Minimum-cost matching in a random bipartite graph with random costs

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Abstract

Let $G = G_{n,n,p}$ be the random bipartite graph on $n+n$ vertices, where each $e \in [n]^2$ appears as an edge independently with probability p. Suppose that each edge e is given an independent uniform exponential rate one cost. Let $C(G)$ denote the expected length of the minimum cost perfect matching. We show that w.h.p. if $d = np \gg (\log n)^2$ then $\mathbf{E}[C(G)] = (1 + o(1))\frac{\pi^2}{6p}$. This generalises the well-known result for the case $G = K_{n,n}$.

1 Introduction

There are many results concerning the optimal value of combinatorial optimization problems with random costs. Sometimes the costs are associated with n points generated uniformly at random in the unit square $[0, 1]^2$. In which case the most celebrated result is due to Beardwood, Halton and Hattersley [3] who showed that the minimum length of a tour through the points a.s. grew as $\beta n^{1/2}$ for some still unknown β. For more on this and related topics see Steele [18].

The optimisation problem in [3] is defined by the distances between the points. So, it is defined by a random matrix where the entries are highly correlated. There have been many examples considered where the matrix of costs contains independent entries. Aside from the Travelling Salesperson Problem, the most studied problems in combinatorial optimization are perhaps, the shortest path problem; the minimum spanning tree problem and the matching problem. As a first example, consider the shortest path problem in the complete graph K_n where the edge lengths are independent exponential random variables with rate 1. We denote the exponential random variable with rate λ by $E(\lambda)$. Thus $\mathbf{Pr}(E(\lambda) \geq x) = e^{-\lambda x}$ for $x \in \boldsymbol{R}$. Janson [9] proved (among other things) that if $X_{i,j}$ denotes the shortest distance between vertices i, j in this model then $\mathbf{E}[X_{1,2}] = \frac{H_n}{n}$ where $H_n = \sum_{i=1}^n \frac{1}{i}$.

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As far as the spanning tree problem is concerned, the first relevant result is due to Frieze [6]. He showed that if the edges of the complete graph are given independent uniform $[0,1]$ edge weights, then the (random) minimum length of a spanning tree L_n satisfies $\mathbf{E}[L_n] \to \zeta(3) = \sum_{i=1}^{\infty} \frac{1}{i^3}$ as $n \to \infty$. Further results on this question can be found in Janson [8], Beveridge, Frieze and McDiarmid [4], Frieze, Ruszinko and Thoma [7] and Cooper, Frieze, Ince, Janson and Spencer [5].

In the case of matchings, the nicest results concern the the minimum cost of a matching in a randomly edge-weighted copy of the complete bipartite graph $K_{n,n}$. If C_n denotes the (random) minimum cost of a perfect matching when edges are given independent exponential $E(1)$ random variables then the story begins with Walkup [19] who proved that $\mathbf{E}[C_n] \leq 3$. Later Karp [10] proved that $\mathbf{E}[C_n] \leq 2$. Aldous [1], [2] proved that $\lim_{n\to\infty} \mathbf{E}[C_n] = \zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2}$. Parisi [13] conjectured that in fact $\mathbf{E}[C_n] = \sum_{k=1}^{n} \frac{1}{k^2}$. This was proved independently by Linusson and Wästlund [11] and by Nair, Prabhakar and Sharma [12]. A short elegant proof was given by Wästlund [16], [17].

In the paper [4] on the minimum spanning tree problem, the complete graph was replaced by a d-regular graph G. Under some mild expansion assumptions, it was shown that if $d \to \infty$ then $\zeta(3)$ can be replaced asymptotically by $\frac{n}{d}\zeta(3)$.

Consider a d-regular bipartite graph G on $2N$ vertices. Here $d = d(N) \to \infty$ as $N \to \infty$. Each edge e is assigned a cost $W(e)$, each independently chosen according to the exponential distribution $E(1)$. Denote the total cost of the minimum-cost perfect matching by $C(G)$.

We conjecture the following (under some possibly mild restrictions):

Conjecture 1. *Suppose $d = d(N) \to \infty$ as $N \to \infty$. For any d-regular bipartite G,*

$$\boldsymbol{E}[C(G)] = (1 + o(1))\frac{N}{d}\frac{\pi^2}{6}.$$

Here the $o(1)$ term goes to zero as $N \to \infty$.

In this paper we prove the conjecture for random bipartite graphs. Let $G = G_{n,n,p}$ be the random bipartite graph on $n + n$ vertices, where each $e \in [n]^2$ appears as an edge independently with probability p. Suppose that each edge e is given an independent uniform exponential rate one cost.

