n) = O(1/\sqrt{\omega}).$$

By the previous part of this lemma, the number of blue steps  $t_B$  elapsed at  $t'$  is  $O(n/\omega)$ . This corresponds to a step  $t$  of the edge-process where

$$t = t' + t_B \leq dn(1 + O(1/\omega)).$$

□

**Vacant set size and threshold.** We recall the discussion in Section 6.1 where the size, and number of edges of the vacant set at any red step  $t_R$  are given by (60) and (63) respectively. Theorem 4 (i) then follows from Lemma 20(ii).

Considering the threshold, let  $t^* = u^*n$  be the red step given by  $u^*$  in (64). We prove that at steps  $t^*(1 - \epsilon)$  and  $t^*(1 + \epsilon)$  respectively of the edge-process, the vacant set is super-critical and sub-critical respectively. At red step  $t^*$ ,

$$|\mathcal{R}(t^*)| = R_{2d}(t^*) = (1 + o(1))n \left(\frac{1}{2d-1}\right)^{\frac{d}{d-1}} = \Theta(n),$$

and  $|\mathcal{R}(t)|$  is concentrated. In Section 6.1, using the Molloy-Reed condition and (65), we proved that at red step  $t \leq t^*(1 - \epsilon)$  the giant component  $C_1(t) = \Theta(|\mathcal{R}(t)|) = \Theta(n)$  w.h.p. Similarly at red step  $t^*(1 + \epsilon)$  for some small  $\epsilon > 0$ , the maximum component size of the vacant set at  $t^*(1 + \epsilon)$  is  $O(\log n)$  w.h.p. Let  $d(1 - \delta) = u^*(1 + \epsilon)$ , then the corresponding  $\delta$  is constant. By Lemma 20(ii), red step  $t_R = t^*(1 + \epsilon)$  corresponds to step  $t = t^*(1 + \epsilon + O(1/\omega))$  of the edge-process. Thus at step  $t = t^*(1 + O(\epsilon))$  the maximum component size is  $O(\log n)$  w.h.p. and the graph of the vacant set is subcritical. This completes the proof of Theorem 4 (iii).

**Vertex cover time.** For the proof of Theorem 4 (iii), that  $T_{cov}^V(G) \sim dn$ , we consider the cases  $r = 4$  and  $r \geq 6$  separately.

*Case  $r = 2d$ ,  $d \geq 3$ .* At red step  $t_R = dn(1 - \delta)$  where  $\delta = 1/n^{1/2d}$ , then  $R_{2d}(t_R) = \Omega(n^{1-1/2d})$ . By Lemma 20(ii) the corresponding step of the edge-process is  $t' = dn(1 + O(1/\omega))$ . However



by Lemma 20(iii), at step  $t_1 = dn(1 + O(1/\omega))$  of the edge-process  $R_{2d}(t_1) = 0$  and the vacant set is empty. Thus the vertex cover time  $T_{cov}^V(G) \sim dn$ .

*Case  $r = 4$ .* The cover time can be deduced from the proof of Lemma 21 (see below) that  $\hat{t} \sim dn$  is the threshold for the vacant net. The relevant facts from Lemma 21 are the following. At  $t = \hat{t}(1 - o(1))$  there are vertices of red degree 4 w.h.p. For some  $t \leq \hat{t}(1 + o(1))$  the last vertex of red degree 4 disappears. Thus, for  $r = 4$  the vacant set becomes empty at some  $t \sim \hat{t} \sim dn$ . We remark that the vacant net could still be nonempty, but if so it will consist of isolated cycles.

**Vacant net. Supercritical regime.** From Section 6.1 the threshold for the vacant net is at red step  $\hat{t} \sim dn$ . Choose a red step  $t_R$ , where  $t_R = dn(1 - \delta)$ ,  $\delta \geq 0$  constant. By Lemma 20(i), the vertex set  $S(t_R)$  of the vacant net is of size  $|S(t_R)| \geq \delta n/2$ . By Lemma 20(ii), the corresponding step  $t = t_R + t_B$  of the edge-process is  $t_R(1 + o(1))$  w.h.p.

