

TRAVELING IN RANDOMLY EMBEDDED RANDOM GRAPHS.

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ABSTRACT. We consider the problem of traveling among random points in Euclidean space, when only a random fraction of the pairs are joined by traversable connections. In particular, we show a threshold for a pair of points to be connected by a geodesic of length arbitrarily close to their Euclidean distance, and analyze the minimum length Traveling Salesperson Tour, extending the Beardwood-Halton-Hammersley theorem to this setting.

1. INTRODUCTION

The classical Beardwood-Halton-Hammersley theorem [2] (see also Steele [17]) concerns the minimum cost Traveling Salesperson Tour through n random points in Euclidean space. In particular, it guarantees the existence of an absolute (though still unknown) constant β_d such that if x_1, x_2, \dots , is a random sequence of points in the d -dimensional cube $[0, 1]^d$, the length $T(\mathcal{X}_{n,1})$ of a minimum tour through x_1, \dots, x_n satisfies

$$(1) \quad T(\mathcal{X}_{n,1}) \sim \beta_d n^{\frac{d-1}{d}} \text{ a.s.}$$

The present paper is concerned still with the problem of traveling among random points in Euclidean space. In our case, however, we suppose that only a (random) subset of the pairs of points are joined by traversable connections, independent of the geometry of the point set.

In particular, we study random embeddings of the Erdős-Rényi-Gilbert random graph $G_{n,p}$ into the d -dimensional cube $[0, 1]^d$. We let \mathcal{X}_n denote a random embedding of $[n] = \{1, \dots, n\}$ into $[0, 1]^d$, where each vertex $i \in [n]$ is mapped (independently) to a random point $X_i \in [0, 1]^d$, and we denote by $\mathcal{X}_{n,p}$ the random graph whose vertex set is \mathcal{X}_n and whose pairs of vertices are joined by edges each with independent probability p . Edges are weighted by the Euclidean distance between their points, and we are interested in the total edge-weight required to travel about the graph.

This model has received much less attention than the standard model of a random geometric graph, defined as the intersection graph of unit balls with random centers $X_i, i \in [n]$, see Penrose [13]. We are only aware of the papers by Mehrabian [10] and

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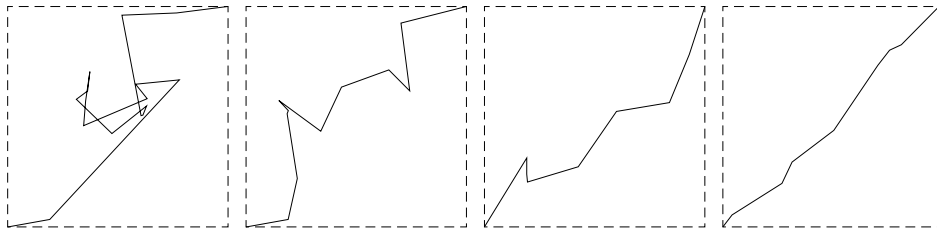


FIGURE 1. Paths in an instance of $\mathcal{X}_{n,p}$ for $d = 2$, $n = 2^{30}$, and $p = \frac{10}{n}$, $\frac{25}{n}$, $\frac{50}{n}$, and $\frac{200}{n}$, respectively. In each case, the path drawn is the shortest route between the vertices x and y which are closest to the SW and NE corners of the square. (See Q. 2, Section 5.)

Mehrabian and Wormald [11] who studied the *stretch factor* of $\mathcal{X}_{n,p}$. In particular, let $\|x-y\|$ denote the Euclidean distance between vertices x, y , and $\text{dist}(x, y)$ denote their distance in $\mathcal{X}_{n,p}$. They showed (considering the case $d = 2$) that unless p is close to 1, the stretch factor

$$\sup_{x,y \in \mathcal{X}_{n,p}} \frac{\text{dist}(x, y)}{\|x - y\|}$$

tends to ∞ with n .

As a counterpoint to this, our first result shows a very different phenomenon when we pay attention to additive rather than multiplicative errors. In particular, for $p \gg \frac{\log^d n}{n}$, the distance between a typical pair of vertices is arbitrarily close to their Euclidean distance, while for $p \ll \frac{\log^d n}{n}$, the distance between a typical pair of vertices in \mathcal{X}_n is arbitrarily large (Figure 1).

Theorem 1.1. *Let $\omega = \omega(n) \rightarrow \infty$. We have:*

(a) *For $p \leq \frac{1}{\omega^d (\log \log n)^{2d}} \frac{\log^d n}{n}$ and fixed u, v ,*

$$\text{dist}(u, v) \geq \frac{\omega}{8d\epsilon^d} \quad \text{a.a.s.}^1$$

(b) *For $p \geq \frac{\omega \log^d n}{n}$, we have a.a.s. that uniformly for all vertices u, v ,*

$$\text{dist}(u, v) = \|u - v\| + o(1).$$

Theorem 1.1 means that, even for p quite small, it is not that much more expensive to travel from one vertex of $\mathcal{X}_{n,p}$ to another than it is to travel directly between them in the plane. On the other hand, there is a dramatic dependence on p if the goal is to travel among *all* points. Let $T(\mathcal{X}_{n,p})$ denote the length of a minimum length tour in $\mathcal{X}_{n,p}$ hitting every vertex exactly once, i.e. a Traveling Salesperson tour.

¹A sequence of events \mathcal{E}_n occurs *asymptotically almost surely* (a.a.s.) if $\lim_{n \rightarrow \infty} \Pr(\neg \mathcal{E}_n) = 0$.

Theorem 1.2. *There exists a sufficiently large constant $K > 0$ such that for all $p = p(n)$ such that $p \geq \frac{K \log n}{n}$, $d \geq 2$, we have that*

$$(2) \quad T(\mathcal{X}_{n,p}) = \Theta \left(\frac{n^{\frac{d-1}{d}}}{p^{1/d}} \right) \quad a.a.s.$$

(Recall that $f(n) = \Theta(g(n))$ means that $f(n)$ is bounded between positive constant multiples of $g(n)$ for sufficiently large n .) As the threshold for $G_{n,p}$ to be Hamiltonian is at $p = \frac{\log n + \log \log n + \omega(n)}{n}$, this theorem covers nearly the entire range for p for which a TSP tour exists a.a.s.

Finally, we extend the asymptotically tight BHH theorem to the case of $\mathcal{X}_{n,p}$ for any constant p . To formulate an “almost surely” statement, we let $\mathcal{X}_{\mathbb{N},p}$ denote a random graph on a random embedding of \mathbb{N} into $[0, 1]^d$, where each pair $\{i, j\}$ is present as an edge with independent probability p , and consider $\mathcal{X}_{n,p}$ as the restriction of $\mathcal{X}_{\mathbb{N},p}$ to the first n vertices $\{1, \dots, n\}$.

Theorem 1.3. *If $d \geq 2$ and $p > 0$ is constant, then there exists $\beta_p^d > 0$ such that*

$$T(\mathcal{X}_{n,p}) \sim \beta_p^d n^{\frac{d-1}{d}} \quad a.s.$$

Karp’s algorithm [9] for a finding an approximate tour through \mathcal{X}_n extends to the case $\mathcal{X}_{n,p}$, p constant as well:

Theorem 1.4. *For fixed $d \geq 2$ and p constant, then there is an algorithm that a.s. finds a tour in $\mathcal{X}_{n,p}$ of value $(1 + o(1))\beta_p^d n^{(d-1)/d}$ in polynomial time, for all $n \in \mathbb{N}$.*

2. TRAVELING BETWEEN PAIRS

In this section, we prove Theorem 1.1. Let ν_d denote the volume of a d -dimensional unit ball; recall that ν_d is bounded ($\nu_d \leq \nu_5 < 6$ for all d).

Proof of Theorem 1.1(a). Let $\varepsilon = \frac{1}{\log \log n}$ and let \mathcal{A}_k be the event that there exists a path of length $k \geq k_0 = \frac{\log n}{2d \log \log n}$ from u to v that uses $\leq \varepsilon k$ edges of length at least $\ell_1 = \frac{\omega(\log \log n)^2}{4e^d \log n}$. Then

$$(3) \quad \begin{aligned} \Pr(\exists k : \mathcal{A}_k) &\leq \sum_{k \geq k_0} (k-1)! \binom{n}{k-1} p^k \binom{k}{(1-\varepsilon)k} \left(\nu_d \left(\frac{\omega(\log \log n)^2}{4e^d \log n} \right)^d \right)^{(1-\varepsilon)k} \\ &\leq \frac{1}{n} \sum_{k \geq k_0} \left(\frac{\nu_d \log^{d\varepsilon} n}{(4e^d)^{d(1-\varepsilon)}} \cdot \left(\frac{e}{\varepsilon} \right)^\varepsilon \right)^k = o(1). \end{aligned}$$

Explanation of (28): Choose the $k-1$ interior vertices of the possible path and order them in $(k-1)! \binom{n}{k-1}$ ways as $(u_1, u_2, \dots, u_{k-1})$. Then p^k is the probability that the edges exist in $G_{n,p}$. Now choose the short edges $e_i = (u_{i-1}, u_i)$, $i \in I$ in $\binom{k}{(1-\varepsilon)k}$

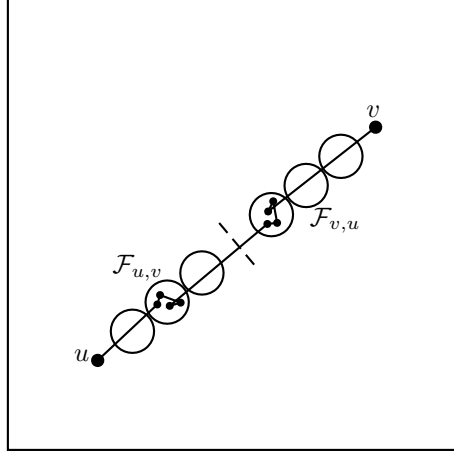


FIGURE 2. Finding a short path.

ways and bound the probability that these edges are short by $\left(\nu_d \left(\frac{\omega(\log \log n)^2}{4e^d \log n}\right)^d\right)^{(1-\varepsilon)k}$ viz. the probability that u_i is mapped to the ball of radius ℓ_1 , center u_{i-1} for $i \in I$.

Now a.a.s. the shortest path in $G_{n,p}$ from u to v requires at least k_0 edges: Indeed the expected number of paths of length at most k_0 from u to v can be bounded by

$$\sum_{k=1}^{k_0} (k-1)! \binom{n}{k-1} p^k \leq \frac{1}{n} \sum_{k=1}^{k_0} \left(\frac{\log^d n}{\omega^d (\log \log n)^{2d}} \right)^k = o(1).$$

So a.a.s.

$$\text{dist}(u, v) \geq \varepsilon k_0 \ell_1 = \frac{\varepsilon \log n}{2d \log \log n} \cdot \frac{\omega(\log \log n)^2}{4e^d \log n} = \frac{\omega}{8de^d}.$$

□

Proof of Theorem 1.1(b). Fix some small $\gamma > 0$. We begin by considering the case of vertices u, v at distance $\|u-v\| \geq \gamma$. Letting $\delta = \frac{1}{\log n}$, there is a constant C such that, for sufficiently large n relative to γ , we can find a set \mathcal{B} of $\geq \frac{2C}{\delta}$ disjoint balls of radius δ centered on the line from u to v , such that $\frac{C}{\delta}$ of the balls are closer to u than v , and $\frac{C}{\delta}$ balls are closer to v than u (Figure 2). Denote these two families of $\frac{C}{\delta}$ balls by $\mathcal{F}_{u,v}$ and $\mathcal{F}_{v,u}$.