Theorem 1. *If $d = np = \omega(\log n)^2$ where $\omega \to \infty$ then w.h.p. $\boldsymbol{E}[C(G)] \approx \frac{\pi^2}{6p}$.*

Here $A_n \approx B_n$ iff $A_n = (1 + o(1))B_n$ as $n \to \infty$ and the event $\mathcal{E}_n$ occurs with high probability (w.h.p.) if $\mathbf{Pr}(\mathcal{E}_n) = 1 - o(1)$ as $n \to \infty$.

Applying results of Talagrand [14] we can prove the following concentration result.

Theorem 2. *Let $\varepsilon > 0$ be fixed, then*

$$\mathbf{Pr}\left(\left|C(G) - \frac{\pi^2}{6p}\right| \geq \frac{\varepsilon}{p}\right) \leq n^{-K},$$

for any constant $K > 0$.

2 Proof of Theorem 1

We find that the proofs in [16], [17] can be adapted to our current situation. Suppose that the vertices of G consist of $A = \{a_i, i \in [n]\}$ and $B = \{b_j, j \in [n]\}$. Let $C(n,r)$ denote the expected cost of the minimum cost matching

$$M_r = \{(a_i, \phi_r(a_i)) : i = 1, 2, \dots, r\} \text{ of } A_r = \{a_1, a_2, \dots, a_r\} \text{ into } B.$$

We will prove that w.h.p.

$$C(n,r) - C(n,r-1) \approx \frac{1}{p} \sum_{i=0}^{r-1} \frac{1}{r(n-i)}. \tag{1}$$

for $r = 1, 2, \dots, n-m$ where

$$m = \left(\frac{n}{\omega^{1/2} \log n} \right).$$

Using this we argue that

$$\mathbf{E}[C(G)] = C(n,n) = (C(n,n) - C(n,n-m+1)) + \frac{1+o(1)}{p} \sum_{r=1}^{n-m} \sum_{i=0}^{r-1} \frac{1}{r(n-i)}. \tag{2}$$

We will then show that

$$\sum_{r=1}^{n-m} \sum_{i=0}^{r-1} \frac{1}{r(n-i)} \approx \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{3}$$

$$C(n,n) - C(n,n-m+1) = o(p^{-1}). \tag{4}$$

Theorem 1 follows from these two statements.

Let $B_r = \{\phi_r(a_i) : i = 1, 2, \dots, r\}$.

Lemma 1. *B_r is a random r-subset of B.*

Proof. Let L denote the $n \times n$ matrix of edge costs, where $L(i,j) = W(a_i, b_j)$ and $L(i,j) = \infty$ if edge (a_i, b_j) does not exist in G. For a permutation π of B let L_π be defined by $L_\pi(i,j) = L(i, \pi(j))$. Let X, Y be two distinct r-subsets of B and let π be any permutation of B that takes X into Y. Then we have

$$\mathbf{Pr}(B_r(L) = X) = \mathbf{Pr}(B_r(L_\pi) = \pi(X)) = \mathbf{Pr}(B_r(L_\pi) = Y) = \mathbf{Pr}(B_r(L) = Y),$$

where the last equality follows from the fact that L and L_π have the same distribution. □

We use the above lemma to bound degrees. For $v \in A$ let $d_r(v) = |\{w \in B \setminus B_r : (v,w) \in E(G)\}|$. Then we have the following lemma:

Lemma 2.

$$|d_r(v) - (n-r)p| \leq \omega^{-1/5}(n-r)p \text{ w.h.p. for } v \in A,\ 0 \leq r \leq n-m.$$

Proof. This follows from Lemma 1 i.e. $B \setminus B_r$ is a random set and the Chernoff bounds viz.

$$\mathbf{Pr}(|d_r(v) - (n-r)p| \geq \omega^{-1/5}(n-r)p) \leq 2e^{-\omega^{-2/5}(n-r)p/3} \leq 2n^{-\omega^{1/10}}.$$

□

We can now use the ideas of [16], [17]. We add a special vertex b_{n+1} to B, with edges to all n vertices of A. Each edge adjacent to b_{n+1} is assigned an $E(\lambda)$ cost independently, $\lambda > 0$. We now consider M_r to be a minimum cost matching of A_r into $B^* = B \cup \{b_{n+1}\}$. We denote this matching by M_r^* and we let B_r^* denote the corresponding set of vertices of B^* that are covered by M_r^*.

Lemma 3. *Suppose $r < n - m$. Then*

$$\mathbf{Pr}(b_{n+1} \in B_r^* \mid b_{n+1} \notin B_{r-1}^*) = \frac{\lambda}{p(n-r+1)(1+\varepsilon_r)+\lambda} \tag{5}$$

where $|\varepsilon_r| \leq \omega^{-1/5}$.

