Looking for vertex number one

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Abstract

Given an instance of the preferential attachment graph $G_n = ([n], E_n)$, we would like to find vertex 1, using only 'local' information about the graph; that is, by exploring the neighborhoods of small sets of vertices. Borgs et. al gave a $O(\log^4 n)$ algorithm, which is local in the sense that at each step, it needs only search the neighborhood of a set of vertices of size $O(\log^4 n)$. We give a faster algorithm to find vertex 1, which is local in the strongest sense of operating only on neighborhoods of sets of size 1.

1 Introduction

The Preferential Attachment Graph G_n was first discussed by Barabási and Albert [2] and then rigorously analysed by Bollobás, Riordan, Spencer and Tusnády [3]. It is perhaps the simplest model of a natural process that produces a graph with a power law degree sequence.

The Preferential Attachment Graph can be viewed as a sequence of random graphs $G_1, G_2, \ldots, G_n$ where G_{t+1} is obtained from G_t as follows: Given G_t, we add vertex $t+1$ and m random edges $\{e_i = (t+1, u_i) : 1 \leq i \leq m\}$ incident with vertex $t+1$. Here the constant m is a parameter of the model. The vertices u_i are not chosen uniformly from V_t, instead they are chosen proportional to their degrees. This tends to generate some very high degree vertices, compared with what one would expect in Erdős-Rényi models with the same edge-density. We refer to $u_1, u_2, \ldots, u_m$ as the *left* choices of vertex $t+1$. We also say that $t+1$ is a *right* neighbor of u_i for $i = 1, 2, \ldots, m$.

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We consider the problem of searching through the preferential attachment graph looking for vertex number 1, using only local information. This was considered by Borgs, Brautbar, Chayes, Khanna and Lucire [5] in the context of the Preferential Attachment Graph $G_n = (V_n, E_n)$. Here $V_n = [n] = \{1, 2, \ldots, n\}$. They present the following local algorithm that searches for vertex 1, in a graph which may be too large to hold in memory in its entirety.

1: Initialize a list $\mathcal{L}$ to contain an arbitrary node u in the graph.

2: **while** $\mathcal{L}$ does not contain node 1 **do**

3: Add a node of maximum degree in $N(\mathcal{L})$ to $\mathcal{L}$ **od**;

4: return L

Here for vertex set $\mathcal{L}$, we let $N(\mathcal{L}) = \{w \notin \mathcal{L} : \exists v \in \mathcal{L} \ s.t. \ \{v, w\} \in E_n\}$.

They show that w.h.p. the algorithm succeeds in reaching vertex 1 in $O(\log^4 n)$ steps. (We assume that an algorithm can recognize vertex 1 when it is reached.) In [5], they also show how a local algorithm to find vertex 1 can be used to give local algorithms for some other problems.

We should note that, as the maximum degree in G_n is $n^{1/2-o(1)}$ w.h.p., one cannot hope to have a polylog(n) time algorithm if we have to check the degrees of the neighbors as we progress. Thus the algorithm above operates on the assumption that we can find the highest-degree neighbor of a vertex in $O(1)$ time. This would be the case, for example, if the neighborhood of a vertex is stored as a linked-list which is sorted by degrees. In the same situation, we would also be able to find, say, the 10 highest-degree neighbors of a vertex in constant time. In this setting, each of steps 2-7 of the following *Degree Climbing Algorithm* takes constant time.

Algorithm **DCA**:
The algorithm generates a sequence of vertices $v_1, v_2, \ldots,$ until vertex 1 is reached.

1: Let v_1 be a vertex of degree at least $\log^{1+15/m} n$ chosen from (close to) the distribution π where $\pi_v = \frac{d_n(v)}{2mn}$. (See Remark 1.3 to see how to implement this step).

2: $i \leftarrow 1$.

3: **repeat**

4: Let $W = \{w_1, w_2, \ldots, w_{m/2}\}$ be the $m/2$ neighbors of v_i of largest degree, excluding v_{i-1}.

5: Choose v_{i+1} randomly from W.

6: $i \leftarrow i + 1$.

7: **until** $d(v_i) \geq \frac{n^{1/2}}{\log^{1/100} n}$.

8: Starting from v_T, where T is the value of t at this point, do a random walk on the vertices of degree at least $\frac{n^{1/2}}{\log^{1/20} n}$ until vertex 1 is reached.

Note that our DCA algorithm is a local algorithm in a strong sense: the algorithm only requires access to the current vertex and its neighborhood. (Unlike the algorithm from [5], it does not need access to the neighborhood of the entire set $v_1, \ldots v_t$ of vertices visited so far.) Our main result is the following:

Theorem 1.1. *Given $\varepsilon > 0$ there exists m_ε suct that if $m \geq m_\varepsilon$ then w.h.p. algorithm DCA finds vertex 1 in G_n in $O(\log^{1+\varepsilon} n)$ time.*

The bulk of our proof consists of showing that the execution of Steps 2–7 requires only time $O(\log n)$ w.h.p. This analysis requires a careful accounting of conditional probabilities. This is facilitated by the conditional model of the preferential attachment graph due to Bollobás and Riordan [4], which we recast in terms of sums of independent copies of the mean one exponential random variable.

We begin with Step 1 because our analysis relies on the algorithm having a good start, i.e. finding a vertex $v_1 \leq \frac{n}{\log^{2+10/m} n}$. Though this step is easily implemented as in Remark 1.3, it is in this step that we pay the biggest price in running time. It seems likely that our proof can be improved to make Step 1 unnecessary:

Conjecture 1.2. *Algorithm DCA finds vertex 1 in G_n in $O(\log n)$ time, w.h.p.*

Remark 1.3. *Step 1 can be implemented, starting at any vertex, via a random walk. It is known that w.h.p. the mixing time of such a walk is $O(\log n)$, see Mihail, Papadimitriou and Saberi [11]. Also, the stationary measure of vertices of degree at least $\log^{1+15/m} n$ is $\Omega\left(\frac{1}{\log^{1+24/m} n}\right)$, as will be shown in Remark A.5 below. So a random walk will w.h.p. find v_1 in $O(\log^{1+25/m} n)$ time.*

Furthermore, if the degree of v_1 is at least $\log^{1+15/m} n$ then we see in Remark A.5 that w.h.p.

$$v_1 \leq \frac{n}{\log^{2+15/m} n}. \tag{1}$$

We will assume this bound on v_1.

2 Outline of paper

Our proofs will use various threshold parameters. For convenience, we collect all of these here in table form, giving a brief (and imprecise) description of the role of these parameters in our proof.

Parameter	Definition	Role in proof
$\omega :=$	$\log\log n$	An arbitrarily chosen slowly growing fucntion.
$\lambda_0 :=$	$\dfrac{1}{\log^{20/m} n}$	A (usually valid) lower bound on important parameters (η_i).
$n_0 :=$	$\dfrac{\lambda_0^2 n}{\omega \log^2 n}$	When $i \le n_0$ and $\eta_i \ge \lambda_0$ the vertex degree $d_n(i) \sim \eta_i \left(\frac{n}{i}\right)^{1/2}$.
$n_1 :=$	$\dfrac{\lambda_0^3 n}{\log^2 n}$	Main loop never visits $v > n_1$.
$n_2 :=$	$\log^{1/100} n$	Main loop never visits $v \le n_2$.

In Section 3 we reformulate the construction of Bollobás and Riordan [4] in terms of sums of independent copies of the exponential random variable of mean one. This enables us to show that for $i \le n_0$ (see Lemma A.4), the degree $d_n(i)$ can w.h.p. be expressed asymptotically as $\eta_i \left(\frac{n}{i}\right)^{1/2}$. Here η_i is the sum of m independent exponential random variables, subject to some mild conditioning.

Section 5 is the heart of the paper. The aim is to show that if v_{t-1} is not too small, then the ratio v_t/v_{t-1} is bounded above by 9/10 in conditional expectation. We deduce from this that w.h.p. the main loop Steps 3–6 only takes $O(\log n)$ rounds. The idea is to determine a degree Δ such that most of v_{t-1}'s left neighbors have degree at least Δ, while only few of v_{t-1}'s right neighbors have degree at least Δ. In this way, v_t is likely to be significantly smaller than v_{t-1}.

There are many technicalities involved in dealing with the conditioning incurred by running DCA and this explains the length of the proof.

Once we find a vertex v_T of high enough degree, then we know that w.h.p. v_T is not very large and lies in a small connected subgraph of vertices of high degree that contains vertex one. Then a simple argument based on the worst-case covertime of a graph suffices to show that only $o(\log n)$ more steps are required.

3 Conditional Model

Bollobás and Riordan [4] gave an ingenious construction equivalent to the preferential attachment graph model. We choose $x_1, x_2, \dots, x_{2mn}$ independently and uniformly from $[0,1]$. We then let $\{\ell_i, r_i\} = \{x_{2i-1}, x_{2i}\}$ where $\ell_i < r_i$ for $i = 1, 2, \dots, mn$. We then sort the r_i in increasing order $R_1 < R_2 < \dots < R_{mn}$ and let $R_0 = 0$. We then let

$$W_j = R_{mj} \text{ and } w_j = W_j - W_{j-1} \text{ and } I_j = (W_{j-1}, W_j] \tag{2}$$

for $j = 1, 2, \dots, n$. Given this we can define G_n as follows: It has vertex set $V_n = [n]$ and an edge $\{x, y\}$, $x \le y$ for each pair ℓ_i, r_i, where $\ell_i \in I_x$ and $r_i \in I_y$.

We can generate the sequence $R_1, R_2, \ldots, R_{mn}$ by letting

$$R_i = \left(\frac{\Upsilon_i}{\Upsilon_{mn+1}}\right)^{1/2}. \tag{3}$$

where

$$\Upsilon_N = \xi_1 + \xi_2 + \cdots + \xi_N \text{ for } N \geq 1$$

and $\xi_1, \xi_2, \ldots, \xi_{mn+1}$ are independent exponential mean one random variables i.e. $\mathbf{Pr}(\xi_i \geq x) = e^{-x}$ for all i.
Explanation: The order statistics of N independent uniform $[0,1]$ random variables can be expressed as the ratios $\Upsilon_i/\Upsilon_{N+1}$ for $1 \leq i \leq N$. Then we note that $r_1^2, r_2^2, \ldots, r_{mn}^2$ are independent uniform $[0,1]$ random variables.

We will analyze the our algorithm under the assumption that the graph G_n exhibits certain typical properties. To this end, we define

$$\eta_i := \xi_{(i-1)m+1} + \xi_{(i-1)m+2} + \cdots + \xi_{im},$$

and let $\mathcal{E}$ denote the event that the following properties hold for G_n: For a proof that these hold w.h.p., see the appendix.

(P1) $W_i \sim \left(\frac{i}{n}\right)^{1/2}$ for $\omega \leq i \leq n$.

(P2) $w_i \sim \frac{\eta_i}{2m(in)^{1/2}}$ for $\omega \leq i \leq n$.