Given a ball $B \in \mathcal{F}_{\{u,v\}} = \mathcal{F}_{u,v} \cup \mathcal{F}_{v,u}$, the induced subgraph G_B on vertices of \mathcal{X} lying in B is a copy of $G_{N,p}$, where $N = N(B)$ is the number of vertices lying in B . Let

$$\mathcal{S}_B \text{ be the event that } N(B) \notin \left[\frac{N_0}{2}, 2N_0 \right] \text{ where } N_0 = \nu_d \delta^d n.$$

The Chernoff bounds imply that for $B \in \mathcal{F}_{\{u,v\}}$,

$$(4) \quad \Pr(\neg \mathcal{S}_B) \leq e^{-\Omega(n\delta^d)} = e^{-n^{1-o(1)}}.$$

This gives us that a.a.s. \mathcal{S}_B holds for all pairs $u, v \in \mathcal{X}$ and all B :

- (A) All subgraphs G_B for $B \in \mathcal{F}_{\{u,v\}}$ have a giant component X_B , containing at least $N_0/3$ vertices.

Indeed, the expected average degree in G_B is $Np = \Omega(\omega) \rightarrow \infty$ and at this value the giant component is almost all of B a.a.s. In particular, since $\neg\mathcal{S}_B$ holds, that

$$(5) \quad \Pr(\exists B : |X_B| \leq N_0/3) \leq ne^{-\Omega(N_0)} \leq ne^{-\Omega(\delta^d n)} = o(1).$$

- (B) There is an edge between X_B and $X_{B'}$ for all $B, B' \in \mathcal{F}_{\{u,v\}}$.

Indeed, the probability that there is no edge between $X_B, X_{B'}$, given (A), is at most

$$(6) \quad (1-p)^{N_0^2/9} \leq e^{-\Omega(\delta^{2d} n^2 p)} \leq e^{-n^{1-o(1)}}.$$

This can be inflated by $n^2 \cdot (C \log n)^2$ to account for all pairs u, v and all pairs B, B' .

- (C) For each $B \in \mathcal{F}_{\{u,v\}}$, the graph diameter $\text{diam}(X_B)$ (the maximum number of edges in any shortest path in X_B) satisfies

$$(7) \quad \Pr\left(\text{diam}(X_B) > \frac{100 \log N}{\log Np}\right) \leq n^{-3}.$$

This can be inflated by $n^2 \cdot (2C \log n)$ to account for pairs u, v and the choice of $B \in \mathcal{F}_{\{u,v\}}$. Fernholz and Ramachandran [4] and Riordan and Wormald [16] gave tight estimates for the diameter of the giant component, but we need this cruder estimate with a lower probability of being exceeded. We will prove this later in Lemma 2.1.

Part (C) implies that with high probability, for any u, v at distance $\geq \gamma$ and all $B \in \mathcal{F}_{\{u,v\}}$ and vertices $x, y \in X_B$,

$$(8) \quad \text{dist}(x, y) \leq 100\delta \times \frac{\log N}{\log Np} \leq \frac{100 \log n - d(\log \omega + \log \log n) + O(1)}{\log \omega - O(1)} = o(1).$$

As the giant components X_B ($B \in \mathcal{F}_{u,v}$) contain in total at least $\frac{C}{\delta} \cdot \frac{N_0}{3} = \frac{C\nu_d n}{3\delta^{d-1}}$ vertices, the probability that u has no neighbor in these giant components is at most

$$(1-p)^{\frac{C\nu_d n}{3\delta^{d-1}}} \leq e^{-\frac{C\nu_d np}{3\delta^{d-1}}} = n^{-\omega C\nu_d/3}.$$

In particular, the probability is small after multiplication by n^2 , and thus a.a.s., for all pairs $u, v \in X_{n,p}$, u has a neighbor in X_B for some $B \in \mathcal{F}_{u,v}$ and v has a neighbor in $X_{B'}$ for some $B' \in \mathcal{F}_{v,u}$. Now by part (B) and equation (8), we can find a path

$$(9) \quad u, w_0, w_1, \dots, w_s, z_t, z_{t-1}, \dots, z_1, z_0, v$$

from u to v where the w_i 's are all in some X_B for $B \in \mathcal{F}_{u,v}$ and the total Euclidean length of the path w_0, \dots, w_s tends to zero with n , and the z_i 's are all in some $X_{B'}$ for some $B' \in \mathcal{F}_{v,u}$, and the total Euclidean length of the path z_0, \dots, z_t tends to zero with n . Meanwhile, the Euclidean segments corresponding to the three edges

u, w_0, w_s, z_t , and z_0, v lie within δ of disjoint segments of the line segment from u to v , and thus have total length $\leq \|u - v\| + 6\delta$, giving

$$(10) \quad \text{dist}(u, v) \leq \|u - v\| + 6\delta + o(1) = \|u - v\| + o(1).$$

We must also handle vertices u, v with $\|u - v\| < \gamma$. We have that

$$(11) \quad \Pr(\exists v, B : v \text{ is not adjacent to } B) \leq n^2(1-p)^{N_0p/3}$$

A fortiori, a.a.s. all vertices u, v are adjacent to some vertex in any ball of radius γ . In particular, we can find $w \sim u$ within distance $\frac{5}{2}\gamma$ of u , $z \sim v$ within distance $\frac{5}{2}\gamma$ of v , such that

$$\gamma \leq \|w - z\| \leq 5\gamma,$$

implying via (10) that

$$(12) \quad \text{dist}(u, v) \leq 6\gamma + 6\delta.$$

In particular, $\text{dist}(u, v) - \|u - v\|$ is bounded by a constant which can be made arbitrarily small by making n large. \square

We complete the proof of Theorem 1.1 by proving

Lemma 2.1. *Suppose that $Np = \omega \rightarrow \infty, \omega = O(\log N)$ and let K denote the unique giant component of size $N - o(N)$ in $G_{N,p}$, that q.s.² exists. Then for L large,*

$$\Pr\left(\text{diam}(K) \geq \frac{L \log N}{\log Np}\right) \leq O(N^{-L/20}).$$

Proof. Let $\mathcal{B}(k)$ be the event that there exists a set S of k vertices in $G_{N,p}$ that induces a connected subgraph and in which more than half of the vertices have less than $\omega/2$ neighbors outside S . Also, let $\mathcal{B}(k_1, k_2) = \bigcup_{k=k_1}^{k_2} \mathcal{B}_k$. Then for $k = o(N)$ we have

$$(13) \quad \Pr(\mathcal{B}(k)) \leq \binom{N}{k} p^{k-1} k^{k-2} 2^k \left(\sum_{i=0}^{\omega/2} \binom{N-k}{i} p^i (1-p)^{N-k-i} \right)^{k/2}$$

$$(14) \quad \leq p^{-1} (2e\omega e^{-\omega/3})^k \leq N e^{-k\omega/4}.$$

Explanation of 13: $\binom{N}{k}$ bounds the number of choices for S . We then choose a spanning tree T for S in k^{k-2} ways. We multiply by p^{k-1} , the probability that T exists. We then choose half the vertices X of S in at most 2^k ways and then multiply by the probability that each $x \in X$ has at most $\omega/2$ neighbors in $[N] \setminus S$.

If $\kappa = \kappa(L) = \frac{L \log N}{\log Np}$ then (14) implies that $\Pr(\mathcal{B}(\kappa)) \leq N^{1-L/10}$.

Next let $\mathcal{D}(k) = \mathcal{D}_N(k)$ be the event that there exists a set S of size k for which the number of edges $e(S)$ contained in S satisfies $e(S) \geq 2k$. Then,

$$\Pr(\mathcal{D}(k)) \leq \binom{N}{k} \binom{\binom{k}{2}}{2k} p^{2k} \leq \left(\frac{Ne}{k} \cdot \left(\frac{k\omega}{2N} \right)^2 \right)^k = \left(\frac{ke^3\omega^2}{2N} \right)^k.$$

²A sequence of events \mathcal{E}_n occurs *quite surely* q.s. if $\Pr(\neg\mathcal{E}_n) = O(n^{-\omega(1)})$.

Since $\omega = O(\log n)$ we have that q.s.

$$(15) \quad \exists k \in [\kappa(1), N^{3/4}] \text{ such that } \mathcal{D}(k) \text{ occurs.}$$

Suppose then that $\mathcal{B}(k_1, k_2) \cup \mathcal{D}(k_1, k_2)$ does not occur, where $k_1 = \kappa(L/4)$ and $k_2 = N^{3/4}$. Fix a pair of vertices v, w and first do a breadth first search (BFS) from $v \in K$ and create sets S_0, S_1, \dots, S_{k_1} where S_i is the set of vertices at distance i from v . We continue this construction unless we find that for some i , we have $w \in S_i$. Failing this, we must have $S_{k_1} \neq \emptyset$ and $|S_{\leq k_1}| \geq k_1$ where $S_{\leq t} = \bigcup_{i=0}^t S_i$ for $t \geq 0$. We continue this construction for $t \geq k_1$ and we see that $k_1 \leq |S_{\leq t}| \leq N^{2/3}$ implies that $|S_{t+1}| \geq \omega|S_t|/4$. This is because only vertices in S_t have neighbors outside $S_{\leq t}$ and we have assumed that $\mathcal{B}(|S_{\leq t}|)$ does not occur and because of (15). Thus if $|S_{t+1}| < \omega|S_t|/4$ then $S_{\leq t+1}$ has at most $\omega N^{2/3}/4$ vertices and more than $\omega N^{2/3}/2$ edges.