Proof. Assume that $b_{n+1} \notin B_{r-1}^*$. M_r^* is obtained from M_{r-1}^* by finding an augmenting path $P = (a_r, \ldots, a_\sigma, b_\tau)$ from a_r to $B^* \setminus B_{r-1}^*$ of minimum additional cost. Let $\alpha = W(\sigma, \tau)$. We condition on (i) σ, (ii) the lengths of all edges other than $(a_\sigma, b_j), b_j \in B^* \setminus B_{r-1}^*$ and (iii) $\min\left\{A(\sigma, j) : b_j \in B^* \setminus B_{r-1}^*\right\} = \alpha$. With this conditioning $M_{r-1} = M_{r-1}^*$ will be fixed and so will $P' = (a_r, \ldots, a_\sigma)$. We can now use the following fact: Let $X_1, X_2, \ldots, X_M$ be independent exponential random variables of rates $\alpha_1, \alpha_2, \ldots, \alpha_M$. Then the probability that X_i is the smallest of them is $\alpha_i/(\alpha_1 + \alpha_2 + \cdots + \alpha_M)$. Furthermore, the probability stays the same if we condition on the value of $\min\{X_1, X_2, \ldots, X_M\}$. Thus

$$\mathbf{Pr}(b_{n+1} \in B_r^* \mid b_{n+1} \notin B_{r-1}^*) = \frac{\lambda}{d_{r-1}(a_\sigma)+\lambda}.$$

□

Corollary 1.

$$\mathbf{Pr}(b_{n+1} \in B_r^*) = \frac{1}{p}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}\right)(1+\varepsilon_k)\lambda + O(\lambda^2) \tag{6}$$

as $\lambda \to 0$, where $|\varepsilon_r| \leq \omega^{-1/5}$.

Proof. Let $\nu(j) = p^{-1}(n-j)(1+\varepsilon_j)$, $|\varepsilon_j| \leq \omega^{-1/5}$. Then the probability is given by

$$\begin{aligned}
& 1 - \frac{\nu(0)}{\nu(0)+\lambda} \cdot \frac{\nu(1)}{\nu(1)+\lambda} \cdots \frac{\nu(r-1)}{\nu(r-1)+\lambda} \\
= \; & 1 - \left(1 + \frac{\lambda}{\nu(0)}\right)^{-1} \cdots \left(1 + \frac{\lambda}{\nu(r-1)}\right)^{-1} \\
= \; & \left(\frac{1}{\nu(0)} + \frac{1}{\nu(1)} + \cdots + \frac{1}{\nu(r-1)}\right)\lambda + O(\lambda^2) \\
= \; & \frac{1}{p}\left(\frac{1}{n(1+\varepsilon_0)} + \frac{1}{(n-1)(1+\varepsilon_1)} + \cdots + \frac{1}{(n-r+1)(1+\varepsilon_{r-1})}\right)\lambda + O(\lambda^2)
\end{aligned}$$

and each error factor satisfies $|1 - 1/(1+\varepsilon_j)| \leq \omega^{-1/5}$. □

Lemma 4. *If $r \leq n - m$ then*

$$\boldsymbol{E}[C(n,r) - C(n,r-1)] = \frac{1}{rp}\sum_{i=0}^{r-1}\frac{1+\varepsilon_k}{n-i} \tag{7}$$

where $|\varepsilon_k| \leq \omega^{-1/5}$.

Proof. Let X be the cost of M_r and let Y be the cost of M_{r-1}. Let w denote the cost of the edge (a_r, b_{n+1}), and let I denote the indicator variable for the event that the cost of the cheapest A_r-assignment that contains this edge is smaller than the cost of the cheapest A_r-assignment that does not use b_{n+1}. In other words, I is the indicator variable for the event $\{Y + w < X\}$.

It follows from Corollary 1 and symmetry (to obtain the factor $1/r$) that the probability that $(a_r, b_{n+1}) \in M_r^*$ is given by

$$\frac{1}{rp}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}\right)(1+\varepsilon_k)\lambda + O(\lambda^2) \tag{8}$$

as $\lambda \to 0$, since each edge adjacent to b_{n+1} is equally likely to participate in M_{r-1}. If $(a_r, b_{n+1}) \in M_r^*$ then $w < X - Y$. Conversely, if $w < X - Y$ and no other edge from b_{n+1} has cost smaller than $X - Y$, then $(a_r, b_{n+1}) \in M_r^*$, and when $\lambda \to 0$, the probability that there are two distinct edges from b_{n+1} of cost smaller than $X - Y$ is of order $O(\lambda^2)$.

