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Optimal Investment and Pricing in Models where the Underlying Asset May Default

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Abstract

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The need for the pricing and hedging of credit events has increased since the financial crisis. For example, large banks are now mandated to compute prices of credit risk for all over-the-counter contracts. Such prices are known by the acronym CVA (Credit Valuation Adjustment), or more generally, XVA. Industry practitioners typically use risk-neutral pricing for such computations, the validity of which is questioned in incomplete markets. In our research, we consider an incomplete market where investment returns and variances are driven by a partially hedgeable factor process, modelled by a multi-dimensional diffusion. Additionally, the issuer of the stock may default, with the default intensity also driven by the factor process. Investors can freely trade the stock to hedge their positions in this market, and do so to maximize their utility. However, in the event of default, the investors lose their position in the stock. In this setting, we price defaultable claims using utility indifference pricing for an exponential investor. Due to the Markovian structure of the problem, we rely on PDE theory rather than BSDE theory to solve the utility maximization problem. This leads to explicit candidate solutions which we verify using the well-developed duality theory. As an application of our optimal investment result,

we define, and compute, the dynamic utility indifference price for insurance against the defaultable stock.

This work is dedicated to the memory of my mother.

Contents

CHAPTER 1

Introduction

The need for the pricing and hedging of credit events has increased since the financial crisis. For example, large banks are now mandated to compute prices of credit risk for all over-the-counter contracts. Such prices are known by the acronym CVA (Credit Valuation Adjustment), or more generally, XVA. For banks, the computations of such prices tend to be intensive because they are always of 1. a cross-asset-class nature between credit and the asset class of original contracts, and, 2. an option-pricing nature, even for linear products, because what is in question is essentially the present value of the positive/negative part of the future exposure subject to credit risk. See [**BMP13**] for comprehensive treatment of this matter from industry practitioners' view.

From a theoretical standpoint, the problem boils down to pricing contingent claims when either party of the contract can default. Industry practitioners typically use risk-neutral pricing for such computations, the validity of which, however, is questioned in incomplete markets. In such markets, the choice of the risk-neutral measure is not clear. Thus the meaning of hedging using greeks computed in the chosen measure is even less clear.

In our research, we consider an incomplete market where stock returns and variances are driven by a partially hedgeable factor process, modeled by a multi-dimensional diffusion. Additionally, the issuer of the stock may default, with the default intensity also driven by the factor process (known as intensity-based modeling of default, see [**Duf05**]). Investors can freely trade the stock to hedge their positions in this market, and do so to maximize the expected exponential utility of their final net wealth and the endowment of a claim at maturity, that may depend on the value of the factor process as well. However, in the event of default, the investors lose their position in the stock and the claim payoff. Note that our model is set up in the physical measure, rather than in a specific risk-neutral measure, and that the hedging is explicitly done in the liquid stock market. An immediate application of our optimal investment result is the utility indifference pricing of such defaultable claims.

There is abundant literature on the optimal investment problem under exponential utilities. Thus, rather than giving a complete literature review, we would like to explain where our research stands in relation to the previous studies.

For the optimal investment problem under exponential utilities, the abstract duality theory provides the definitive results regarding the existence and uniqueness of the optimal investment strategies in a general locally bounded semimartingale context [**Fri00, GR02, DGR$^+$02, KS02**]. However, for given models of the market, it is still difficult to obtain explicit solutions to the optimal investment problem. One way to obtain explicit solutions is through backward stochastic differential equations (BSDEs). The main difficulties in this approach are quadratic drivers in BSDEs due to market incompleteness, and the presence of jumps. Among many papers regarding this topic, the closest to our current setting is [**LQ$^+$11**], although it explicitly excludes the possibility that the stock process jumps to zero. In the paper, they first restrict the range of the strategy to compact sets to reduce the driver of their BDSE from quadratic to Lipschitz continuous. They then obtain the solutions to such localized BSDEs. The optimal value function is obtained through taking the limit of the localized solutions. This idea of reducing quadratic BSDEs to Lipschitz continuous drivers and then taking a limit of thus obtained solutions can be traced back to [**Mor09**].

When the model is Markovian, as in our case, we can write down the Hamilton-Jacobi-Bellman (HJB) equation for the optimal investment problem, which yields a semilinear parabolic partial differential equation (PDE). We use the classical parabolic PDE theory in Hölder spaces to solve the PDE, as opposed to seeking viscosity solutions or solutions in a weak sense. The solutions in Hölder spaces already have enough regularity to guarantee the existence of candidate optimal strategies. Therefore we can use the aforementioned duality theory results to verify our solutions. This program was already carried out in a stochastic volatility market, but without default, under exponential utilities in [**BK05**]. The paper relies on [**FS06**, IV.4] for the existence of classical solutions to PDEs, but this result cannot be used for our problem because of the non-polynomial term of the value function G in the Hamiltonian of our HJB equation. Instead, we use the nonlinear parabolic PDE theory in [**Lie96**] together with some estimations in [**Fri13**] to solve our PDE. To our knowledge, this is the first result to solve the HJB equation for the optimal investment problem in a defaultable market using the classical parabolic PDE theory, and complete the verification steps from the duality theory.

The thesis is organized as follows. Chapter 2 introduces the market model and the investor in the market, and states our main optimal investment result, Theorem 7, where the value function is given in terms of the solution to the semilinear parabolic PDE. Before the proof of Theorem 7, we explore a simple case where the coefficients of the model are constant, so that we can obtain an almost closed-form solution to the optimal investment problem. We embark on the proof of Theorem 7 in Chapter 4. There, we are able to show the existence of classical solutions to our semilinear parabolic PDE under rather general conditions, but to complete the verification we need more stringent conditions in which our factor process is an extension of the multi-dimensional Ornstein-Uhlenbeck (OU) process. In Chapter 5, we define and compute the dynamic utility indifference price for loss-insurance against the stock defaulting. This connects the problem in [**SZ07**], where the investor receives full pre-default market value on her stock holdings on liquidation without having any default protection, to our problem with no default protection. The investor can choose to fully protect herself against default by purchasing insurance, and the price process of insurance is endogenously set so that the investor is indifferent between holding or not holding this insurance at any time before the time of default. The price of this insurance is explicitly computed in Theorem 44 with an example in the simple case of constant coefficients.

CHAPTER 2

Model and Main Result

2.1. Model Setup

We now introduce the model we will be using for the optimal investment problem. We assume the risk-less, or safe, asset price is identically equal to one. As for the risky asset, instantaneous returns and variances are driven by an exogenous factor process X which is only partially hedgeable. Thus, even absent defaults, there are unspanned risks and hence the model is incomplete. Furthermore, we assume the stock may default, with the default intensity also governed by the factor process.

To precisely define the asset dynamics, we first identify the factor process. To do this, let $(\Omega, \mathcal{F}, P)$ denote a complete probability space, and assume it is rich enough to support an $n+1$ dimensional standard Brownian motion $W = (W^0, W^1, \ldots, W^n)$. The risky asset will be driven by W whereas the factor process will be driven by $(W^1, W^2, \ldots, W^n)$. Denote by $\mathbb{F}$ the P-augmentation of the natural filtration $\mathbb{F}^W$ so that $\mathbb{F}$ satisfies the usual conditions of right-continuity and completeness.

Let $C_b(\mathbb{R}^n)$ be the set of bounded continuous functions on $\mathbb{R}^n$, and $C_b^1(\mathbb{R}^n)$ be the set of continuously differentiable functions on $\mathbb{R}^n$ where the first-order derivative is bounded (thus the function value is not necessarily bounded).

The factor process X is a solution to a stochastic differential equation (SDE) taking values in $\mathbb{R}^n$. The dynamics for X are

$$\begin{aligned} dX_t^i &= b^i(X_t)\,dt + \sum_{j=1}^{n} c^{ij}\,dW_t^j; \quad 1 \le i \le n, \\ X_0 &= x \in \mathbb{R}^n. \end{aligned} \tag{1}$$

For the drift function $b : \mathbb{R}^n \to \mathbb{R}^n$ and the volatility c above, we assume

Assumption 1. $b^i \in C_b^1(\mathbb{R}^n)$ for $1 \le i \le n$ and c is an $n \times n$-invertible constant matrix. As such, for $a \triangleq cc^\top/2$, we can pick constants $\Lambda \ge \lambda > 0$ such that for all ξ in $\mathbb{R}^n$, $\Lambda|\xi|^2 \ge a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2$.

Assumption 1 yields a unique non-explosive strong solution X to the SDE in (1) and hence X is $\mathbb{F}$ adapted (more specifically, it is $\mathbb{F}^{W^1,W^2,\dots,W^n}$-adapted).

Having established the well-posedness of X, we now turn to the default time τ for the risky asset. We wish for τ to have $\mathbb{F}$ intensity $(\gamma_t)_{t\ge 0}$, where $\gamma_t = \gamma(X_t)$ for an exogenously specified function $\gamma : \mathbb{R}^n \to (0,\infty)$. More precisely, we assume

Assumption 2. $\gamma \in C_b(\mathbb{R}^n) \cap C_b^1(\mathbb{R}^n)$ and $\inf_{x\in\mathbb{R}^n} \gamma(x) > 0$.

Given the candidate intensity function γ there are numerous methods by which to construct the default time. Rather than defining τ abstractly and then enforcing, for example, the H-hypothesis [**EJY00**], we construct τ directly. Specifically, we assume $(\Omega, \mathcal{F}, P)$ supports a random variable $U \sim U(0,1)$ which is independent of W. Then set

$$\tau = \inf\left\{ t \ge 0 \;:\; \int_0^t \gamma(X_u)du = -\log(U) \right\}, \tag{2}$$

so that

$$F_t \triangleq P(\tau \le t|\mathcal{F}_t) = 1 - e^{-\Gamma_t},$$

where

$$\Gamma_t \triangleq \int_0^t \gamma(X_u)du.$$

Using the default time τ we create the indicator process H via $H_t \triangleq 1_{\tau\le t}$ and the enlarged filtration $\mathbb{G}$ via $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is the P-augmented version of the natural filtration associated to H. Note that this setup clearly implies the H hypothesis of [**EJY00**] that every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale. Furthermore, we know that $\mathbb{G}$ satisfies the usual conditions and

$$M_t \triangleq H_t - \Gamma_{\tau\wedge t},$$

is a $(\mathbb{G}, P)$-martingale.

As a last step, we introduce the Brownian motion which will drive the risky asset. To do so, define B by

$$B_\cdot = \sum_{j=1}^{n} \int_0^\cdot \rho^j(X_t)\, dW_t^j + \int_0^\cdot \sqrt{1 - |\rho(X_t)|^2}\, dW_t^0. \tag{3}$$

Above, the correlation $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is a given function satisfying

Assumption 3. $\rho_i \in C_b^1(\mathbb{R}^n)$ for $1 \le i \le n$ satisfying $0 \le |\rho(x)| \le 1$ in $\mathbb{R}^n$.

Clearly, B is a Brownian motion adapted to $\mathbb{F}$. With all the notation and assumptions in place, the risky asset S has dynamics

$$\begin{aligned} \frac{dS_t}{S_{t-}} &= 1_{t\le\tau}(\mu(X_t)dt + \sigma(X_t)dB_t) - dH_t; \\ &= 1_{t\le\tau}[(\mu(X_t) - \gamma(X_t))dt + \sigma(X_t)dB_t] - dM_t. \end{aligned} \tag{4}$$

In other words, S follows a strictly positive continuous diffusion process until τ, at which time it jumps to zero and stays there. Thus S is a locally bounded, $\mathbb{G}$-adapted semimartingale. Regarding the coefficients μ, σ we assume

Assumption 4. $\mu, \sigma \in C_b(\mathbb{R}^n) \cap C_b^1(\mathbb{R}^n)$ and $\inf_{x\in\mathbb{R}^n} \sigma(x) > 0$.

Finally, to ease notation going forward, we consolidate Assumptions 1 – 4 into one assumption, which will be in force in the sequel:

Assumption 5. Assumptions 1 – 4 hold.

2.2. Optimal Investment Problem and the Main Result

2.2.1. Investor Preferences and Wealth Processes. We assume the investor has preferences modeled by the exponential utility function

$$U(w) = -\exp(-\alpha w); \qquad w \in \mathbb{R},$$

so that $\alpha > 0$ measures the absolute risk aversion. Now, fix a finite time horizon $T > 0$. The investor's goal is to maximize her expected utility from terminal wealth by trading in the risky and safe assets, taking into account the fact the stock may default. To precisely define the class of acceptable wealth processes, it is first necessary to define the class of equivalent local martingale measures. As such, we set

$$\mathcal{M}_e \triangleq \{Q \sim P \text{ on } \mathcal{G}_T : S \text{ is a local martingale under } Q\}. \tag{5}$$

For $Q \in \mathcal{M}_e$ define the relative entropy of Q with respect to P by

$$H(Q|P) \triangleq \mathbb{E}\left[\frac{dQ}{dP}\log\left(\frac{dQ}{dP}\right)\right]^{1}, \tag{6}$$

and let

$$\mathcal{M}_{e,f} \triangleq \{Q \in \mathcal{M}_e \ : \ H(Q|P) < \infty\}.$$

Note that Assumption 5 insures

$$\mathcal{M}_{e,f} \neq \emptyset, \tag{7}$$

and in fact we will construct a concrete element in $\mathcal{M}_{e,f}$ in Section 4.4.

The relation (7) is intimately related to the lack of arbitrage in the market and from [**Fri00, KS02**] it is well known that (7) implies there exists a unique $\overline{Q}^0 \in \mathcal{M}_{e,f}$ such that

$$H(\overline{Q}^0|P) = \min_{Q\in\mathcal{M}_{e,f}} H(Q|P) < \infty. \tag{8}$$

[1]Unless otherwise mentioned, all Radon-Nikodym derivatives will be on $\mathcal{G}_T$.

$\overline{Q}^0$ is known as the minimal entropy martingale measure, or MEMM, under P.