Thus if L is large, then we find that there exists $t \leq k_1 + \kappa(3/4)$ such that $|S_t| \geq N^{2/3}$. Now apply the same argument for BFS from w to create sets T_0, T_1, \dots, T_s , where either we reach v or find that $|T_s| \geq N^{2/3}$ where $s \leq k_1 + \kappa(3/4)$. At this point the edges between S_t and T_s are unconditioned and the probability there is no $S_t : T_s$ edge is at most $(1-p)^{N^{4/3}} = O(e^{-\Omega(N^{1/3})})$. \square

3. TRAVELING AMONG ALL VERTICES

Our first aim is to prove Theorem 1.3; this will be accomplished in Section 3.2, below. In fact, we will prove the following general statement, which will also be useful in the proof of Theorem 1.2:

Theorem 3.1. *Let $\mathcal{Y}_1^d \subset [0, 1]^d$ denote a set of points chosen from any fixed distribution, such that the cardinality $Y = |\mathcal{Y}_1^d|$ satisfies $\mathbf{E}(Y) = \mu > 0$ and $\Pr(Y \geq k) \leq C\rho^k$ for all k , for some $C > 0, \rho < 1$. Let \mathcal{Y}_t^d denote a random set of points in $[0, t]^d$ obtained from t^d independent copies $\mathcal{Y}_1^d + x$ ($x \in \{0, \dots, t-1\}^d$).*

If $p > 0$ is constant, $d \geq 2$, and $\mathcal{Y}_{t,p}^d$ denotes the random graph on \mathcal{Y}_t^d with independent edge probabilities p , then $\exists \beta > 0$ (depending on p and the process generating \mathcal{Y}_1^d) such that

- (i) $T(\mathcal{Y}_{t,p}^d) \sim \beta t^d$ a.s., and
- (ii) $T(\mathcal{Y}_{t,p}^d) \leq \beta t^d + o(t^d)$ q.s.³

The restriction $\Pr(|\mathcal{Y}_1^d| \geq k) \leq \rho^k$ simply ensures that we have exponential tail bounds on the number of points in a large number of independent copies of \mathcal{Y}_1^d :

Observation 3.2. *For the total number T_n of points in n independent copies of \mathcal{Y}_1^d , we have*

$$(16) \quad \Pr(|T_n - \mu n| > \delta \mu n) < e^{-A_\rho \delta^2 \mu^2 n}. \quad \square$$

³In this context $O(n^{-\omega(1)})$ is replaced by $O(t^{-\omega(1)})$.

This is a straightforward consequence, but we do not have a reference and so we give a sketch proof in the appendix.

Note that the conditions on the distribution of \mathcal{Y}_t^d are satisfied for a Poisson cloud of intensity 1, and it is via this case that we will derive Theorem 1.3. Other examples for which these conditions hold include the case where \mathcal{Y}_t^d is simply a suitable grid of points, or is a random subset of a suitable grid of points in $[0, t]^d$, and we will make use of this latter case of Theorem 3.1 in our proof of Theorem 1.2.

Our proof is by induction on d . For technical reasons (see also Question 4 of Section 5) Theorems 3.1 and 1.3 are given just for $d \geq 2$, and before beginning with the induction, we must carry out a separate argument to bound the length of the tour in 1 dimension.

3.1. Bounding the expected tour length in 1 dimension. We begin with the following simple lemma.

Lemma 3.3. *Let σ be a permutation of $[n]$, and let $\ell(\sigma)$ be $\sum_{i=1}^{n-1} |\sigma_{i+1} - \sigma_i|$. Then*

$$(17) \quad \ell(\sigma) < \sigma_n + 3 \cdot \text{inv}(\sigma),$$

where $\text{inv}(\sigma)$ is the number of inversions in σ .

Proof. We prove this by induction on n . It is trivially true for $n = 1$ since in this case $\ell(\sigma) = 0$. Assume now that $n > 1$, and given a permutation σ of $[n]$, consider permutation σ' of $[n-1]$ obtained by truncation:

$$\sigma'_i = \begin{cases} \sigma_i & \text{if } \sigma_i < \sigma_n \\ \sigma_i - 1 & \text{if } \sigma_i > \sigma_n \end{cases}$$

We have by induction that

$$(18) \quad \ell(\sigma') \leq \sigma'_{n-1} + 3 \cdot \text{inv}(\sigma').$$

Now observe that

$$\begin{aligned} \ell(\sigma) &= \ell(\sigma') + |\sigma_n - \sigma_{n-1}| + |\{i \mid \sigma_i < \sigma_n < \sigma_{i+1} \text{ OR } \sigma_i > \sigma_n > \sigma_{i+1}\}| \\ &\leq \ell(\sigma') + |\sigma_n - \sigma_{n-1}| + \text{inv}(\sigma) - \text{inv}(\sigma'), \end{aligned}$$

and, recalling that $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$,

$$\text{inv}(\sigma) - \text{inv}(\sigma') = n - \sigma_n.$$

Since $\sigma'_{n-1} \leq \sigma_{n-1}$, (18) gives that

$$\begin{aligned} \ell(\sigma) &\leq \sigma_{n-1} + 3 \cdot \text{inv}(\sigma') + |\sigma_n - \sigma_{n-1}| + \text{inv}(\sigma) - \text{inv}(\sigma') \\ &= \sigma_{n-1} + \text{inv}(\sigma') + 2(\text{inv}(\sigma) - n + \sigma_n) + |\sigma_n - \sigma_{n-1}| + \text{inv}(\sigma) - \text{inv}(\sigma') \\ &= \sigma_{n-1} + 3 \cdot \text{inv}(\sigma) - 2n + 2\sigma_n + |\sigma_n - \sigma_{n-1}| \\ &= \sigma_n + 3 \cdot \text{inv}(\sigma) - (2n - \sigma_{n-1} - \sigma_n - |\sigma_n - \sigma_{n-1}|) \\ &\leq \sigma_n + 3 \cdot \text{inv}(\sigma). \end{aligned} \quad \square$$

For the 1-dimension case of Theorem 1.3, we have, roughly speaking, a 1-dimensional string of points joined by some random edges. Lemma 3.3 allows us to prove the following lemma, which begins to approximate this situation.

Lemma 3.4. *Consider the random graph $G = G_{n,p}$ on the vertex set $[n]$ with constant p , where each edge $\{i, j\} \in E(G)$ is given length $|i - j| \in \mathbb{N}$. Let Z denote the minimum length of a Hamilton cycle in G starting at vertex 1, assuming one exists. If no such cycle exists let $Z = n^2$. Then there exists a constant A_p such that*

$$\mathbf{E}(Z) \leq A_p n \text{ and } Z \leq \frac{2A_p n}{p}, \text{ q.s.}$$

Proof. We first write $G = G_1 \cup G_2 \cup G_3$ where the G_i are independent copies of G_{n,p_1} , where $1 - p = (1 - p_1)^3$. We will first construct a long path in G_1 via the following algorithm: We start with $v_1 = 1$. Then for $j \geq 1$ we let

$$\phi(j) = \min_{k \in [n]} \{k : k \notin \{v_1, v_2, \dots, v_j\} \text{ and } v_j \sim k\}$$

and let $v_{j+1} = \phi(j)$ i.e. we move from v_j to the lowest indexed k that has not been previously visited. We repeat this until we reach j_0 such that $\phi(j_0)$ is undefined. This defines a path P_1 of length $\Lambda_1 = \sum_{j=1}^{j_0-1} |v_{j+1} - v_j|$. It is convenient to extend the sequence v_1, \dots, v_{j_0} by v_{j_0+1}, \dots, v_n where the latter is $[n] \setminus \{v_1, \dots, v_{j_0}\}$ in increasing order. Now think of v_1, v_2, \dots, v_n as a permutation of $[n]$. Then Lemma 3.3 implies that the length Λ_1 of the initial part corresponding to the path is at most $\ell(v) < n + 3 \cdot \text{inv}(v)$.

Observe that $\Pr(j_0 \leq n - k) \leq n(1 - p_1)^k$. This is because at j_0 we find that v_{j_0} has no neighbors in the set of unvisited vertices and the existence of such edges is unconditioned at this point. So,

$$(19) \quad j_0 \leq n - \frac{\log^2 n}{p_1} \text{ q.s.}$$

Now let $\alpha_j = |\{i < j : v_i > v_j\}|$, $j = 1, 2, \dots, n$ so that $\text{inv}(v) = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Let $L_j = \max\{v_i : 1 \leq i \leq j\}$. Then if $i < j$ and $v_i > v_j$ we must have $j \leq v_j < v_i \leq L_j$. So,

$$(20) \quad \alpha_j \leq \Delta_j = L_j - j.$$

Furthermore, we will need

$$(21) \quad |v_{i+1} - v_i| \leq |v_{i+1} - (i+1)| + |v_i - i| + 1 \leq \Delta_{i+1} + \Delta_i + 1 \quad \text{for } 1 \leq i < j_0.$$

It is important therefore to analyze the sequence Δ_j , $1 \leq j \leq j_0$. We observe that

$$(22) \quad \Pr(L_{j+1} = L_j + u) \begin{cases} = 1 - (1 - p_1)^{\Delta_j} & u = 0. \\ = p_1(1 - p_1)^{\Delta_j + u - 1} & u > 0. \end{cases}$$

Furthermore, these probabilities hold regardless of previous edge exposures. This is because edges incident with v_j and vertices not on P_1 have not been exposed.

It will follow from (22) that

$$(23) \quad \Delta_j \leq \frac{\log^2 n}{p_1}, \forall j, \text{ q.s.}$$

$$(24) \quad \mathbf{E} \left(\sum_{j=1}^{j_0} \Delta_j \right) \leq \frac{n}{p_1}.$$

$$(25) \quad \sum_{j=1}^{j_0} \Delta_j \leq \frac{2n}{p_1}, \text{ q.s.}$$

We will prove (23), (24), (25) momentarily, but first let us use them to finish the proof of the lemma.

It follows from Lemma 3.3, (20) and (24) that

$$\mathbf{E} \Lambda_1 \leq A_1 n,$$

where $A_1 = 1 + \frac{3}{p_1}$.

It remains to show that there is a Hamilton cycle of length not much greater than Λ_1 .

Let $J = \{v_{j_0+1}, \dots, v_m\}$. We will use the edges of G_2 to insert J into the path P_1 . Let $v_j \in J_0$. Assume that $v_j \geq n/2$, the argument for $v_j < n/2$ is similar. We examine $k = v_j - 1, v_j - 2, \dots$ in turn until we find a k such that (i) $(v_j, v_j - k) \in E(G_2)$, $v_j - k = v_\ell \notin J$ and (ii) $(v_j, v_{\ell-1}) \in E(G_2)$. We will find such a k q.s. after examining at most $\log^2 n$ possibilities. Using (21) and (23) we see that replacing the edge $(v_{\ell-1}, v_\ell)$ by a path $v_{\ell-1}, v_j, v_\ell$ q.s. incorporates v_j into our path at a cost of at most $O\left(\log^2 n + \frac{\log^2 n}{p_1}\right)$ and (19) implies that there is room to insert all vertices in J in this way, without using the same v_ℓ more than once. This gives us a Hamilton path x_1, x_2, \dots, x_n in $G_1 \cup G_2$ q.s. and the total added cost over the cost of P_1 is q.s. $O(\log^4 n)$. There is only an exponentially small probability that we cannot find G_3 -edges $\{x_1, x_{j+1}\}, \{x_j, x_n\}$ which now give us a Hamilton cycle; since the maximum value of Z is just n^2 , this gives $\mathbf{E}(Z) \leq A_p n$, as desired.