On the other hand, w is $E(\lambda)$ distributed, so

$$\mathbf{E}[I] = \Pr\{w < X - Y\} = \mathbf{E}\left[1 - e^{-\lambda(X-Y)}\right] = 1 - \mathbf{E}\left[e^{-\lambda(X-Y)}\right]. \tag{9}$$

Hence $\mathbf{E}[I]$, regarded as a function of λ, is essentially the Laplace transform of $X - Y$. In particular $\mathbf{E}[X-Y]$ is the derivative of $\mathbf{E}[I]$ evaluated at $\lambda = 0$, so

$$\mathbf{E}[X-Y] = \frac{d}{d\lambda}\mathbf{E}[I]\bigg|_{\lambda=0} = \frac{1}{rp}\left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{n-r+1}\right)(1+\varepsilon_k) \tag{10}$$

where $|\varepsilon_k| \leq \omega^{-1/5}$. Now clearly, as $\lambda \to 0$, $\mathbf{E}[X] \to C(n,r)$ and $\mathbf{E}[Y] = C(n, r-1)$ and the lemma follows. $\square$

This confirms (2) and we turn to (3). We use the following expression from Young [20].

$$\sum_{i=1}^{n}\frac{1}{i} = \log n + \gamma + \frac{1}{2n} + O(n^{-2}), \qquad \text{where } \gamma \text{ is Euler's constant.} \tag{11}$$

Let $m_1 = \omega^{1/4}m$. Observe first that

$$\sum_{i=0}^{m_1}\frac{1}{n-i}\sum_{r=i+1}^{n-m}\frac{1}{r} \leq o(1) + \sum_{i=n^{3/4}}^{m_1}\frac{1}{n-i}\sum_{r=i+1}^{n-m}\frac{1}{r} \leq$$
$$o(1) + \frac{2}{n}\sum_{i=n^{3/4}}^{m_1}\left(\log\left(\frac{n}{i}\right) + \frac{1}{2(n-m)} + O(n^{-3/2})\right) \leq o(1) + \frac{2}{n}\log\left(\frac{n^{m_1}}{m_1!}\right)$$
$$\leq o(1) + \frac{2m_1}{n}\log\left(\frac{ne}{m_1}\right) = o(1). \tag{12}$$

Then,

$$\begin{aligned}
\sum_{r=1}^{n-m}\sum_{i=0}^{r-1}\frac{1}{r(n-i)} &= \sum_{i=0}^{n-m-1}\frac{1}{n-i}\sum_{r=i+1}^{n-m}\frac{1}{r},\\
&= \sum_{i=m_1}^{n-m-1}\frac{1}{n-i}\sum_{r=i+1}^{n-m}\frac{1}{r}+o(1),\\
&= \sum_{i=m_1}^{n-m-1}\frac{1}{n-i}\left(\log\left(\frac{n-m}{i}\right)+\frac{1}{2(n-m)}-\frac{1}{2i}+O(i^{-2})\right)+o(1),\\
&= \sum_{i=m_1}^{n-m-1}\frac{1}{n-i}\log\left(\frac{n-m}{i}\right)+o(1),\\
&= \sum_{j=m+1}^{n-m_1}\frac{1}{j}\log\left(\frac{n-m}{n-j}\right)+o(1), \qquad (13)\\
&= \int_{x=m+1}^{n-m_1}\frac{1}{x}\log\left(\frac{n-m}{n-x}\right)dx+o(1).
\end{aligned}$$

We can replace the sum in (1) by an integral because the terms are all $o(1)$ and the sequence of summands is unimodal.

Continuing, we have

$$\begin{aligned}
&\int_{x=m+1}^{n-m_1}\frac{1}{x}\log\left(\frac{n-m}{n-x}\right)dx\\
&= -\int_{x=m+1}^{n-m_1}\frac{1}{x}\log\left(1-\frac{x-m}{n-m}\right)dx\\
&= \sum_{k=1}^{\infty}\int_{x=m+1}^{n-m_1}\frac{1}{x}\frac{(x-m)^k}{k(n-m)^k}dx\\
&= \int_{y=1}^{n-m-m_1}\frac{1}{y+m}\frac{y^k}{k(n-m)^k}dy. \qquad (14)
\end{aligned}$$

Observe next that

$$\int_{y=1}^{n-m-m_1}\frac{1}{y+m}\frac{y^k}{k(n-m)^k}dy \le \int_{y=0}^{n-m-m_1}\frac{y^{k-1}}{k(n-m)^k}dy \le \frac{1}{k^2}.$$

So,

$$0 \le \sum_{k=\log n}^{\infty}\int_{x=m+1}^{n-m_1}\frac{1}{x}\frac{(x-m)^k}{k(n-m)^k}dx \le \sum_{k=\log n}^{\infty}\frac{1}{k^2}=o(1). \qquad (15)$$

If $1\le k\le \log n$ then we write

$$\int_{y=1}^{n-m-m_1}\frac{1}{y+m}\frac{y^k}{k(n-m)^k}dy = \int_{y=1}^{n-m-m_1}\frac{(y+m)^{k-1}}{k(n-m)^k}dy+\int_{y=1}^{n-m-m_1}\frac{y^k-(y+m)^k}{(y+m)k(n-m)^k}dy.$$