**Vacant net. Subcritical regime.** Because the vacant net becomes sub-linear in size near  $dn \sim \hat{t}$ , the time taken by the blue walk to reach unvisited edges increases rapidly. Thus more work is needed to prove the vacant net has maximum component size  $O(\log n)$  at some step  $t = dn(1 + o(1))$  of the edge-process.

**Lemma 21.** *There is a step  $t$  of the edge-process, where  $t = dn(1 + o(1))$  such that w.h.p. at step  $t$  all components of the vacant net have size  $O(\log n)$ .*

*Proof.* The proof is in three parts. In the first part we count up the number of blue steps occurring before red time  $t_1 = dn(1 - \delta_1)$  where  $\delta_1 = \sqrt{\omega \log n/n}$ . At  $t_1$  the vacant net consists mainly of vertices of red degree 2, with a few vertices of red degree 4. In the second part, we prove that after a further  $t_B = o(n)$  steps of the blue walk we have removed all vertices of red degree 4, thus destroying any complex components of the vacant net. The vacant net now consists entirely of red cycles. In the third part we use a further  $t_B = o(n)$  steps of the blue walk to remove any red cycles of length at least  $\log n$ .

**Part 1.** Let  $t_1$  be red step  $dn(1 - \delta_1)$  where  $\delta_1 = \sqrt{\omega \log n/n}$ . By Lemma 20(ii) the corresponding step of the edge-process is  $t = dn(1 + O(1/\omega))$ . At any red step  $t_R = dn(1 - \delta)$ , the maximum component size is at most the number of red edges  $dn\delta$ . Thus at step  $t$  of the edge-process corresponding to  $t_1$  the giant component is of size

$$C_1(t) = O(n\delta_1) = O(\sqrt{\omega n \log n}).$$

By Lemma 20(iii) the vacant net  $\hat{\Gamma}(t_1)$  consists of  $R_{2i}(t_1) = n_{2i}$  vertices of red degree  $2i$ . For some  $c_2$  constant, w.h.p.

$$n_2 = c_2 \sqrt{n\omega \log n}, \quad n_4 \leq \omega \log n, \quad n_{2i} = 0, \quad i \geq 3. \quad (72)$$

**Part 2.** It follows from (72) that at red step  $t_1$  the vacant net consists of 2-cycles (cycles with vertices of red degree 2) and complex components with vertices of degree 2 and 4. Such components are Eulerian, and can be decomposed (non-uniquely) into (2, 4)-cycles (cycles where all vertices have red degree 2 or 4 in the vacant net). We prove that after  $t_B = o(n)$  further blue steps, the blue walk has visited every (2, 4)-cycle in the vacant net  $\widehat{\Gamma}(t_1)$ . If so, the vacant net is either empty or consists entirely of red 2-cycles. To assume otherwise leads to a contradiction.

We count (2, 4)-cycles in the configuration model. Let  $\Phi(m) = (2m)!/m!2^m$ . Using  $\binom{2k}{k}/2^{2k} = \Theta(1/\sqrt{k+1})$ , it follows that

$$\frac{m!}{(m-s)!} 2^s \frac{\Phi(m-s)}{\Phi(m)} = \Theta\left(\sqrt{\frac{m}{m-s+1}}\right). \quad (73)$$

Let  $C(i, a, b)$  be the number of (2, 4)-cycles of length  $i = a + b$  and containing  $a$  vertices of red degree 2, and  $b$  vertices of red degree 4. Thus

$$\mathbf{EC}(i, a, b) = \binom{n_2}{a} \binom{n_4}{b} \frac{(i-1)!}{2} \binom{4}{2}^b 2^i \frac{\Phi(n_2 + 2n_4 - i)}{\Phi(n_2 + 2n_4)}.$$