Proof of (23): First of all we note that (22) that

$$\Pr \left(\exists j : L_{j+1} \geq L_j + \frac{\log^2 n}{4p_1} \right) \leq (1 - p_1)^{\log^2 n / 4p_1} \leq e^{-\log^2 n / 4}.$$

So if there exists j with $\Delta_j \geq \frac{\log^2 n}{p_1}$ then q.s. there must be k such that $\Delta_k \in \left[\frac{\log^2 n}{2p_1}, \frac{3\log^2 n}{4p_1} \right]$. But then (22) implies that with probability $1 - O(e^{-\log^2 n / 2})$, $L_{k+r} = L_k$ for $r \leq n$ and this completes the proof of (23).

Proof of (24), (25): It follows from (22) that the sum in (24) is bounded by the sum of n independent geometric random variables with success probability p_1 . This gives both the bound on expectation and the q.s. bound. \square

We have:

Corollary 3.5. *Suppose that we replace the length of edge (i, j) in Lemma 3.4 by $\xi_i + \dots + \xi_{j-1}$ where $\xi_1, \xi_2, \dots, \xi_n$ are random variables with mean bounded above by μ and exponential tails. If ξ_1, \dots, ξ_n are independent of $G_{n,p}$ then $\mathbf{E}(Z) \leq \frac{A_p \mu n}{p}$.*

Proof. The bound on the expectation follows directly from Lemma 3.4 and the linearity of expectation. \square

Let us observe now that we get an upper bound $\mathbf{E}(T(\mathcal{Y}_{t,p}^1)) \leq A_p t$ on the length of a tour in 1 dimension. We have

$$\mathbf{E}(T(\mathcal{Y}_{t,p}^1)) = \sum_{n=0}^{\infty} \Pr(|\mathcal{Y}_{t,p}^1| = n) \mathbf{E}(\mathcal{Y}_{t,p}^1 | |\mathcal{Y}_{t,p}^1| = n).$$

When conditioning on $|\mathcal{Y}_{t,p}^1| = n$, we let $p_1 < p_2 < \dots < p_n \subset [0, t]$ be the points in $\mathcal{Y}_{t,p}^1$. We choose $k \in \{0, n-1\}$ uniformly randomly and let $\xi_i = \|p_{k+i+1} - p_{k+i}\|$, where the indices of the p_j are evaluated modulo n . We now have $\mu(\xi_i) \leq \frac{2t}{n}$ for all i , and Corollary 3.5 gives that

$$\mathbf{E}(\mathcal{Y}_{t,p}^1 | |\mathcal{Y}_{t,p}^1| = n) \leq \frac{A_p n}{p} \cdot \frac{2t}{n} = O(t),$$

and thus

$$(26) \quad \mathbf{E}(\mathcal{Y}_{t,p}^1) \leq A_p t.$$

3.2. The asymptotic tour length. Our proof of Theorem 3.1 will use recursion, by dividing the $[t]^d$ cube into smaller parts. However, since our divisions of the cube most not cross boundaries of the elemental regions \mathcal{Y}_1^d , we cannot restrict ourselves to subdivisions into perfect cubes (in general, the integer t may not have the divisors we like).

To this end, if $L = T_1 \times T_2 \times \dots \times T_d$ where each T_i is either $[0, t]$ or $[0, t-1]$, we say L is a d -dimensional *near-cube* with sidelengths in $\{t-1, t\}$. For $0 \leq d' \leq d$, we define the canonical example $L_d^{d'} := [0, t]^{d'} \times [0, t-1]^{d-d'}$ for notational convenience, and let

$$\Phi_p^{d,d'}(t) = \mathbf{E}\left(T(\mathcal{Y}_{t,p}^d \cap L_d^{d'})\right).$$

so that

$$\Phi_p^d(t) := \Phi_p^{d,d}(t) = \Phi_p^{d,0}(t+1).$$

In the unlikely event that $\mathcal{Y}_{t,p}^d \cap L_d^{d'}$ is not Hamiltonian, we take $T(\mathcal{Y}_{t,p}^d \cap L_d^{d'}) = t^{d+1} \sqrt{d}$, for technical reasons.

Our first goal is an asymptotic formula for Φ :

Lemma 3.6. *There exists $\beta_p > 0$ such that*

$$\Phi_p^{d,d'}(t) \sim \beta_p t^d.$$

The proof is by induction on $d \geq 2$. We prove the base case $d = 2$ along with the general case. We begin with a technical lemma.

Lemma 3.7. *There is a constant $F_{p,d} > 0$ such that*

$$(27) \quad \Phi_p^{d,d'}(t) \leq \Phi_p^{d,d'-1}(t) + F_{p,d}t^{d-1}$$

for all t sufficiently large. In particular, there is a constant $A_{p,d} > 0$ such that

$$(28) \quad \Phi_p^d(t+h) \leq \Phi_p^d(t) + A_{p,d}ht^{d-1}$$

for sufficiently large t and $1 \leq h \leq t$.

Proof. We let S denote the subgraph of $\mathcal{Y}_{t,p}^d \cap L_d^{d'}$ induced by the difference $L_d^{d'} \setminus L_d^{d'-1}$.

By ignoring the d' th coordinate, we obtain the $(d-1)$ dimensional set $\pi(S)$, for which induction on d (or line (26) if $d=2$) implies an expected tour $T(S)$ of length $\Phi_p^{d-1,d'-1}(t) \leq \beta_p^{d-1}t^{d-1}$, and so

$$\Phi_p^{d-1,d'-1}(t) \leq D_{p,d-1}t^{d-1}$$

for some constant $D_{p,d-1}$, for sufficiently large t .

We have that

$$\mathbf{E}(T(S)) \leq \mathbf{E}(T(\pi(S))) + d^{1/2} \mathbf{E}(|V(S)|) \leq D_{p,d-1}t^{d-1} + d^{1/2}t^{d-1}.$$

The first inequality stems from the fact that the points in $L_d^{d'} \setminus L_d^{d'-1}$ have a d' coordinate in $[t-1, t]$.

Now if $\mathcal{Y}_{t,p}^d \cap L_d^{d'-1}$ and S are both Hamiltonian, then we have

$$(29) \quad T(\mathcal{Y}_{t,p}^d \cap L_d^{d'}) \leq T(\mathcal{Y}_{t,p}^d \cap L_d^{d'-1}) + T(S) + O_d(t)$$

which gives us the Lemma, by linearity of expectation. We have (29) because we can patch together the minimum cost Hamilton cycle in $\mathcal{Y}_{t,p}^d \cap L_d^{d'-1}$ and the minimum cost path P in S as follows: Let u_1, v_1 be the endpoints of P . If there is an edge u, v of H such that $(u_1, u), (v_1, v)$ is an edge in $\mathcal{Y}_{t,p}^d$ then we can create a cycle H_1 through $\mathcal{Y}_{t,p}^d \cap L_d^{d'-1} \cup P$ at an extra cost of at most $2d^{1/2}t$. The probability there is no such edge is at most $(1-p^2)^{t/2}$, which is negligible given the maximum value of $T(\mathcal{Y}_{t,p}^d \cap L_d^{d'})$.

On the other hand, the probability that either of $\mathcal{Y}_{t,p}^d \cap L_d^{d'-1}$ or S is not Hamiltonian is exponentially small in t , which is again negligible given the maximum value of $T(\mathcal{Y}_{t,p}^d \cap L_d^{d'})$. \square

Our argument is an adaptation of that in Beardwood, Halton and Hammersley [2] or Steele [17], with modifications to address difficulties introduced by the random set of available edges. First we introduce the concept of a decomposition into near-cubes. (Allowing near-cube decompositions is necessary for the end of the proof, beginning with Lemma 3.10).

We say that a partition of $L_d^{d'}$ into m^d near-cubes S_α with sidelengths in $\{u, u+1\}$ indexed by $\alpha \in [m]^d$ is a *decomposition* if for each $1 \leq b \leq d$, there is an integer

M_b such that, letting

$$f_b(a) = \begin{cases} a \cdot u & \text{if } a < M_b \\ a \cdot u + (a - M_b) & \text{if } a \geq M_b. \end{cases}$$

we have that

$$S_\alpha = [f_1(\alpha_1 - 1), f_1(\alpha_1)] \times [f_2(\alpha_2 - 1), f_2(\alpha_2)] \times \cdots \times [f_d(\alpha_d - 1), f_d(\alpha_d)].$$

Observe that so long as $u < t^{1/2}$, $L_d^{d'}$ always has a decomposition into near-cubes with sidelengths in $\{u, u + 1\}$.

First we note that tours in not-too-small near-cubes of a decomposition can be pasted together into a large tour at a reasonable cost:

Lemma 3.8. *Fix $\delta > 0$, and suppose $t = mu$ for $u = t^\gamma$ for $\delta < \gamma \leq 1$ ($m, u \in \mathbb{Z}$), and suppose S_α ($\alpha \in [m]^d$) is a decomposition of $L_d^{d'}$. We let $\mathcal{Y}_{t,p}^{d,\alpha} := \mathcal{Y}_{t,p}^d \cap S_\alpha$. We have*

$$T(\mathcal{Y}_{t,p}^d \cap L_d^{d'}) \leq \sum_{\alpha \in [m]^d} T(\mathcal{Y}_{t,p}^{d,\alpha}) + 4m^d u \sqrt{d} \quad \text{with probability at least } 1 - e^{-\Omega(u^d p)}.$$

Proof. Let \mathcal{B}, \mathcal{C} denote the events

$$\begin{aligned} \mathcal{B} &= \left\{ \exists \alpha : \mathcal{Y}_{t,p}^{d,\alpha} \text{ is not Hamiltonian} \right\} \\ \mathcal{C} &= \left\{ \exists \alpha : \left| |\mathcal{Y}_{t,p}^{d,\alpha}| - u^d \right| \geq \delta u^d \right\}, \end{aligned}$$

and let $\mathcal{E} = \mathcal{B} \cup \mathcal{C}$.

Now $\Pr(\mathcal{B}) \leq m^d e^{-\Omega(u^d p)}$ and, by Observation 3.2, $\Pr(\mathcal{C}) \leq m^d e^{-\Omega(u^d)}$ and so $\Pr(\mathcal{E}) \leq e^{-\Omega(u^d p)}$. Assume therefore that $\neg \mathcal{E}$ holds. Each subsquare S_α will contain a minimum length tour H_α . We now order the subcubes $\{S_\alpha\}$ as T_1, \dots, T_{m^d} , such that for $S_\alpha = T_i$ and $S_\beta = T_{i+1}$, we always have that the Hamming distance between α and β is 1. Our goal is to inductively assemble a tour through the subcubes T_1, T_2, \dots, T_j from the smaller tours H_α with a small number of additions and deletions of edges.