(P3) $\lambda_0 \leq \eta_i \leq 20m \log\log n$ for $i \in [\log^{10} n]$.

(P4) $\eta_i \leq \log n$ for $i \in [n]$.

(P5) If $\omega \leq i < j \leq n$ then $\mathbf{Pr}(\text{edge } ij \text{ exists}) \sim \frac{\eta_i}{2(ij)^{1/2}}$.

(P6) If $\omega \leq i < j \leq n$ then $\mathbf{Pr}(\text{edge } ij \text{ exists}) \leq \frac{\log n}{2(ij)^{1/2}}$. This holds regardless of the existence or not of other edges.

(P7) $\eta_i \geq \lambda_0$ and $i \leq n_0$ implies that $d_n(i) \sim \eta_i \left(\frac{n}{i}\right)^{1/2}$.

(P8) $\omega \leq i \leq \log^{10} n$ implies that $d_n(i) \sim \eta_i \left(\frac{n}{i}\right)^{1/2}$.

(P9) $i \leq n_0$ implies that $d_n(i) \leq (1 + o(1)) \max\{1, \eta_i\} \left(\frac{n}{i}\right)^{1/2}$.

(P10) $d_n(i) \geq \frac{n^{1/2}}{\log^{1/20} n}$ implies that $i \leq \log^{1/39} n$.

(P11) $B_i := \sum_{v=\omega}^{i} X_v = o\left(\frac{W_i}{\log n}\right)$ for all $i \geq \log^{10} n$, where $X_v := \frac{\lambda_0 \cdot 1_{\eta_v \leq \lambda_0}}{2m(vn)^{1/2}}$

Note that **(P2)** implies that B_i asymptotically bounds the total length $\sum_{v=\omega}^{i} w_v \times 1_{\eta_v \le \lambda_0}$ of the intervals $I_v, v \le i$ that have $\eta_v \le \lambda_0$. **(P11)** will ensure that w.h.p. the algorithm will only encounter vertices for which the bound $\eta_v \ge \lambda_0$ is valid. In particular, from **(P7)**, **(P9)**, $\eta_i \left(\frac{n}{i}\right)^{1/2}$ will always be a good approximation to the degree of vertex i in our analysis.

In Section 5.4, we prove that $\mathbf{Pr}(\neg\mathcal{E}) = o(1)$, allowing us to condition on $\mathcal{E}$ in the analysis of the algorithm.

4 A structural lemma

We need to deal with small cycles.

Lemma 4.1. *Let a cycle C be ρ_N-*small *if $|C| \le \rho_N = 1000\log_{10/9}\log N$.*

(a) Let $M = e^{20000(\log_{10/9}\log N)^2}$. Then w.h.p, no two ρ_N-small cycles C_1, C_2 where at least one of them $C_1 \subseteq [M, N]$ in G_N are within distance ρ_N of each other.

(b) W.h.p. the number of vertices within distance ρ_N of a ρ_N-small cycle is $N^{o(1)}$.

Proof. (a) We observe that the existence of two small cycles close together implies the existence of a path P of length at most $3\rho_N$ such that the two endpoints both have neighbors in P, other than their P-neighbors. The expected number of such paths can be bounded by

$$\sum_{t=4}^{3\rho_N} \sum_{i_1,i_2,\ldots,i_t} \prod_{i=1}^{t} \frac{\log N}{(i_{r-1}i_r)^{1/2}} \left(\sum_{1\le r,s\le t} \frac{\log N}{i_r^{1/2}} \times \frac{\log N}{i_s^{1/2}} \right) \le \frac{\log^2 N}{\max\{i_r, i_s\}^{1/2}} \sum_{t=4}^{3\rho_N} t^2 \left(\log N \sum_{i=1}^{N} \frac{1}{i} \right)^t \le \frac{10\rho_N^2}{M^{1/2}} (\log N)^{6\rho_N+2} = o(1).$$

Here $\frac{\log N}{(ij)^{1/2}}$ bounds the probability that the edge ij exists, regardless of other edges, see Lemma A.2(f).

(b) We observe that the number of vertices $x \ge M$ within ρ_N of a small cycle is bounded by the number of paths P with endpoint x of length at most $2\rho_N$ such that the other endpoint y has a neighbor in P, other than its P-neighbors. The expected number of such paths can be bounded by

$$\sum_{t=4}^{2\rho_N} \sum_{i_1,i_2,\ldots,i_t} \prod_{i=1}^{t} \frac{\log N}{(i_{r-1}i_r)^{1/2}} \sum_{1\le s\le t} \frac{\log N}{i_s^{1/2}} \le \frac{\log N}{i_s^{1/2}} \sum_{t=4}^{2\rho_N} t \left(\log N \sum_{i=1}^{N} \frac{1}{i} \right)^t \le 5\rho_N^2 (\log N)^{4\rho_N+1} \le N^{o(1)}.$$

Now use the Markov inequality. $\square$

5 Analysis of the main loop

The main loop consists of Steps 3–6. Let $P_s = (v_1, v_2, \ldots, v_s)$ for $s \geq 1$ and let P_T be the sequence of vertices followed by the algorithm. Let $\rho_i = v_i/v_{i-1}$. We will argue that if $v_{i-1} \notin \{v_1, v_2, \ldots, v_{i-2}\}$ and $v_{i-1} < n_1$ then

$$\mathbf{E}(\rho_i \mid P_{i-1}, \mathcal{E}) \leq \frac{9}{10}. \tag{4}$$

Now, roughly speaking, if $r = 2\log_{10/9} n$ and μ is the number of steps in the main loop, and there are no repetitions of vertices, then we would have

$$\mathbf{Pr}(\mu \geq r) \leq \mathbf{Pr}\left(\rho_1\rho_2\cdots\rho_r \geq \frac{1}{n}\right) \leq n\,\mathbf{E}(\rho_1\rho_2\cdots\rho_r) \leq \frac{1}{n}$$

and so w.h.p. the algorithm will complete the main loop within $2\log_{10/9} n$ steps. To make this precise, we will also need to ensure that the algorithm stays below n_1, and handle the possibility of cycles (which we will accomplish with Lemma 4.1).

We will use a method of deferred decisions, exposing various parameters of G_n as we proceed.

We do this by considering two separate entities which are exploring the graph G_n. Apart from *Algorithm* which is attempting to find vertex 1, we imagine there is *Analyzer*, who exposes bits of information.

As Algorithm progresses, we allow Analyzer access to more and more information. In particular, until Step 7 has been reached, whenever Algorithm chooses vertex v_t:

E1 Analyzer learns the index v_t and the interval I_{v_t}, as defind in (2).

E2 Analyzer learns the left-choices $\lambda(v_t, 1), \lambda(v_t, 2), \ldots, \lambda(v_t, m)$ and the corresponding left neighbors $u_1, u_2, \ldots, u_m$. Recall that, a priori, the $\lambda(v_t, i)$ are random points chosen uniformly from the interval $[0, \rho(v_t, k)]$ for some $\rho(v_t, k) \in I_{v_t}$ for $k = 1, 2, \ldots, m$ (see Section 3). The analyzer also learns the intervals I_{u_i} and the values η_{u_i} for $i = 1, 2, \ldots, m$.

E3 Analyzer learns the the list $u'_1, u'_2, \ldots, u'_s$ of all vertices u'_i which have the property that $\lambda(u'_i, k) \in I_{v_t}$ for some $1 \leq k \leq m$.

These are the right-neighbors of v_t. Analyzer also learns $I_{u'_k}$ and $\eta_{u'_k}$ for $k = 1, 2, \ldots, s$.

Note that at a typical point in the running of the algorithm, Algorithm and Analyzer have access to incomparable sets of information. For example, Analyzer knows the indices of vertices examined by Algorithm (knowledge which would make Algorithm's task trivial). On the other hand, Algorithm knows the precise degrees of neighbors of its current vertex (or at least, the precise list of the $m/2$ neighboring vertices of maximum degree).

Our proof will analyze the progress of the Algorithm from Analyzer's perspective. In particular, we will use the random variables exposed in E1-E3 to prove that Algorithm reaches Step 7 w.h.p. Limiting the information available to the Analyzer is useful in reducing the necesary conditioning. (I.e., we must now prove that Algorithm succeeds using only E1-E3, but also now need only condition on E1-E3.)

Note that our proofs below will sometimes include bounds on probabilities of events involving random variables not exposed in E1-E3 (for example, events concerning the degree of a vertex). In all cases, however, the probabilities of these events are bounded by the probabilities of events involving only the variables from E1-E3 using inequalities (such as those in Lemma A.4) which hold simultaneously for all vertices in the range of interest in the underlying graph, given $\mathcal{E}$. In particular, though Analyzer has not exposed random variables such as the degrees of vertices:

1. At each step of the analysis, he can compute bounds on some events involving random variables exposed in E1-E3 (as these are exposed only when they first appear in calculations, there is control over the effect of conditioning);
2. He knows that given $\mathcal{E}$, all of these events (simultaneously) give bounds on corresponding events he is interested in (perhaps involving other random variables, such as the degree, which we have not exposed). See (26) and (27) for an example of this.

Remark 5.1. *We emphasize that at any point in the running of DCA, it is not the case that the distribution of random variables which Analyzer has not yet learned have identical distributions througout the process. For example, having learned the left neighbors of all visited vertices so far conditions down slightly the lengths of intervals I_v he has not yet seen yet, for vertices v which are not a known left-neighbor of any vertex. Our proof will have to handle this kind of extra conditioning.*

5.1 Analyzing one step

We let $N_L(i)$ denote the left neighbors of i i.e. those $j < i$ that are neighbors of i. The distribution of all parameters not exposed so far is random, subject to conditioning on what we know (Remark 5.1) and the continued occurrence of $\mathcal{E}$.

We let

$$M_1 = e^{20000(\log_{10/9} \log n)^2}$$

and

$$\rho_t := \frac{v_t}{v_{t-1}}.$$

Our analysis of one step of the main loop consists of the following Lemma:

Lemma 5.2. *Suppose that*

$$M_1 \leq v_t \leq n_1.$$

Then,

$$\mathbf{E}(\rho_t \mid \; P_{t-1}, \mathcal{E}, \mathcal{L}) \leq \frac{9}{10},$$

where $\mathcal{L} = \{v_k \neq v_\ell : 1 \leq k < \ell \leq T\}$.

Before the proof, let us observe the following:

Observation 5.3. *It follows from* ***(P11)*** *that w.h.p. for $t = O(\log n)$, all of the left neighbors w of $v_t \geq \log^{10} n$ will satisfy $\eta_w \geq \lambda_0$. (The probability that there exists $t \leq T$ such that $\eta_w < \lambda_0$ is bounded by $O\left(mT \max_t \left(\frac{B_{v_t}}{W_{v_t}}\right)\right) = o(1)$).* $\square$

Proof of Lemma 5.2. To simplify the analysis, we will sometimes ignore conditioning on P_{t-1} in our computation of $\mathbf{E}(\rho_t \mid \mathcal{E}, \mathcal{L})$. When these calculations are completed, we will show how to correct them for the extra conditioning. For the rest of the proof of this lemma, we let $i = v_{t-1}$ and denote by j a candidate for v_t.