Let  $m = n_2 + 2n_4$ ,  $s = i$  in (73). Then,

$$\begin{aligned} \mathbf{EC}(i, a, b) &= \Theta\left(\sqrt{\frac{n_2 + 2n_4}{n_2 + 2n_4 - i}}\right) \frac{1}{i} \binom{i}{b} 6^b n_4^b \frac{(n_2)_a}{(n_2 + 2n_4)_i} \\ &= \Theta\left(\sqrt{\frac{n_2 + 2n_4}{n_2 + 2n_4 - i}}\right) \frac{1}{i} \binom{i}{b} 6^b \left(\frac{n_4}{n_2 + 2n_4}\right)^b e^{O\left(\frac{(a+b)b}{n_2 + 2n_4}\right)} \\ &= \Theta\left(\sqrt{\frac{n_2 + 2n_4}{n_2 + 2n_4 - i}}\right) \frac{O(1)}{i} \left(\frac{ie^{1+O(a/n_2)} 6n_4}{b n_2}\right)^b. \end{aligned}$$

Thus for 2-cycles (case  $b = 0$ ) we have  $\mathbf{EC}(i, i, 0) = O(1/i)$ . If  $b > 0$  then for some  $\beta < 1/7e$ ,

$$\sum_{i < \beta n_2/n_4} C(i, a, b) = o(1),$$

and thus w.h.p. all (2, 4)-cycles are size at least  $\Theta(n_2/n_4)$ . The expected number of all (2, 4)-cycles is

$$\sum_{a \leq n_2} \sum_{b \leq n_4} \sum_{i \leq n_2 + n_4} \mathbf{EC}(i, a, b) = O\left(n_2^{3/2} n_4^{1/2} (cn_4)^{n_4}\right).$$

Thus w.h.p. the total number  $L(t_1)$  of such cycles of all sizes is at most  $L(t_1) = O(n_2^{3/2} (n_4)^{n_4+1} \log n)$ .

Let  $\mathcal{E}(t_b)$  be the event that

$$\mathcal{E}(t_B) = \{\text{After } t_B + T_G \text{ further blue steps, there exists an unvisited (2, 4)-cycle.}\}$$

Let  $t_B$  be given by

$$t_B = \frac{n_4}{n_2} K n \log n \log \log n \leq n^{2/3}.$$

Using (71), conditional on  $n_4 \leq \omega \log n$  and  $\omega \leq \log \log n$ , for some  $\alpha > 0$  constant we have

$$\Pr(\mathcal{E}(t_B)) \leq \Theta(n_2^2 n_4^{n_4+1}) e^{-\alpha \log n \log \log n} = o(1). \quad (74)$$

**Part 3.** Let  $t_2 = dn(1 - \delta_2)$  be the red time reached by the edge-process after the further  $t_B$  blue steps made in Part 2 of the proof. The precise value of  $\delta_2$  is unknown, but the vacant net  $\widehat{\Gamma}(t_2)$  consists only of 2-cycles. The existence of a vertex of red degree 4 contradicts  $\Pr(\mathcal{E}(t_B)) = o(1)$  in (74). Thus  $\widehat{\Gamma}(t_2)$  is a random 2-regular graph. As  $\widehat{\Gamma}(t_1)$  has  $n_2 + n_4 = n_2(1 + o(1))$  vertices of positive red degree, and  $\widehat{\Gamma}(t_2)$  is a subgraph of  $\widehat{\Gamma}(t_1)$ , it also has at most this many vertices of red degree 2. By the result for  $\mathbf{EC}(i, i, 0) = O(1/i)$  in Part 2, in expectation,  $\widehat{\Gamma}(t_2)$  has  $\mathbf{EC}(i) = O(1/i)$  cycles of length  $i$ . Thus

$$\Pr(\text{There are more than } s^2 \mathbf{EC}(s) \text{ cycles size } s \text{ for any } s \geq \log n) \leq \sum_{s \geq \log n} \frac{1}{s^2} = O\left(\frac{1}{\log n}\right).$$