Assume inductively that for some $1 \leq j < m^d$ we have added and deleted edges and found a single cycle C_j through the points in T_1, \dots, T_j in such a way that (i) the added edges have total length at most $4\sqrt{d}ju$ and (ii) we delete one edge from $\tau(T_1)$, $\tau(T_j)$ and two edges from each $\tau(T_i)$, $2 \leq i \leq j - 1$. To add the points of T_{j+1} to create C_{j+1} we delete one edge (u, v) of $\tau(T_j) \cap C_j$ and one edge (x, y) of $\tau(T_{j+1})$ such that both edges $\{u, x\}, \{v, y\}$ are in the edge set of $\mathcal{Y}_{t,p}^d$. Such a pair of edges will satisfy (i) and (ii) and the probability we cannot find such a pair is at most $(1 - p^2)^{(u^d/2-1)u^d/2}$. Thus with probability at least $1 - e^{-\Omega(u^d p)}$ we build the cycle C_{m^d} with a total length of added edges $\leq 4\sqrt{d}m^d u$. \square

Linearity of expectation (and the polynomial upper bound $t^{d+1}\sqrt{d}$ on $T(\mathcal{Y}_{t,p}^d)$) now gives a short-range recursive bound on $\Phi_p^d(t)$ when t factors reasonably well:

Lemma 3.9. *For all large u and $1 \leq m \leq u^{10}$ ($m, u \in \mathbb{N}$),*

$$\Phi_p^d(mu) \leq m^d(\Phi_p^d(u) + B_d u)$$

for some constant B_d . □

Note that here we are using a decomposition of $[mu]^d$ into m^d subcubes with side-length u ; near-cubes are not required.

To get an asymptotic expression for $\Phi_p^d(t)$ we now let

$$\beta = \liminf_t \frac{\Phi_p^d(t)}{t^d}.$$

Choose u_0 large and such that

$$\frac{\Phi_p^d(u_0)}{u_0^d} \leq \beta + \varepsilon$$

and then define the sequence $u_k, k \geq -1$ by $u_{-1} = u_0$ and $u_{k+1} = u_k^{10}$ for $k \geq 0$. Assume inductively that for some $i \geq 0$ that

$$(30) \quad \frac{\Phi_p^d(u_i)}{u_i^d} \leq \beta + \varepsilon + \sum_{j=-1}^{i-2} \left(\frac{A_{p,d}}{u_j} + \frac{B_{p,d}}{u_j^{d-1}} \right).$$

This is true for $i = 0$, and then for $i \geq 0$ and $0 \leq u \leq u_i$ and $d \leq m \in [u_{i-1}, u_{i+1}]$ we have

$$(31) \quad \begin{aligned} \frac{\Phi_p^d(mu_i + u)}{(mu_i + u)^d} &\leq \frac{\Phi_p^d(mu_i) + A_{p,d}u(mu_i)^{d-1}}{(mu_i)^d} \\ &\leq \frac{m^d(\Phi_p^d(u_i) + B_{p,d}u_i) + A_{p,d}u(mu_i)^{d-1}}{(mu_i)^d} \\ &\leq \beta + \varepsilon + \sum_{j=-1}^{i-2} \left(\frac{A_{p,d}}{u_j} + \frac{B_{p,d}}{u_j^{d-1}} \right) + \frac{B_{p,d}}{u_i^{d-1}} + \frac{A_{p,d}}{m} \\ &\leq \beta + \varepsilon + \sum_{j=-1}^{i-1} \left(\frac{A_{p,d}}{u_j} + \frac{B_{p,d}}{u_j^{d-1}} \right). \end{aligned}$$

Putting $m = u_{i+1}/u_i$ and $u = 0$ into (31) completes the induction. We deduce from (30) and (31) that for $i \geq 0$ we have

$$(32) \quad \frac{\Phi_p^d(t)}{t^d} \leq \beta + \varepsilon + \sum_{j=-1}^{\infty} \left(\frac{A_{p,d}}{u_j} + \frac{B_{p,d}}{u_j^{d-1}} \right) \leq \beta + 2\varepsilon \quad \text{for } t \in J_i = [u_{i-1}u_i, u_i(u_{i+1}+1)]$$

Now $\bigcup_{i=0}^{\infty} J_i = [u_0^2, \infty]$ and since ε is arbitrary, we deduce that

$$(33) \quad \beta = \lim_{t \rightarrow \infty} \frac{\Phi_p^d(t)}{t^d},$$

We can conclude that

$$\Phi_p^d(t) \sim \beta t^d,$$

which, together with Lemma 3.7, completes the proof of Lemma 3.6, once we show that $\beta > 0$ in (33). To this end, we let ρ denote $\mathbf{Pr}(|\mathcal{Y}_1^d| \geq 1)$, so that $\mathbf{E}(|\mathcal{Y}_t^d|) \geq \rho t^d$.

We say $x \in \{0, \dots, t-1\}^d$ is *occupied* if there is a point in the copy $\mathcal{Y}_1^d + x$. Observing that a unit cube $[0, 1]^d + x$ ($x \in \{0, \dots, t-1\}^d$) is at distance at least 1 from all but $3^d - 1$ other cubes $[0, 1]^d + y$, we certainly have that the minimum tour length through \mathcal{Y}_t^d is at least $\frac{\mathcal{O}}{3^d - 1}$, where \mathcal{O} is the number of occupied x . Linearity of expectation now gives that $\beta > \rho/(3^d - 1)$, completing the proof of Lemma 3.6.

Before continuing, we prove the following much cruder version of Part (ii) of Theorem 3.1:

Lemma 3.10. *For any fixed $\varepsilon > 0$, $T(\mathcal{Y}_{t,p}^d) \leq t^{d+\varepsilon}$ q.s.*

Proof. We let $m = \lfloor t^{1-\varepsilon/2} \rfloor$, $u = \lfloor t/m \rfloor$, and let $\{\mathcal{Y}_{\tau,p}^{d,\alpha}\}$ be a decomposition of $\mathcal{Y}_{t,p}^d$ into m^d near-cubes with sidelengths in $\{u, u+1\}$. We have that q.s. each $\mathcal{Y}_{\tau,p}^{d,\alpha}$ has (i) $\approx u^d$ points, and (ii) a Hamilton cycle H_α . We can therefore q.s. bound all $T(\mathcal{Y}_{\tau,p}^{d,\alpha})$ by $du \cdot u^d$, and Lemma 3.8 gives that q.s. $T(\mathcal{Y}_{t,p}^d) \leq 4dut^d + 4m^d u \sqrt{d}$. \square

To prove Theorem 3.1, we now consider a decomposition $\{S_\alpha\}$ ($\alpha \in [m]^d$) of \mathcal{Y}_t^d into m^d near-cubes of side-lengths in $\{u, u+1\}$, for $\gamma = 1 - \frac{\varepsilon}{2}$, $m = \lfloor t^\gamma \rfloor$, and $u = \lfloor t/m \rfloor$.

Lemma 3.6 gives that

$$\mathbf{E} T(\mathcal{Y}_{t,p}^{d,\alpha}) \sim \beta_p u^d \sim \beta_p t^{(1-\gamma)d}.$$

Let

$$\mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d) = \sum_{\alpha \in [m]^d} \min \left\{ T(\mathcal{Y}_{t,p}^{d,\alpha}), 2dt^{(1-\gamma)(d+\varepsilon)} \right\}.$$

Note that $\mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d)$ is the sum of $t^{\gamma d}$ identically distributed bounded random variables.

Applying Hoeffding's theorem we see that for any t , we have

$$\mathbf{Pr}(|\mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d) - m^d \mathbf{E}(T(\mathcal{Y}_{u,p}^d))| \geq T) \leq 2 \exp \left(-\frac{2T^2}{4m^d d^2 t^{2(1-\gamma)(d+\varepsilon)}} \right).$$

Putting $T = t^{d\varepsilon}$ for small ε , we see that

$$(34) \quad \mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d) = \beta_p t^d + o(t^d) \quad q.s.$$

Now, since q.s. $T(\mathcal{Y}_{t,p}^{d,\alpha}) \leq 2dt^{(1-\gamma)(d+\varepsilon)}$ for all α by Lemma 3.10, we have that q.s. $\mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d) = \sum_\alpha T(\mathcal{Y}_{t,p}^{d,\alpha})$, so that Lemma 3.8 implies that

$$(35) \quad T(\mathcal{Y}_{t,p}^d) \leq \mathcal{S}_\gamma(\mathcal{Y}_{t,p}^d) + \delta_2 \text{ where } \delta_2 = o(t^d) \quad q.s.$$

It follows from (34) and (35) and the fact that $\mathbf{Pr}(|\mathcal{Y}_t^d| = t^d) = \Omega(t^{-d/2})$ that

$$(36) \quad T(\mathcal{Y}_{t,p}^d) \leq \beta_p t^d + o(t^d) \quad q.s.$$

which proves part (ii) of Theorem 3.1.

Of course, we have from Lemma 3.6 that

$$(37) \quad \mathbf{E}(T(\mathcal{Y}_{t,p}^d)) = \beta_p^d t^d + \delta_1 \text{ where } \delta_1 = o(t^d),$$

and we show next that that this together with (35) implies part (i) of Theorem 3.1, that:

$$(38) \quad T = T(\mathcal{Y}_{t,p}^d) = \beta_p t^d + o(t^d) \quad a.a.s.$$

We choose $0 \leq \delta_3 = o(t^{\frac{d-1}{d}})$ such that $0 \leq \delta_2, |\delta_1| = o(\delta_3)$. Let $I = [\beta t^{\frac{d-1}{d}} - \delta_3, \beta t^{\frac{d-1}{d}} + \delta_2]$. Then we have

$$\begin{aligned} \beta t^{\frac{d-1}{d}} + \delta_1 &= \mathbf{E}(T(\mathcal{Y}_{t,p}^d) \mid T(\mathcal{Y}_{t,p}^d) \geq \beta t^{\frac{d-1}{d}} + \delta_2) \mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \geq \beta t^{\frac{d-1}{d}} + \delta_2) \\ &\quad + \mathbf{E}(T(\mathcal{Y}_{t,p}^d) \mid T(\mathcal{Y}_{t,p}^d) \in I) \mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \in I) + \\ &\quad \mathbf{E}(T(\mathcal{Y}_{t,p}^d) \mid T(\mathcal{Y}_{t,p}^d) \leq \beta t^{\frac{d-1}{d}} - \delta_3) \mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \leq \beta t^{\frac{d-1}{d}} - \delta_3). \end{aligned}$$

Now $\varepsilon_1 = \mathbf{E}(T(\mathcal{Y}_{t,p}^d) \mid T(\mathcal{Y}_{t,p}^d) \geq \beta t^{\frac{d-1}{d}} + \delta_2) \mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \geq \beta t^{\frac{d-1}{d}} + \delta_2) = O(t^{-\omega(1)})$ since $|\mathcal{Y}_{t,p}^d| \leq 2d^{1/2}t^d$ and $\mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \geq \beta t^{\frac{d-1}{d}} + \delta_2) = O(t^{-\omega(1)})$.