We will begin by proving

$$\mathbf{E}(\rho_t \mid \ P_{t-1}, \mathcal{E}, \mathcal{D}_t) \leq \frac{9}{10}, \tag{5}$$

where $\mathcal{D}_t$ is the event that

$$v_{t-1} \notin \{v_1, v_2, \ldots, v_{t-2}\} \cup N(v_{t-3}) \cup N_2(\{v_1, v_2, \ldots, v_{t-4}\}). \tag{6}$$

Here $N(S)$ is the set of neighbors and $N_2(S)$ is the set of vertices within distance two of $S \subseteq [n]$. So $\mathcal{D}_t$ is a more restrictive version of $\mathcal{L}$.

The calculation of the ratio ρ_t takes contributions from two cases: that where v_t is a left-neighbor of v_{t-1}, and that where v_t is a right-neighbor of v_{t-1}. Note that in the former case, since G_i has total degree $2mi$ and in addition $G_{i/2}$ has total degree im, each random choice j by vertex i has probability at least $1/2$ of satisfying $j \leq i/2$. This implies that

$$\mathbf{E}(\rho_t \mid v_t < v_{t-1}) \leq \frac{3}{4}. \tag{7}$$

Next we wish to check that $v_t < v_{t-1}$ is reasonably likely. (Later, we will bound $\mathbf{E}(\rho_t \mid v_t > v_{t-1})$.)

Let Λ denote the degree of a candidate $j \in N_L(i)$; we wish Λ to be large to bound $\mathbf{Pr}(v_t = j)$ from below. For $\lambda_0 \leq \gamma \leq 1 - 1/m$ we define

$$\Delta_i(\gamma) := m + \gamma m \left(\frac{n}{i}\right)^{1/2}.$$

This is a degree threshold. For a suitable parameter γ, we wish it to be known to Analyzer that there should be many left-neighbors but few right-neighbors which have degree $> \Delta_i(\gamma)$. Given **(P1)**, **(P2)**, we have

$$W_i \sim \left(\frac{i}{n}\right)^{1/2} \tag{8}$$

and for $j < i$ we have

$$w_j \sim \frac{\eta_j}{2m(jn)^{1/2}} \tag{9}$$

and so

$$\frac{w_j}{W_i} \sim \frac{\eta_i}{2m(ij)^{1/2}}. \tag{10}$$

Under the assumption that $\mathcal{D}_t$ holds, the left choices $\lambda(i,k), k = 1,2,\ldots,m$ will be uniformly chosen from $[0, W_i]$, except for a few intervals forbidden by $\mathcal{D}_t$. Lemma 6.1 shows that the amount S_i that is forbidden is $o(W_i)$ w.h.p.

Let us first consider the degrees of the neighbors j of i for which $j > i$. Let

$$D_i(\gamma) = \{i < j < n_1 : d_n(j) \geq \Delta_i(\gamma)\}$$

and

$$J^i_\kappa(\gamma) = \begin{cases} [i, \gamma^{-2} i] \cap D_i(\gamma) & \kappa = 0. \\ [\kappa^2\gamma^{-2} i, (\kappa+1)^2\gamma^{-2} i] \cap D_i(\gamma) & \kappa \geq 1. \end{cases}$$

As part of the proof we will verify that w.h.p. $v_t < n_1$ throughout the execution of DCA and so the upper bound $j < n_1$ in the definition of $D_i(\gamma)$ is justified.

We let

$$r^{i,\kappa}_\gamma = |J^i_\kappa(\gamma)| \text{ and } r^i_\gamma = \sum_{\kappa\geq 0} r^i_\gamma.$$

And we let

$$s^i_\gamma = \sum_{\kappa\geq 0}\sum_{j\in J^i_\kappa(\gamma)} j.$$

Lemma 5.4. *If $\gamma < 1$ then*

$$\mathbf{E}(r^i_\gamma) \leq \frac{2m}{\gamma} \quad \textit{and} \quad \mathbf{E}(s^i_\gamma) \leq \frac{2mi}{\gamma^3}. \tag{11}$$

And if ε is a small positive constant and $m \gg 1/\varepsilon$, we have

$$\mathbf{E}(r^i_{1-\varepsilon}) \leq 2\varepsilon m \quad \textit{and} \quad \mathbf{E}(s^i_{1-\varepsilon}) \leq 2\varepsilon mi. \tag{12}$$

Finally, for sufficiently large m and $\kappa > m$, say, we have

$$\mathbf{E}(r^{i,\kappa}_\gamma) \leq \tfrac{1}{\gamma} e^{-m\kappa/3}. \tag{13}$$

Proof. Here we can write

$$\mathbf{E}(r^{i,\kappa}_\gamma \mid \mathcal{E},\mathcal{L})$$

$$\leq (1+o(1)) \int_{\eta_i=0}^\infty \left(\frac{\eta_i^{m-1} e^{-\eta_i}}{(m-1)!} \cdot \sum_{j\in J^i_\kappa(\gamma)} \frac{mw_i}{W_j} \int_{\eta_j=(1-o(1))\gamma m(j/i)^{1/2}}^\infty \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \right) d\eta_i \tag{14}$$

$$\leq (1+o(1)) \int_{\eta_i=0}^\infty \left(\frac{\eta_i^{m-1} e^{-\eta_i}}{(m-1)!} \cdot \sum_{j\in J^i_\kappa(\gamma)} \frac{\eta_i}{2(ij)^{1/2}} \int_{\eta_j=(1-o(1))\gamma m(j/i)^{1/2}}^\infty \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \right) d\eta_i \tag{15}$$

$$= (1+o(1))\, \mathbf{E}(\eta_i) \sum_{j\in J^i_\kappa(\gamma)} \frac{1}{2(ij)^{1/2}} \int_{\eta_j=(1-o(1))\gamma m(j/i)^{1/2}}^\infty \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \tag{16}$$

Explanation of (15)**:** The factor $(1+o(1))$ accounts for the conditioning on $\mathcal{E}$, through $\mathbf{Pr}(\mathcal{A} \mid \mathcal{B}) \le \mathbf{Pr}(\mathcal{A})/\mathbf{Pr}(\mathcal{B})$ for events $\mathcal{A}, \mathcal{B}$. We "fix" η_i and sum over relevant j and "fix" η_j. We multiply by the densities of η_i, η_j and integrate. The factor $\frac{\eta_i}{2(ij)^{1/2}} \approx \frac{mw_i}{W_j}$ is asymptotically equal to the expected number of times j chooses i as a neighbor. We integrate over $\eta_j \ge \gamma m(j/i)^{1/2}$ to get $d_n(j) \ge \Delta_i(\gamma)$, given $\mathcal{E}_2$.

Thus

$$\mathbf{E}(r_\gamma^{i,\kappa} \mid \mathcal{E}, \mathcal{L}) \le (1+o(1)) \sum_{j \in J_\kappa^i(\gamma)} \frac{m}{2(ij)^{1/2}} I_j$$

where

$$I_j = \int_{\eta_j=(1-o(1))\gamma m(j/i)^{1/2}}^{\infty} \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \le 1.$$

Note that some conditioning has been omitted in the computation of $\mathbf{E}(\eta_i)$ as m. This will be accounted for later.

Thus, if m is large then

$$\mathbf{E}\left(r_0^i\right) \le (1+o(1)) \sum_{j \in J_0^i(\gamma)} \frac{m(1+o(1))}{2(ij)^{1/2}} \le (1+o(1)) m \frac{1-\gamma}{\gamma}. \tag{17}$$

Continuing,

$$\begin{aligned}
\mathbf{E}(r_\gamma^{i,1} \mid \mathcal{E}, \mathcal{L}) &\le \sum_{j \in J_1^i(\gamma)} \frac{m(1+o(1))}{2(ij)^{1/2}} \int_{\eta_j=(1-o(1))\gamma m(j/i)^{1/2}}^{\infty} \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \\
&\le \frac{m(1+o(1))}{2i^{1/2}} \int_{x=\gamma^{-2} i}^{2\gamma^{-2} i} \frac{1}{x^{1/2}} \int_{\eta_j=(1-o(1))\gamma m(x/i)^{1/2}}^{\infty} \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j dx \\
&= \frac{m(1+o(1))}{2i^{1/2}} \int_{y=0}^{2^{1/2}-1} \frac{2i^{1/2}}{\gamma} \int_{\eta_j=(1-o(1))m(1+y)}^{\infty} \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j dy \\
&\le \frac{m(1+o(1))}{2\gamma} \int_{y=0}^{\infty} (e^{-my^2/3} + 2e^{-m(1+y)/2}) dy, \qquad \text{from Lemma A.1(c),(d),} \\
&= \frac{3^{1/2} m(1+o(1))}{4(m-1)^{1/2} \pi^{1/2} \gamma} + \frac{2}{e^{m/2} \gamma}. \qquad (18) \\
\mathbf{E}(r_\gamma^{i,\kappa} \mid \mathcal{E}, \mathcal{L}) &\le \sum_{j \in J_\kappa^i(\gamma)} \frac{m(1+o(1))}{2(ij)^{1/2}} \int_{\eta_j=(1-o(1))\kappa m/\gamma}^{\infty} \frac{\eta_j^{m-1} e^{-\eta_j}}{(m-1)!} d\eta_j \\
&\le \sum_{j \in J_\kappa^i} \frac{m(1+o(1)) e^{-(\kappa-1)m/2}}{(ij)^{1/2}} \\
&\le \frac{2m(\kappa+1+o(1)) e^{-(\kappa-1)m/2}}{\gamma}, \qquad (19)
\end{aligned}$$

for $\kappa \ge 2$.

Equation (13) follows immediately from (19).