Having defined the class of equivalent local martingale measures with finite relative entropy we may now define the class of trading strategies available to the investor. Let π be a $\mathbb{G}$-predictable, S-integrable processes, where π_t denotes the dollar amount invested in S at time t. For an initial capital w, the self-financing wealth process $\mathcal{W}^\pi$ associated to π has dynamics

$$\begin{aligned} \mathcal{W}^\pi_\cdot &= w + \int_0^\cdot \pi_t \frac{dS_t}{S_{t-}}; \\ &= w + \int_0^\cdot \pi_t \mathbb{1}_{t\le\tau}(\mu(X_t)dt + \sigma(X_t)dB_t) - \int_0^\cdot \pi_t dH_t; \\ &= w + \int_0^\cdot \pi_t \mathbb{1}_{t\le\tau}\left((\mu(X_t) - \gamma(X_t))dt + \sigma(X_t)dB_t - dM_t\right). \end{aligned} \tag{9}$$

The acceptable class of trading strategies is defined as

$$\mathcal{A}_{\mathbb{G}} = \{\pi : \mathcal{W}^\pi \text{ is a } Q \text{ super-martingale for all } Q \in \mathcal{M}_{e,f}\}. \tag{10}$$

Having defined the class of trading strategies, we now consider when the investor, in addition to trading in the underlying assets, also has a non-traded random endowment with payoff $\varphi(X_T)$ provided that $\tau > T$. Regarding φ we assume

Assumption 6. $\varphi \in C^{2+\beta}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\beta \in (0,1)$, i.e., φ has second-order derivatives which are β-Hölder continuous on any compact set in $\mathbb{R}^n$. Furthermore, we assume

$$\sup_{\mathbb{R}^n} |\varphi| < \infty, \quad \limsup_{|x|\to\infty} \frac{|D\varphi|}{|x|} < \infty.$$

The investors goal is to maximize her expected utility by trading in the underlying market and owning the defaultable claim: i.e. to identify

$$v(x,w;\varphi) = \sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[U\left(\mathcal{W}^\pi_T + \varphi(X_T)1_{\tau>T}\right)\right], \tag{11}$$

where the x and w are initial values of the processes X and $\mathcal{W}^\pi$ defined in (1) and (9), respectively. Since $v(x,w;\varphi) = e^{-\alpha w} v(x,0;\varphi)$, we take $w = 0$ throughout. At an abstract level, this problem is well understood. Indeed, under the given assumptions it was shown in [**KS02**] that an optimizer $\overline{\pi}^\varphi \in \mathcal{A}_{\mathbb{G}}$ exists and is unique. Furthermore, there is a measure $\overline{Q}^\varphi \in \mathcal{M}_{e,f}$ so that, with $\overline{\mathcal{W}}^\varphi = \mathcal{W}^{\overline{\pi}^\varphi}$ denoting the optimal wealth process, $\overline{\mathcal{W}}^\varphi$ and $\overline{Q}^\varphi$ satisfy the first-order conditions

$$
\begin{aligned}
\frac{d\overline{Q}^\varphi}{dP} &= \frac{e^{-\alpha \overline{\mathcal{W}}^\varphi_T - \alpha\varphi(X_T)1_{\tau>T}}}{\mathbb{E}\left[e^{-\alpha \overline{\mathcal{W}}^\varphi_T - \alpha\varphi(X_T)1_{\tau>T}}\right]}; \\
&= e^{-\alpha\left(\overline{\mathcal{W}}^\varphi_T + \varphi(X_T)1_{\tau>T} - G(x;\varphi)\right)},
\end{aligned}
\tag{12}
$$

where we have set $G(x;\varphi)$ as the certainly equivalent to $v(x;\varphi) \triangleq v(x,0;\varphi)$ so that $v(x;\varphi) = U(G(x;\varphi))$. In fact, $\overline{\mathcal{W}}^\varphi$ is a Q uniformly integrable Q-martingale for all $Q \in \mathcal{M}_{e,f}$.

Despite these general facts, there is still much to be learned by studying this problem: first and foremost, what do the optimal strategies look like? How do they differ from the strategies obtained in the absence of default? How may strategies be computed? What is the indifference price for the defaultable option? To answer these questions, we seek to identify the value function $v(;\varphi)$ with a partial differential equation. Thus in Chapter 4 we

(1) Use the dynamic programming principle to (informally) obtain the HJB equation for the value function,
(2) State our main PDE existence result regarding solutions to the HJB equation,
(3) State our main verification result where solutions to the HJB equation are shown to be the value function.

We remark that in the above verification steps, we are able to obtain smooth solutions to the HJB equation under the general conditions outlined in Section 4.2. However, the verification argument requires us to restrict the model to more stringent conditions;

Assumptions 5 and 6. In particular, under Assumption 1 within Assumption 5, the factor process X is an extension of the multi-dimensional OU process in $\mathbb{R}^n$.

We are now ready to state our main result.

Theorem 7 (Optimal Investment). *Under Assumptions 5 and 6, we have*

$$\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[U(\mathcal{W}_T^{\pi}+\varphi(X_T)\mathbb{1}_{\tau>T})\right]=U(G(0,x;\varphi))=-e^{-\alpha G(0,x;\varphi)},$$

where the supremum is attained by the optimal control $\overline{\pi}^{\varphi}$ in $\mathcal{A}_{\mathbb{G}}$ given in (54) *below, and $G(;\varphi)^2\in C^{1,2}([0,T]\times\mathbb{R}^n)$ is a pre-default certainty equivalent for the investor, satisfying the PDE* (25) *below. Moreover, $\overline{\mathcal{W}}^{\varphi}$ is a Q-uniformly integrable $(\mathbb{G},Q)$-martingale for all $Q\in\mathcal{M}_{e,f}$.*

Proof. See Chapter 4. □

Corollary 8 (Utility Indifference Pricing for Defaultable Claims). *Under Assumptions 5 and 6, the buyer's indifference price for the defaultable claim $\varphi(X_T)\mathbb{1}_{\tau>T}$ is $G(0,x;\varphi)-G(0,x;0)$.*

Proof. The buyer's indifference price for the defaultable claim $\varphi(X_T)\mathbb{1}_{\tau>T}$, p_B^{φ}, should satisfy

$$\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[U(\mathcal{W}_T^{\pi}-p_B^{\varphi}+\varphi(X_T)\mathbb{1}_{\tau>T})\right]=\sup_{\pi\in\mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[U(\mathcal{W}_T^{\pi})\right].$$

From Theorem 7, the above relation becomes

$$U(-p_B^{\varphi}+G(0,x;\varphi))=U(G(0,x;0)),$$

which yields the relation $p_B^{\varphi}=G(0,x;\varphi)-G(0,x;0)$. □

Remark 9. Taking $\varphi\equiv 1$ in Corollary 8 will give the price of one unit of a defaultable bond. We will price it explicitly for the simple case of constant coefficients in Remark 14, Chapter 3.

[2] $G(t,x;\varphi)$ is the extension of the certainty equivalent introduced in (12) as $G(x;\varphi)=G(0,x;\varphi)$.

CHAPTER 3

Simple Case

Before proving Theorem 7, we look at a simpler version of the problem where,

Assumption 10 (Simple Case). $\mu \in \mathbb{R}, \sigma > 0$, and $\gamma > 0$ are constant and the Brownian filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion W. Thus there is no factor process X. For the payoff φ, we assume $\varphi \in \mathbb{R}$ is constant.

Here, the problem becomes considerably simpler because the default time τ defined in (2) is independent of the Brownian motion W. The HJB equation gives rise to an ordinary differential equation (ODE) rather than PDE, which allows a rather explicit solution to our optimal investment problem. More importantly, we can compare this result with the celebrated Merton problem, and observe that there is a non-vanishing difference in the optimal strategies as the time horizon increases, due to stock defaulting.

This simple case result is not new in the literature: for example, [**LR12**] discusses when the coefficients are not just constants but time-dependent deterministic functions. They obtain a coupled system of ODEs by the duality argument, which is basically the same as our ODE (13) below derived from the HJB equation. Yet, it is worthwhile to state the result here for analytical tractability. We revisit this simple model when we price the insurance coupon rate in Section 5.3.

Under Assumption 10, the PDE (25) below for the pre-default certainty equivalent turns into an ODE because the x-dependency is lost. Thus, $G(t;\varphi) = G(t,x;\varphi)$ now satisfies

$$
\begin{aligned}
G_t(t;\varphi) - \frac{\sigma^2}{2\alpha}\left(\theta^2(G(t;\varphi)) + 2\theta(G(t;\varphi))\right) + \bar{\gamma} = 0; \quad 0 \le t < T,\\
G(T;\varphi) = \varphi,
\end{aligned}
\tag{13}
$$

where (see (24) and (31) below),

$$
\theta(z) = W\left(\frac{\gamma}{\sigma^2} e^{\frac{\mu}{\sigma^2} + \alpha z}\right), \quad \bar{\gamma} = \frac{1}{\alpha}\left(\gamma + \frac{\mu^2}{2\sigma^2}\right). \tag{14}
$$

Above, $W(\cdot)$ is the product-log function or Lambert W function, satisfying the relation $x = W(x)e^{W(x)}$. It is straightforward to see W is well defined and infinitely differentiable on $(0,\infty)$.

Notice that Assumption 10 implies Assumptions 5 and 6. Thus the conclusion of Theorem 7 holds. Therefore we have

Theorem 11. *Under Assumption 10,*

$$
\sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[U(\mathcal{W}_T^{\pi} + \varphi \mathbb{1}_{\tau > T})\right] = U(G(0;\varphi)) = -e^{-\alpha G(0;\varphi)},
$$

where the supremum is attained by the optimal control $\overline{\pi}^{\varphi}$ in $\mathcal{A}_{\mathbb{G}}$ given in (16) *below, and $G(;\varphi) \in C^1[0,T]$ is a pre-default certainty equivalent for the investor, satisfying the ODE* (13). *Moreover, $\overline{\mathcal{W}}^{\varphi}$ is a Q-uniformly integrable $(\mathbb{G}, Q)$-martingale for all $Q \in \mathcal{M}_{e,f}$.*

We can obtain a rather closed solution of (13) as follows. As in (26) in Section 4.2 below, we first reverse the time in (13) to change the terminal condition to an initial one. As such, we define

$$
u(t) = G(T-t;\varphi); \quad t \le T,
$$

where φ is dropped for simplicity. Using (13), we see that u solves the ODE,

$$-u'(t) - \frac{\sigma^2}{2\alpha}\left(\theta^2(u(t)) + 2\theta(u(t))\right) + \bar{\gamma} = 0; \quad 0 < t \le T,$$
$$u(0) = \varphi.$$

Next, set

$$p(t) \triangleq \theta(u(t)) = \theta(G(T-t;\varphi)), \tag{15}$$

so that the optimal position in the stock, $\overline{\pi}$, becomes (see (23) below),

$$\overline{\pi}^{\varphi}(t) = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - \theta(G(t;\varphi))\right) = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - p(T-t)\right); \quad t \le T. \tag{16}$$

Thus the function p acts as a deviation from the optimal position in stock from the Merton problem, where the exponential investor holds $\mu/(\alpha\sigma^2)$-dollar amount of stock to maximize her utility.

From Lemma 54 below, we have

$$\theta'(z) = \frac{\alpha\theta(z)}{1+\theta(z)}.$$

Thus the ODE for $p(t)$ becomes

$$\begin{aligned} p'(t) &= u'(t)\theta'(u(t)) \\ &= \frac{\alpha p(t)}{1+p(t)}\left[-\frac{\sigma^2}{2\alpha}\left(p^2(t) + 2p(t)\right) + \bar{\gamma}\right] \\ &= -\frac{\sigma^2}{2}\frac{p(t)(p(t)-p_+)(p(t)-p_-)}{1+p(t)}; \quad 0 < t \le T, \end{aligned} \tag{17}$$

where

$$p_{\pm} \triangleq -1 \pm \sqrt{1 + 2\frac{\gamma}{\sigma^2} + \frac{\mu^2}{\sigma^4}},$$

and the initial condition is given by

$$p(0) = \theta(u(0)) = W\left(\frac{\gamma}{\sigma^2}e^{\frac{\mu}{\sigma^2}+\alpha\varphi}\right) > 0.$$

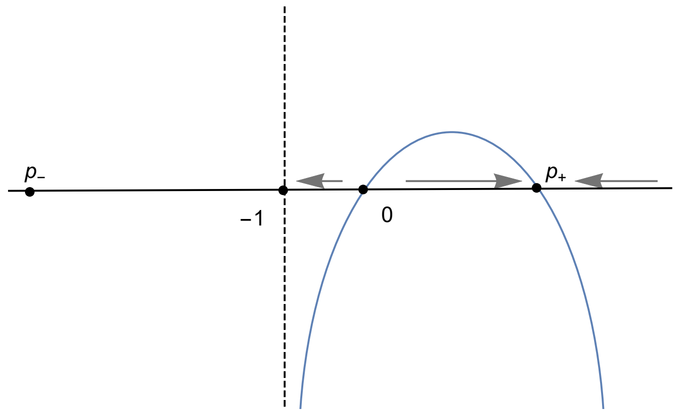

FIGURE 1. Sketch of the flow of the ODE (17). It has two fixed points, 0 (unstable) and p_+ (stable).

Figure 1 shows the flow of the ODE (17). Noticing the shape of the curve in the figure, we see that there is a unique solution for $p(t)$ that starts at $p(0) > 0$ and converges to p_+ as $t \to \infty$.

This result in turn gives $G(;\varphi) \in C^1[0,T]$ as,

$$G(t;\varphi) = \frac{1}{\alpha}\log\left[\frac{p(T-t)e^{p(T-t)}}{\frac{\gamma}{\sigma^2}e^{\frac{\mu}{\sigma^2}}}\right]. \tag{18}$$

from the definition of θ in (14) and (15).

Remark 12. When $\varphi \equiv 0$, we can judge the sign of the optimal position in stock (16) from the values μ and γ as follows. Lemma 57 below with $x = \gamma/\sigma^2$ and $y = \mu/\sigma^2$ gives $p(0) = W(\gamma/\sigma^2 e^{\mu/\sigma^2}) \le p_+$, where the equality holds only when $\mu = \gamma$. Therefore, in light of the flow of ODE from (17), we observe the relations that for $t \in [0,T]$,

$$\begin{cases} \gamma > \mu : & \mu/\sigma^2 < p(0) \le p(t) \le p(T) < p_+ < \gamma/\sigma^2 \longrightarrow \overline{\pi}^0 < 0, \\ \mu > \gamma : & \gamma/\sigma^2 < p(0) \le p(t) \le p(T) < p_+ < \mu/\sigma^2 \longrightarrow \overline{\pi}^0 > 0, \\ \mu = \gamma : & p(t) \equiv \mu/\sigma^2 \longrightarrow \overline{\pi}^0 \equiv 0. \end{cases}$$

In other words, when the default intensity exceeds the return of the stock, the investor is better off betting on the default, thus she shorts the stock. By contrast, when the return of the stock exceeds the default intensity, she longs the stock. Indeed $\mu - \gamma$ acts as an effective drift of the risky asset when μ/σ^2 and γ/σ^2 are equally small compared to 1, because in that case we have $p_+ \approx \gamma/\sigma^2$. Thus for long horizons,

$$\lim_{T\to\infty} \overline{\pi}^0(0) = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - p_+\right) \approx \frac{\mu-\gamma}{\alpha\sigma^2},$$

which is the optimal position in stock in the Merton problem when the stock drift is $\mu - \gamma$.