Now

$$\int_{y=1}^{n-m-m_1}\frac{(y+m)^{k-1}}{k(n-m)^k}dy = \frac{1}{k^2}\frac{(n-m_1)^k-(m+1)^k}{(n-m)^k} = \frac{1}{k^2}+O\left(\frac{1}{k\omega^{1/4}\log n}\right). \qquad (16)$$

If $k = 1$ then
$$\int_{y=1}^{n-m-m_1} \frac{(y+m)^k - y^k}{(y+m)k(n-m)^k} dy = \frac{m \log(n-m_1)}{n-m} = o(1).$$
And if $2 \le k \le \log n$ then
$$\begin{aligned}\int_{y=1}^{n-m-m_1} \frac{(y+m)^k - y^k}{(y+m)k(n-m)^k} dy &= \sum_{l=1}^{k} \int_{y=1}^{n-m-m_1} \binom{k}{l} \frac{y^{k-l} m^l}{(y+m)k(n-m)^k} dy \\ &\le \sum_{l=1}^{k} \int_{y=0}^{n-m-m_1} \binom{k}{l} \frac{y^{k-l-1} m^l}{k(n-m)^k} dy \\ &= \sum_{l=1}^{k} \binom{k}{l} \frac{m^l (n-m-m_1)^{k-l}}{k(k-l)(n-m)^k} \\ &= O\left(\frac{km}{k(k-1)n}\right) = O\left(\frac{1}{k\omega^{1/2}\log n}\right)\end{aligned}$$
It follows that
$$0 \le \sum_{k=1}^{\log n} \int_{y=1}^{n-m-m_1} \frac{(y+m)^k - y^k}{(y+m)k(n-m)^k} dy = o(1) + O\left(\sum_{k=2}^{\log n} \frac{1}{k\omega^{1/2}\log n}\right) = o(1). \tag{17}$$
equation (3) now follows from (14), (15), (16) and (17).

Turning to (4) we prove the following lemma:

Lemma 5. *If $r \ge n-m$ then $0 \le C(n, r+1) - C(n,r) = O\left(\frac{\log n}{np}\right)$.*

This will prove that
$$0 \le C(n,n) - C(n-m+1) = O\left(\frac{m \log n}{np}\right) = O\left(\frac{n}{\omega^{1/2} np}\right) = o\left(\frac{1}{p}\right)$$
and complete the proof of (4) and hence Theorem 1.

2.1 Proof of Lemma 5

Let $w(e)$ denote the weight of edge e in G. Let $V_r = A_{r+1} \cup B$ and let G_r be the subgraph of G induced by V_r. For a vertex $v \in V_r$ order the neighbors $u_1, u_2, \dots,$ of v in G_r so that $w(v, u_i) \le w(v, u_{i+1})$. Define the k-neighborhood $N_k(v) = \{u_1, u_2, \dots, u_k\}$.

Let the k-neighborhood of a set be the union of the k-neighborhoods of its vertices. In particular, for $S \subseteq A_{r+1}$, $T \subseteq B$,
$$N_k(S) = \{b \in B :\ \exists a \in S : y \in N_k(a)\}, \tag{18}$$
$$N_k(T) = \{a \in A_{r+1} :\ \exists b \in T : a \in N_k(b)\}. \tag{19}$$

Given a function ϕ defining a matching M of A_r into B, we define the following digraph: let $\vec{\Gamma}_r = (V_r, \vec{X})$ where $\vec{X}$ is an orientation of
$$X = \{\{a,b\} \in G : a \in A_{r+1}, b \in N_{40}(a)\} \cup \{\{a,b\} \in G : b \in B, a \in N_{40}(b)\} \cup \{(\phi(a_i), a_i) : i = 1, 2, \dots, r\}.$$

An edge $e \in M$ is oriented from B to A and has weight $w_r(e) = -w(e)$. The remaining edges are oriented from A to B and have weight equal to their weight in G.

The arcs of directed paths in $\vec{\Gamma}_r$ are alternately forwards $A \to B$ and backwards $B \to A$ and so they correspond to alternating paths with respect to the matching M. It helps to know (Lemma 6, next) that given $a \in A_{r+1}, b \in B$ we can find an alternating path from a to b with $O(\log n)$ edges. The ab-diameter will be the maximum over $a \in A_{r+1}, b \in B$ of the length of a shortest path from a to b.