Furthermore, it follows from (17) – (19) that if ε is a small positive constant and if $m \gg 1/\varepsilon$ is sufficiently large and $\gamma = 1-\varepsilon$ then

$$\mathbf{E}(r^i_{1-\varepsilon}) \leq (1+o(1))\left(m\frac{1-\gamma}{\gamma}+\frac{m^{1/2}}{\gamma}+\sum_{\kappa\geq 2}\frac{2m(\kappa+1+o(1))e^{-(\kappa-1)m/2}}{\gamma}\right) \leq 2\varepsilon m. \tag{20}$$

$$\mathbf{E}\left(\frac{s^i_{1-\varepsilon}}{i}\right)$$
$$\leq (1+o(1))\left(m\frac{1-\gamma}{\gamma}\frac{1}{\gamma^2}+\frac{m^{1/2}}{\gamma}\frac{4}{\gamma^2}+\sum_{\kappa\geq 2}\frac{2m(\kappa+1+o(1))e^{-(\kappa-1)m/2}}{\gamma}\frac{(\kappa+1)^2}{\gamma^2}\right) \leq 2\varepsilon m. \tag{21}$$

The lemma follows from (19), (20) and (21). □

Remark 5.5. *In some estimates for $r^{i,\kappa}_\gamma$ we are concerned only with $\gamma \geq 1-\varepsilon$. In which case we can assume that $r^{i,\kappa}_\gamma = 0$ for $\kappa > \log\log n$. Indeed, if $T = O(\log n)$ then (13) implies that*

$$\mathbf{Pr}(\exists i \in \{v_1, v_2, \ldots, v_T\}, \kappa \geq \log\log n : r^{i,\kappa}_\gamma > 0) = O(\log n) \times \sum_{\kappa \geq \log\log n} (\log n)^{-m\kappa/3} = o(1). \tag{22}$$

We can therefore assume that

$$n_2 \leq v_{t-1} \leq n_1 \text{ implies that } v_t \leq 2v_{t-1}(\log\log n)^2. \tag{23}$$

The above argument relies on us resolving some conditioning problems. In particular it relies on Lemma 6.1. This will become clear through Remark 5.8 below. We continue with a bound on v_t/v_{t-1} that only depends on $\mathcal{E}$.

We now consider $j < i$ and let

$$\gamma^*_i = \max\left\{\gamma : |\{j \in N_L(i) : d_n(i) < \Delta_i(\gamma)\}| \leq m/2\right\}. \tag{24}$$

Remark 5.6. *For $\gamma \geq \gamma^*$, we have that $\frac{2}{m}\frac{1}{i}s^i_\gamma$ is an upper bound on the ratio $\rho_t = \frac{j}{i}$, conditioned on the event that $j > i$, since each right neighbor whose index is included in the sum s^i_γ has probability $\leq \frac{1}{m/2}$ of being chosen by the algorithm.*

We now derive an upper bound for $\mathbf{Pr}(\gamma^* \leq \theta)$ when $\theta \leq 1-1/m$. So we first consider

$$L_0(i,\theta) = \{j \leq i : d_n(j) \leq \Delta_i(\theta)\} \subseteq L_1(i,\theta) = \left\{j \leq i : \eta_j \leq (1+o(1))m\theta\left(\frac{j}{i}\right)^{1/2}\right\}, \quad \text{w.h.p.}$$

Note that if $\gamma^* \leq \theta$ then i has at least $m/2$ left neighbors in $L_0(i,\theta)$.

We need to estimate $w_0(i,\theta)=\sum_{j\in L_0(i,\theta} w_j$ and given $\mathcal{E}$ we can approximate this by

$$w_1(i,\theta)=\frac{(1+o(1))Z}{2n^{1/2}} \text{ where } Z=\sum_{j=1}^{i}\frac{\eta_j}{j^{1/2}}1_{\eta_j\leq(1+o(1))\theta m(j/i)^{1/2}}.$$

Note first that

$$\mathbf{E}(Z)\leq\sum_{j=1}^{i}\frac{1}{j^{1/2}}\int_{\eta=0}^{\theta m}\frac{\eta^m e^{-\eta}}{(m-1)!}d\eta\leq 2i^{1/2}m^2(\theta e^{1-\theta})^m.$$

We have used (44) to bound the integral.

Now Z is the sum of independent non-negative random variables, each bounded by m. It follows from Hoeffding's inequality [9] that

$$\mathbf{Pr}(Z\geq(1+\delta)\,\mathbf{E}(Z))\leq e^{-\delta^2\,\mathbf{E}(Z)/3m},$$

for $0\leq\delta\leq 1$.

It follows that if $i\geq M_1\geq\log^K n$ for any constant $K>0$ and $\theta\geq 1/\log n$ then

$$\mathbf{Pr}\left(\exists i:\ w_1(i,\theta)\geq\frac{3m^2i^{1/2}(\theta e^{1-\theta})^m}{n^{1/2}}\right)\leq n^{-2}.$$

But if $w_1(i,\theta)\leq\frac{3m^2i^{1/2}(\theta e^{1-\theta})^m}{n^{1/2}}$ then the number of left neigbors of i that are in $L_0(i,\theta)$ is bounded in distribution by the binomial $Bin(m,4m^2(\theta e^{1-\theta})^m)$. Here we have used Observation 5.3 and replaced $3+o(1)$ by 4.

Thus, we see that if $\theta\geq\theta_0=1/\log n$ then

$$\mathbf{Pr}(\gamma_i^*\leq\theta\mid P_{t-1},\mathcal{E},\mathcal{D}_t)\leq\binom{m}{m/2}(4m^2(\theta e^{1-\theta})^m)^{m/2}\leq(16m^2(\theta e^{1-\theta})^m)^{m/2}. \tag{25}$$

For $\theta<\theta_0$ we proceed as follows:

$$\begin{aligned}
\mathbf{Pr}(\gamma_i^*\leq\theta\mid\mathcal{E})&\leq\mathbf{Pr}(\exists j\in N_L(i):d_n(j)\leq\Delta_i(\theta)),\\
&\leq\frac{m(1+o(1))}{W_i}\sum_{j=1}^{i}w_j\,\mathbf{Pr}(d_n(j)\leq\Delta_i(\theta)\mid\mathcal{E},\mathcal{L}), &(26)\\
&\leq\frac{m(1+o(1))}{2mi^{1/2}}\sum_{j=1}^{i}\frac{1}{j^{1/2}}\,\mathbf{Pr}\left(\eta_j\leq(1+o(1))m\theta\left(\frac{j}{i}\right)^{1/2}\middle|\mathcal{E}\right), &(27)\\
&\leq\frac{1+o(1)}{2i^{1/2}}\sum_{j=1}^{i}\frac{1}{j^{1/2}}m(\theta e^{1-\theta})^m,\qquad(44), &(28)\\
&\leq(1+o(1))m(\theta e^{1-\theta})^m. &(29)
\end{aligned}$$

Because m is large, we will use the larger of the two estimates (25) and (29) and write

$$\mathbf{Pr}(\gamma_i^*\leq\theta\mid P_{t-1},\mathcal{E},\mathcal{D}_t)\leq 2m(\theta e^{1-\theta})^m. \tag{30}$$

We can now estimate $\mathbf{E}(\rho_t)$. Let $f(\theta) = \mathrm{d}P(\gamma^* \leq \theta)$ where γ^* is as in (24). Then,

$$
\begin{aligned}
\mathbf{E}(\rho_t) &= \int_{\theta=0}^{1-\varepsilon} \mathbf{E}(\rho_t \mid \gamma_i^* = \theta) f(\theta) d\theta + \int_{\theta=1-\varepsilon}^{1} \mathbf{E}(\rho_t \mid \gamma_i^* = \theta) f(\theta) d\theta \\
&\leq \int_{\theta=0}^{1-\varepsilon} \left(1 + \frac{4}{\theta^3}\right) f(\theta) d\theta + \int_{\theta=1-\varepsilon}^{1} \left(\frac{2\varepsilon m}{m/2} + \frac{m/2 - 2\varepsilon m}{m/2} \cdot \frac{3}{4}\right) f(\theta) d\theta \qquad (31)\\
&\leq \int_{\theta=0}^{1-\varepsilon} \left(1 + \frac{4}{\theta^3}\right) f(\theta) d\theta + \frac{7}{8} \\
&\leq \left[\left(1 + \frac{4}{\theta^3}\right) \mathbf{Pr}(\gamma_i^* \leq \theta)\right]_0^{1-\varepsilon} + \int_{\theta=0}^{1-\varepsilon} \frac{12}{\theta^4} \mathbf{Pr}(\gamma_i^* \leq \theta) d\theta + \frac{7}{8} \\
&\leq \frac{4}{(1-\varepsilon)^2} 2m((1-\varepsilon)e^{\varepsilon})^m + \int_{\theta=0}^{1-\varepsilon} \frac{12}{\theta^4} 2m((1-\varepsilon)e^{\varepsilon})^m d\theta + \frac{7}{8} \\
&\leq \frac{8}{9}.
\end{aligned}
$$

Explanation of (31)**:** When $\theta < 1 - \varepsilon$ we obtain the term $4/\theta^3$ from Remark 5.6 and (11). We add 1 to account for the possibility of choosing a left neighbor in this case. When $\theta \geq 1 - \varepsilon$ the term $2\varepsilon m/(m/2)$ comes from the same remark and from the first inequality (12), which bounds the probabilty that $v_{t+1} > i$. The 3/4 comes from (7).

Remark 5.7. *Note that the above calculations shows that*

$$\mathbf{Pr}(v_t > v_{t-1} \mid \mathcal{E}, \mathcal{D}_t, P_{t-1}) \leq 4\varepsilon + \mathbf{Pr}(\gamma^* \leq 1 - \varepsilon) \leq 5\varepsilon.$$

It remains to account for the conditioning on P_{t-1}, and then also account consider the contribution to the expectation from the event $\mathcal{L} \setminus \mathcal{D}_t$.

We begin with the conditioning on P_{t-1}. There are three ways in which knowledge of the path P_{t-1} can alter the distribution of random variables considered in our calculation.

Remark 5.8. *(i) We learn that some vertices are not included in the left choices of other vertices. In particular, when we construct the J_κ^i we learn that v_t is not a left neighbor of certain vertices. This affects* (14) *and* (26)*. This is dealt with in Lemma 6.1. The lemma shows that the conditioning only affect the $o(1)$ term. Also, vertex v not being chosen as the left neighbor of a v_i conditions the value η_v downwards. This affects equation* (28)*. Lemma 6.1 also shows that for any fixed v, $\mathbf{Pr}(v \text{ not chosen}) = 1 - o(1)$ and so the net effect on* (28) *is another $1 + o(1)$ factor.*

(ii) If $i = v_{t-1} \in N_L(v_{t-2})$ then η_i will be conditioned upwards. This affects (15) *and* (16) *and* (22)*. In particular the value of $\mathbf{E}(\eta_i)$ needs to be increased. Lemma 6.2 shows that it is only necessary to inflate this by at most $3m^2$. The effect of this is in the estimates for the $\mathbf{E}(r_\kappa^i)$. Because these estimates drop exponentially fast with m, multiplying them by $3m^2$ has a negligble effect. It will perhaps increase the needed size of m.*

(iii) If $j = v_{t-2} \in N_L(i)$ then then η_j will be conditioned upwards. Lemma 6.2 shows that it is only necessary to inflate the estimated value of $\mathbf{E}(\eta_j)$ by at most $3m^2$ too.

This completes the proof of (5). Now, considering the case $\mathcal{L} \setminus \mathcal{D}_t$, we deal with the possibility that DCA visits a vertex v_ℓ within distance two of a previously visited vertex v_k ($k < \ell - 2$). Let us call this a *clash* between k and ℓ.

Let a vertex v be *unusual* if it has a left neighbor w where $w \leq \frac{v}{\log^{10} n}$. We show in Lemma 6.3 that w.h.p. none of $v_1, v_2, \ldots, v_T$ are unusual.