Remark 13. We observe that there is a non-vanishing difference in the optimal strategies from the Merton problem in the long-time-horizon, small-default-intensity limit. First, we see from (16) and $W(0) = 0$ that

$$\lim_{\gamma\to 0+} \overline{\pi}^\varphi(0) = \lim_{\gamma\to 0+} \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - p(T)\Big|_{p(0)=W\left(\frac{\gamma}{\sigma^2}e^{\mu/\sigma^2+\alpha\varphi}\right)}\right) = \frac{\mu}{\alpha\sigma^2},$$

which is the same as the Merton case, because for any fixed time horizon, taking the default intensity to zero makes the market default-free. Thus the problem boils down to the Merton problem.

However, for long horizons,

$$\begin{aligned}\lim_{\gamma\to 0+}\lim_{T\to\infty} \overline{\pi}^\varphi(0) &= \lim_{\gamma\to 0+}\lim_{T\to\infty} \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - p(T)\Big|_{p(0)=W\left(\frac{\gamma}{\sigma^2}e^{\mu/\sigma^2+\alpha\varphi}\right)}\right)\\ &= \lim_{\gamma\to 0+} \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - p_+\right) = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - \left(\sqrt{1+\frac{\mu^2}{\sigma^4}} - 1\right)\right).\end{aligned}$$

This manifests the fact that, however small the default intensity is, the position in the Merton problem $\mu/(\alpha\sigma^2)$ is too high. The stock will almost surely default in the long-time-horizon limit. Thus the investor prepares for it by holding a smaller amount of stock than she would in Merton case.

Remark 14 (Defaultable Bond Pricing). Using Corollary 8 for the simple case with $\varphi \equiv 1$, we can compute the buyer's indifference price of one unit of a defaultable bond, p_B^1, as

$$p_B^1 = G(0;1) - G(0;0).$$

Figure 2 shows the associated yield to maturity of the defaultable bond, defined by $-\log(p_B^1(T))/T$ for each $T > 0$, under parameters $\mu = .04$, $\sigma = .3$, $\gamma = .03$ and $\alpha = 1$. For comparison, γ the default intensity of τ under the physical measure is plotted, too.

Figure 3 shows the behavior of p_B^1 with varied α, or more precisely, with the varied risk tolerance parameter $1/\alpha$. The increasing and concave nature of the price curve can be understood by the duality formulation of the buyer's indifference price [**IJS05**]:

$$p_B^1 = \inf_{Q \in \mathcal{M}_{e,f}} \left\{ \mathbb{E}^Q \mathbb{1}_{\tau > T} + \frac{1}{\alpha} \Big(H(Q|P) - H(\overline{Q}^0|P) \Big) \right\} \tag{19}$$

where $\overline{Q}^0$ is defined in (8). Moreover, by [**Bec03**, Proposition 3.2], we know that p_B^1 converges to the "risk-neutral" price under MEMM in the zero risk-aversion limit, i.e.,

$$\lim_{1/\alpha \to \infty} p_B^1 = \mathbb{E}^{\overline{Q}^0} \mathbb{1}_{\tau > T} = e^{-\int_0^T \sigma^2 \theta(G(t)) dt},$$

where the last equality follows from the fact that the default intensity of τ under $\overline{Q}^0$ becomes the deterministic process $\overline{\gamma}(t) = \sigma^2 \theta(G(t))$ (see (65) below). Computing the last integration numerically, we obtain

$$\lim_{1/\alpha \to \infty} p_B^1 = 0.967942,$$

which is the horizontal asymptote for the curve in Figure 3. On the other hand, when taking $1/\alpha \to 0$, we can bring the limit inside the infimum in (19) to get ([**DGR$^+$02**, Proposition 5.1]),

$$\lim_{1/\alpha \to 0+} p_B^1 = \inf_{Q \in \mathcal{M}_{e,f}} \mathbb{E}^Q \mathbb{1}_{\tau > T} = 0.$$

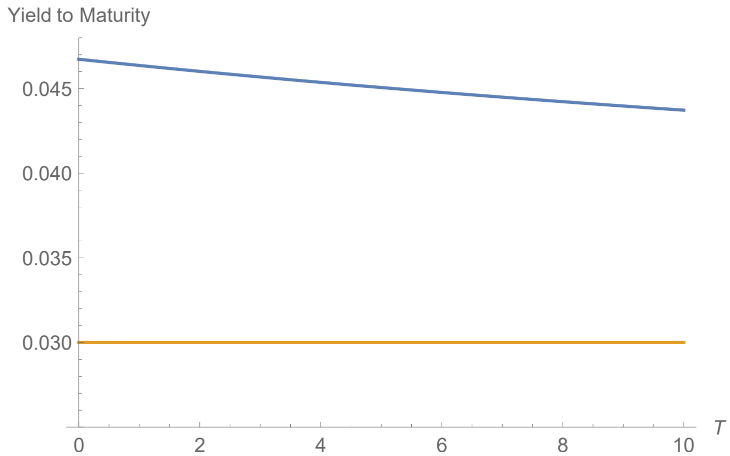

FIGURE 2. Above: Yield to maturity of the defaultable bond when $\mu = .04, \sigma = .3, \gamma = .03$, and $\alpha = 1$. Below: γ, which is the default intensity of τ under the physical measure.

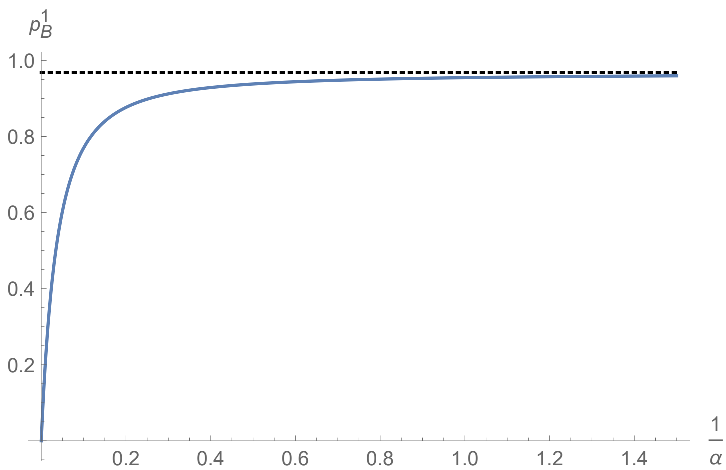

FIGURE 3. Defaultable bond price under various risk tolerance parameter $1/\alpha$ when $\mu = .04, \sigma = .3, \gamma = .03, T = 1$. The horizontal asymptote is at 0.967942.

CHAPTER 4

Proof of the Main Result: Theorem 7

We now embark on the proof of our main result, Theorem 7. The semilinear parabolic PDE for the pre-default certainty equivalent is derived from the HJB equation in Section 4.1. We state and solve our Cauchy problem for the pre-default certainty equivalent in Sections 4.2 and 4.3. Using the obtained solution, we construct explicit candidates for the optimal wealth process and the martingale measure, then explore the implications by the well-developed duality result to those candidates in Section 4.4. The verification is completed in Section 4.6.

We assume Assumption 5 and 6 for the entire chapter except Sections 4.2 and 4.3, where we can solve the Cauchy problem under fairly general conditions, given in Assumptions 20, 21, and 22.

4.1. HJB to PDE

We will denote by $\mathcal{A}_{\mathbb{F}} \subset \mathcal{A}_{\mathbb{G}}$ the class where π is $\mathbb{F}$-predictable. Using the dynamic programming principle, we can (informally) derive the pre-default expected utility at time

t as,

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{G}}}\mathbb{E}\left[\mathbb{1}_{\tau>t}U(\mathcal{W}_T^\pi+\varphi(X_T)\mathbb{1}_{\tau>T})|\mathcal{G}_t\right]\\
=&\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{G}}}\mathbb{E}\left[\mathbb{1}_{\tau>t}U(\mathcal{W}^\pi_{(\tau\wedge T)-}-\pi_\tau\mathbb{1}_{\tau\le T}+\varphi(X_T)\mathbb{1}_{\tau>T})|\mathcal{G}_t\right];\\
=&\mathbb{1}_{\tau>t}e^{\Gamma_t}\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{F}}}\mathbb{E}\left[\int_t^\infty U(\mathcal{W}^\pi_{(u\wedge T)-}-\pi_u\mathbb{1}_{u\le T}+\varphi(X_{u\wedge T})\mathbb{1}_{u>T})dF_u\,\middle|\,\mathcal{F}_t\right];\\
=&\mathbb{1}_{\tau>t}e^{\Gamma_t}\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{F}}}\mathbb{E}\left[\int_t^T U(\mathcal{W}^\pi_{u-}-\pi_u)dF_u+(1-F_T)U(\mathcal{W}^\pi_T+\varphi(X_T))\,\middle|\,\mathcal{F}_t\right];\\
=&\mathbb{1}_{\tau>t}\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{F}}}\mathbb{E}\left[\int_t^T U(\mathcal{W}^\pi_{u-}-\pi_u)e^{-\int_t^u\gamma_s ds}\gamma_u du+e^{-\int_t^T\gamma_s ds}U(\mathcal{W}^\pi_T+\varphi(X_T))\,\middle|\,\mathcal{F}_t\right],
\end{aligned}$$

where in the second equality, we used [**BR04**, Proposition 5.1.1.(ii)] noting that $U(\mathcal{W}^\pi_{(\tau\wedge T)-}-\pi_\tau\mathbb{1}_{\tau\le T}+\varphi(X_T)\mathbb{1}_{\tau>T})$ is the time τ value of an $\mathbb{F}$-predictable process[1] $U(\mathcal{W}^\pi_{(\cdot\wedge T)-}-\pi_\cdot\mathbb{1}_{\cdot\le T}+\varphi(X_{\cdot\wedge T})\mathbb{1}_{\cdot>T})$. By the Markovian nature of the model, the expression in the last line prompts us to define the pre-default value function $v(t,x,w;\varphi)$[2] as

$$\operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{\mathbb{G}}}\mathbb{E}\left[\mathbb{1}_{\tau>t}U(\mathcal{W}_T^\pi+\varphi(X_T)\mathbb{1}_{\tau>T})|\mathcal{G}_t\right]=\mathbb{1}_{\tau>t}v(t,X_t,\mathcal{W}_t;\varphi),$$

where

$$\begin{aligned}
&v(t,x,w;\varphi)\\
=&\sup_{\pi\in\mathcal{A}_{\mathbb{F}}}\mathbb{E}\left[\int_t^T U(\mathcal{W}^\pi_{u-}-\pi_u)e^{-\int_t^u\gamma(X_s)ds}\gamma(X_u)du+e^{-\int_t^T\gamma(X_s)ds}U(\mathcal{W}^\pi_T+\varphi(X_T))\,\middle|\,X_t=x,\mathcal{W}_t=w\right].
\end{aligned}$$

This is exactly of the form to which the extension result of [**Pha09**, Section 3.4.2] is applicable. We can thus derive the HJB equation as (here and in all that follows, the

[1] $\mathcal{W}^\pi_{(\cdot\wedge T)-}$ is $\mathbb{G}$-predictable but we can take a unique $\mathbb{F}$-predictable process that coincide with $\mathcal{W}^\pi_{(\cdot\wedge T)-}$ on $[0,\tau)$. Look at the remark after [**BR04**, Corollary 5.3.1].

[2] $v(t,x,w;\varphi)$ is the extension of $v(x,w;\varphi)$ introduced in (11) as $v(x,w;\varphi)=v(0,x,w;\varphi)$.

appearance of the same index used twice in a term indicates summation from 1 to n),

$$v_t + \max_{\pi} \Big[-\gamma v + \pi\mu v_w + b^i v_{x_i} + \frac{\pi^2}{2}\sigma^2 v_{ww} + \pi\sigma c^{ij}\rho^j v_{x_i,w} + \frac{(cc^\top)^{ij}}{2} v_{x_i,x_j} - \gamma e^{-\alpha(w-\pi)} \Big] = 0, \quad (20)$$

with the terminal condition,

$$v(T, x, w; \varphi) = -e^{-\alpha(w+\varphi(x))}.$$

As introduced in Theorem 7, let $G(t, x; \varphi)$ be the pre-default certainty equivalent at time t with X starting at $X_t = x$ for the terminal payoff of $\varphi(X_T)$, so that

$$v(t, x, w; \varphi) = -e^{-\alpha w} e^{-\alpha G(t,x;\varphi)}.$$

Substituting into (20), we have[3]

$$G_t + \frac{(cc^\top)^{ij}}{2} D_{ij}G - \frac{\alpha}{2}(cc^\top)^{ij} D_iGD_jG + \max_{\pi} \Big[(b^i - \alpha\sigma\pi c^{ij}\rho^j)D_iG - \frac{\alpha}{2}\sigma^2\pi^2 + \mu\pi + \frac{\gamma}{\alpha}\left(1 - e^{\alpha(G+\pi)}\right) \Big] = 0, \quad (21)$$

with the terminal condition,

$$G(T, x; \varphi) = \varphi(x); \quad x \in \mathbb{R}^n.$$

The maximum of (21) is attained when

$$-\mu + \alpha\sigma^2\pi + \alpha\sigma c^{ij}\rho^j D_iG + \gamma e^{\alpha(G+\pi)} = 0. \quad (22)$$

We introduce θ by setting

$$\pi = \frac{1}{\alpha}\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} c^{ij}\rho^j D_iG - \theta \right). \quad (23)$$

[3] We use notations $D_iG = G_{x_i}, D_{ij}G = G_{x_i,x_j}$, and $DG = (G_{x_1}, G_{x_2}, \ldots, G_{x_n})$.

Thus θ denotes the deviation of the optimal strategy from an incomplete but no default market case (at least in a formal level because G here already takes account of the default).

Substituting (23) back into (22), we see that θ must satisfy

$$\theta e^{\theta} = \frac{\gamma}{\sigma^2} \exp\left(\frac{\mu}{\sigma^2} + \alpha G - \frac{\alpha}{\sigma} c^{ij} \rho^j D_i G\right),$$

or $\theta = \theta(x, G, DG)$ at the maximum in (21), where

$$\theta(x, z, p) \triangleq W\left[\frac{\gamma(x)}{\sigma^2(x)} \exp\left(\frac{\mu(x)}{\sigma^2(x)} + \alpha z - \frac{\alpha}{\sigma(x)} c^{ij}(x) \rho^j(x) p_i\right)\right]; \quad (x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n, \tag{24}$$

and $W(\cdot)$ is the product-log function already introduced in (14).