Condition on the number of cycles size  $s$  being at most  $s^2 \mathbf{EC}(s) = O(s)$ . Using (71), for some constant  $\alpha > 0$ , the probability  $P_s(t)$  that some cycle size  $s$  remains unvisited after  $t$  steps of the blue walk is

$$P_s(t) = O\left(se^{-t\alpha \frac{s}{n}}\right).$$

Let  $\mathcal{F}$  be the event that some red cycle of size at least  $\log n$  is unvisited after

$$t_B = \frac{K}{\alpha} \frac{n}{\log n} \log \log n$$

further blue steps. Thus for  $K \geq 3$ ,

$$\begin{aligned} \Pr(\mathcal{F}) &\leq \sum_{s \geq \log n} P_s(t_B) \\ &\leq O(1/\log n) + \sum_{s \geq \log n} s \exp\left(-\frac{Ks \log \log n}{\log n}\right) \\ &= O(1/\log n). \end{aligned}$$

□

## 7 Acknowledgement

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## 8 Appendix

### 8.1 Experimental results for the unvisited edge process

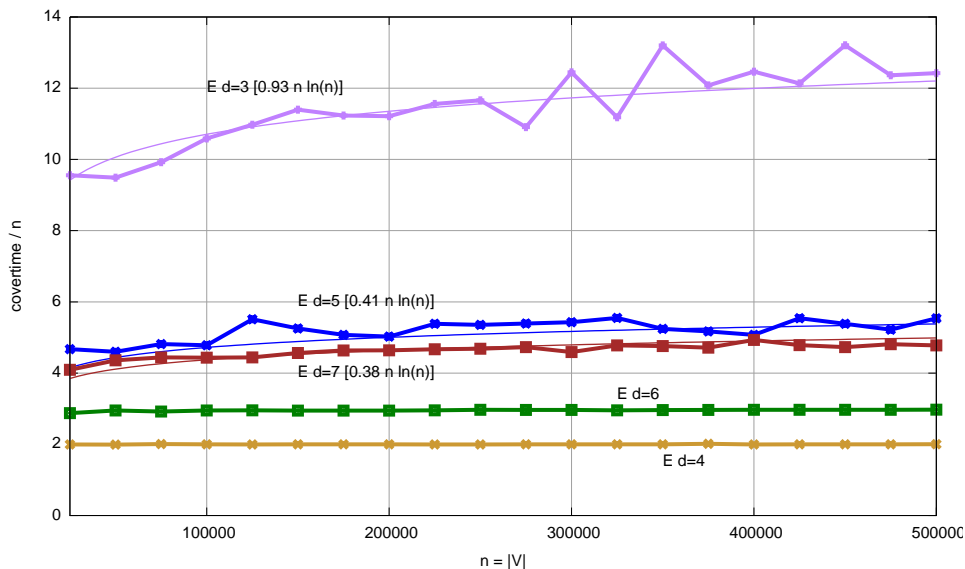


Figure 1: Vertex cover time of the unvisited edge process on  $d$ -regular graphs as function of  $n = |V|$ . All cover times are normalized by dividing by the vertex set size  $n$ . The plot shows  $d = 3, 4, 5, 6, 7$ . See [3] for details.

### 8.2 Estimates of $R_v$ for nice vertices

Recall the definition of  $\mathcal{N}$ , the set of nice vertices of  $G$  as given in Section 2. For a nice vertex  $v$ , the following lemma relates the value of  $R_v$  as given in (16) to the probability of a first return to  $v$  in the graph obtained by extending the subgraph  $H$  of depth  $\ell_1$  around  $v$  to an infinite  $r$ -regular tree  $\mathcal{T}$  rooted at  $v$ . Note that, we do not require the root  $v$  of  $\mathcal{T}$  to have degree  $r$ .

**Lemma 22.** *Let  $v$  be a vertex of degree  $d(v) \geq 1$  whose subgraph  $H$  to distance  $\ell_1$  in a graph  $G$  induces of a tree in which all vertices except  $v$  have degree  $r$ . Then*

$$R_v = (1 + o(1)) \frac{1}{1 - f} \quad \text{where } f = \frac{1}{r - 1}, \quad (75)$$

where  $f$  is the probability of a first return to  $v$  in  $\mathcal{T}$ , the extension of  $H$  to an infinite  $r$ -regular tree. The  $o(1)$  term in (75) is  $o(\log^{-K} n)$  for any positive constant  $K$ .