Suppose first that there is a clash between k and ℓ where $\ell - k > \rho_n = 1000 \log_{10/9} \log n$ and that there are no clashes between k and $\ell - 1$. Then, by (5)

$$\mathbf{E}\left(\frac{v_\ell}{v_k}\right) \leq \frac{1}{\log^{1000} n}.$$

So,

$$\mathbf{Pr}\left(\frac{v_\ell}{v_k} \geq \frac{1}{\log^{50} n}\right) \leq \frac{1}{\log^{50} n}. \tag{32}$$

We can inflate the R.H.S. of (32) by $O(\log^2 n)$ to account for the number of possible choices for k, ℓ. We can therefore assume that $v_\ell / v_k \leq 1/\log^{50} n$.

This rules out $v_k = v_\ell$. If v_ℓ, v_k are neighbors then Remark 5.5 implies that w.h.p. v_k does not affect the choice of $v_{\ell+1}$. Suppose then that v_k, v_ℓ are neighbors of some vertex x. Then either $x \leq \frac{v_k}{\log^{25} n}$ and then v_k is unusual. Or, $x \geq v_\ell \log^{25} n$ and then Remark 5.5 implies that x does not affect the choice of $v_{\ell+1}$. So w.h.p. either there are no clashes of this type or (5) is unaffected by the clash. We call this the *long path property*.

Suppose now that $\ell - k \leq \rho_n$. $v_k = v_\ell$ is ruled out by our assumption that $\mathcal{L}$ occurs. Suppose next that v_k and v_ℓ are neighbors. Let us now consider the effect of v_k on the choice of $v_{\ell+1}$. If we compute $v_{\ell+1}$, ignoring v_k then with a minor adjustment to our calculations for (5) we get that $\mathbf{E}(v_{\ell+1}/v_\ell) \leq 0.89 + O(1/m)$. The 0.89 comes from not rounding up to 0.9 and the $O(1/m)$ accounts for excluding v_k. Remember here that we are conditioning on $\mathcal{L}$ and so we are conditioning on $v_{\ell+1} \neq v_k$. The same argument applies in the case where v_k, v_ℓ have a common neighbor x. We are conditioning on $v_{\ell+1} \neq x$. $\square$

5.2 The Main Loop

Let

$$X = \{j \leq T_1 : \ v_j \in \{v_1, v_2, \ldots, v_{j-1}\}\} = \{j_1 < j_2 < \cdots < j_s\}.$$

In the following imagine first that $N = n_1$, we need to change it later to $M_1 \log^{10} n$. We consider that the Main Loop is run in two Phases. We claim that w.h.p.

Q1 $j \in X$ implies that $j > \rho_N$.

Q2 $j, k \in X$ implies that $|j - k| > \rho_N$.

Q3 $j \notin X$ implies that $\mathbf{E}(v_{j+1} \mid P_j, \mathcal{E}) \leq (0.9)v_j$.

Q4 $v_j \leq N$ for $j \geq 0$.

Assume for the moment that $\mathcal{Q} = \{\mathbf{Q1}, \mathbf{Q2}, \mathbf{Q3}, \mathbf{Q4}\}$ are all true. We let

$$r(i) = \begin{cases} 0 & i < j_1 \\ |\{j \in X : j < i\}| & i \geq j_1 \end{cases} \tag{33}$$

and let $\mathcal{B}_i$ denote the event

$$\mathcal{B}_i = \left\{ \frac{v_j}{N} \leq \frac{(0.9)^{j-r(j)}(\log\log N)^{2(r(j)+1)}}{\omega_0}, 1 \leq j \leq i \right\},$$

where $\omega_0 \to \infty$. Here $\mathcal{B}_0$ occurs w.h.p. if we take $\omega_0 = n_1/v_1$ and v_1 satisfies (1).

Then $\mathcal{Q}$ and (23) imply that for $i > 0$ we have

$$\mathbf{E}\left(\frac{v_i}{N} \,\middle|\, \mathcal{B}_i \right) \leq \frac{(0.9)^{i-r(i)}(\log\log N)^{2(r(i)+1)}}{\omega_0}. \tag{34}$$

So, from the Markov inequality, we have

$$\mathbf{Pr}(\neg\mathcal{B}_{i+1} \mid \mathcal{B}_i) \leq \frac{(0.9)^{i-r(i)}(\log\log N)^{2(r(i)+1)}}{\omega_0}.$$

Now we have

$$\mathbf{Pr}(\neg\mathcal{B}_{i+1}) \leq \mathbf{Pr}(\neg\mathcal{B}_{i+1} \mid \mathcal{B}_i) + \mathbf{Pr}(\neg\mathcal{B}_i)$$

and so for any fixed $\tau > 0$ we have

$$\mathbf{Pr}(\neg\mathcal{B}_\tau) \leq \sum_{i=1}^{\tau} \frac{(0.9)^{i-r(i)}(\log\log N)^{2(r(i)+1)}}{\omega_0}.$$

But $\mathbf{Q1}, \mathbf{Q2}$ imply that $r(i) \leq i/\rho_N$. In which case $(\log\log N)^{r(i)/i} \leq \exp\left\{\frac{\log\log\log N}{\rho_N}\right\} \leq 1.01$ and so

$$\mathbf{Pr}(\neg\mathcal{B}_\tau) \leq \sum_{i=1}^{\tau} \frac{(0.95)^i}{\omega_0} = o(1). \tag{35}$$

It follows that after $O(\log n)$ steps DCA will w.h.p. reach a vertex $v \in [1, M_1]$. This ends the first phase and begins the second phase. We can then repeat this argument with $N = M_1 \log^{10} N$ to argue that w.h.p. DCA will reach $v_t \leq M_2 = e^{20000(\log\log N)^2} = e^{O((\log\log\log n)^2} \leq \log^{1/20} n$.

We now verify $\mathcal{Q}$. The reader should be aware that we have to verify it for two phases: Phase 1, where $v_t \geq M_1$ and Phase 2, where we maintain $v_t \leq M_1 \log^4 N$. In Phase 2 we can take $\omega_0 = \log^6 N$. Let the sequence of vertices visited in Phase 2 be $w_1, w_2, \ldots$. In the second Phase w_j replaces v_j in the definitions.

For **Q1** in Phase 1, we observe that in G_n there are w.h.p. $n^{1-o(1)}$ vertices with degree at least $\log^{1+15/m} n$ and hence the total degree D_0 of these vertices is $n^{1-o(1)}$. To see this observe that from **(P7)** we have that if $\nu_0 = n\eta_0^2/\log^{2+30/m} n$ then w.h.p.

$$D_0 \geq \log^{1+15/m} n \sum_{i=1}^{\nu_0} 1_{\eta_i \geq \eta_0}. \tag{36}$$

The sum in (36) has expectation $(1-o(1))\nu_0$ (see (45)) and Hoeffding's theorem implies that this sum is concentrated around its mean. This implies that $D_0 \geq n^{1-o(1)}$ w.h.p.

Now w.h.p. G_n has maximum degree $n^{1/2+o(1)}$ and applying Lemma 4.1(b) we see that w.h.p. the total degree of vertices that would violate **Q1** can be bounded by $n^{1/2+o(1)}$. It follows that **Q1** fails with probability at most $n^{-1/2+o(1)}$.

In Phase 2, we have less control over the start vertex w_1. We avoid this problem by scrapping the assumption of **Q1** for this phase. We only really needed **Q1** to avoid an early move to a vertex $v > n_1$. In Phase 2 we know that $w_1 \leq N/\log^{10} n$. Without **Q1** we replace (34) by

$$\mathbf{E}\left(\frac{w_i}{N}\,\middle|\,\mathcal{B}_i\right) \leq \frac{(0.9)^{i-r(i)}(\log\log N)^{2(r(i)+1)}}{\omega_0}.$$

The proof of (35) goes through unchanged.

Property **Q2** follows from Lemma 4.1(a). Property **Q3** follows from Lemma 5.2 and Property **Q4** will be proved in Section 7. It is important to realise that we are replacing n by N in the use of (23) in Phase 2. Our definition of N then implies that w.r.t. G_N, we start at a vertex with index less than $N/\log N$.

Using **(P3)** and **(P7)** we see that

$$\mathbf{Pr}\left(\exists i \in [\log^{1/4} n] : d_n(i) \notin \left(\frac{n}{i}\right)^{1/2}\left[\frac{1}{\log^{1/10} n}, 2\log\log n\right]\right) = o(1)$$

for large m.

In summary, it follows that w.h.p. DCA reaches Step 7 in $O(\log n)$ time. Also, at this time $v_T \leq \log^{1/49} n$. This follows from **(P3)** and **(P7)**. The random walk will w.h.p. take place on $[\log^{1/39} n]$. This follows from **(P7)** and Lemma **(P10)** and justifies using n_2 as a lower bound on vertices visited during the main loop. Vertex 1 will be in the same component as v_t in the subgraph of G_n induced by vertices of degree at least $\frac{n^{1/2}}{\log^{1/20} n}$. This is because there is a path from v_T to vertex 1 through vertices in $[v_T]$ only and furthermore it follows from Lemmas A.2(d) and A.4(a) that w.h.p. every vertex on this path has degree at least $\frac{n^{1/2}}{\log^{1/20} n}$. The expected time to visit all vertices of a graph with ν vertices is $O(\nu^3)$, see for example Aleliunas, Karp, Lipton, Lovász and Rackoff [1]. Consequently, vertex 1 will be reached in a furrther $o(\log n)$ steps w.h.p.

This completes the proof of Theorem 1.1. $\square$

6 Lemmas to address conditioning

Let

$$S_{T,i} = \sum_{t=1}^{T} w_{v_t} 1_{v_t \le i}.$$

We replace $\frac{w_i}{W_j}$ in (14) by $\theta \in \left[\frac{w_i}{W_j}, \frac{w_i}{W_j - S_{T,j}}\right]$ as an upper bound for the probability of an edge ij, when $i < j$. In (26) we treat the ratio $\frac{w_j}{W_i}$ in a similar manner.

We now prove that $S_{T,j} = o(W_j)$ for $j \ge \log^{1/10} n$, showing that this does not affect (14) and (26). We also show that

Lemma 6.1.

(a) $S_{T,j} = o(W_j)$ for $j \ge \log^{1/10} n$, w.h.p.

(b) For any fixed v, $\mathbf{Pr}(v$ is not a left choice of a $v_i \mid \mathcal{E}) = 1 - o(1)$.