With $\theta = \theta(x, G, DG)$, we can set π as in (23) to get the following partial differential equation for G,

$$\begin{aligned} 0 = G_t + \frac{(cc^\top)^{ij}}{2} D_{ij} G - \frac{\alpha}{2}\left[(cc^\top)^{ij} D_i G D_j G - (c^{ij} \rho^j D_i G)^2\right] + (b^i - \frac{\mu}{\sigma} c^{ij} \rho_j) D_i G \\ - \frac{\sigma^2}{2\alpha}(\theta^2 + 2\theta) + \frac{1}{\alpha}\left(\gamma + \frac{\mu^2}{2\sigma^2}\right); \quad \text{in } [0, T) \times \mathbb{R}^n, \\ G(T, x; \varphi) = \varphi(x); \quad x \in \mathbb{R}^n. \end{aligned} \tag{25}$$

4.2. Setup and Statement of the Cauchy Problem

In the previous section, we obtained the parabolic PDE for the pre-default certainty equivalent, $G(t, x; \varphi)$, in (25). In this section, we rewrite (25) in a form suitable to applying the classical parabolic PDE theory, and state the existence result for the PDE in Theorem 26. We are able to prove the Cauchy existence result, Theorem 26, under Assumptions 20, 22, and 21 stated below, which are more general than Assumptions 5 and 6 required for our main optimal investment result.

The precise assumptions to solve the Cauchy problem are now listed. They are in place only in Section 4.2 and 4.3. Each of the assumptions is a general version of an aforementioned assumption from Section 2.1, as noted right next to the assumption number.

For our generalized spacial domain $D \subset \mathbb{R}^n$, fix $\beta \in (0,1)$ and assume

Assumption 15. $D = \cup_{m=1}^{\infty} D_m$ where for each $m = 1, 2, \ldots$, D_m is an open connected subset of $\mathbb{R}^n$, which is bounded and has a $C^{2+\beta}$-boundary. Furthermore, $\bar{D}_m \subset D_{m+1}$ for $m = 1, 2, \ldots$, thus D is also connected but not necessarily bounded.

which clearly includes the case $D = \mathbb{R}^n$ as in our main setup in Section 2.1. Thus the parabolic PDE (25) is solved in the domain

$$\Omega \triangleq (0,T) \times D.$$

Regarding b, c, γ, ρ, μ, and σ from Section 2.1, assume they are defined on D and satisfy:

Assumption 16 (cf. Assumption 1). $b^i : D \to \mathbb{R}$ and $c^{ij} : D \to \mathbb{R}$ are continuously differentiable for $1 \leq i, j \leq n$. Furthermore, the $n \times n$-matrix $c(x)$ is invertible for all $x \in D$. As such, $a(x) \triangleq c(x)c^{\top}(x)/2$ is locally elliptic in that for each m, there is a $\lambda_m > 0$ so that for all $\xi \in \mathbb{R}^n$ and $x \in D_m$ we have $\Lambda_m \xi^{\top}\xi \geq \xi^{\top}a(x)\xi \geq \lambda_m \xi^{\top}\xi$. ($\Lambda_m$ exists because a is bounded on D_m.)

Assumption 17 (cf. Assumption 2). $\gamma : D \to (0, \infty)$ is continuously differentiable. As such, for all m, $\inf_{x \in D_m} \gamma(x) > 0$.

Assumption 18 (cf. Assumption 3). $\rho : D \to \mathbb{R}^n$ is continuously differentiable and for all $x \in D$ we have $0 \leq |\rho(x)| \leq 1$.

Assumption 19 (cf. Assumption 4). $\mu, \sigma : D \to \mathbb{R}$ are both continuously differentiable with $\sigma(x) > 0$ for all $x \in D$. As such, for all m, $\inf_{x \in D_m} \sigma(x) > 0$.

To ease notation going forward, we consolidate Assumptions 15 – 19 into one assumption:

Assumption 20 (cf. Assumption 5). Assumptions 15 – 19 hold.

For the terminal condition, we assume

Assumption 21 (cf. Assumption 6). $\varphi : D \to \mathbb{R}$ in $C^{2+\beta}_{\text{loc}}$, i.e., φ has second-order derivatives which are β-Hölder continuous in D_m for each m. Furthermore, we assume

$$\sup_{x \in D} |\varphi(x)| < \infty.$$

We first reverse the time in (25) to change the terminal boundary condition into to an initial one. As such, we define

$$u(t,x) = G(T-t, x; \varphi); \quad t \le T,\ x \in D, \tag{26}$$

where φ is dropped for simplicity. Using (25), we see that u solves the PDE,

$$\begin{aligned} 0 = -u_t + \frac{(cc^\top)^{ij}}{2} D_{ij}u - \frac{\alpha}{2}\left[(cc^\top)^{ij} D_i u D_j u - (c^{ij}\rho^j D_i u)^2\right] \\ + \left(b^i - \frac{\mu}{\sigma} c^{ij}\rho^j\right) D_i u - \frac{\sigma^2}{2\alpha}(\theta^2 + 2\theta) + \frac{1}{\alpha}\left(\gamma + \frac{\mu^2}{2\sigma^2}\right); \quad \text{in } (0,T] \times D, \\ u(0,x) = \varphi(x); \quad x \in D, \end{aligned} \tag{27}$$

where $\theta = \theta(x, u, Du)$ (see (24) for the definition of $\theta(x,z,p)$, which is now defined in $D \times \mathbb{R} \times \mathbb{R}^n$).

To conform to the notations in [**Lie96**], we write (27) in the general form

$$Pu \triangleq -u_t + a^{ij}(x) D_{ij}u + a(x, u, Du) = 0, \tag{28}$$

where we drop the obvious arguments (t, x) for solutions to PDEs and their derivatives here and in all that follows. Above,

$$
\begin{aligned}
a^{ij}(x) &= \frac{1}{2}(cc^\top(x))^{ij}, && (29)\\
a(x,z,p) &\triangleq -\frac{\alpha}{2}\left[(cc^\top(x))^{ij}p_ip_j - (c^{ij}(x)\rho^j(x)p_i)^2\right] + \left(b^i(x) - \frac{\mu(x)}{\sigma(x)}c^{ij}(x)\rho^j(x)\right)p_i;\\
&\quad - \frac{\sigma^2(x)}{2\alpha}\left(\theta^2(x,z,p) + 2\theta(x,z,p)\right) + \frac{1}{\alpha}\left(\gamma(x) + \frac{\mu^2(x)}{2\sigma^2(x)}\right);\\
&= -\frac{\alpha}{2}p^\top\overline{a}(x)p + p^\top\overline{b}(x) - \frac{\sigma^2(x)}{2\alpha}\left(\theta^2(x,z,p) + 2\theta(x,z,p)\right) + \overline{\gamma}(x), && (30)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{a}(x) &\triangleq c(x)(I - \rho\rho^\top(x))c^\top(x),\\
\overline{b}(x) &\triangleq b(x) - \frac{\mu(x)}{\sigma(x)}c(x)\rho(x), && (31)\\
\overline{\gamma}(x) &\triangleq \frac{1}{\alpha}\left(\gamma(x) + \frac{\mu^2(x)}{2\sigma^2(x)}\right).
\end{aligned}
$$

Note that $\overline{a}^{ij}(x), \overline{b}^i(x)$ and $\overline{\gamma}(x)$ are all continuously differentiable in D, and so is $a(x,z,p)$ in $D \times \mathbb{R} \times \mathbb{R}^n$ under Assumption 20.

We further assume that $a(x,z,p)$ defined in (30) satisfies:

Assumption 22.

$$
\sup_{x\in D} a(x,0,0) = \sup_{x\in D}\left[-\frac{\sigma^2(x)}{2\alpha}\left(\theta^2(x,0,0) + 2\theta(x,0,0)\right) + \frac{1}{\alpha}\left(\gamma(x) + \frac{\mu^2(x)}{2\sigma^2(x)}\right)\right] \le K
$$

for some positive constant K, where

$$
\theta(x,0,0) = W\left[\frac{\gamma(x)}{\sigma^2(x)}\exp\left(\frac{\mu(x)}{\sigma^2(x)}\right)\right]
$$

by (24).

Note that the above assumption is met under Assumption 5 because both γ and μ^2/σ^2 are bounded from above in $D = \mathbb{R}^n$. Assumption 22 is the condition that guarantees the uniform boundedness of the local solutions. Look at Section 4.3.1 for the detailed discussion.

The solution to the parabolic PDE (27) is obtained as a limit of a localized version of the problem in

$$\Omega_m \triangleq (0,T)\times D_m, \quad m=1,2,\dots.$$

For each m, let Γ_m be the parabolic boundary of Ω_m, i.e., the union of the bottom $\{(t,x): t=0, x\in D_m\}$ and the side $\{(t,x): 0\le t\le T,\ x\in\partial D_m\}$.

The parabolic distance ρ between (t_1,x_1) and (t_2,x_2) in Ω (or more generally in $\mathbb{R}^{n+1}$) is defined as

$$\rho\left((t_1,x_1),(t_2,x_2)\right) \triangleq \max(|t_1-t_2|^{1/2}, |x_1-x_2|).$$

If f is a function in a domain $Q\subset\Omega$, we denote

$$[f]_{\beta;Q} \triangleq \sup_{\substack{(t_1,x_1)\neq(t_2,x_2)\\(t_i,x_i)\in Q}} \frac{|f(t_1,x_1)-f(t_2,x_2)|}{\rho^\beta\left((t_1,x_2),(t_2,x_2)\right)}.$$

We also set

$$|f|_{0;Q} \triangleq \sup_Q |f|.$$

We then introduce the parabolic Hölder space,

Definition 23. $H_{2+\beta}(Q)$ is the Banach space of functions f that are continuous in Q, together with all derivatives of the form f_t, f_x, f_{xx}, and have a finite norm

$$|f|_{2+\beta;Q} \triangleq \sum_{|\alpha|+2j\le 2} \sup |D_x^\alpha D_t^j f|_{0;Q} + \sum_{|\alpha|+2j=2} [D_x^\alpha D_t^j f]_{\beta;Q} + \sum_{|\alpha|=1} \langle D_x^\alpha f\rangle_{1+\beta;Q},$$

where

$$\langle D_x^\alpha f\rangle_{1+\beta;Q} \triangleq \sup_{\substack{t_1\neq t_2\\(t_i,x)\in Q}} \frac{|D_x^\alpha f(t_1,x)-D_x^\alpha f(t_2,x)|}{|t_1-t_2|^{(1+\beta)/2}}.$$

Remark 24. This definition coincides with $H_{2+\beta}(Q)$ in [**Lie96**] or $H^{2+\beta,1+\beta/2}(\overline{Q})$ in [**LSU88**]. Also, notice that $f \in H_{2+\beta}(Q)$ is uniformly continuous in Q with its corresponding derivatives. Thus we can speak about the values of f and its derivatives on ∂Q without ambiguity. With regards to Assumption 21, we see that $\varphi \in H_{2+\beta}(\Omega_m)$ for each m.

To localize our Cauchy problem, we extensively use mollifiers [**Eva10**]:

Definition 25. $\chi_m : \mathbb{R}^n \to [0,1]$ is the C^∞-mollifier supported on $\overline{D_m}$ such that $\chi_m = 1$ on $\overline{D_{m-1}}$ and $\chi_m, D_x\chi_m, D_{xx}\chi_m \to 0$ as $x \to \partial D_m$.

With all assumptions and definitions provided, we are ready to state our Cauchy problem. The existence of a solution to this problem is proved in Section 4.3.

Theorem 26 (Cauchy Problem). *Under the Assumptions 20, 21, and 22, there exists $u \in C^{1,2}([0,T] \times D)$ that solves*

$$\begin{cases} Pu &= 0 \quad \text{in } \Omega, \\ u|_{t=0} &= \varphi. \end{cases} \tag{32}$$

Moreover, u satisfies $u_0 \le u \le u_1$ in Ω for some constants u_0 and u_1, and $u \in H_{2+\beta}(\Omega_m)$ for each $m = 1, 2, \ldots$.

Remark 27. Because of the way we obtain u in Section 4.3.4 below as a limit of a converging subsequence in a precompact subset, there is no guarantee of uniqueness for the solutions in Theorem 26. In fact, through this construction, we do not obtain any information regarding the behaviors of u and Du as they approach $(0,T) \times \partial D$. Such information is crucial in applying the uniqueness results of parabolic PDEs (as we need the the value of u on the parabolic boundary), and in verifying that u represents a pre-default certainty equivalent of the investor in our optimal investment problem (as we need to estimate Du to prove certain local martingales are true martingales). However, once the verification is completed, the uniqueness of the solution is guaranteed by the duality theory results [**KS02**].

4.3. Existence of the solution to the Cauchy Problem in Ω

Since the spacial domain D is not necessarily bounded in Theorem 26, we cannot directly cite the classical semi-linear parabolic PDE results (e.g. [**Lie96**]) which deal with problems in bounded domains with parabolic boundary conditions provided. Therefore, we break the proof of Theorem 26 into four steps, Propositions 28 – 31. Each step assumes the result from the previous step. The idea of the proof is: we construct a local solution in each bounded domain Ω_k in Proposition 28, and, with certain interior boundedness results from Propositions 29 and 30, we construct the solution with the desired properties in the whole domain Ω in Proposition 31. Assumptions 20, 21, and 22 prevail throughout these four propositions.

In this section, $C(m)$ represents a constant that only depends on the values of $b, c, \mu, \sigma, \gamma, \rho$, and φ in D_m, and possibly on the geometries of $D_1, D_2, \dots D_m$. There is no relation in the multiple appearances of $C(m)$.

Proposition 28 (Section 4.3.1). *For each $k \in \mathbb{N}$, there exists $u^{(k)} \in H_{2+\beta}(\Omega_k)$ solving*

$$Pu^{(k)} = 0 \text{ in } \Omega_k, \quad u^{(k)}|_{t=0} = \varphi \text{ in } D_k. \tag{33}$$

Moreover, the solutions are uniformly bounded, i.e., we can pick constants u_0 and u_1 such that $u_0 \le u^{(k)} \le u_1$ in Ω_k for any $k \in \mathbb{N}$.

Proposition 29 (Section 4.3.2). *For $(u^{(k)})_k$ obtained in Proposition 28 and for each $m \in \mathbb{N}$,*

$$\sup_{k \ge m+1} |Du^{(k)}|_{0;\Omega_m} \le C(m+1).$$

Proposition 30 (Section 4.3.3). *For $(u^{(k)})_k$ obtained in Proposition 28 and for each $m \in \mathbb{N}$,*

$$\sup_{k \geq m+2} \left([u^{(k)}]_{\beta;\Omega_m} + [Du^{(k)}]_{\beta;\Omega_m} \right) \leq C(m+2).$$

Proposition 31 (Section 4.3.4). *For $(u^{(k)})_k$ obtained in Proposition 28 and for each $m \in \mathbb{N}$,*

$$\sup_{k \geq m+3} |u^{(k)}|_{2+\beta;\Omega_m} \leq C(m+3).$$

Consequently, there exists $u \in C^{1,2}([0,T] \times D), u_0 \leq u \leq u_1$ which satisfies (32).