**Proof** Let  $H$  denote the subgraph of  $G$  induced by the set of vertices at distance at most  $\ell_1$  from  $v$ . This is a tree and we can embed it into an infinite  $r$ -regular tree  $\mathcal{T}$  rooted at  $v$ . Let  $W_v$  be the walk on  $G$  starting from  $v$ , and let  $\mathcal{X}$  be the walk on  $\mathcal{T}$ , starting from  $v$ .

Let  $X_0 = 0$ , and let  $X_t$  be the distance of  $\mathcal{X}$  from the root vertex  $v$  at step  $t$ . Let  $D_0 = 0$ , and let  $D_t$  be the distance from  $v$  of  $W$  in  $G$  at step  $t$ . Note that we can couple  $W_v, \mathcal{X}$  so that  $D_t = X_t$  up until the first time that  $D_t > \ell_1$ .

The values of  $X_t$  are as follows:  $X_0 = 0$ ,  $X_1 = 1$ , and if  $X_t = 0$  then  $X_{t+1} = 1$ . If  $X_t > 0$  then

$$X_t = \begin{cases} X_{t-1} - 1 & \text{with probability } q = \frac{1}{r} \\ X_{t-1} + 1 & \text{with probability } p = \frac{r-1}{r}. \end{cases} \quad (76)$$

The following result (see e.g. [14]) is for a random walk on the line  $= \{0, \dots, a\}$  with absorbing states  $\{0, a\}$ , and transition probabilities  $q, p$  for moves left and right respectively. Starting at vertex  $z$ , the probability of absorption at the origin 0 is

$$\rho(z, a) = \frac{(q/p)^z - (q/p)^a}{1 - (q/p)^a} \leq \left(\frac{q}{p}\right)^z, \quad (77)$$

provided  $q \leq p$ .

Let  $U_\infty = \{\exists t \geq 1 : X_t = 0\}$ , i.e. the event that the particle ever returns to the root vertex in  $\mathcal{T}$ . It follows from (77) with  $z = 1$  and  $a = \infty$  that

$$f = \Pr(U_\infty) = \frac{1}{r-1}. \quad (78)$$

It follows that the expected number of visits by  $\mathcal{X}$  to  $v$  is

$$\frac{1}{1-f}.$$

We write

$$R_v = \sum_{t=0}^T r_t \text{ and } \rho = \sum_{t=0}^{\infty} \rho_t$$

where  $\rho_t = \Pr(X_t = v)$ . Now  $r_t = \rho_t$  for  $t \leq \ell_1$  and part (a) follows once we prove that

$$\sum_{t=\ell_1+1}^T r_t = o(1) \text{ and } \sum_{t=\ell_1+1}^{\infty} \rho_t = o(1). \quad (79)$$

The first equation of (79) follows from

$$\left| r_t - \frac{1}{n} \right| \leq \lambda_{\max}^t \quad (80)$$

where  $\lambda_{\max}$  is the second largest eigenvalue of the walk. This follows from (11).

The second equation of (79) is proved in Lemma 7 of [9] where it is shown that

$$\sum_{t=\ell_1+1}^{\infty} \rho_t \leq \sum_{2j=\ell_1+1}^{\infty} \binom{2j}{j} \frac{(r-1)^j}{r^{2j}} \leq \sum_{2j=\ell_1+1}^{\infty} \left( \frac{4(r-1)}{r^2} \right)^j. \quad (81)$$