Proof. (a) Assume first that $j \ge \log^6 n$. It follows from **(P1)**, **(P2)**, **(P4)** that w.h.p.

$$S_{T,j} \le (1 + o(1)) \left(\frac{T \log n}{2mn^{1/2}} + \frac{\omega^{1/2}}{n^{1/2}} \right) = O\left(\frac{\log^2 n}{n^{1/2}} \right). \tag{37}$$

$$W_j \ge (1 - o(1)) \left(\frac{j}{n} \right)^{1/2} = \Omega\left(\frac{\log^3 n}{n^{1/2}} \right). \tag{38}$$

This verifies (a) in this case. Now assume that $j \le \log^6 n$. We observe first that **(P3)** implies that w.h.p.

$$\max\left\{\eta_{v_t} :\ v_t \le j\right\} \le 20m \log\log n. \tag{39}$$

It follows from (39) that w.h.p.

$$S_{T,j} \le (20 + o(1)) m \log\log n \sum_{t=1}^{T} \frac{1}{2m(v_t n)^{1/2}}.$$

Let $\tau = \min\left\{t : v_t \le \log^{10} n\right\}$. We claim that w.h.p.

$$T - \tau \le 20 \log_{10/9} \log n. \tag{40}$$

Given (40) we have

$$S_{T,j} \le \frac{(20 + o(1)) mT \log\log n}{2mn^{1/2} \log^5 n} + \frac{20 \log_{10/9} \log n}{2mn^{1/2}} \ll W_j = (1 + o(1)) \left(\frac{j}{n} \right)^{1/2}$$

for $j \geq \log^{1/10} n$.

But (40) follows directly from the Markov inequality and the fact that $\mathbf{E}(v_{\tau+k}) \leq \left(\frac{9}{10}\right)^k \log^{10} n$.

(b)

$$\begin{aligned}\mathbf{Pr}(v \text{ is a left choice of a } v_i \mid \mathcal{E}) &\leq (1+o(1)) \sum_{i=1}^{T} \frac{w_v}{W_{v_i}} \\ &\leq (1+o(1)) \left(\sum_{i:v_i \geq \log^{10} n}^{T} \frac{\eta_v}{2m(vv_i)^{1/2}} + \sum_{i:v_i < \log^{10} n}^{T} \frac{\eta_v}{2m(vv_i)^{1/2}} \right). \qquad (41)\end{aligned}$$

We can deal with the first sum without using Lemma 5.2, thus avoiding a circular argument. It follows from **(P3)**, **(P4)** that w.h.p.

$$\sum_{i:v_i \geq \log^{10} n}^{T} \frac{\eta_v}{2m(vv_i)^{1/2}} \leq \frac{T \log n}{\log^5 n} = o(1).$$

We can now apply Lemma 5.2 up until $v_t \leq \log^{10} n$ and so we see that w.h.p. we reach $v_t \leq M_1$ in $O(\log n)$ steps.

Consider the next $\log^{1/100} n$ steps where $\log^{1/40} n \leq v_t \leq \log^{10} n$. It follows from **(P3)** that $\eta_{v_t} \leq 20m \log\log n$ for these vertices. Thus the contribution of these vertices to the second sum in (41) can w.h.p. be bounded by

$$\frac{\log^{1/100} n \times 20m \log\log n}{2m \log^{1/40} n} = o(1).$$

We have not used Lemma 5.2 to prove this and so we can apply it to the proof of the lemma. But since $(10/9)^{\log^{1/100} n} \gg \log^{10} n$ we see that w.h.p. the Main Loop of Algorithm DCA will complete before we have taken $\log^{1/100} n$ steps. $\square$

Lemma 6.2. *Let $i = v_t$ and $j = v_{t-1}$. Then*

$$\bar{E} = \mathbf{E}(\eta_i \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}) \leq 3m^2.$$

Proof. We first assume that $i < j$. In this case we have

$$\bar{E} \leq \mathbf{E}\left(\sum_{v \in N_L(j)} \eta_v \,\middle|\, \mathcal{E} \right) \leq (2+o(1))m^2.$$

Explanation: $N_L(j)$ is obtained through the random choice of m points in W_j. In addition, because of $\mathcal{D}_{t-1}$, we know that these points do not belong to any interval I_v where we have learnt anything about η_v.

We now assume that $i > j$. In this case we have

$$\bar{E} = \mathbf{E}(\eta_i \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}) \le 3m^2.$$

Let $f(\gamma) = dP(\gamma^* \le \gamma \mid \mathcal{E})$. We write

$$\bar{E} \le \int_{\gamma=0}^{1} f(\gamma) \sum_{v>j} \frac{\mathbf{E}(\eta_j \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E})}{2m(vn)^{1/2}} \int_{\eta_v=(1-o(1))\gamma m(v/j)^{1/2}}^{\infty} \frac{\eta_v^m e^{-\eta_v}}{(m-1)!} d\eta_v d\gamma. \tag{42}$$

Explanation: Having fixed $\gamma^* = \gamma$, we sum over $v > j$, the probability that $j \in N_L(v)$ and times the expected value of η_v for η_v such that $d_n(v) \ge \Delta_j(\gamma)$.

We split the evaluation of the RHS of (42) into several parts.

$$\begin{aligned}
E_1 &= \int_{\gamma=0}^{1-\varepsilon} f(\gamma) \sum_{v>j} \frac{\mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E})}{2m(vn)^{1/2}} \int_{\eta_v=(1-o(1))\gamma m(v/j)^{1/2}}^{\infty} \frac{\eta_v^m e^{-\eta_v}}{(m-1)!} d\eta_v, \\
&\le \mathbf{Pr}(\gamma^* \le 1-\varepsilon) \times \mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}), \\
&\le 2m((1-\varepsilon)e^{\varepsilon})^m \times \mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}), \qquad \text{see (30)}. \\
E_2 &= \int_{\gamma=1-\varepsilon}^{1} \phi(\gamma) \sum_{j<v\le(1+\varepsilon)^2 j} \frac{\mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E})}{2m(vn)^{1/2}} \int_{\eta_v=(1-o(1))\gamma m(v/j)^{1/2}}^{\infty} \frac{\eta_v^m e^{-\eta_v}}{(m-1)!} d\eta_v \\
&\le \mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}). \\
E_3 &= \int_{\gamma=1-\varepsilon}^{1} f(\gamma) \sum_{v>(1+\varepsilon)^2 j} \frac{\mathbf{E}(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E})}{2m(vn)^{1/2}} \int_{\eta_v=(1-o(1))\gamma m(v/j)^{1/2}}^{\infty} \frac{\eta_v^m e^{-\eta_v}}{(m-1)!} d\eta_v \\
&\le E(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}) \times m^2 e^{-m\varepsilon^2/4}, \qquad \text{see Lemma A.1(a)}.
\end{aligned}$$

Thus

$$\bar{E} = E_1 + E_2 + E_3 \le (1 + 2m^2 e^{-m\varepsilon^2/4}) E(\eta_{v_{t-1}} \mid P_{t-1}, \mathcal{D}_{t-1}, \mathcal{E}).$$

Applying the inequality of Remark 5.7 and induction on t, we see that

$$\bar{E} \le (1-5\varepsilon) \times (2+o(1))m^2 + 5\varepsilon \times 3(1 + 2m^2 e^{-m\varepsilon^2/4})m^2 \le 3m^2.$$

□

Recall that a vertex v is unusual if it has a left neighbor w where $w \le \frac{v}{\log^{10} n}$.

Lemma 6.3. *W.h.p., none of $v_1, v_2, \ldots, v_T$ are unusual.*

Proof. Let $i = v_t$ and let $i_0 = \frac{i}{\log^{10} n}$. Assuming $\mathcal{E}$ we find that $W_{i_0}/W_i \le \frac{1+o(1)}{\log^5 n}$. So the probability that i chooses a left neighbor $j \le i_0$ is $O\left(\frac{1}{\log^5 n}\right)$. Now inflate by $O(\log n)$ to account for all possible values of t. □

7 Bounding v_t from above

With $\gamma = 1 - \varepsilon$ we have from (13) that

$$\mathbf{Pr}\left(\frac{v_{i+1}}{v_i} \geq \frac{\kappa^2}{\gamma^2}\right) \leq e^{-m\kappa/5}.$$

We first consider Phase One of the Main Loop:
Beginning moves: So, with $\kappa = \gamma\omega_1$ for some $\omega_1 \to \infty$, we have

$$\mathbf{Pr}\left(\frac{v_{i+1}}{v_i} \geq \omega_1\right) \leq e^{-m\gamma\omega_1^{1/2}/3}.$$

And so

$$\mathbf{Pr}\left(\frac{v_{i+k_1}}{v_i} \geq \omega_1^{k_1}\right) \leq k_1 e^{-m\gamma\omega_1^{1/2}/3}.$$

Now choose k_1, ω_1 such that

$$\omega_1^{k_1} = (\log n)^{o(1)} \text{ or } k_1 \log \omega_1 = o(\log \log n).$$

On the other hand,

$$\mathbf{E}\left(\frac{v_{i+k_1}}{v_i}\right) \leq \left(\frac{9}{10}\right)^{k_1}$$

and this implies that

$$\mathbf{Pr}\left(\frac{v_{i+k_1}}{v_i} \geq \left(\frac{9}{10}\right)^{k_1/2}\right) \leq \left(\frac{9}{10}\right)^{k_1/2}.$$

Let now

$$k_1 = (\log \log n)^{1/2} \text{ and } \omega_1 = \log \log n.$$

If a *round* takes k_1 steps and *success* means reducing

$$\frac{v_{i+k_1}}{v_i} \leq \left(\frac{9}{10}\right)^{k_1/2} \text{ and } \max_{l=1,\ldots,k_1} \frac{v_i + l}{v_i} \leq \omega_1^{k_1}$$

then

$$\mathbf{Pr}(failure) \leq k_1 e^{-m\gamma\omega_1^{1/2}/3} + \left(\frac{9}{10}\right)^{k_1/2} \leq e^{-k_1/20}.$$

Now consider the first $\rho_1 = e^{k_1/40}$ rounds. Then, w.h.p. we have success in each round. So we can claim that w.h.p.

R1 $\dfrac{v_r}{v_1} \leq \left(\dfrac{9}{10}\right)^{\rho_1 k_1/2}$.

R2 $1 \leq i \leq \rho_1 k_1$ implies that $v_i \leq v_1 \omega_1^{k_1} < n_1$.

For **R2** we are assuming that (1) holds.

Subsequent moves: We can now repeat the argument but with k_2, ω_2 in place of k_1, ω_1. We now we need only insist that

$$\omega_2^{k_2} = o\left(\left(\frac{10}{9}\right)^{\rho_1 k_1/2}\right)$$

so that we cannot reach past v_1 in the first round.

So now we let

$$k_2 = e^{(\log\log n)^{1/2}/2} \text{ and } \omega_2 = \log n.$$

Note now that

$$k_2 e^{k_2/40} \gg \log n$$

and so we will not have to complete $e^{k_2/40}$ rounds to complete Phase One of the main loop of DCA.