4.3.1. Proof of Proposition 28. First we prove the following lemma for $a(x,z,p)$ defined in (30).

Lemma 32. $a(x,z,0) \geq 0$ *for* $z \leq 0$*. Thus* $za(x,z,0) < 0$ *for* $z < 0$*.*

Proof. $a(x,z,0)$ depends on z only through $\theta(x,z,0)$ by (30). Note also that, since the product-log function is an increasing function on $(0,\infty)$, $\theta(x,z,0)$ is increasing in z by (24), thus $a(x,z,0)$ itself is a decreasing function of z. Therefore, to show the lemma, we only need to prove

$$a(x,0,0) \geq 0, \tag{34}$$

where

$$a(x,0,0) = -\frac{\sigma^2(x)}{2\alpha}\left(\theta^2(x,0,0) + 2\theta(x,0,0)\right) + \frac{\sigma^2(x)}{2\alpha}\left[\left(\frac{\mu(x)}{\sigma^2(x)}\right)^2 + 2\frac{\gamma(x)}{\sigma^2(x)}\right].$$

Here, we can use Corollary 58 setting $x = \gamma/\sigma^2$ and $y = \mu/\sigma^2$ to show $\theta^2 + 2\theta \leq (\mu/\sigma^2)^2 + 2\gamma/\sigma^2$, thus (34) holds. □

Coming back to the proof of Theorem 28, we first fix $k \geq 2$. Even though (33) has no spatial boundary condition, we need to specify a parabolic boundary condition on Γ_k to obtain the solution to (33). In fact, we will construct $\Phi_k \in H_{2+\beta}(\Omega_k)$ and then take our parabolic boundary condition as the value of Φ_k uniquely extended to Γ_k.

$\Phi_k(t,x)$ should take the same value as $\varphi(x)$ for $x \in D_{k-1}$, but it requires a special care near the spacial boundary. In particular, the behavior of Φ_k at the "corner" $\{t=0\} \times \partial D_k$ is crucial because all values of u, u_t, u_x, and u_{xx} are determined by Φ_k there.

With this in mind, we set

$$\Phi_k(t,x) \triangleq \chi_k(x)\varphi(x),$$

and define a new parabolic operator

$$P^{(k)}u(t,x) \triangleq -u_t(t,x) + a^{ij}(x)D_{ij}u(t,x) + \chi_k(x)a\left(x, u(t,x), Du(t,x)\right).$$

Notice that Φ_k satisfies the *compatibility condition of the first order* for the parabolic operator $P^{(k)}$, i.e.,

$$P^{(k)}\Phi_k = 0 \quad \text{on } \{t=0\} \times \partial D_k. \tag{35}$$

We now seek for the existence of the solution to the following parabolic system,

$$\begin{cases} P^{(k)}u &= 0 \quad \text{in } \Omega_k, \\ u|_{\Gamma_k} &= \Phi_k|_{\Gamma_k}. \end{cases} \tag{36}$$

Theorem 12.16 in [**Lie96**][4] gives the existence of the solution to (36). As for the condition (12.27) within the theorem, it is straightforward to see $a_p^{ij} = a_z^{ij} = 0$ and a_x^{ij} is bounded on Ω_k. By (30) and Corollary 56 below, we have $a = O(|p|^2)$ as $|p| \to \infty$ (see the beginning of Appendix A for the definition of $O(|p|^2)$). Thus (12.27) is satisfied.

---

[4][**Lie96**, Theorem 12.16] actually proves that our solution u to (36) is in the class $H_{2+\beta}^{(-1-\gamma)}(\Omega_k)$, where $\gamma \in (0,1)$ is a constant determined by the regularity of parabolic boundary Γ_k and the regularity of function Φ_k. The class $H_{2+\beta}^{(-1-\gamma)}$ is defined in Chapter 4 of [**Lie96**] as the class of functions that are $H_{2+\beta}$ in the region strictly away from the parabolic boundary but may blow up near the parabolic boundary.
In our case, since the parabolic boundary Γ_k is $H_{2+\beta}$ and $\Phi_k \in H_{2+\beta}(\Omega_k)$ satisfies the compatibility condition of the first order, (35), the comment at the end of [**Lie96**, Theorem 12.14] also applies to Theorem 12.16, thus we have $u^{(k)} \in H_{2+\beta}(\Omega_k)$ for the solution of (36). This comment is based on the additional hypothesis in [**Lie96**, Theorem 8.2], which can be further traced back to the linear result with the compatibility condition, [**Lie96**, Theorem 5.14].

We also need to check the condition (12.26) within the theorem, which is the condition for the maximum principle as stated in [**Lie96**, Theorem 9.5]. Namely, if we can find constants C_1 and C_2 such that

$$z \cdot \chi_k(x)a(x,z,0) \le C_1 z^2 + C_2, \tag{37}$$

then [**Lie96**, Theorem 9.5] implies that the solution to (36) satisfies

$$\sup_{\Omega_k} u^+ \le e^{(C_1+1)T}(\sup_{\Gamma_k} u^+ + C_2^{1/2}).$$

Applying the same theorem to the PDE satisfied by $v \triangleq -u$:

$$-v_t + a^{ij}(x)D_{ij}v - \chi_k(x)a(x,-v,-Dv) = 0,$$

we see that the same condition (37) leads to

$$\sup_{\Omega_k} v^+ \le e^{(C_1+1)T}(\sup_{\Gamma_k} v^+ + C_2^{1/2}).$$

Finally, combining the bounds on u^+ and v^+ yields

$$\sup_{\Omega_k} |u| \le e^{(C_1+1)T}(\sup_{\Gamma_k} |u| + C_2^{1/2}) \le e^{(C_1+1)T}(\sup_{D} |\varphi| + C_2^{1/2}). \tag{38}$$

Coming back to (37), we can evaluate $z \cdot \chi_k(x)a(x,z,0)$ as follows. By Lemma 32, (37) always holds when $z < 0$. For $z > 0$, observe

$$z \cdot \chi_k(x)a(x,z,0) \le za(x,0,0) \le Kz$$

where the last inequality follows from Assumption 22. Thus we can indeed choose constants C_1 and C_2, only dependent on K, such that (37) holds. Therefore the right-hand side of (38) is a finite constant not depending on k.

Therefore, from [**Lie96**, Theorem 12.16], there exists a solution $u \in H_{2+\beta}(\Omega_k)$ for (36), which satisfies $u_0 \le u \le u_1$ in Ω_k, where u_0 and u_1 do not depend on k. We see that the restriction of the solution u to Ω_{k-1} satisfies (33). Therefore we set $u^{(k-1)} \triangleq u|_{\Omega_{k-1}}$ to have

$$|u^{(k-1)}|_{2+\beta;\Omega_{k-1}} \leq |u|_{2+\beta;\Omega_k} < \infty,$$

i.e., $u^{(k-1)} \in H_{2+\beta}(\Omega_{k-1})$. Thus Proposition 28 is proved.

4.3.2. Proof of Proposition 29. Fix $m \geq 2$ and $k \geq m+1$. Since we are only interested in the gradient strictly away from the "side" (note $D_m \subset\subset D_k$), we can use the local gradient bound results from [**Lie96**, Section 11.3], in particular Theorem 11.3 (b).

The Bernstein $\mathcal{E}$ function is defined in [**Lie96**, (8.3)] as

$$\mathcal{E}(x,p) \triangleq a^{ij}(x)p_i p_j. \tag{39}$$

From Assumption 16, we have

$$\mathcal{E}(x,p) \geq \lambda_m |p|^2; \quad \forall x \in D_m, \ \forall p \in \mathbb{R}^n.$$

We also use some differential operators defined in [**Lie96**, Chapter 11]:

$$\delta(p) = D_z + |p|^{-2} p \cdot D_x, \quad \overline{\delta}(p) = p \cdot D_p. \tag{40}$$

As noted right after [**Lie96**, (11.4)], we take

$$a_*^{ij} = a^{ij}, \quad f_j = 0.$$

Note also that v should be read $|p|^2$ as defined right after [**Lie96**, (11.2)].

Then, for the domain Ω_m, the quantities A, B and C defined in (11.7) [**Lie96**], become

$$\begin{aligned}
A_m(x,z,p) &\triangleq \frac{1}{\mathcal{E}(x,p)}\left(\frac{|p|^2}{2\lambda_m}\sum\nolimits_{i,j}\left(\overline{\delta}(p)a^{ij}(x)\right)^2 + (\overline{\delta}(p)-1)\mathcal{E}(x,p)\right); \\
&= \frac{1}{\mathcal{E}(x,p)}\left(0 + (p\cdot D_p - 1)a^{ij}(x)p_ip_j\right); \\
&= 1, \qquad (41)
\end{aligned}$$

$$B_m(x,z,p) \triangleq \frac{1}{\mathcal{E}(x,p)}\left(\delta(p)\mathcal{E}(x,p) + (\overline{\delta}(p)-1)a(x,z,p)\right), \qquad (42)$$

$$C_m(x,z,p) \triangleq \frac{1}{\mathcal{E}(x,p)}\left(\frac{|p|^2}{2\lambda_m}\sum\nolimits_{i,j}(\delta(p)a^{ij}(x))^2 + \delta(p)a(x,z,p)\right). \qquad (43)$$

The estimation of $B_m(x,z,p)$ and $C_m(x,z,p)$ are done in Lemmas 48 and 49, respectively. From those estimations, we obtain

$$A_m^\infty, B_m^\infty, C_m^\infty = \limsup_{|p|\to\infty}\sup_{\Omega_m\times[u_0,u_1]} A_m, B_m, C_m. \qquad (44)$$

are all finite with

$$A_m^\infty = 1, \quad C_m^\infty = 0.$$

For [**Lie96**, (11.17)], we only need to check (11.17b) since $a^{ij}_{*,p} = a^{ij}_{,p} = 0$ and $f = 0$. As for (11.17b), we take $\theta = 1$ and we introduce the following quantity D for the domain Ω_m:

$$D_m(x,z,p) \triangleq \frac{1}{\mathcal{E}(x,p)}\left(|p|^2\Lambda_m + |p|(|\mathcal{E}_p(x,p)| + |a_p(x,z,p)|)\right). \qquad (45)$$

The estimation of $D_m(x,z,p)$ is done in Lemma 50. We have

$$D_m^\infty = \limsup_{|p|\to\infty}\sup_{\Omega_m\times[u_0,u_1]} D_m, \qquad (46)$$

which is also finite.

We are now ready to apply [**Lie96**, Theorem 11.3 (b)] to the domain Ω_m. Set $r = \mathrm{dist}(D_{m-1}, \partial D_m)$ so that, as long as we choose $x \in D_{m-1}$, we have the cylinder $(0,T)\times$

$B(x,R)$ contained within Ω_m. Hence we have, by Theorem 11.3 (b)

$$\sup_{\Omega_{m-1}} |Du^{(k)}| \leq C_3 \left(1 + \frac{\operatorname{osc} u}{r}\right),$$

where C_3 depends on $\sup_{D_m} |D\varphi|$, $A_\infty^{(m)}$, $B_\infty^{(m)}$, $C_\infty^{(m)}$, $D_\infty^{(m)}$ and their limit behaviors in (44) and (46). Here, $\operatorname{osc} u$ is the oscillation of u in Ω_m as defined in [**Lie96**, Section 4.1], which is in our case bounded by $u_1 - u_0$ (see Theorem 28).

Noticing the right-hand side of the above inequality does not depend on k, we have

$$\sup_{k\geq m} |Du^{(k)}|_{0;\Omega_{m-1}} \leq C(m),$$

which is the statement of this theorem after reindexing.

4.3.3. Proof of Proposition 30. Fix $m \geq 2$ and $k \geq m+1$. We use [**Fri13**, Theorem 4, Section 2, Chapter 7] in Ω_m to get the estimation. Instead of applying the theorem directly to $u^{(k)}$, we apply it to the truncated version of the function $u^{(k)}$ defined by

$$v \triangleq \chi_m(x)\left(u^{(k)} - \varphi(x)\right) = \chi_m(x)u^{(k)} - \Phi_m(x). \tag{47}$$

Note that $v \in H_{2+\beta}(\Omega_m)$ and $v = 0$ on Γ_m.

Since $u^{(k)}$ satisfies the PDE

$$-u_t^{(k)} + a^{ij}(x)D_{ij}u^{(k)} = -a(x, u^{(k)}, Du^{(k)}),$$

the PDE satisfied by v becomes

$$-v_t + a^{ij}(x)D_{ij}v = f(t,x), \tag{48}$$

where

$$
\begin{aligned}
f(t,x) \triangleq & -\chi_m(x)a(x,u^{(k)},Du^{(k)}) \\
& + a^{ij}(x)\left(u^{(k)}D_{ij}\chi_m + D_i\chi_m D_j u^{(k)} + D_j\chi_m D_i u^{(k)}\right) - a^{ij}(x)D_{ij}(\chi_m\varphi). \quad (49)
\end{aligned}
$$

Since $\chi_m, D_x\chi_m, D_{xx}\chi_m \to 0$ as $x \to \partial D_m$ by Definition 25, f vanishes at $\{t = 0\} \times \partial D_m$. Moreover, since

- $|\chi_m|_{0;\Omega_m} + |D_x\chi_m|_{0;\Omega_m} + |D_{xx}\chi_m|_{0;\Omega_m} \le C(m)$ by Definition 25,
- $u_0 \le u^{(k)} \le u_1$ by Proposition 28,
- $|Du^{(k)}|_{0;\Omega_m} \le C(m+1)$ by Proposition 29,
- $\varphi \in H_{2+\beta}(\Omega_m)$ by Assumption 21,

we have

$$
|f|_{0;\Omega_m} \le C(m+1). \quad (50)
$$

Therefore, by [**Fri13**, Theorem 4, Section 2, Chapter 7], we have

$$
[v]_{\beta,\Omega_m} + [Dv]_{\beta,\Omega_m} \le C_4|f|_{0,\Omega_m}, \quad (51)
$$

where $C_4 = C(m)$[5]. Since $u^{(k)} = \varphi + v$ in Ω_{m-1} by (47), we have by triangle inequality,

$$
\begin{aligned}
[u^{(k)}]_{\beta,\Omega_{m-1}} + [Du^{(k)}]_{\beta,\Omega_{m-1}} &\le [\varphi]_{\beta,\Omega_{m-1}} + [v]_{\beta,\Omega_{m-1}} + [D\varphi]_{\beta,\Omega_{m-1}} + [Dv]_{\beta,\Omega_{m-1}}; \\
&\le [\varphi]_{\beta,\Omega_{m-1}} + [D\varphi]_{\beta,\Omega_{m-1}} + [v]_{\beta,\Omega_m} + [Dv]_{\beta,\Omega_m}; \\
&\le |\varphi|_{2+\beta,\Omega_{m-1}} + C_4|f|_{0,\Omega_m},
\end{aligned}
$$

where the first inequality is from the triangle inequality and the last one from (51). Since the last line is $C(m+1)$ by (50), with reindexing, we have proved the proposition.