So, if $\lambda = \mathbf{Pr}(T(\mathcal{Y}_{t,p}^d) \in I)$ then we have

$$\beta t^{\frac{d-1}{d}} + \delta_1 \leq \varepsilon_1 + (\beta t^{\frac{d-1}{d}} + \delta_2)\lambda + (\beta t^{\frac{d-1}{d}} - \delta_3)(1 - \lambda)$$

or

$$\lambda \geq \frac{\delta_1 - \varepsilon_1 + \delta_3}{\delta_2 + \delta_3} = 1 - o(1),$$

and this proves (38) completing the proof of Theorem 3.1. \square

To derive Theorem 1.3, we now let $\mathcal{W}_{t,p}^d$ be the graph on the set of points in $[0, t]^d$ which is the result of a Poisson process of intensity 1. Our task is now to control the variance of $T(\mathcal{W}_{t,p}^d)$. Here we follow Steele's argument [17] with only small modifications.

Let \mathcal{E}_t denote the event that

$$T(\mathcal{W}_{2t,p}^d) \leq \sum_{\alpha \in [2]^d} T(\mathcal{W}_{t,p}^{d,\alpha}) + 2^{d+2}t\sqrt{d}.$$

Observe that Lemma 3.8 implies that

$$(39) \quad \mathbf{Pr}(\neg \mathcal{E}_t) \leq e^{-\Omega(t^d p)}.$$

We define the random variable $\lambda(t) = T(\mathcal{W}_{t,p}^d) + 10\sqrt{dt}$, and let λ_i denote independent copies. Conditioning on \mathcal{E}_t , we have

$$(40) \quad \lambda_0(2t) \leq \sum_{i=1}^{2^d} \lambda_i(t) - 4\sqrt{dt} \leq \sum_{i=1}^{2^d} \lambda_i(t).$$

In particular, (39) implies that there is enough room that, letting $\Upsilon(t) = \mathbf{E}(\lambda(t))$ and $\Psi(t) = \mathbf{E}(\lambda(t)^2)$, we have for sufficiently large t that

$$\Psi(2t) \leq 2^d \Psi(t) + 2^d(2^d - 1)\Upsilon^2(t)$$

and for

$$\mathcal{V}(t) := \mathbf{Var}(T(\mathcal{W}_{t,p}^d)) = \Psi(t) - \Upsilon(t)^2,$$

we have

$$\frac{\mathcal{V}(2t)}{(2t)^{2d}} - \frac{1}{2^d} \frac{\mathcal{V}(t)}{t^{2d}} \leq \frac{\Upsilon^2(t)}{t^{2d}} - \frac{\Upsilon^2(2t)}{(2t)^{2d}}.$$

Now summing over $t = 2^k t_0$ for $k = 0, \dots, M-1$ gives

$$\sum_{k=1}^M \frac{\mathcal{V}(2^k t)}{(2^k t)^{2d}} - \frac{1}{2^d} \sum_{k=0}^{M-1} \frac{\mathcal{V}(2^k t)}{(2^k t)^{2d}} \leq \frac{\Upsilon^2(t)}{t^{2d}} - \frac{\Upsilon^2(2^M t)}{(2^M t)^{2d}} \leq \frac{\Upsilon^2(t)}{t^{2d}}$$

and so, solving for the first sum, we find

$$(41) \quad \sum_{k=1}^M \frac{\mathcal{V}(2^k t)}{(2^k t)^{2d}} \leq \left(1 - \frac{1}{2^d}\right) \left(\frac{\mathcal{V}(t)}{t^{2d}} + \frac{\Upsilon^2(t)}{t^{2d}}\right) < \infty.$$

Still following Steele, we let $N(t)$ be the Poisson counting process on $[0, \infty)$. We fix a random embedding \mathcal{U} of \mathbb{N} in $[0, 1]^d$ as u_1, u_2, \dots and a random graph \mathcal{U}_p where each edge is included with independent probability p . We let $\mathcal{U}_{n,p}$ denote the restriction of this graph to the first n natural numbers. In particular, note that $\mathcal{U}_{N((t^d)),p}$ is equivalent to $\mathcal{W}_{t,p}$, scaled from $[0, t]^d$ to $[0, 1]^d$. Thus, applying Chebychev's inequality to (41) gives that

$$(42) \quad \sum_{k=0}^{\infty} \Pr \left(\left| \frac{t^{2k} T(\mathcal{U}_{N((t2^k)^d),p})}{(t2^k)^d} - \beta_p^d \right| > \varepsilon \right) < \infty$$

and so for $t > 0$ that

$$(43) \quad \lim_{k \rightarrow \infty} \frac{T(\mathcal{U}_{N((t2^k)^d),p})}{(t2^k)^{d-1}} = \beta \quad a.s.$$

Now choosing some large integer ℓ , we have that (43) holds simultaneously for all the (finitely many) integers $t \in S_P = [2^\ell, 2^{\ell+1})$; and $r \in \mathbb{R}$, we have that $r \in [2^k t, 2^k(t+1))$ for $t \in S_\ell$ and some k .

Unlike the classical case $p = 1$, in our setting, we do not have monotonicity of $T(\mathcal{U}_{n,p})$. Nevertheless, we show a kind of continuity of the tour length through $T(\mathcal{U}_{n,p})$:

Lemma 3.11. *For all $\varepsilon > 0$, $\exists \delta > 0$ such that for all $0 \leq k < \delta n$, we have*

$$(44) \quad T(\mathcal{U}_{n+k,p}) < T(\mathcal{U}_{n,p}) + \varepsilon n^{\frac{d-1}{d}}, \quad q.s.$$

Proof. We consider cases according to the size of k .

Case 1: $k \leq n^{\frac{1}{3}}$.

Note that we have $T(\mathcal{U}_{n+1,p}) < T(\mathcal{U}_{n,p}) + \sqrt{d}$ q.s., since we can q.s. find an edge in the minimum tour though $\mathcal{U}_{n,p}$ whose endpoints are both adjacent to $(n+1)$. $n^{\frac{1}{3}}$ applications of this inequality now give (44).

Case 2: $k > n^{\frac{1}{3}}$.

In this case the restriction \mathcal{R} of $\mathcal{U}_{n+k,p}$ to $\{n+1, \dots, k\}$ is q.s. (with respect to n) Hamiltonian [3]. In particular, by Theorem 3.1, we can q.s. find a tour T though

\mathcal{R} of length $\leq 2\beta_p^d k^{\frac{d-1}{d}}$. Finally, there are q.s., edges $\{x, y\}$ and $\{w, z\}$ on the minimum tours through $\mathcal{U}_{n,p}$ and \mathcal{R} , respectively, such that $x \sim w$ and $y \sim z$ in $\mathcal{U}_{n+k,p}$, giving a tour of length

$$T(\mathcal{U}_{n+k,p}) \leq T(\mathcal{U}_{n,p}) + 2\beta_p^d k^{\frac{d-1}{d}} + 4\sqrt{d}. \quad \square$$

Applying Lemma 3.11 and the fact that $N((1+\delta)r^d) < (1+2\delta)N(r^d)$ q.s (with respect to r). gives that for some $\varepsilon_\ell > 0$ which can be made arbitrarily small by increasing ℓ , we have q.s.

$$T(\mathcal{U}_{N((t+1)2^k)^d,p}) - \varepsilon_\ell r^{d-1} < T(\mathcal{U}_{N(r^d),p}) < T(\mathcal{U}_{N((t2^k)^d),p}) + \varepsilon_\ell (t2^k)^{d-1},$$

and so dividing by r^{d-1} and taking limits we find that a.s.

$$(\beta - \varepsilon_\ell)(1 + \frac{1}{2^p})^{d-1} \leq \liminf_{r \rightarrow \infty} \frac{T(\mathcal{U}_{N(r^d)})}{r^{d-1}} \leq \limsup_{r \rightarrow \infty} \frac{T(\mathcal{U}_{N(r^d)})}{r^{d-1}} \leq \frac{\beta + \varepsilon_\ell}{(1 + \frac{1}{2^p})^{d-1}}.$$

Since ℓ may be arbitrarily large, we find that

$$\lim_{r \rightarrow \infty} \frac{T(\mathcal{U}_{N(r^d)})}{r^{d-1}} = \beta.$$

Now the elementary renewal theorem guarantees that

$$N^{-1}(n) \sim n, \quad \text{a.s.}$$

So we have a.s.

$$\lim_{r \rightarrow \infty} \frac{T(\mathcal{U}_{n,p})}{n^{\frac{d-1}{d}}} = \lim_{r \rightarrow \infty} \frac{T(\mathcal{U}_{N(N^{-1}(n)),p})}{(N^{-1}(n))^{\frac{d-1}{d}}} \frac{(N^{-1}(n))^{\frac{d-1}{d}}}{n^{\frac{d-1}{d}}} = \beta \cdot 1 = \beta.$$

3.3. The case $p(n) \rightarrow 0$. We will in fact show that (2) holds q.s. for $np \geq \omega \log n$, for some $\omega \rightarrow \infty$. That we also get the statement of Theorem 3.3 can be seen by following the proof carefully, but this also follows as a consequence directly from the appendix in Johansson, Kahn and Vu [8].

We first show that q.s.

$$(45) \quad T(\mathcal{X}_{n,p}) = \Omega(n^{(d-1)/d}/p^{1/d}).$$

Let Y_1 denote the number of vertices whose closest $G_{n,p}$ -neighbor is within $\frac{1}{(np)^{1/d}}$. Observe first that if $r = 1/(np)^{1/d}$ then with probability $\geq (1 - \nu_d r^d p)^{n-1} \approx e^{-\nu_d}$, there are no points within distance $1/(np)^{1/d}$ of any fixed $v \in \mathcal{X}_{n,p}$. Thus $\mathbf{E}(Y_1) \geq ne^{-\nu_d}/2$ and one can use the Azuma-Hoeffding inequality to show that Y_1 is concentrated around its mean. Thus q.s. $T(\mathcal{X}_{n,p}) \geq n^{(d-1)/d} e^{-\nu_d}/4p^{1/d}$, proving (45).

We will for convenience prove

Theorem 3.12. *Let $\mathcal{Y}_1^d \subset [0, 1]^d$ denote a set of points chosen via a Poisson process of intensity one in $[0, t]^d$ where $t = n^{1/d}$. Then there exists a constant γ_p^d such that*

$$T(\mathcal{Y}_{t,p}^d) \leq \gamma_p^d \frac{t^d}{p^{1/d}} \quad \text{q.s.}$$

Proof. We consider independent copies of \mathcal{Y}_{t,p_i}^d , $i = 1, 2, \dots, k+1$. We will let $p_0 = p_1 = p/3$ and $p_i = p_1/2^i$, $i = 1, 2, \dots, k = \log_2 t$ and define p_{k+1} so that $1-p = \prod_{j=1}^{k+1} (1-p_j)$. Observe that with this choice, we have that $\mathcal{Y}_{t,p}^d$ decomposes as $\mathcal{Y}_{t,p}^d = \bigcup_{i=0}^{k+1} G_i$, where the G_i are spanning subgraphs given by independent instances of \mathcal{Y}_{t,p_i}^d .