For Phase 2 we replace n by $N = M_1 = e^{20000(\log_{10/9}\log n)^2}$. We use $k_1 = (\log\log N)^{1/2}$ and $\omega_1 = \log\log N$. We then let $k_2 = \frac{\log\log n}{\log\log\log n}$ and $\omega_2 = \log\log n$ so that $\omega_2^{k_2} = \log n$. The preceding analysis shows that w.h.p. we reduce v_t by $\log^{-c} n$ in each round, for some $c > 0$ and that $v_t \leq M_1 \log n$ throughout.

8 Concluding remarks

We have described an algorithm that finds a distinguished vertex using $O(1)$ space. It would be nice to improve the running time to $O(\log n)$. Unless we can find a vertex of degree $\log^{1+\varepsilon} n$ more quickly, this will involve dealing with vertices of high index, where the degrees are not so concentrated. It would also be nice not to have to assume that m is large.

It would be nice to extend the result to other more general models of web graphs e.g. Cooper and Frieze [6]. In this case, we would not be able to use the model described in Section 3.

As a final observation, the algorithm DCA could be used to find the vertex of largest degree. Leastwise, if we replace Step 8 by "Do the random walk for $\log n$ steps and output the vertex of largest degree encountered" then w.h.p. this will produce a vertex of highest degree. This is because $\log n$ will be enough time to visit all vertices $v \leq \log^{1/39} n$, wherein the maximum degree vertex lies.

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A Proof of Properties P1–P10.

We note that the probability density $\phi(x)$ of the sum η of m independent exponential mean one random variables is given by

$$\phi(x) := \frac{x^{m-1}e^{-x}}{(m-1)!}.$$

That is,

$$\mathbf{Pr}(a \leq \eta \leq b) = \int_a^b \phi(y)dy. \tag{43}$$

This is a standard result. It can be verified by induction on m. See for example Exercise 4.14.10 of Grimmett and Stirzaker [8]. Although we will frequently need to bound the probability (43), this integral cannot be evaluated exactly in general, and thus we will often use simple bounds on $\phi(\eta)$. For example, since $\phi(\eta)$ is maximized at $\eta = m-1$, taking $\phi(mx)$ $(x \le 1 - 1/m)$ as an upper bound on $\phi(y)$ for $y \in [0, mx]$ in (43) gives us:

$$\mathbf{Pr}(\eta \le mx) \le m(xe^{1-x})^m. \tag{44}$$

Note that the upper bound in (44) is exponentially small in m since $xe^{1-x} < 1$ for $x \neq 1$.

We will also use the trivial bound

$$\mathbf{Pr}(\eta \le x) \le x^{m-1}. \tag{45}$$

We will use the following estimates that can easily be verified from (43):

Lemma A.1.

(a) If $\eta = (1+\alpha)m$ for $0 < \alpha \le 1$ then

$$\frac{\eta^{m-1}e^{-\eta}}{(m-1)!} \le (1+\alpha)^m e^{-m\alpha} \le e^{-m\alpha^2/3}.$$

(b) If $\eta \ge 2m$ then

$$\frac{\eta^m e^{-\eta}}{(m-1)!} \le m\left(\frac{e^{1-\eta/m}}{m}\right)^m \le e^{-\eta/2}.$$

(c) It follows from (a) that

$$\int_{\eta=(1+\alpha)m}^{2m} \frac{\eta^{m-1}e^{-\eta}}{(m-1)!} d\eta \le me^{-m\alpha^2/3} \qquad 0 \le \alpha \le 1.$$

(d) It follows from (b) that

$$\int_{\eta=\beta m}^{\infty} \frac{\eta^{m-1}e^{-\eta}}{(m-1)!} d\eta \le \int_{\eta=\beta m}^{\infty} e^{-\eta/2} d\eta = 2e^{-\beta m/2}, \qquad 2 \le \beta.$$

(e) If Z is the sum of N independent exponential mean one random variables then

$$\mathbf{Pr}(|Z-N| \ge \alpha N) \le 2e^{-\alpha^2 N/3} \text{ for } 0 < \alpha < 1.$$

(f)

$$\mathbf{Pr}(Z \ge \beta N) \le \left(\frac{1}{\beta e^{\beta-1}}\right)^N \text{ for } \beta \ge 2.$$

$\square$

Lemma A.2. *Let L be a large constant and $\mathcal{E}_1$ be the event that*

$$\Upsilon_k \in k\left[1 \pm \frac{L\theta_k^{1/2}}{3k^{1/2}}\right] \quad \text{for} \quad \frac{k}{m} \in \{\omega, \omega+1, \ldots, n\} \ \text{ or } k = mn+1 \tag{46}$$

where Υ_k is as in (3) and

$$\theta_k = \begin{cases} k^{1/2} & k \le n^{2/5} \\ \frac{k^{3/2}\log n}{n^{1/2}} & n^{2/5} < k \le n_0 \\ \frac{n}{\omega^{3/2}\log^2 n} & n_0 < k. \end{cases}$$

Then we have:

- ***(a)*** $\mathbf{Pr}(\neg\mathcal{E}_1) = o(1)$.
- ***(b)*** *Let $\eta_i = \xi_{(i-1)m+1} + \xi_{(i-1)m+2} + \cdots + \xi_{im}$. If $\mathcal{E}_1$ occurs then*

  (1) $W_i \sim \left(\frac{i}{n}\right)^{1/2}$ *for $\omega \le i \le n$, and*

  (2) $w_i \sim \frac{\eta_i}{2m(in)^{1/2}}$ *for $\omega \le i \le n$.*
- ***(c)*** $\eta_i \le \log n$ *for $i \in [n]$ w.h.p.*
- ***(d)*** $\lambda_0 \le \eta_i \le 20m\log\log n$ *for $i \in [\log^{10} n]$ w.h.p.*
- ***(e)*** *If $\omega \le i < j \le n$ then $\mathbf{Pr}$(edge ij exists) $\sim \frac{\eta_i}{2(ij)^{1/2}}$.*
- ***(f)*** *If $\omega \le i < j \le n$ then $\mathbf{Pr}$(edge ij exists) $\le \frac{\log n}{2(ij)^{1/2}}$. This holds regardless of the existence or not of other edges.*

Proof. (a) Applying Lemma A.1(e) to (3) for $i \ge 1$ we see that

$$\begin{aligned}
&\mathbf{Pr}(\neg\mathcal{E}_1)\\
&\le 2\sum_{k=\omega}^{n} \exp\left\{-\frac{L^2\theta_k}{27}\right\} + 2\exp\left\{-\frac{L^2\theta_{mn+1}}{27}\right\}\\
&= 2\sum_{k=\omega}^{n^{2/5}} \exp\left\{-\frac{L^2k^{1/2}}{27}\right\} + 2\sum_{k=n^{2/5}+1}^{n_0} \exp\left\{-\frac{L^2k^{3/2}\log n}{27n^{1/2}}\right\} + 2\sum_{k=n_0+1}^{n+1} \exp\left\{-\frac{L^2 n}{27\omega^{3/2}\log^2 n}\right\}\\
&= o(1).
\end{aligned}$$

(b) For this we use

$$W_i = \left(\frac{\Upsilon_{mi}}{\Upsilon_{mn+1}}\right)^{1/2}.$$

Then,

$$W_i \notin \left(\frac{i}{n}\right)^{1/2}\left[1 \pm \frac{L\theta_i^{1/2}}{i^{1/2}}\right]$$

implies that either

$$\Upsilon_{mn+1} \notin (mn+1)\left[1 \pm \frac{L\theta_i^{1/2}}{3(mn+1)^{1/2}}\right] \text{ or } \Upsilon_{mi} \notin mi\left[1 \pm \frac{L\theta_i^{1/2}}{3i^{1/2}}\right].$$

These events are ruled out by the occurence of $\mathcal{E}_1$.

We now estimate the w_i's. We use $(1+x)^{1/2} \leq 1 + \frac{x}{2}$ for $0 \leq |x| \leq 1$. Then,

$$\begin{aligned}
w_i &= \left(\frac{\Upsilon_{mi}}{\Upsilon_{mn+1}}\right)^{1/2} - \left(\frac{\Upsilon_{m(i-1)}}{\Upsilon_{mn+1}}\right)^{1/2} \\
&= \left(\frac{\Upsilon_{m(i-1)}}{\Upsilon_{mn+1}}\right)^{1/2}\left(\left(1 + \frac{\eta_i}{\Upsilon_{m(i-1)}}\right)^{1/2} - 1\right) \\
&\leq \frac{\left(m(i-1)\left(1 + \frac{L\theta_i^{1/2}}{3m^{1/2}(i-1)^{1/2}}\right)\right)^{1/2}}{\left((mn+1)\left(1 - \frac{L\theta_i^{1/2}}{3(mn+1)^{1/2}}\right)\right)^{1/2}} \frac{\eta_i}{2m(i-1)\left(1 - \frac{L\theta_i^{1/2}}{3m^{1/2}(i-1)^{1/2}}\right)} \\
&\leq \frac{\eta_i}{2m(in)^{1/2}}\left(1 + \frac{2L\theta_i^{1/2}}{m^{1/2}i^{1/2}}\right).
\end{aligned}$$

A similar calculation gives

$$w_i \geq \frac{\eta_i}{2m(in)^{1/2}}\left(1 - \frac{2L\theta_i^{1/2}}{m^{1/2}i^{1/2}}\right).$$

(c) This follows from Lemma A.1(f).

(d) The upper bound follows from Lemma A.1(f). For the lower bound, we observe by (45) that the expected number of $i \leq \log^{10} n$ with $\eta_i \leq \lambda_0$ is at most $\log^{10} n \times \lambda_0^m = o(1)$.

(e) This follows from (b) and the fact that the edge exists with probability asymptotic to $1 - \left(1 - \frac{w_i}{W_j}\right)^m$.

(f) This follows from (e) and (c). Furthermore, this probability holds independent of the left choices of other vertices. □

At this point we have verified **(P1)**, **(P2)**, **(P3)**, **(P5)**, **(P6)**.

At this point we let $\mathcal{E}_2 = \mathcal{E}_1$ intersected with the high probability events described in Lemma A.2(c),(d).

We need to control the proliferation of vertices v with $\eta_v \leq \lambda_0$. For $i \in [n]$ we let

$$B_i = \sum_{v=\omega}^{i} X_v \text{ where } X_v = \frac{\eta_v \times 1_{\eta_v \leq \lambda_0}}{2m(vn)^{1/2}}.$$

Note that given $\mathcal{E}_2$, B_i is asymptotically equal to the total length $\sum_{v=\omega}^{i} w_v \times 1_{\eta_v \leq \lambda_0}$ of the intervals $I_v, v \leq i$ that have $\eta_v \leq \lambda_0$.