[5]For the conditions of the [**Fri13**, Theorem 4, Section 2, Chapter 7], their H_0 is the same as our elliptic constant λ_m whereas continuous differentiability of $a = cc^\top/2$ and $C^{2+\beta}$-boundary of D_m guarantee the existence of H_1 and H_2 (see Assumption 15 and 16). The constant C_4 only depends on H_0, H_1, H_2, β, and Ω_m.

4.3.4. Proof of Theorem 31. Application of the well-known linear parabolic PDE existence result (for example [**Lie96**, Theorem 5.14]) to (48) with the boundary condition $v|_{\Gamma_m} = 0$ yields,

$$|v|_{2+\beta;\Omega_m} \leq C_5 |f|_{\beta,\Omega_m} \triangleq C_5 \left(|f|_{0,\Omega_m} + [f]_{\beta,\Omega_m} \right),$$

where C_5 only depends on $|a^{ij}|_{\beta;\Omega_m}, \Omega_m$, and β. In the right-most term, $|f|_{0,\Omega_m}$ is already estimated in (50). Furthermore, the result of Preposition 30, together with the fact $\chi_m, \varphi \in H^{2+\beta}(\Omega_m)$, allows us to estimate $[f]_{\beta,\Omega_m}$ through (49) to have

$$[f]_{\beta,\Omega_m} \leq C(m+2).$$

Thus we have

$$|v|_{2+\beta;\Omega_m} \leq C(m+2).$$

Similarly to the argument following (51) in Section 4.3.3, restriction of the domain Ω_m to Ω_{m-1} gives,

$$|u|_{2+\beta,\Omega_{m-1}} \leq C(m+2),$$

which is the first statement of Theorem 31 after reindexing.

By employing the usual diagonal argument, we can extract from $(u^{(k)})$ a subsequence $(u^{(k_l)})$ that converges together with the derivatives $u_t^{(k_l)}, u_x^{(k_l)}, u_{xx}^{(k_l)}$ at each point of D to some function u and its corresponding derivatives. It is clear that u does not exceed the bound $[u_0, u_1]$ and belongs to $H_{2+\beta}(\Omega_m)$ for each m. Thus u is the solution of the Cauchy problem (32), with its derivatives u_t, u_x, u_{xx} allowing continuous extensions to $[0,T] \times D$.

4.4. Optimal Investment Problem

Using the solution to the Cauchy problem from the last section, we construct explicitly a candidate optimal strategy and a candidate martingale measure in this section, both of which are already alluded to in Section 2.2. We will then apply the well-developed duality result, Theorem 34, to those explicit candidates to answer our main optimal investment problem. Assumptions 5 and 6 are in place.

Remark 33. Assumptions 5 and 6 imply Assumptions 20 and 21, respectively. Moreover, Assumption 5 implies Assumption 22, the extra condition that guarantees the uniform boundedness of local solutions in Section 4.3.1. Thus, we already know from Theorem 26 that there exists $u \in C^{1,2}([0,T] \times \mathbb{R}^n)$ satisfying (32) with $u_0 \le u \le u_1$.

Recall that the pre-default certainty equivalent with the terminal payoff φ, $G(t,x;\varphi)$, is related to the solution to Cauchy problem u in Theorem 26 through (26). As such, we have $G(;\varphi) \in C^{1,2}([0,T] \times \mathbb{R}^n)$ which satisfies (21):

$$G_t + \frac{(cc^\top)^{ij}}{2} D_{ij}G - \frac{\alpha}{2}(cc^\top)^{ij} D_iGD_jG \\ + (b^i - \alpha\sigma\overline{\pi}c^{ij}\rho^j)D_iG - \frac{\alpha}{2}\sigma^2(\overline{\pi})^2 + \mu\overline{\pi} + \frac{\gamma}{\alpha}\left(1 - e^{\alpha(G+\overline{\pi})}\right) = 0, \quad (52)$$

where (see (23)),

$$\overline{\pi}(t,x;\varphi) \triangleq \frac{1}{\alpha}\left(\frac{\mu(x)}{\sigma^2(x)} - \frac{\alpha}{\sigma(x)}c^{ij}(x)\rho^j(x)D_iG(t,x;\varphi) - \theta(x, G(t,x;\varphi), DG(t,x;\varphi))\right); \\ t \le T,\ x \in \mathbb{R}^n. \quad (53)$$

Let our candidate optimal strategy[6] be

$$\overline{\pi}_t^\varphi \triangleq \overline{\pi}(t, X_t;\varphi); \quad t \le T, \quad (54)$$

and our candidate martingale measure $\overline{Q}^\varphi$ be

$$\frac{d\overline{Q}^\varphi}{dP} \triangleq e^{-\alpha\left(\overline{\mathcal{W}}_T^\varphi + \varphi(X_T)\mathbb{1}_{\tau > T} - G(0,x;\varphi)\right)}, \quad (55)$$

although $\overline{Q}^\varphi$ defined this way may not be a probability measure at this point.

[6]Although the candidate optimal strategy $\overline{\pi}^\varphi$ is defined up until T, its value is only relevant to our problem for $t \le \tau \wedge T$.

Let P^φ be defined by

$$\frac{dP^\varphi}{dP} \triangleq \frac{e^{-\alpha\varphi(X_T)\mathbb{1}_{\tau>T}}}{\mathbb{E}\left[e^{-\alpha\varphi(X_T)\mathbb{1}_{\tau>T}}\right]}. \tag{56}$$

Observing that $\overline{\mathcal{W}}^\varphi$, the wealth process driven by the candidate optimal strategy $\overline{\pi}^\varphi$ in (54), and $\overline{Q}^\varphi$ defined in (55) satisfy the first-order conditions of the duality, i.e.,

$$\frac{d\overline{Q}^\varphi}{dP^\varphi} \propto \exp(-\alpha\overline{\mathcal{W}}^\varphi_T),$$

we have the following result.

Theorem 34. *Suppose*[7]

(i) $\overline{Q}^\varphi \in \mathcal{M}_{e,f}$,

(ii) $(\overline{\mathcal{W}}^\varphi_t)_{t\leq T}$ *is a true* $\overline{Q}^\varphi$*-martingale and a* Q*-supermartingale for all* $Q \in \mathcal{M}_{e,f}$.

Then, $\overline{Q}^\varphi$ *is the unique MEMM under* P^φ. *Moreover,* $(\overline{\mathcal{W}}^\varphi_t)_{t\leq T}$ *is a* Q*-uniformly integrable* Q*-martingale for all* $Q \in \mathcal{M}_{e,f}$. *Thus the duality gap is closed and we have the optimal investment result in Theorem 7.*

Proof. See Propositions 3.1 and 3.3 in [**KS02**]. □

Therefore Theorem 7 is proved once the two conditions in Theorem 34 are verified. We will postpone it until Section 4.6. The next section lays out preparations for Section 4.6.

4.5. Girsanov Theorem and Related Estimations

Before we proceed, we review the Girsanov-type result in our setting. For the proof of the next theorem, see Proposition 5.3.1 and the remark following Corollary 5.3.1 in [**BR04**].

[7]The two conditions in Theorem 34 do not seem to involve P^φ. The actual condition for the finite relative entropy in (i) is $H(Q|P^\varphi) < \infty$ but this is equivalent to $H(Q|P) < \infty$, because the payoff φ is assumed to be bounded under Assumption 21. Look at the discussion in [**DGR$^+$02**, Section 2] for details. Therefore, we do not make distinction between the underlying measure P or P^φ when we talk about the finiteness of relative entropies.

Theorem 35. *For any measure $Q \sim P$ on $(\Omega, \mathcal{G}_T)$, we have the representation*

$$\begin{aligned}\frac{dQ}{dP} &= \mathcal{E}_T\left(\sum_{0\le j\le n}\int_0^\cdot \beta_t^j dW_t^j + \int_{0+}^\cdot \kappa_t dM_t\right);\\ &= \mathcal{E}_T\left(\int_{0+}^\cdot \kappa_t dM_t\right)\cdot\prod_{j=0}^n \mathcal{E}_T\left(\int_0^\cdot \beta_t^j dW_t^j\right)\end{aligned}$$

where β and κ are $\mathbb{G}$- and $\mathbb{F}$-predictable processes, respectively.

By the Girsanov theorem,

$$W_\cdot^Q = W_\cdot - \int_0^\cdot \beta_t dt$$

is a $(\mathbb{G}, Q)$-Brownian motion, and

$$M_\cdot^Q = H_\cdot - \int_0^{\cdot\wedge\tau} \gamma_t(1+\kappa_t)dt$$

is a $(\mathbb{G}, Q)$-martingale.

From (4) and using Theorem 35, we have a concrete description of the set of equivalent local-martingale measures $\mathcal{M}_e$ defined in (5).

Lemma 36. *For $Q \sim P$, we have*

$Q \in \mathcal{M}_e$ *if and only if*

$$\mu_t + \sigma_t\left(\sum_{j=1}^n \rho_t^j\beta_t^j + \sqrt{1-|\rho_t|^2}\beta_t^0\right) - \gamma_t(1+\kappa_t) = 0 \quad \textit{for } t \le \tau\wedge T, \tag{57}$$

where the processes β and κ are as defined in Theorem 35.

Proof. Fix $Q \sim P$. From (3),

$$B_\cdot^Q \triangleq B_\cdot - \sum_{j=1}^n \int_0^\cdot \rho^j(X_t)\beta_t^j dt - \int_0^\cdot \sqrt{1-|\rho(X_t)|^2}\beta_t^0 dt \tag{58}$$

is a $(\mathbb{G}, Q)$-Brownian motion by Theorem 35.

Again by Theorem 35, the dynamics (4) become

$$\frac{dS_t}{S_{t-}} = \mathbb{1}_{t\le\tau}\left[\left(\mu_t + \sigma_t\left(\sum\nolimits_{j=1}^n \rho_t^j\beta_t^j + \sqrt{1-|\rho_t|^2}\beta_t^0\right) - \gamma_t(1+\kappa_t)\right)dt + \sigma_t dB_t^Q\right] - dM_t^Q.$$

Since stochastic integrals of adapted processes with càglàd paths with respect to local martingales are local martingales by [**Pro03**, Theorem 33, Chap.IV], the two stochastic integrals with respect to B^Q and M^Q above are $(\mathbb{G}, Q)$-local martingales. Thus S is a $(\mathbb{G}, Q)$-local martingale if and only if the dt-integration part is a $(\mathbb{G}, Q)$-local martingale. Since the continuous local martingales with finite variation paths have to be constant (see [**RY13**, Proposition (1.2), Chap. IV]), and S_{t-} is strictly positive for $t \le \tau \wedge T$, we reach the conclusion. □

In order to apply Lemma 36 to $\overline{Q}^\varphi$ defined in (55), we first define a possible density process $(Z_t)_{t\le T}$ for $\overline{Q}^\varphi$ by,

$$\begin{aligned} Z_t &= e^{-\alpha(\overline{\mathcal{W}}_t^\varphi - G(0,x;\varphi))}\cdot\left(H_t + (1-H_t)e^{-\alpha G(t,X_t;\varphi)}\right) \\ &= \begin{cases} \exp\left[-\alpha(\overline{\mathcal{W}}_t^\varphi - G(0,x;\varphi) + G(t,X_t;\varphi))\right]; & t<\tau, \\ \exp\left[-\alpha(\overline{\mathcal{W}}_t^\varphi - G(0,x;\varphi))\right]; & \tau \le t. \end{cases} \end{aligned} \tag{59}$$

Note that $Z_0 = 1$ and $Z_T = d\overline{Q}^\varphi/dP$ hold.

Before computing the differential of Z, first observe the differentials:

$$\begin{aligned} d\left(e^{-\alpha\overline{\mathcal{W}}_t^\varphi}\right) &= e^{-\alpha\overline{\mathcal{W}}_{t-}^\varphi}\mathbb{1}_{t\le\tau}\left[-\alpha\overline{\pi}_t^\varphi(\mu_t dt + \sigma_t dB_t) + \frac{\alpha^2}{2}\sigma_t^2(\overline{\pi}_t^\varphi)^2 dt\right] \\ &\qquad + \left(e^{-\alpha(\overline{\mathcal{W}}_{t-}^\varphi - \overline{\pi}_t^\varphi)} - e^{-\alpha\overline{\mathcal{W}}_{t-}^\varphi}\right)dH_t, \\ d\left(e^{-\alpha G}\right) &= e^{-\alpha G}\Bigg[-\alpha G_t dt - \alpha D_i G(b_t^i dt + c^{ij}dW_t^j \\ &\qquad + \frac{1}{2}(cc^\top)^{ij}(\alpha^2 D_i G D_j G - \alpha D_{ij}G)dt\Bigg], \end{aligned} \tag{60}$$

where G, G_t, D_iG, and $D_{ij}G$ are all evaluated at $(t, X_t; \varphi)$.