Lemma 6. *W.h.p., for every ϕ, the (unweighted) ab-diameter of $\vec{\Gamma}_r$ is at most $k_0 = \lceil 3\log_4 n \rceil$.*

Proof. For $S \subseteq A_{r+1}$, $T \subseteq B$, let

$$N(S) = \{b \in B : \exists a \in S \text{ such that } (a,b) \in \vec{X}\},$$
$$N(T) = \{a \in A_{r+1} : \exists b \in T \text{ such that } (a,b) \in \vec{X}\}.$$

We first prove an expansion property: that **whp**, for all $S \subseteq A_{r+1}$ with $|S| \le \lceil n/5 \rceil$, $|N(S)| \ge 4|S|$. (Note that $N(S), N(T)$ involve edges oriented from A to B and so do not depend on ϕ.)

$$\begin{aligned}
\mathbf{Pr}(\exists S : |S| \le \lceil n/5 \rceil, |N(S)| < 4|S|) &\le o(1) + \sum_{s=1}^{\lceil n/5 \rceil} \binom{r+1}{s} \binom{n}{4s} \left(\frac{\binom{4s}{40}}{\binom{n}{40}} \right)^s \\
&\le \sum_{s=1}^{\lceil n/5 \rceil} \left(\frac{ne}{s}\right)^s \left(\frac{ne}{4s}\right)^{4s} \left(\frac{4s}{n}\right)^{40s} \\
&= \sum_{s=1}^{\lceil n/5 \rceil} \left(\frac{e^5 4^{36} s^{35}}{n^{35}} \right)^s \\
&= o(1). \qquad (20)
\end{aligned}$$

Explanation: *The $o(1)$ term accounts for the probability that each vertex has at least 40 neighbors in $\vec{\Gamma}_r$. Condition on this. Over all possible ways of choosing s vertices and $4s$ "targets", we take the probability that for each of the s vertices, all 40 out-edges fall among the $4s$ out of the n possibilities.*

Similarly, w.h.p., for all $T \subseteq B$ with $|T| \le \lceil n/5 \rceil$, $|N(T)| \ge 4|T|$. Thus by the union bound, w.h.p. both these events hold. In the remainder of this proof we assume that we are in this "good" case, in which all small sets S and T have large vertex expansion.

Now, choose an arbitrary $a \in A_{r+1}$, and define $S_0, S_1, S_2, \ldots$ as the endpoints of all alternating paths starting from a and of lengths $0, 2, 4, \ldots$. That is,

$$S_0 = \{a\} \text{ and } S_i = \phi^{-1}(N(S_{i-1})).$$

Since we are in the good case, $|S_i| \ge 4|S_{i-1}|$ provided $|S_{i-1}| \le n/5$, and so there exists a smallest index i_S such that $|S_{i_S-1}| > n/5$, and $i_S - 1 \le \log_4(n/5) \le \log_4 n - 1$. Arbitrarily discard vertices from S_{i_S-1} to create a smaller set S'_{i_S-1} with $|S'_{i_S-1}| = \lceil n/5 \rceil$, so that $S'_{i_S} = N(S'_{i_S-1})$ has cardinality $|S'_{i_S}| \ge 4|S'_{i_S-1}| \ge 4n/5$.

Similarly, for an arbitrary $b \in B$, define $T_0, T_1, \ldots,$ by

$$T_0 = \{b\} \text{ and } T_i = \phi(N(T_{i-1})).$$

Again, we will find an index $i_T \le \log_4 n$ whose modified set has cardinality $|T'_{i_T}| \ge 4n/5$.

With both $|S'_{i_S}|$ and $|T'_{i_T}|$ larger than $n/2$, there must be some $a' \in S'_{i_S}$ for which $b' = \phi(a') \in T'_{i_T}$. This establishes the existence of an alternating walk and hence (removing any cycles) an alternating path of length at most $2(i_S + i_T) \le 2\log_4 n$ from a to b in $\vec{\Gamma}_r$. $\square$

We will need the following lemma,

Lemma 7. *Suppose that $k_1+k_2+\cdots+k_M \le a\log N$, and $X_1, X_2, \ldots, X_M$ are independent random variables with Y_i distributed as the k_ith minimum of N independent exponential rate one random variables. If $\mu > 1$ then*

$$\mathbf{Pr}\left(X_1+\cdots+X_M \ge \frac{\mu a\log N}{N-a\log N}\right) \le N^{a(1+\log\mu-\mu)}.$$

Proof. Let $Y_{(k)}$ denote the kth smallest of $Y_1, Y_2, \ldots, Y_N$, where we assume that $k = O(\log N)$. Then the density function $f_k(x)$ of $Y_{(k)}$ is

$$f_k(x) = \binom{N}{k} k(1-e^{-x})^{k-1} e^{-x(N-k+1)}dx$$

and hence the ith moment of $Y_{(k)}$ is given by

$$\begin{aligned}
\mathbf{E}\left[Y_{(k)}^i\right] &= \int_0^\infty \binom{N}{k} kx^i(1-e^{-x})^{k-1}e^{-x(N-k+1)}dx \\
&\le \int_0^\infty \binom{N}{k} kx^{i+k-1}e^{-x(N-k+1)}dx \\
&= \binom{N}{k} k\frac{(i+k-1)!}{(N-k+1)^{i+k}} \\
&\le \left(1+O\left(\frac{k^2}{N}\right)\right)\frac{k(k+1)\cdots(i+k-1)}{(N-k+1)^i}.
\end{aligned}$$