Lemma A.3. *W.h.p., $B_i = o\left(\frac{W_i}{\log n}\right)$ for all $i \geq \log^3 n$.*

Proof. It follows from (45) that

$$\mathbf{E}(B_i \mid \mathcal{E}_2) \leq (1 + o(1) \mathbf{E}(B_i) \leq \sum_{v=\omega}^{i} \frac{\lambda_0^m}{2m(vn)^{1/2}} \leq \frac{\lambda_0^m}{m} \left(\frac{i}{n}\right)^{1/2}.$$

Now B_i is the sum of independent bounded random variables X_v. Applying Hoeffding's theorem [9] we see that

$$\mathbf{Pr}\left(B_i \geq 2\frac{\lambda_0^m}{m}\left(\frac{i}{n}\right)^{1/2}\right) \leq \exp\left\{-\frac{\lambda_0^{2m} i}{m^2 n}\left(\sum_{v=\omega}^{i} \frac{\lambda_0^2}{4m^2 vn}\right)^{-1}\right\} \leq \exp\left\{-\frac{\lambda_0^{2m} i}{\lambda_0^2 \log n}\right\} = o(n^{-1}).$$

$\square$

This verifies **(P11)**.

At this point we let $\mathcal{E}_3$ be the intersection of $\mathcal{E}_2$ with the high probability events described in Lemma A.3.

Suppose that we fix the values for $W_1, W_2, \ldots, W_n$. Then the degree $d_n(i)$ of vertex i can be expressed

$$d_n(i) = m + \sum_{j=i}^{n} \sum_{k=1}^{m} \zeta_{j,k}$$

where the $\zeta_{j,k}$ are independent Bernouilli random variables such that

$$\mathbf{Pr}(\zeta_{j,k} = 1) \in \left[\frac{w_i}{W_j}, \frac{w_i}{W_{j-1}}\right].$$

So, putting

$$\bar{d}_n(i) = \mathbf{E}(d_n(i) \mid \mathcal{E}_3)$$

we have

$$mw_i \left(1 + \sum_{j=i}^{n} \frac{1}{W_j}\right) \leq \bar{d}_n(i) - m \leq mw_i \left(1 + \sum_{j=i-1}^{n} \frac{1}{W_j}\right).$$

Now assuming $\mathcal{E}_3$, we have for $i \geq \omega$,

$$\sum_{j=i}^{n} \frac{1}{W_j} \geq \sum_{j=i}^{n} \left(\frac{n}{j}\right)^{1/2} \left(1 - \frac{2L\theta_j^{1/2}}{j^{1/2}}\right). \tag{47}$$

But

$$\begin{aligned}\sum_{j=\omega}^{n} \frac{\theta_j^{1/2}}{j} &\leq \sum_{j=\omega}^{n^{2/5}} \frac{1}{j^{3/4}} + \sum_{j=n^{2/5}+1}^{n/\omega \log^2 n} \frac{\log^{1/2} n}{n^{1/4} j^{1/4}} + \sum_{j=n/\omega \log^2 n}^{n} \frac{n^{1/2}}{j\omega^{3/2} \log^2 n} \\ &\leq 4n^{1/10} + \frac{4n^{1/2}}{3\omega^{3/4} \log n} + \frac{3n^{1/2} \log\log n}{\omega^{3/2} \log^2 n} \\ &\leq \frac{2n^{1/2}}{\omega^{3/2} \log n}.\end{aligned} \tag{48}$$

It follows that

$$\begin{aligned}\bar{d}_n(i) &\geq m + mw_i n^{1/2} \left(1 + 2(n^{1/2} - (i+1)^{1/2}) - \frac{2Ln^{1/2}}{\omega^{3/2} \log n}\right) \\ &\geq m + \eta_i \left(\frac{n}{i}\right)^{1/2} \left(1 - \left(\frac{i}{n}\right)^{1/2} - \frac{2L}{\omega^{3/2} \log n} - \frac{2L\theta_i^{1/2}}{m^{1/2} i^{1/2}}\right).\end{aligned}$$

A similar calculation gives a similar upper bound for $\bar{d}_n(i)$ and this proves that

$$i \geq \omega \text{ implies that } \bar{d}_n(i) \in m + \eta_i \left(\frac{n}{i}\right)^{1/2} \left[1 - \left(\frac{i}{n}\right)^{1/2} \pm \frac{4L}{m\omega^{3/4}}\right]. \tag{49}$$

Now $d_n(i) - m$ is the sum of independent $0,1$ random variables. So, from Hoeffding [9], for $0 \leq \delta \leq 1$,

$$\mathbf{Pr}(|d_n(i) - \bar{d}_n(i)| \geq \delta(\bar{d}_n(i) - m) \mid \mathcal{E}_3) \leq 2\,\mathbf{Pr}(\mathcal{E}_3)^{-1} e^{-\delta^2(\bar{d}_n(i)-m)/3} \leq (2+o(1))e^{-\delta^2(\bar{d}_n(i)-m)/3}. \tag{50}$$

And furthermore, for $\beta > 0$,

$$\mathbf{Pr}(d_n(i) - \bar{d}_n(i) \geq \beta(\bar{d}_n(i) - m) \mid \mathcal{E}_3) \leq 2\left(\frac{e}{\beta}\right)^{\beta(\bar{d}_n(i)-m)}. \tag{51}$$

The next lemma summarises what we need to know about $d_n(i)$.

Lemma A.4. *With high probability:*

(a) $\eta_i \geq \lambda_0$ *and* $i \leq n_0$ *implies that* $d_n(i) \sim \eta_i \left(\frac{n}{i}\right)^{1/2}$.

(b) $\omega \leq i \leq \log^{10} n$ *implies that* $d_n(i) \sim \eta_i \left(\frac{n}{i}\right)^{1/2}$.

(c) $i \leq n_0$ *implies* $d_n(i) \leq \max\{1, \eta_i\} \left(\frac{n}{i}\right)^{1/2}$.

(d) $d_n(i) \geq \frac{n^{1/2}}{\log^{1/20} n}$ *implies* $i \leq \frac{n}{\log^{1/39} n}$.

(e) $n_0 \leq i \leq n$ *implies that* $d_n(i) \leq \log^6 n$.

Proof. Putting $\delta = \frac{L}{\omega^{1/4}}$ into (50) we get that for $i \leq n/\omega^2$,

$$\mathbf{Pr}\left(d_n(i) \notin \eta_i \left(\frac{n}{i}\right)^{1/2} \left[1 \pm \frac{5L}{m\omega^{3/4}}\right] \,\middle|\, \mathcal{E}_3, \eta_i\right) \leq \frac{2}{\mathbf{Pr}(\mathcal{E}_3)} \exp\left\{-\frac{L^2 \eta_i n^{1/2}}{i^{1/2}\omega^{3/2}}\right\}. \tag{52}$$

(a) Here we have

$$\exp\left\{-\frac{L^2 \eta_i n^{1/2}}{i^{1/2}\omega^{1/2}}\right\} \leq n^{-L^2}.$$

(b) This follows from (a) and Lemma A.2(d).

(c) We substitute $\max\{1, \eta_i\}$ for η_i into the probability bound for the upper limit on $d_n(i)$ in (52).

(d) We can use (c) and Lemma A.2(c)(d).

(e) We can assume that $\eta_i \leq \log n$ and then $d_n(i) - m$ is stochastically dominated by the binomial $Bin\left(mn, \frac{\log^4 n}{mn}\right)$ and the claim follows. To explain the domination, we see that there are fewer than mn times when a choice can add to $d_n(i)$. And in each case the probability of increasing $d_n(i)$ can be bounded by $\frac{\log n}{\Upsilon_{mn_0}} \leq \frac{\log^4 n}{mn}$, w.h.p.. $\square$

This confirms that **(P7)**, **(P8)**, **(P9)**, **(P10)** hold w.h.p.

Finally, we check the statements made in Remark 1.3.

Remark A.5. *We first check the claim about the steady state. Here w.h.p. all vertices* $i \leq i_0 = \frac{\lambda_0^2 n}{\log^{2+32/m}} < n_0$ *have degree at least* $\ell_0 = \log^{1+15/m} n$ *and so w.h.p. the steady state probability of having degree at least* ℓ_0 *is at least*

$$\frac{1+o(1)}{2mn} \sum_{i=1}^{i_0} \frac{\lambda_0 n^{1/2}}{i^{1/2}} \sim \frac{\lambda_0 i_0^{1/2}}{mn^{1/2}} = \frac{1}{m \log^{1+24/m} n}. \tag{53}$$

This implies that we can find v_1 *in* $O(\log^{1+25/m} n)$ *time w.h.p.*

We now check the claim (1). Here we can assume that v_1 *is drawn with probability proportional to degree, but conditioned on having degree at least* ℓ_0*. Putting* $i_1 = \frac{n}{\log^{2+15/m} n}$ *we write*

$$\mathbf{Pr}(v_1 > i_1) \leq \mathbf{Pr}(\eta_{v_1} \geq 20m \log\log n) + \mathbf{Pr}(v_1 > i_1 \mid \eta_{v_1} < 20m \log\log n). \tag{54}$$

We first bound the contribution to $\mathbf{Pr}(\eta_{v_1} \geq 20m \log\log n)$ *from vertices* $i_1 \leq i \leq n_0$*. It follows from Lemma A.4(a) that this is at most*

$$\frac{1+o(1)}{2mn} \sum_{i=i_1}^{n_0} \eta_i \left(\frac{n}{i}\right)^{1/2} \times 1_{\eta_i \geq \log^{15/2m} n}.$$

This sum has expectation bounded by

$$\frac{1+o(1)}{2mn}\sum_{i=i_1}^{n_0}\left(\frac{n}{i}\right)^{1/2}\int_{\eta\geq\log^{15/2m} n}\frac{\eta^m e^{-\eta}}{(m-1)!}d\eta = O(\log^{-2} n).$$

The final estimate can be deduced for example from Lemma A.1(d), with $m-1$ rplaced by m. Comparing $O(\log^{-2} n)$ with the RHS of (53) *we see that $i_1 \leq i \leq n_0$ contribute $o(1)$ to $\mathbf{Pr}(\eta_{v_1} \geq 20m\log\log n)$.*

For $i \geq n_0$ we bound the contribution (see Lemma A.4(e)),

$$\frac{1}{2mn}\sum_{i=n_0}^{n}\log^6 n \times 1_{\eta_i\geq 20m\log\log n}.$$

This also has expectation $O(\log^{-2} n)$ and we have shown that $\mathbf{Pr}(\eta_{v_1} \geq 20m\log\log n) = o(1)$. We now turn to the second term in the RHS of (54).

It follows from (49) *that if $i > i_1$ and $\eta_i \leq 20m\log\log n$ then*

$$\bar{d}_n(i) \leq 40m\log\log n\left(\frac{n}{i}\right)^{1/2} \leq \log^{1+8/m} n.$$

Applying (51) *with $\beta \geq \ell_0/(\log^{1+8/m} n - m) \geq \log^{6/m}$ we see that*

$$\mathbf{Pr}(\exists i > i_1 :\ \eta_i \leq 20m\log\log n \textit{ and } d_n(i) \geq \ell_0) \leq n\left(\frac{e}{\log^{6/m} n}\right)^{\ell_0} = o(1).$$

Plugging this into (54) *completes our proof.*