We continue by constructing a large cycle, using only the edges of G_1 . We choose ε small and then choose K sufficiently large for subsequent claims. In preparation for an inductive argument we let $t_1 = t$, $T_1 = t_1^d$, $m_1 = \lfloor (T_1 p_1 / K)^{1/d} \rfloor$ and consider the partition $\Delta_1 = \{S_\alpha\}$ ($\alpha \in [m_1]^d$) of $[0, t]^d$ into m_1^d subcubes of side length $u = t/m$. (Note that t will not change throughout the induction). Now each S_α contains $\approx K/p_1$ vertices, in expectation and so it has at least $(1-\varepsilon)K/p_1$ vertices with probability $1 - e^{-\Omega(K/p_1)} = 1 - o(1)$. Let α be *heavy* if S_α has at least this many vertices, and *light* otherwise. Let Γ_α be the subgraph of G_1 induced by S_α . If α is heavy then for any $\varepsilon > 0$ we can if K is sufficiently large find with probability at least $1 - e^{-\Omega(K/p_1)} = 1 - o(1)$, a cycle C_α in Γ_α containing at least $(1-\varepsilon)^2 K/p_1$ vertices. This is because when α is heavy, Γ_α has expected average degree at least $(1-\varepsilon)K$. We say that a heavy α is *typical* if Γ_α contains a cycle with $(1-\varepsilon)|S_\alpha \cap \mathcal{X}|$ edges; otherwise it is *atypical*.

We now let N denote the set of vertices in $\bigcup C_\alpha$, where the union is taken over all typical heavy α . Our aim is to use Theorem 3.1(ii) to prove that we can q.s. merge the vertices N into a single cycle C_1 , without too much extra cost, and using only the edges of G_1 . Letting $q_\alpha = \Pr(S_\alpha \text{ is normal}) \geq 1 - \varepsilon$, we make each typical heavy α *available* for this round with independent probability $\frac{1-\varepsilon}{1-q_\alpha}$, so that the probability that any given α is available is exactly $1 - \varepsilon$. (This is of course *rejection sampling*.) Now we can let $Y = \mathcal{Y}_1^d$ in Theorem 3.1 be a process which places a single point at the center of $[0, 1]^d$ with probability $1 - \varepsilon$, or produces an empty set with probability ε . Let now Y_α ($\alpha \in t^d$) be the independent copies of Y which give \mathcal{Y}_t^d . Given two cycles C_1, C_2 in a graph G we say that edges $u_i = (x_i, y_i) \in C_i$, $i = 1, 2$ are a *patchable pair* if $f_x = (x_1, x_2)$ and $f_y = (y_1, y_2)$ are also edges of G . Given $x \in Y_\alpha, y \in Y_\beta$, we let $x \sim y$ whenever there exist *two disjoint* patchable pairs $\sigma_{\alpha,\beta}$ between C_α, C_β . Observe that an edge between two vertices of \mathcal{Y}_1^d is then present with probability

$$q_{\alpha,\beta} \geq \Pr(\text{Bin}(K^2/100p_1^2, p_1^2) \geq 2) \geq 1 - \varepsilon.$$

In particular, this graph contains a copy of $\mathcal{Y}_{1,(1-\varepsilon)}^d$, for which Theorem 3.1(ii) gives that q.s. we have a tour of length $\leq B_1 m_1^d$ for some constant B_1 ; in particular, there is a path $P = (\alpha_1, \alpha_2, \dots, \alpha_M)$ through the typical heavy α with at most this length. Using P , we now merge its cycles $C_{\alpha_i}, i = 1, 2, \dots, M$ into a single cycle.

Suppose now that we have merged $C_{\alpha_1}, C_{\alpha_2}, \dots, C_{\alpha_j}$ into a single cycle C_j and have used one choice from $\sigma_{\alpha_{j-1}, \alpha_j}$ to patch C_{α_j} into C_{j-1} . We initially had two choices for patching $C_{\alpha_{j+1}}$ into C_{α_j} , one may be lost, but one at least will be available. Thus we can q.s. use G_1 to create a cycle H_1 from $C_{\alpha_1}, C_{\alpha_2}$, by adding

only patchable pairs of edges, giving a total length of at most

$$(46) \quad 2T_1 \times \frac{2t_1 d^{1/2}}{m_1} + a_1 m_1^d \times \frac{2t_1 d^{1/2}}{m_1} \leq \frac{3T_1 d^{1/2}}{p_1^{1/d}}.$$

The first term in (46) is a bound on the total length of the cycles C_α where α is available, assuming that $|\mathcal{Y}_{t,p}^d| \leq 2t^d$. The second smaller term is the q.s. cost of patching these cycles into H_1 .

Having constructed H_1 , we will consider coarser and coarser subdivisions \mathcal{D}_i of $[0, t]^d$ into m_i^d subcubes, and argue inductively that we can q.s. construct, for each $1 \leq i \leq \ell$ for suitable ℓ , vertex disjoint cycles H_1, H_2, \dots, H_ℓ satisfying:

- (1) $T_i \leq 3\varepsilon T_{i-1}$ for $i \geq 2$, where $T_j = t^d - \sum_{i=1}^{j-1} |H_i|$,
- (2) the set of points in the α th subcube in the decomposition \mathcal{D}_i occupied by vertices which fail to participate in H_i is given by a process which occurs independently in each subcube in \mathcal{D}_i , and
- (3) the total length of each H_i is at most $\frac{3T_i d^{1/2}}{p_i^{1/d}}$.

Note that H_1 , above, satisfies these conditions for $\ell = 1$.

Assume inductively that we have constructed such a sequence H_1, H_2, \dots, H_{j-1} ($j \geq 2$). We will now use the G_j edges to construct another cycle H_j . Suppose now that the set \mathcal{T}_j of points that are not in $\bigcup_{i=1}^{j-1} H_i$ satisfies $T_j = |\mathcal{T}_j| \geq t^{d-1}/\log t$. We let $m_j = (T_j p_j / K)^{1/d}$ and $t_j = T_j^{1/d}$. The expected number of points in a subcube will be K/p_j but we have not exercised any control over its distribution. For $i \geq 2$, we let $\alpha \in [m_i]^d$ be heavy if S_α contains at least $\varepsilon K/p_j$ points. Now we want K to be large enough so that εK is large and that a heavy subcube has a cycle of size $(1 - \varepsilon)|\mathcal{T}_j \cap S_\alpha|$ with probability at least $1 - \varepsilon$, in which case, again, it is *typical*. We define Γ_j as the set of typical heavy pairs $\{\alpha, \beta\}$ for which there are at least two disjoint patchable pairs between the corresponding large cycles. Applying the argument above with T_j, t_j, m_j, Γ_j replacing T_1, t_1, m_1, Γ_1 (note that 2, above, ensures that Theorem 3.1 applies) we can q.s. find a cycle H_j with at least $(1 - 3\varepsilon)T_j$ vertices and length at most $\frac{3T_j d^{1/2}}{p_j^{1/d}}$, giving induction hypothesis part 3.

Part 1 is satisfied since the light subcubes only contribute ε fraction of points to \mathcal{T}_j , and we q.s. take a $(1 - \varepsilon)$ fraction of the heavy subcubes. Finally, Part 2 is satisfied since participation in H_j is determined exclusively by the set of adjacency relations in $G_j \cap \mathcal{T}_j$, which is independent of the positions of the vertices.

Thus we are guaranteed a sequence H_1, H_2, \dots, H_ℓ as above, such that $T_{\ell+1} < t^{d-1}/\log t$. The total length of H_1, H_2, \dots, H_ℓ is at most

$$(47) \quad \sum_{i=1}^{\ell} \frac{3T_i d^{1/2}}{p_i^{1/d}} \leq \frac{3^{1+1/d} t^d}{p^{1/d}} \sum_{i=1}^{\infty} 3^i \cdot 2^{i/d} \varepsilon^{i-1} = O\left(\frac{t^d}{p^{1/d}}\right).$$

We can now use G_0 to finish the proof. It will be convenient to write $G_0 = \bigcup_{i=0}^2 A_i$ where $A_i, i = 1, 2, 3$ are independent copies of $\mathcal{Y}_{t,q}^d$ where $1 - p_0 = (1 - q)^3$. Also, let $R = \{x_1, x_2, \dots, x_r\} = \mathcal{Y}_{t,p}^d \setminus \bigcup_{i=1}^{\ell} H_i$.

We first create a Hamilton path containing all vertices, only using the edges of $A_1 \cup A_2$ and the extension-rotation algorithm introduced by Pósa [14]. We begin by deleting an arbitrary edge from H_1 to create a path P_1 . Suppose inductively that we have found a path P_j through $Y_j = H_1 \cup \dots \cup H_{\rho_j} \cup X_j$ where $X_j \subseteq R$ at an added cost of $O(jt)$. We let V_j denote the vertices of P_j and promise that $V_{\ell+r} = \mathcal{Y}_{t,p}^d$. We also note that $|V_j| \geq |V_1| = \Omega(t^d)$ for $j \geq 1$.

At each stage of our process to create P_{j+1} we will construct a collection $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_r\}$ of paths through V_j . Let $Z_{\mathcal{Q}}$ denote the set of endpoints of the paths in \mathcal{Q} . Round j of the process starts with P_j and is finished when we have constructed P_{j+1} .

If at any point in round j we find a path Q in \mathcal{Q} with an endpoint x that is an A_2 -neighbor of a vertex in $y \notin V_j$ then we will make a *simple extension* and proceed to the next round. If $x \in H_i$ then we delete one of the edges in H_i incident with y to create a path Q' and then use the edge (x, y) to concatenate Q, Q' to make P_{j+1} . If $x \in R$ then $P_{j+1} = Q + y$.

If $Q = (v_1, v_2, \dots, v_s) \in \mathcal{Q}$ and $(v_s, v_1) \in A_1$ then we can take any $y \notin V_j$ and with probability at least $1 - (1-q)^s = 1 - O(t^{-\omega(1)})$ find an edge $(y, v_i) \in A_2$. If there is a cycle H_i with $H_i \cap V_j = \emptyset$ then we choose $y \in H_i$ and delete one edge of H_i incident with y to create a path Q' and then we can take $P_{j+1} = (Q', v_i, v_{i-1}, \dots, v_{i+1})$ and proceed to the next round. Failing this, we choose any $y \in R \setminus V_j$ and let $P_{j+1} = (y, v_i, v_{i-1}, \dots, v_{i+1})$ and proceed to the next round. Note that this is the first time we will have examined the A_2 edges incident with y . We call this a *cycle extension*.