Thus

$$R_v = \rho + O(T\lambda_{\max}^{\ell_1} + T/n + (8/9)^{\ell_1})$$

□

**Remark.** We can use the method of Lemma 22 to calculate  $R_u$  for a vertex  $u = \gamma(S)$  in a graph  $H$  obtained from  $G$  by contracting a finite set of vertices  $S$  to a single vertex  $u = \gamma(S)$ , either directly, or after subdividing sets of edges incident with these vertices. We assume that all vertices in  $S$  have a unique neighbour  $w$  in  $N(S)$ , and that  $w$  is tree-like to depth  $\ell = \ell_1$  in  $G - S$ . It follows that, in  $H$ ,

$$R_u = (1 + o(1)) \frac{1}{1 - f_u},$$

where  $f_u$  is the probability of first return to  $u$  in the graph  $\mathcal{T}(S)$  obtained by extending the  $r$ -regular trees rooted at vertices of  $N(S)$  to infinity, and then contracting  $S$  to  $u = \gamma(S)$ .

### 8.3 Mixing time of chain $\mathcal{M}$

**Lemma 23.** *For  $G \in G_r$ ,  $r \geq 3$  constant, w.h.p.  $T_{\mathcal{M}} = O(\log n)$ .*

*Proof.* Mihail [19] gives the following conductance based measure of convergence for a strongly aperiodic walk with transition matrix  $P$  on a  $d$ -regular digraph  $D = (V, A)$ . For vertices  $e, f \in V$ ,

$$|P_e^t(f) - \pi_f| \leq (1 - \alpha^2)^{t/2}. \quad (82)$$

Here,

$$\alpha = \frac{1}{2d} \min_{|B| \leq |V|/2} \frac{|C(B)|}{|B|},$$

and  $B \subseteq V$  and  $C(B) = \{a \in A : a = (e, f), e \in B, f \in \overline{B}\}$ . The proof in [19] assumes the walk is lazy (i.e. for our model the non-backtracking walk on the underlying graph is lazy).

In Lemma 24 (below) we prove there is an  $\epsilon > 0$  constant such that w.h.p.  $\alpha \geq \epsilon/4r$ . The result that  $T_{\mathcal{M}} = O(\log n)$  follows from using this in (82). □

**Lemma 24.** *For  $G \in G_r$ , there is an  $\epsilon > 0$  constant such that w.h.p.  $\alpha \geq \epsilon/4r$ .*



*Proof.* For our chain  $\mathcal{M}$ ,  $d = r - 1$ , and  $|V_M| = rn$  where  $V_M$  is the set of oriented arcs of the underlying graph  $G$ . Suppose that  $B \subseteq V_M$  is a set of vertices of  $\mathcal{M}$  (directed arcs of  $G$ ). Let  $R = \overline{B} = V_M - B$  denote the rest of the arcs. Thus  $|B| + |R| = rn$ . Assume that  $|R| \geq |B|$ . We need to estimate  $C(B)$ . For a vertex  $v \in V(G)$ , let  $d_R^+(v)$  etc. be the  $R$ -out-degree of  $v$  (i.e.  $d_R^+(v) = |\{\{v, w\} \in E(G) : (v, w) \in R\}|$ ). Next let

$$W_0 = \{w : d_B^+(w) = r - 1 \text{ and } d_B^-(w) = 1\},$$

$$W_{1,s} = \{w : d_B^+(w) = r, d_B^-(w) = s\} \text{ and } W_1 = \bigcup_{s=0}^r W_{1,s}.$$

If  $(v, w) \in B$  and  $w \notin W_0 \cup W_1$  there is always an edge  $(w, x)$ ,  $x \neq v$  such that  $(w, x) \in R$ . If  $e = (v, w) \in B$  and  $f = (w, x) \in R$ ,  $x \neq v$  then the transition from  $e$  to  $f$  is non-backtracking, and arc  $(e, f)$  contributes to  $C(B)$ . We can bound  $|C(B)|$  from below by

$$|C(B)| \geq \sum_{(v,w) \in B} (1 - 1_{w \in W_0 \cup W_1}).$$

Enumerating  $W_0 \cup W_1$  by in-degree gives

$$\begin{aligned} \sum_{(v,w) \in B} (1 - 1_{w \in W_0 \cup W_1}) &= |B| - |W_0| - \sum_{s=0}^r \sum_{w \in W_{1,s}} d_B^-(w) \\ &= |B| - |W_0| - \sum_{s=0}^r s |W_{1,s}|. \end{aligned} \tag{83}$$