Thus,

$$
\begin{aligned}
dZ_t = {} & \mathbb{1}_{t\le\tau} Z_{t-}(-\alpha)\Big[G_t + D_i G b_t^i + \frac{1}{2}(cc^\top)^{ij}(D_{ij}G - \alpha D_i G D_j G) + \mu_t \overline{\pi}_t^\varphi - \frac{\alpha}{2}\sigma_t^2(\overline{\pi}_t^\varphi)^2 \\
& - \alpha\sigma_t\overline{\pi}_t^\varphi \rho_t^j D_i G c^{ij}\Big] dt + \mathbb{1}_{t\le\tau} Z_{t-}\left(-\alpha\overline{\pi}_t^\varphi \sigma_t dB_t - \alpha D_i G c^{ij} dW_t^j\right) + (Z_t - Z_{t-})\, dH_t; \\
= {} & \mathbb{1}_{t\le\tau} Z_{t-}\gamma_t\left(1 - e^{\alpha(G+\overline{\pi}_t^\varphi)}\right) dt + \mathbb{1}_{t\le\tau} Z_{t-}\left(-\alpha\overline{\pi}_t^\varphi \sigma_t dB_t - \alpha D_i G c^{ij} dW_t^j\right) \\
& \qquad + Z_{t-}\left(\frac{Z_t}{Z_{t-}} - 1\right) dH_t; \\
= {} & \mathbb{1}_{t\le\tau} Z_{t-}\left(-\alpha\overline{\pi}_t^\varphi \sigma_t dB_t - \alpha D_i G c^{ij} dW_t^j\right) + Z_{t-}\left(e^{\alpha(G+\overline{\pi}_t^\varphi)} - 1\right) dM_t; \\
= {} & Z_{t-}\Big[\,\mathbb{1}_{t\le\tau}\left(-\alpha\overline{\pi}_t^\varphi \sigma_t \rho_t^j - \alpha D_i G c^{ij}\right) dW_t^j - \mathbb{1}_{t\le\tau}\alpha\overline{\pi}_t\sigma\sqrt{1-|\rho_t|^2}\, dW_t^0 \\
& \qquad + \left(\frac{\sigma_t^2}{\gamma_t}\theta_t - 1\right) dM_t\Big],
\end{aligned}
$$

where (52) is used for the second equality, and the relation

$$
e^{\alpha(G(t,x;\varphi)+\overline{\pi}(t,x;\varphi))} = \frac{\sigma^2(x)}{\gamma(x)}\theta(x, G(t,x;\varphi), DG(t,x;\varphi)),
$$

derived from (23) and (24), is used for the last equality. The form of dZ shows that Z is a local martingale under P, again by [**Pro03**, Theorem 33, Chap.IV]. Furthermore,

Lemma 37. *If $(Z_t)_{t\le T}$ in (59) is a true martingale under P, then $\overline{Q}^\varphi \in \mathcal{M}_e$.*

Proof. If $(Z_t)_{t\le T}$ is a true martingale under P, then $\overline{Q}^\varphi$ is indeed a probability measure equivalent to P. Moreover, by Theorem 35, we see that both

$$
\overline{W}^j_\cdot \triangleq W^j_\cdot - \int_0^\cdot \overline{\beta}_t^j dt, \quad 0 \le j \le n, \tag{61}
$$

and

$$
\overline{M}_\cdot \triangleq H_\cdot - \int_0^{\cdot\wedge\tau} \gamma_t(1+\overline{\kappa}_t) dt,
$$

are local martingales under $\overline{Q}^{\varphi}$, where

$$\begin{cases} \overline{\beta}_t^j & \triangleq \mathbb{1}_{t\leq\tau}(-\alpha\overline{\pi}_t^{\varphi}\sigma_t\rho_t^j - \alpha D_iGc^{ij}); \quad 1\leq j\leq n, \\ \overline{\beta}_t^0 & \triangleq -\mathbb{1}_{t\leq\tau}\alpha\overline{\pi}_t^{\varphi}\sigma_t\sqrt{1-|\rho_t|^2}, \\ \overline{\kappa}_t & \triangleq \frac{\sigma_t^2}{\gamma_t}\theta_t - 1. \end{cases} \tag{62}$$

Thus for $t\leq\tau\wedge T$,

$$\begin{aligned} &\mu_t + \sigma_t\left(\sum\nolimits_{j=1}^n \rho_t^j\overline{\beta}_t^j + \sqrt{1-|\rho_t|^2}\overline{\beta}_t^0\right) - \gamma_t(1+\kappa_t) \\ =&\mu_t + \sigma_t\left(-\alpha\overline{\pi}_t^{\varphi}\sigma_t|\rho_t|^2 - \alpha c^{ij}\rho_t^jD_iG - \alpha\overline{\pi}_t^{\varphi}\sigma_t(1-|\rho_t|^2)\right) - \gamma_t e^{\alpha(G+\overline{\pi}_t^{\varphi})}; \\ =&\mu_t - \alpha\sigma_t^2\overline{\pi}_t^{\varphi} - \alpha\sigma_t c^{ij}\rho_t^jD_iG - \gamma_t e^{\alpha(G+\overline{\pi}_t^{\varphi})}; \\ =&0, \end{aligned}$$

where the last equality is from (22). Thus by Lemma 36, we have $\overline{Q}^{\varphi}\in\mathcal{M}_e$. □

Lemma 37 assumes that $(Z_t)_{t\leq T}$ is a P-martingale. There are some known results as to when an exponential martingale with jumps is a true martingale, e.g., [**PS$^+$08**]. However, the result there is not readily applicable because it requires $\overline{\kappa}\geq -1+\varepsilon$ for positive ε, which we do not have in general.

However, the following lemma shows in our case that the jump component is typically a true martingale.

Lemma 38. *Let $(Z_t^{\kappa})_{t\leq T}$ be defined by $Z_0^{\kappa}=1$ and*

$$dZ_t^{\kappa} = Z_{t-}^{\kappa}\kappa_t dM_t = Z_{t-}^{\kappa}\kappa_t(dH_t - \gamma_t dt),$$

where κ is a $\mathbb{F}$-predictable process such that $\kappa > -1$ and $\kappa\gamma$ integrable on $[0,T]$. Then $Z_T^{\kappa} > 0$ and $\mathbb{E}Z_T^{\kappa} = 1$.

Proof. Solving the stochastic differential equation for Z^κ, we obtain

$$Z_T^\kappa = \begin{cases} e^{-\int_0^T \kappa_u \gamma_u du} & \tau > T, \\ (1+\kappa_\tau) e^{-\int_0^\tau \kappa_u \gamma_u du} & \tau \leq T. \end{cases}$$

Thus Z_T^κ is clearly positive.

For the expectation of Z_T^κ:

$$\mathbb{E}\left[Z_T^\kappa\right] = \mathbb{E}\left[e^{-\int_0^T \kappa_u \gamma_u du} \mathbb{1}_{\tau > T}\right] + \mathbb{E}\left[(1+\kappa_\tau) e^{-\int_0^\tau \kappa_u \gamma_u du} \mathbb{1}_{\tau \leq T}\right].$$

The first term is,

$$\begin{aligned} \mathbb{E}\left[e^{-\int_0^T \kappa_u \gamma_u du} \mathbb{1}_{\tau > T}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{-\int_0^T \kappa_u \gamma_u du} \mathbb{1}_{\tau > T} \middle| \mathcal{F}_T\right]\right]; \\ &= \mathbb{E}\left[e^{-\int_0^T \kappa_u \gamma_u du} P\left(\tau > T | \mathcal{F}_T\right)\right]; \\ &= \mathbb{E}\left[e^{-\int_0^T \kappa_u \gamma_u du} e^{-\int_0^T \gamma_u du}\right]. \end{aligned}$$

For the second term, using [**BR04**, Proposition 5.1.1 (ii)][8] with a non-negative $\mathbb{F}$-predictable process Y defined by

$$Y_t = (1+\kappa_t) e^{-\int_0^t \kappa_u \gamma_u du},$$

[8]In fact, [**BR04**, Proposition 5.1.1 (ii)] is proved for bounded Y. To show the result holds for non-negative Y, we first use the proposition [**BR04**, Proposition 5.1.1 (ii)] for the predictable bounded process $Y \wedge n$, then take $n \to \infty$ and the monotone convergence theorem proves the claim for non-negative Y.

we have

$$\begin{aligned}\mathbb{E}\left[Y_\tau \mathbb{1}_{\tau\le T}\right] &= \mathbb{E}\left[\int_0^T Y_s dF_s\right];\\ &= \mathbb{E}\left[\int_0^T (1+\kappa_s)e^{-\int_0^s \kappa_u\gamma_u du}\gamma_s e^{-\int_0^s \gamma_u}ds\right];\\ &= \mathbb{E}\left[\int_0^T (1+\kappa_s)\gamma_s e^{-\int_0^s (1+\kappa_u)\gamma_u du}ds\right];\\ &= \mathbb{E}\left[1-e^{-\int_0^T (1+\kappa_u)\gamma_u du}\right].\end{aligned}$$

Hence we have $\mathbb{E}\left[Z_T^\kappa\right]=1$. □

To fully prove $(Z_t)_{t\le T}$ is a P-martingale and therefore to complete the proof of Theorem 7, we need the following estimations.

Proposition 39. *Under Assumptions 5 and 6, the solution to* (32) *satisfies*

$$|Du(t,x)| \le C_1(1+|x|) \quad \textit{for } (t,x)\in[0,T]\times\mathbb{R}^n,$$

or, similarly, by (26) *we have*

$$|DG(t,x;\varphi)| \le C_1(1+|x|) \quad \textit{for } (t,x)\in[0,T]\times\mathbb{R}^n.$$

Proof. The proof carefully follows Sections 11.1 and 11.3 of [**Lie96**], with special attention to the dependencies on the domain Ω_m. See Appendix B.1. □

We introduce functions that are associated with the processes defined in (62):

$$\begin{cases}\overline{\beta}^j(t,x;\varphi) & \triangleq -\alpha\overline{\pi}(t,x;\varphi)\sigma(x)\rho^j(x)-\alpha D_iG(t,x;\varphi)c^{ij}(x); \quad 1\le j\le n,\\ \overline{\beta}^0(t,x;\varphi) & \triangleq -\alpha\overline{\pi}(t,x;\varphi)\sigma(x)\sqrt{1-|\rho(x)|^2},\\ \overline{\gamma}(t,x;\varphi) & \triangleq \sigma^2(x)\theta(x,G(t,x;\varphi),DG(t,x;\varphi)).\end{cases} \tag{63}$$

Note the relations,

$$\overline{\beta}^j_t = \mathbb{1}_{t\le\tau}\overline{\beta}^j(t,X_t;\varphi); \quad 0\le j\le n, \tag{64}$$

and

$$\overline{\gamma}_t \triangleq \overline{\gamma}(t, X_t; \varphi) \tag{65}$$

becomes the intensity of the jump process H under $\overline{Q}^{\varphi}$ by Theorem 35.

Lemma 40. *Under Assumptions 5 and 6, $\overline{\pi}(t, x; \varphi), \overline{\beta}(t, x; \varphi)$, and $\overline{\gamma}(t, x; \varphi)$ defined in* (53) *and* (63)*, satisfy the linear growth condition with respect to $|x|$, i.e., we have positive constants, C_2, C_3, and C_4 such that*

$$|\overline{\pi}(t, x; \varphi)| \leq C_2(1 + |x|),$$

$$|\overline{\beta}(t, x; \varphi)| \leq C_3(1 + |x|),$$

$$|\overline{\gamma}(t, x; \varphi)| \leq C_4(1 + |x|),$$

for $(t, x) \in [0, T] \times \mathbb{R}^n$.

Proof. From Proposition 39, DG grows linearly with $|x|$ under Assumptions 5 and 6. Note also that under those assumptions, we have $\gamma/\sigma^2, \mu/\sigma^2$, and c^{ij}/σ bounded in $\mathbb{R}^n$. Thus $\theta(x, G(t, x), DG(t, x))$ grows linearly with $|x|$ by its definition (24) and Corollary 56. Therefore, it is straightforward to see that $\overline{\pi}$ and $\overline{\gamma}$ satisfy the linear growth condition with respect to $|x|$ in Ω by their definitions.

Because $\overline{\pi}$ and DG grow linearly with respect to $|x|$, so does $\overline{\beta}$, by (63). □

Lemma 41. *Assume Assumption 1, or more generally, suppose there exists a process X that has dynamics*

$$dX_t = b(\omega, t, X_t)dt + c\, dW_t, \quad X_0 = x \in \mathbb{R}^n,$$

where c is a constant $n \times n$-invertible matrix and $b : \Omega \times [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{G}_t \otimes \mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}^n)$-measurable for each t and linearly grows with respect to its spacial argument, i.e.,

$$|b(\omega, t, x)| \leq C(1 + |x|); \quad \forall (\omega, t, x) \in \Omega \times [0, T] \times \mathbb{R}^n,$$

for some positive constant C. Then we have

$$\sup_{t\le T}\mathbb{E}\left[\exp(\varepsilon|X_t|^2)\right]<\infty$$

for some $\varepsilon>0$.

PROOF. See Appendix B.2. □

Finally we are ready to complete the proof of Theorem 7 in the next section.

4.6. Completion of the Proof of Theorem 7

The proof of Theorem 7 will be completed once we verify the conditions in Theorem 34. We first show condition (i). By Lemma 37, we have to show $(Z_t)_{t\le T}$ in (59) is a P-martingale to say $\overline{Q}^{\varphi}\in\mathcal{M}_e$. Furthermore, from Lemma 38, we see that $(\mathcal{E}_t(\overline{\kappa}dM))_{t\le T}$ is a P-martingale. Thus $(Z_t)_{t\le T}$ is a P-martingale if

$$\mathcal{E}_t\left(\sum_{0\le j\le n}\int_0^{\cdot}\overline{\beta}^j_u dW^j_u\right);\quad 0\le t\le T,$$

is a martingale under the new measure $dP^{\overline{\kappa}}/dP\triangleq\mathcal{E}_T(\overline{\kappa}dM)$.

Using a variant of the Novikov condition (see, for instance, [**KS12**, Corollary 3.5.14]), it is sufficient to show that we can pick a positive constant ε such that, for any subinterval $[s,s+\varepsilon]\subset[0,T]$, we have

$$\mathbb{E}^{P^{\overline{\kappa}}}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}_u|^2du}<\infty.$$

By (64), we see

$$\mathbb{E}^{P^{\overline{\kappa}}}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}_u|^2du}\le\mathbb{E}^{P^{\overline{\kappa}}}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}(u,X_u;\varphi)|^2du}.$$

Notice that moving from P to $P^{\overline{\kappa}}$ only changes the distribution of τ but not the Brownian motion W (see Theorem 35). By the weak uniqueness of X, we have

$$\mathbb{E}^{P^{\overline{\kappa}}}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}(u,X_u;\varphi)|^2du}=\mathbb{E}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}(u,X_u;\varphi)|^2du}.$$

Thus we are going to show the finiteness of the right-hand side. By Jensen's inequality, we have

$$\begin{aligned} e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}(u,X_u;\varphi)|^2du} &= e^{\frac{1}{\varepsilon}\int_s^{s+\varepsilon}\frac{\varepsilon}{2}|\overline{\beta}(u,X_u;\varphi)|^2du}; \\ &\le \frac{1}{\varepsilon}\int_s^{s+\varepsilon} e^{\frac{\varepsilon}{2}|\overline{\beta}(u,X_u;\varphi)|^2}du. \end{aligned}$$

Thus

$$\mathbb{E}e^{\frac{1}{2}\int_s^{s+\varepsilon}|\overline{\beta}(u,X_u;\varphi)|^2du} \le \frac{1}{\varepsilon}\int_s^{s+\varepsilon}\mathbb{E}e^{\frac{\varepsilon}{2}|\overline{\beta}(u,X_u;\varphi)|^2}du.$$

From Lemma 40, we obtain

$$\mathbb{E}e^{\frac{\varepsilon}{2}|\overline{\beta}(u,X_u;\varphi)|^2} \le \mathbb{E}\exp(\varepsilon C_3^2|X_u|^2 + \varepsilon C_3^2).$$

But the right-hand side is bounded for some choice of $\varepsilon > 0$ by Lemma 41. Thus $(Z_t)_{t\le T}$ is proved to be a martingale and $\overline{Q}^{\varphi} \in \mathcal{M}_e$.