Thus, if $0 \le t < N-k+1$,

$$\mathbf{E}\left[e^{tY_{(k)}}\right] \le \left(1+O\left(\frac{k^2}{N}\right)\right)\sum_{i=0}^\infty \left(\frac{-t}{N-k+1}\right)^i\binom{-k}{i} = \left(1+O\left(\frac{k^2}{N}\right)\right)\left(1-\frac{t}{N-k+1}\right)^{-k}.$$

If $Z = X_1+X_2+\cdots+X_M$ then if $0 \le t < N-a\log N$,

$$\mathbf{E}\left[e^{tZ}\right] = \prod_{i=1}^M \mathbf{E}\left[e^{tX_i}\right] \le \left(1-\frac{t}{N-a\log N}\right)^{-a\log N}.$$

It follows that

$$\mathbf{Pr}\left(Z \ge \frac{\mu a\log N}{N-a\log N}\right) \le \left(1-\frac{t}{N-a\log N}\right)^{-a\log N}\exp\left\{-\frac{t\mu a\log N}{N-a\log N}\right\}.$$

We put $t = (N-a\log N)(1-1/\mu)$ to minimise the above expression, giving

$$\mathbf{Pr}\left(Z \ge \frac{\mu a\log N}{N-a\log N}\right) \le (\mu e^{1-\mu})^{a\log N}.$$

$\square$

Lemma 8. *W.h.p., for all ϕ, the weighted ab-diameter of $\vec{\Gamma}_r$ is at most $c_1 \frac{\log n}{np}$ for some absolute contant $c_1 > 0$.*

Proof. Let

$$Z_1 = \max\left\{ \sum_{i=0}^{k} w(a_i, b_i) - \sum_{i=0}^{k-1} w(b_i, a_{i+1}) \right\}, \tag{21}$$

where the maximum is over sequences $a_0, b_0, a_1, \dots, a_k, b_k$ where (a_i, b_i) is one of the 40 shortest arcs leaving a_i for $i = 0, 1, \dots, k \le k_0 = \lceil 3 \log_4 n \rceil$, and (b_i, a_{i+1}) is a backwards matching edge.

We compute an upper bound on the probability that Z_1 is large. For any $\zeta > 0$ we have

$$\mathbf{Pr}\left(Z_1 \ge \zeta \frac{\ln n}{np}\right) \le o(1) + \sum_{k=0}^{k_0} ((r+1)n)^{k+1} \left(\frac{1+o(1)}{np}\right)^{k+1} \times$$
$$\int_{y=0}^{\infty} \left[\frac{1}{(k-1)!} \left(\frac{y \ln n}{np}\right)^{k-1} \sum_{\rho_0 + \rho_1 + \cdots + \rho_k \le 40(k+1)} q(\rho_0, \rho_1, \dots, \rho_k; \zeta + y) \right] dy$$

where

$$q(\rho_0, \rho_1, \dots, \rho_k; \eta) = \mathbf{Pr}\left(X_0 + X_1 + \cdots + X_k \ge \eta \frac{\log n}{np}\right),$$

$X_0, X_1, \dots, X_k$ are independent and X_j is distributed as the ρ_jth minimum of r independent exponential random variables. (When $k = 0$ there is no term $\frac{1}{(k-1)!}\left(\frac{y \log n}{n}\right)^{k-1}$).

Explanation: *The $o(1)$ term is for the probability that there is a vertex in V_r that has fewer than $(1 - o(1))np$ neighbors in V_r. We have at most $((r+1)n)^{k+1}$ choices for the sequence $a_0, b_0, a_1, \dots, a_k, b_k$. The term $\frac{1}{(k-1)!}\left(\frac{y \ln n}{np}\right)^{k-1} dy$ bounds the probability that the sum of k independent exponentials, $w(b_0, a_1) + \cdots + w(b_{k-1}, a_k)$, is in $\frac{\ln n}{np}[y, y + dy]$. (The density function for the sum of k independent exponentials is $\frac{x^{k-1} e^{-x}}{(k-1)!}$.) We integrate over y.*
$\frac{1+o(1)}{np}$ is the probability that (a_i, b_i) is the ρ_ith shortest edge leaving a_i, and these events are independent for $0 \le i \le k$. The final summation bounds the probability that the associated edge lengths sum to at least $\frac{(\zeta+y)\ln n}{np}$.