Suppose now that $Q = (v_1, v_2, \dots, v_s) \in \mathcal{Q}$ and $(v_s, v_i) \in A_1$ where $1 < i < s - 1$. The path $Q' = (v_1, \dots, v_i, v_s, v_{s-1}, \dots, v_{i+1})$ is said to be obtained by a rotation. v_1 is the *fixed* endpoint. We partition $\mathcal{Q} = \mathcal{Q}_0 \cup \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_{k_0}$, $k_0 = \log t$ where $\mathcal{Q}_0 = \{P_j\}$ and \mathcal{Q}_i is the set of paths that are obtainable from P_j by exactly i rotations with fixed endpoint v_1 . We let N_i denote the set of endpoints of the paths in \mathcal{Q}_i , other than v_1 , and let $\nu_i = |N_i|$ and let $N_{\mathcal{Q}} = \bigcup_i N_i$. We will prove that q.s.

$$(48) \quad |\nu_i| \leq \frac{1}{100q} \text{ implies that } |\nu_{i+1}| \geq \frac{|\nu_i| t^d q}{300}.$$

It follows from this that q.s. we either end the round through a simple or cycle extension or arrive at a point where the paths in \mathcal{Q} have $\Omega(t^d)$ distinct endpoints. We can take an arbitrary $y \notin V_j$ and find an A_2 neighbor of y among $N_{\mathcal{Q}}$. The probability we cannot find a neighbor is at most $(1-q)^{\Omega(t^d)} = O(t^{-\omega(1)})$. Once we prove (48) we will have shown that we can create a Hamilton path through $\mathcal{Y}_{t,p}^d$ from $H_1, H_2, \dots, H_{\ell}, R$ at an extra cost of $O(d^{1/2}(t\ell + t^{d-1}/\log t \times \log t \times t)) = O(t^d)$. We will not have used any A_3 edges to do this. The second $\log t$ factor comes from the fact that each path is obtained by at most k_0 rotations and each rotation adds one new edge.

Proof of (48): We first prove that in the graph induced by A_1 we have

$$(49) \quad |S| \leq \frac{1}{100q} \text{ implies that } |N_{A_1}(S)| \geq \frac{|S|t^dq}{100}.$$

Here $N_{A_1}(S)$ is the set of vertices not in S that have at least one A_1 -neighbor in S .

Indeed, if $s_0 = \frac{1}{100q} = o(n)$ then

$$\begin{aligned} \Pr(\exists S) &\leq \sum_{s=1}^{s_0} \binom{t^d}{s} \Pr\left(\text{Bin}(t^d - s, 1 - (1 - q)^s) \leq \frac{st^dq}{100}\right) \\ &\leq \sum_{s=1}^{s_0} \binom{t^d}{s} \Pr\left(\text{Bin}\left(t^d - s, \frac{sq}{2}\right) \leq \frac{st^dq}{100}\right) \\ &\leq \sum_{s=1}^{s_0} \left(\frac{t^de}{s} \cdot e^{-\Omega(t^dq)}\right)^s \\ &= O(t^{-\omega(1)}). \end{aligned}$$

Now (48) holds for $i = 0$ because q.s. each vertex in $\mathcal{Y}_{t,p}^d$ is incident with at least $t^dq/2$ A_1 edges. Given (49) for $i = 0, 1, \dots, i-1$ we see that $\nu_1 + \dots + \nu_{i-1} = o(\nu_i)$. In which case (49) implies that

$$\nu_{i+1} \geq \frac{|N_{A_1}(N_i)| - (\nu_0 + \dots + \nu_{i-1})}{2} \geq \frac{t^{\gamma d} \nu_i}{2 + o(1)}$$

completing an inductive proof of (48).

Let P^* be the Hamilton path created above. We now use rotations with v_1 fixed via the edges A_2 to create $\Omega(t^d)$ Hamilton paths with distinct endpoints. We then see that q.s. one of these endpoints is an A_2 -neighbor of v_1 and so we get a tour at an additional cost of $O(d^{1/2}t)$.

This completes the proof of Theorem 3.12. \square

The upper bound in Theorem 1.2 follows as before by (i) replacing $\mathcal{Y}_{t,p}^d$ by $\mathcal{X}_{n,p}^d$, allowable because our upper bound holds q.s. and $\Pr(|\mathcal{Y}_{t,p}^d| = t^d) = \Omega(t^{-d/2})$ and then (ii) scaling by $n^{-1/d}$ so that we have points in $[0, 1]^d$.

4. AN ALGORITHM

To find an approximation to a minimum length tour in $\mathcal{X}_{n,p}$, we can use a simple version of Karp's algorithm [9]. We let $m = (n/K\nu_d \log n)^{1/d}$ for some constant $K > 0$ and partition $[0, 1]^d$ into m^d subcubes of side $1/m$, as in Lemma 3.8. The number of points in each subsquare is distributed as the binomial $B(n, q)$ where $q = K \log n/n$ and so we have a.s. that every subsquare has $K \log n \pm \log n$, assuming K is large enough. The probability that there is no Hamilton cycle in S_α is $O(e^{-Knqp/2})$ and so a.s. every subsquare induces a Hamiltonian subgraph. Using the dynamic programming algorithm of Held and Karp [7] we solve the TSP in each subsquare in time $O(\sigma^2 2^\sigma) \leq n^K$, where $\sigma = \sigma_\alpha = |S_\alpha \cap \mathcal{X}_{n,p}|$. Having

done this, we can with probability of failure bounded by $m^2(1-p^2)^{(K \log n)^2}$ patch all of these cycles into a tour at an extra $O(m^{d-1}) = o(n^{\frac{d-1}{d}})$ cost. The running time of this step is $O(m^d \log^2 n)$ and so the algorithm is polynomial time overall. The cost of the tour is bounded q.s. as in Lemma 3.8. This completes the proof of Theorem 1.3.

5. FURTHER QUESTIONS

Theorem 1.1 shows that there is a definite qualitative change in the diameter of $\mathcal{X}_{n,p}$ at around $p = \frac{\log^d n}{n}$, but our methods leave a $(\log \log n)^{2d}$ size gap for the thresholds.

1. What is the precise threshold for there to be distances in $\mathcal{X}_{n,p}$ which tend to ∞ ? What is the precise threshold for distance in $\mathcal{X}_{n,p}$ to be arbitrarily close to Euclidean distance? What is the behavior of the intermediate regime?

One could also analyze the geometry of the geodesics in $\mathcal{X}_{n,p}$ (Figure 1). For example:

2. Let ℓ be the length of a random edge on the geodesic between fixed points at constant distance in $\mathcal{X}_{n,p}$. What is the distribution of ℓ ?

Improving Theorem 1.2 to give an asymptotic formula for $T(\mathcal{X}_{n,p})$ is another obvious target. It may seem unreasonable to claim such a formula for all (say, decreasing) functions p ; in particular, in this case, the constant in the asymptotic formula would necessarily be universal. The following, however, seems reasonable:

Conjecture 5.1. *If $p = \frac{1}{n^\alpha}$ for some constant $0 < \alpha < 1$ then there exists a constant β_α^d such that a.a.s. $T(\mathcal{X}_{n,p}) \sim \beta_\alpha \frac{n^{\frac{d-1}{d}}}{p^{1/d}}$.*

We note that $T(\mathcal{X}_{n,1})$ is known to be remarkably well-concentrated around its mean; see, for example, the sharp deviation result of Rhee and Talagrand [15].

3. How concentrated is the random variable $T(\mathcal{X}_{n,p})$?

The case of where $p = o(1)$ may be particularly interesting.

Even for the case $p = 1$ covered by the BHH theorem, the constant β_1^d ($d \geq 2$) from Theorem 3.1 is not known. Unlike the case of $p = 1$, the 1-dimensional case is not trivial for our model. In particular, we have proved Theorems 1.3 and 1.2 only for $d \geq 2$. We have ignored the case $d = 1$ not because we consider the technical problems insurmountable, but because we hope that it may be possible to prove a stronger result for $d = 1$, at least for the case of constant p .

4. Determine an explicit constant β_p^1 as a function of (constant) p such that for $d = 1$,

$$\lim_{n \rightarrow \infty} T(\mathcal{X}_{n,p}) = \beta_p^1 n.$$

Our basic motivation has been to understand the constraint imposed on travel among random points by the restriction set of traversable edges which is chosen randomly independently of the geometry of the underlying point-set. While the Erdős-Rényi-Gilbert model is the prototypical example of a random graph, other models such as the Barabási-Albert preferential attachment graph have received wide attention in recent years, due to properties (in particular, the distribution of degrees) they share with real-world networks. In particular, if the random graph one is traveling within is the flight-route map for an airline, the following questions may be the most relevant:

5. If the preferential attachment graph is embedded randomly in the unit square (hypercube), what is the expected diameter? What is the expected size of a minimum-length spanning tree?

Similarly, one could examine a combination of geometry and randomness in determining connections in the embedded graph. Our methods already give something in this direction. In particular, we can define $\mathcal{X}_{n,p,r}$ as the intersection of the graphs $\mathcal{X}_{n,p}$ with the random geometric graph on the vertex set \mathcal{X}_n , where a pair of points are joined by an edge if they are at distance $\leq r$. Following our proof of Theorem 1.3, one sees that we find that

Theorem 5.2. *If $d \geq 2$, $p > 0$ is constant, and $r = r(n) \geq n^{\varepsilon-1/d}$ for some $\varepsilon > 0$, then*

$$T(\mathcal{X}_{n,p,r}) \sim \beta_p^d n^{\frac{d-1}{d}} \quad a.a.s.$$

Of course, the ideas behind Question 5 and Theorem 5.2 could be considered together; note that Flaxman, Frieze and Vera [5] considered a geometric version of a preferential attachment graph.

The proof of Theorem 1.4 is relatively painless. We are reminded that Arora [1] and Mitchell [12] have described more sophisticated polynomial time algorithms that are asymptotically optimal even with the worst-case placing of the points. It would be interesting to see whether these algorithms can handle the random loss of edges.

6. Do the methods of Arora and Mitchell allow efficient approximation of the tour length through $\mathcal{X}_{n,p}$, when the embedding \mathcal{X}_n is *arbitrary*?

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APPENDIX A. PROOF OF (16)

Assume without loss of generality that we have scaled so that $\mu = 1$. Now $e^x \leq 1 + x + x^2 e^x$ when $x \geq 0$ and so for $\lambda > 0$, $e^\lambda < 1/\rho$ we have

$$\mathbf{E}(e^{\lambda Y}) \leq 1 + \lambda + \lambda^2 \left(1 + \frac{2}{(1 - \rho e^\lambda)^3} \right).$$

So, if $Z = Y_1 + Y_2 + \dots + Y_n$ where Y_1, Y_2, \dots, Y_n are independent copies of Y ,

$$\begin{aligned} \Pr(Z \geq n + \delta n) &\leq e^{-\lambda(1+\delta)n} \mathbf{E}(e^{\lambda Y})^n \\ &\leq e^{-\lambda(1+\delta)n} \exp \left\{ \left(\lambda + \lambda^2 \left(1 + \frac{2}{(1 - \rho e^\lambda)^3} \right) \right) n \right\} \\ &\leq e^{-\lambda \delta n} \exp \{ \lambda^2 (1 + 2\varepsilon^{-3}) \} \end{aligned}$$

assuming that

$$(50) \quad e^\lambda \leq (1 - \varepsilon)/\rho.$$

Now choose $\lambda = \delta/(1 + 2\varepsilon^{-3})$ and $\varepsilon = \varepsilon(\delta)$ such that (50) holds. Then

$$\Pr(Z \geq n + \delta n) \leq \exp \left\{ -\frac{\delta^2 n}{2(1 + 2\varepsilon^{-3})} \right\}.$$

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