Enumerating  $B$  by initial and terminal vertices gives

$$\sum_{s=0}^r (r + s) |W_{1,s}| + r |W_0| \leq 2|B|.$$

So,

$$\begin{aligned} \sum_{(v,w) \in B} (1 - 1_{w \in W_0 \cup W_1}) &\geq \frac{1}{2} \sum_{s=0}^r (r + s) |W_{1,s}| + \frac{r}{2} |W_0| - |W_0| - \sum_{s=0}^r s |W_{1,s}| \\ &= \sum_{s=0}^r \left( \frac{r}{2} - \frac{s}{2} \right) |W_{1,s}| + \left( \frac{r}{2} - 1 \right) |W_0|. \end{aligned}$$

**Case 1:**  $\exists 0 \leq s < r$  such that  $|W_{1,s}| \geq \epsilon |B|$  or  $|W_0| \geq \epsilon |B|$ .

In this case,

$$\sum_{(v,w) \in B} (1 - 1_{w \in W_0 \cup W_1}) \geq \frac{\epsilon}{2} |B|.$$

**Case 2:**  $|W_{1,s}| < \epsilon |B|$ ,  $\forall 0 \leq s < r$  and  $|W_0| < \epsilon |B|$  and  $|W_{1,r}| \leq r^{-1} \left( 1 - \frac{r^2}{2} \epsilon \right) |B|$ .

Going back to (83) we get

$$\sum_{(v,w) \in B} (1 - 1_{w \in W_0 \cup W_1}) \geq |B| \left( 1 - \epsilon - \frac{r(r-1)}{2} \epsilon - \left( 1 - \frac{r^2}{2} \epsilon \right) \right) = \frac{\epsilon}{2} |B|.$$

**Case 3:**  $|W_0| < \epsilon |B|$ ,  $|W_1| > r^{-1} \left( 1 - \frac{r^2}{2} \epsilon \right) |B|$  and  $|W_1| \leq \frac{3}{4}n$ .

Let  $e(W_1, \overline{W}_1)$  be the number of edges between  $W_1$  and  $\overline{W}_1$  in the underlying graph  $G$ , and let  $\Phi = \Phi(G)$  be the conductance of  $G$ . Thus

$$e(W_1, \overline{W}_1) \geq \min(|W_1|, |\overline{W}_1|) r \Phi.$$

If  $u \in W_1$ , and  $\{u, v\}$  is an edge of  $G$ , then by definition of  $W_1$ , the arc  $(u, v) \in B$ . Thus if  $v \in \overline{W}_1$ , and  $v \notin W_0$ , there is some  $z \in V$ ,  $z \neq u$  such that  $(v, z) \in R$ . Let  $A$  be the set of such good arcs  $(v, z)$ , then

$$|C(B)| \geq |A| \geq e(W_1, \overline{W}_1) - |W_0|.$$

If  $|W_1| \leq n/2$ ,

$$|A| \geq |W_1| r \Phi - |W_0| \geq ((1 - r^2 \epsilon / 2) \Phi - \epsilon) |B|.$$

If  $n/2 \leq |W_1| \leq 3n/4$ , and  $|B| \leq rn/2$ ,

$$|A| \geq |\overline{W}_1| r \Phi - |W_0| \geq \frac{n}{4} r \Phi - \epsilon |B| \geq (\Phi/2 - \epsilon) |B|.$$

In either case, for  $r^2 \epsilon < 1$ ,  $|C(B)| \geq |B|(\Phi/2 - \epsilon)$ .

**Case 4:**  $|W_1| > \frac{3}{4}n$ .

If  $|B| \leq rn/2$ , this is impossible since we have  $|B| \geq r|W_1| > \frac{3}{4}rn$ . □