It follows from Lemma 7 that if ζ is sufficiently large then, for all $y \ge 0$,

$$q(\rho_1, \dots, \rho_k; \zeta + y) \le (np)^{-(\zeta+y)\log n/(2\log np)} = n^{-(\zeta+y)/2}.$$

Since the number of choices for $\rho_0, \rho_1, \dots, \rho_k$ is at most $\binom{41k+40}{k+1}$ (the number of positive integral solutions to $a_0 + a_1 + \ldots + a_{k+1} \le 40(k+1)$) we have

$$\begin{aligned}
\mathbf{Pr}\left(Z_1 \ge \zeta \frac{\ln n}{np}\right) &\le 2n^{2-\zeta/2} \sum_{k=0}^{k_0} \frac{(\ln n)^{k-1}}{(k-1)!} \binom{41k+40}{k+1} \int_{y=0}^{\infty} y^{k-1} n^{-y/2} dy \\
&\le 2n^{2-\zeta/2} \sum_{k=0}^{k_0} \frac{(\ln n)^{k-1}}{(k-1)!} \left(\frac{84e}{\ln n}\right)^k \Gamma(k) \\
&\le 2n^{2-\zeta/2} (k_0+1)(84e)^{k_0+2} \\
&= o(n^{-4}).
\end{aligned}$$

□

Lemma 8 shows that with probability $1 - o(n^{-2})$ in going from M_r to M_{r+1} we can find an augmenting path of weight at most $\frac{c_1 \log n}{np}$. This completes the proof of Lemma 5 and Theorem 1. (Note that to go from w.h.p. to expectation we use the fact that w.h.p. $w(e) = O(\log n)$ for all $e \in A \times B$,) □

Notice that in the proof of Lemmas 6 and 8 we can certainly make the failure probability less than $n^{-anyconstant}$.

3 Proof of Theorem 2

The proof of Lemma 8 allows us to claim that with probability $1 - O(n^{-anyconstant})$ the maximum length of an edge in the minimum cost perfect matching of G is at most $\mu = c_2 \frac{\log n}{np}$ for some constant $c_2 > 0$. We can now proceed as in Talagrand's proof of concentration for the assignment problem. We let $\widehat{w}(e) = \min \{w(e), \mu\}$ and let $\widehat{C}(G)$ be the assignment cost using $\widehat{w}$ in place of w. We observe that

$$\mathbf{Pr}(\widehat{C}(G) \neq C(G)) = O(n^{-anyconstant}) \tag{22}$$

and so it is enough to prove concentration of $\widehat{C}(G)$.

For this we use the following result of Talagrand [14]: consider a family $\mathcal{F}$ of N-tuples $\boldsymbol{\alpha} = (\alpha_i)_{i \leq N}$ of non-negative real numbers. Let

$$Z = \min_{\boldsymbol{\alpha} \in \mathcal{F}} \sum_{i \leq N} \alpha_i X_i$$

where $X_1, X_2, \ldots, X_N$ are an independent sequence of random variables taking values in $[0, 1]$.

Let $\sigma = \max_{\boldsymbol{\alpha} \in \mathcal{F}} ||\boldsymbol{\alpha}||_2$. Then if M is the median of Z and $u > 0$, we have

$$\mathbf{Pr}(|Z - M| \geq u) \leq 4 \exp \left\{ -\frac{u^2}{4\sigma^2} \right\}. \tag{23}$$

We apply (23) with $N = n^2$ and $X_e = \widehat{w}(e)/\mu$. For $\mathcal{F}$ we take the $n!$ $\{0, 1\}$ vectors corresponding to perfect matchings and scale them by μ. In this way, $\sum_e \alpha_e X_e$ will be the weight of a perfect matching. In this case we have $\sigma^2 \leq n\mu^2$. Applying (23) we obtain

$$\mathbf{Pr} \left(|\widehat{C}(G) - \widehat{M}| \geq \frac{\varepsilon}{p} \right) \leq 4 \exp \left\{ -\frac{\varepsilon^2}{4p^2} \cdot \frac{1}{n\mu^2} \right\} = \exp \left\{ -\frac{\varepsilon^2 n}{(c_2 \log n)^2} \right\}, \tag{24}$$

where $\widehat{M}$ is the median of $\widehat{C}(G)$. Theorem 2 follows easily from (22) and (24).

4 Final remarks

We have generalised the result of [2] to random bipartite graphs. It would seem that in the absence of proving Conjecture 1 we should be able to replace $\omega(\log n)^2$ in Theorem 1 by $\omega \log n$. It would be of interest to prove the analogous result for $G_{n,p}$. Here we would expect to find that the expected cost of a minimum matching was asymptotically $\frac{\pi^2}{12p}$, given that Wästlund has proved this for $p = 1$ in [15].

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