To have $\overline{Q}^{\varphi} \in \mathcal{M}_{e,f}$, we further need to show

$$H(\overline{Q}^{\varphi}|P) = \mathbb{E}^{\overline{Q}^{\varphi}}[-\alpha(\overline{\mathcal{W}}_T^{\varphi} + \varphi(X_T)\mathbb{1}_{\tau>T} - G(0,x;\varphi))]$$

is finite. Since φ and G are both bounded, it suffices to show that $\overline{\mathcal{W}}^{\varphi}$ is a $\overline{Q}^{\varphi}$-martingale (which is actually the first half of condition (ii)). Observe that the dynamics of X under $\overline{Q}^{\varphi}$ becomes

$$\begin{aligned} dX_t &= \left(b(X_t) + c\overline{\beta}_t\right)dt + c\,d\overline{W}; \\ &= \left(b(X_t) + \mathbb{1}_{t\le\tau}c\overline{\beta}(t,X_t;\varphi)\right)dt + c\,d\overline{W}; \\ &= \overline{b}(\omega,t,X_t)dt + c\,d\overline{W}, \end{aligned} \tag{66}$$

where we set

$$\overline{b}(\omega,t,x) \triangleq b(x) + \mathbb{1}_{t\le\tau(\omega)}c\overline{\beta}(t,x;\varphi); \quad (\omega,t,x) \in \Omega\times[0,T]\times\mathbb{R}^n,$$

and $\overline{W}$ is a $\overline{Q}^{\varphi}$-Brownian motion defined in (61). Since $b \in C_b^1$ by Assumption 1 and $\overline{\beta}$ linearly grows by Lemma 40, the drift $\overline{b}$ in (66) also linearly grows with $|x|$. Therefore, by Lemma 41, we have

$$\sup_{t \le T} \mathbb{E}^{\overline{Q}^{\varphi}} \left[\exp(\varepsilon |X_t|^2) \right] < \infty,$$

for some $\varepsilon > 0$, and in particular, $(X_t)_{t \le T}$ has uniformly bounded moments of any order under $\overline{Q}^{\varphi}$, i.e.,

$$\sup_{t \le T} \mathbb{E}^{\overline{Q}^{\varphi}} |X_t|^n < \infty; \quad n = 1, 2, \dots. \tag{67}$$

Note by Lemma 36 that, under any $Q \in \mathcal{M}_e$, the dynamics of $\overline{\mathcal{W}}^{\varphi}$ becomes

$$d\overline{\mathcal{W}}_t^{\varphi} = \overline{\pi}_t^{\varphi} (\mathbb{1}_{t \le \tau} \sigma_t dB_t^Q - dM_t^Q).$$

Thus the expectation of the quadratic variation of $(\overline{\mathcal{W}}_t^{\varphi})_{t \le T}$ under $\overline{Q}^{\varphi} \in \mathcal{M}_e$ can be computed as

$$\begin{aligned}
\mathbb{E}^{\overline{Q}^{\varphi}} [\overline{\mathcal{W}}^{\varphi}]_T &\le \mathbb{E}^{\overline{Q}^{\varphi}} \left[\int_0^T \sigma_s^2 (\overline{\pi}_s^{\varphi})^2 ds + (\overline{\pi}_\tau^{\varphi})^2 \mathbb{1}_{\tau \le T} \right]; \\
&\le \sup_{\mathbb{R}^n} \sigma^2(x) \int_0^T \mathbb{E}^{\overline{Q}^{\varphi}} (\overline{\pi}_s^{\varphi})^2 ds + \int_0^T \mathbb{E}^{\overline{Q}^{\varphi}} (\overline{\pi}_s^{\varphi})^2 \overline{\gamma}_s e^{\int_0^s \overline{\gamma}_u du} ds; \\
&\le C_2^2 \sup_{\mathbb{R}^n} \sigma^2(x) \int_0^T \mathbb{E}^{\overline{Q}^{\varphi}} (1 + |X_s|)^2 ds + C_2^2 C_4 \int_0^T \mathbb{E}^{\overline{Q}^{\varphi}} (1 + |X_s|)^3 ds; \\
&< \infty,
\end{aligned}$$

where the third inequality is by Lemma 40 and the last one by (67). Thus we show that $H(\overline{Q}^{\varphi}|P) < \infty$, i.e., $\overline{Q}^{\varphi} \in \mathcal{M}_{e,f}$.

For condition (ii) in Theorem 34, we are going to show that $\overline{\mathcal{W}}^{\varphi}$ is a true Q-martingale for all $Q \in \mathcal{M}_{e,f}$. Indeed, we can estimate the expectation of the quadratic variation of

$(\overline{\mathcal{W}}_t^\varphi)_{t\le T}$ under any $Q \in \mathcal{M}_{e,f}$ as

$$\begin{aligned}\mathbb{E}^Q[\overline{\mathcal{W}}^\varphi]_T &\le \mathbb{E}^Q\left[\int_0^T \sigma_s^2(\overline{\pi}_s^\varphi)^2 ds + (\overline{\pi}_\tau^\varphi)^2 \mathbb{1}_{\tau\le T}\right];\\ &\le 2C_2^2 \sup_{\mathbb{R}^n}\sigma^2(x)\int_0^T \mathbb{E}^Q(1+|X_s|^2)ds + 2C_2^2\mathbb{E}^Q\mathbb{1}_{\tau\le T}(1+|X_\tau|^2). \end{aligned} \tag{68}$$

For the first term in (68), notice by [**DGR$^+$02**, Lemma 3.5] that for any positive constant ε,

$$\mathbb{E}^Q|X_s|^2 \le \frac{1}{\varepsilon}\left[H(Q|P) + \frac{1}{e}\mathbb{E}\exp(\varepsilon|X_s|^2)\right]. \tag{69}$$

Above, $H(Q|P)$ is finite because of our assumption $Q \in \mathcal{M}_{e,f}$. For the second term in (68), again, for any positive constant ε,

$$\begin{aligned}\mathbb{E}^Q\mathbb{1}_{\tau\le T}|X_\tau|^2 &\le \frac{1}{\varepsilon}\left[H(Q|P) + \frac{1}{e}\mathbb{E}\exp(\varepsilon|X_\tau|^2\mathbb{1}_{\tau\le T})\right];\\ &\le \frac{1}{\varepsilon}\left[H(Q|P) + \frac{1}{e}\mathbb{E}\left(\int_0^T \exp(\varepsilon|X_s|^2)\gamma_s e^{-\int_0^s\gamma_u du}ds\right) + \frac{1}{e}P(\tau > T)\right];\\ &\le \frac{1}{\varepsilon}\left[H(Q|P) + \frac{1}{e}\sup_{\mathbb{R}^n}\gamma(x)\int_0^T \mathbb{E}\exp(\varepsilon|X_s|^2)ds + \frac{1}{e}\right].\end{aligned}$$

Thus the entire $\mathbb{E}^Q[\overline{\mathcal{W}}^\varphi]_T$ in (68) is finite as long as

$$\int_0^T \mathbb{E}\exp(\varepsilon|X_s|^2)ds < \infty,$$

for some $\varepsilon > 0$, but this is again already proved by Lemma 41.

Thus both of the conditions in Theorem 34 are proved, which completes the proof of Theorem 7.

Remark 42. It is not hard to see that our minimal entropy martingale measure $\overline{Q}^\varphi$ satisfies the reverse Hölder inequality under P^φ defined in [**DGR$^+$02**]. First, with regards to P^φ in (56), let the martingale density process be

$$M_t \triangleq \mathbb{E}\left[\left.\frac{dP^\varphi}{dP}\right|\mathcal{G}_t\right]; \quad t \le T,$$

so that $(M_t)_{t\leq T}$ is uniformly bounded from above and below by some positive constants by Assumption 6. We also define the density process of $\overline{Q}^\varphi$ with respect to P^φ by

$$Z_t^{\overline{Q}^\varphi|P^\varphi} \triangleq \mathbb{E}^{P^\varphi}\left[\left.\frac{d\overline{Q}^\varphi}{dP^\varphi}\right|\mathcal{G}_t\right] = \frac{1}{M_t}\mathbb{E}\left[M_T\left.\frac{d\overline{Q}^\varphi}{dP^\varphi}\right|\mathcal{G}_t\right] = \frac{1}{M_t}\mathbb{E}\left[Z_T|\,\mathcal{G}_t\right] = \frac{Z_t}{M_t};\quad t\leq T,$$

where Z is the density process in (59). Then, for any stopping time $\sigma \leq T$, we have

$$\begin{aligned}
&\mathbb{E}^{P^\varphi}\left[\left.\frac{Z_T^{\overline{Q}^\varphi|P^\varphi}}{Z_\sigma^{\overline{Q}^\varphi|P^\varphi}}\log\frac{Z_T^{\overline{Q}^\varphi|P^\varphi}}{Z_\sigma^{\overline{Q}^\varphi|P^\varphi}}\right|\mathcal{G}_\sigma\right]\\
=&\mathbb{E}^{\overline{Q}^\varphi}\left[\left.\log\frac{Z_T^{\overline{Q}^\varphi|P^\varphi}}{Z_\sigma^{\overline{Q}^\varphi|P^\varphi}}\right|\mathcal{G}_\sigma\right] = \mathbb{E}^{\overline{Q}^\varphi}\left[\left.\log\frac{Z_T/M_T}{Z_\sigma/M_\sigma}\right|\mathcal{G}_\sigma\right];\\
=&\mathbb{E}^{\overline{Q}^\varphi}\left[\left.-\alpha\left(\overline{\mathcal{W}}_T^\varphi-\overline{\mathcal{W}}_\sigma^\varphi\right)+\mathbb{1}_{T<\tau}G(T,X_T;\varphi)-\mathbb{1}_{\sigma<\tau}G(\sigma,X_\sigma;\varphi)+\log\frac{M_\sigma}{M_T}\right|\mathcal{G}_\sigma\right];\\
=&\mathbb{E}^{\overline{Q}^\varphi}\left[\left.\mathbb{1}_{T<\tau}G(T,X_T;\varphi)-\mathbb{1}_{\sigma<\tau}G(\sigma,X_\sigma;\varphi)+\log\frac{M_\sigma}{M_T}\right|\mathcal{G}_\sigma\right],
\end{aligned}$$

where the last equality is by the optional sampling theorem applied to the $\overline{Q}^\varphi$-martingale $\overline{\mathcal{W}}^\varphi$. The last term is indeed bounded from above by a constant not depending on σ, because $G(;\varphi)$ is bounded on $[0,T]\times\mathbb{R}^n$ and M_σ/M_T is bounded by a constant only depending on φ.

CHAPTER 5

Pricing Default Insurance

We introduce an insurance against the default of the stock issuer in the market. A dollar amount of the insurance requires a continual payment of f per unit time, and pays back a dollar if the company defaults.

For the holder of π-dollar amount S, it is natural to consider holding π-dollar amount insurance to dynamically hedge against the default. We consider those two strategies:

- Strategy 1. The investor invests only in the stock S for π-dollar amount, which will be entirely lost at the time of default, τ, as in our main problem in Section 2.2. Thus the wealth process dynamics (9):

$$d\mathcal{W}_t^\pi = \pi_t \left(\mathbb{1}_{t\leq\tau}(\mu(X_t)dt + \sigma(X_t)dB_t) - dH_t \right),$$

and the objective:

$$\text{maximize } \mathbb{E}U(\mathcal{W}_T^\pi), \quad \pi \in \mathcal{A}_{\mathbb{G}}.$$

- Strategy 2. The investor invests in the stock S for π-dollar amount, while taking the π-dollar amount position in the insurance so that no loss will happen at the time of default. The investor has to continually pay πf per unit time for the insurance protection until the default. Thus the wealth process dynamics:

$$\begin{aligned} d\tilde{\mathcal{W}}_t^\pi &= \pi_t \mathbb{1}_{t\leq\tau}(\mu(X_t)dt + \sigma(X_t)dB_t) - \mathbb{1}_{t\leq\tau}\pi_t f_t dt; \\ &= \pi_t \mathbb{1}_{t\leq\tau}(\tilde{\mu}_t dt + \sigma(X_t)dB_t), \end{aligned}$$

where

$$\tilde{\mu}_t \triangleq \mu(X_t) - f_t, \tag{70}$$

and the objective:

$$\text{maximize } \mathbb{E}U(\tilde{\mathcal{W}}_T^\pi), \quad \pi \in \tilde{\mathcal{A}}_{\mathbb{G}}.$$

where the precise definition of $\tilde{\mathcal{A}}_{\mathbb{G}}$ is given in Section 5.2.

The goal of this chapter is to find the buyer's indifference coupon process f such that *the investor is indifferent between Strategy 1 and Strategy 2 at any time before $\tau \wedge T$.* The precise result is given in Theorem 44.

Remark 43. The buyer's indifference coupon process f corresponds to the default intensity of τ in the following sense. Under the risk-neutral pricing scheme, the default insurance described above can be priced as

$$\lim_{\Delta t \to 0+} \frac{1}{\Delta t} \mathbb{E}^Q \left[\mathbb{1}_{t<\tau\le t+\Delta t} \middle| \mathcal{G}_t \right] = \gamma_t^Q \mathbb{1}_{t<\tau},$$

where γ^Q is the default intensity of τ under the "risk-neutral measure" $Q \sim P$ of choice.

5.1. Derivation of the Coupon Process f

In this section, we formally derive the buyer's indifference coupon process f introduced above under Assumption 5.

Strategy 1 is the same situation as in our main problem in Section 2.2 with no terminal payoff $\varphi = 0$ (thus we drop φ in all function arguments in this chapter). Readers should be reminded from Theorem 7 (more specifically, Theorem 26) that the pre-default certainty function $G(t,x)$ exists satisfying the PDE (25) with the terminal condition $G(T,x) = 0$.

For Strategy 2, we follow the same steps as in Section 4.1 to get HJB and the consequent PDE. Namely, the same argument that leads to the HJB equation (20) will produce the HJB equation for the pre-default value function $\tilde{v}(t,x,y)$,

$$\tilde{v}_t + \max_{\pi}\Big[-\gamma\tilde{v} + \pi\tilde{\mu}\tilde{v}_w + b^i\tilde{v}_{x_i} + \frac{\pi^2}{2}\sigma^2\tilde{v}_{ww} + \pi\sigma c^{ij}\rho^j\tilde{v}_{x_i,w} + \frac{(cc^\top)^{ij}}{2}\tilde{v}_{x_i,x_j} - \gamma e^{-\alpha w}\Big] = 0, \quad (71)$$

with the terminal condition $\tilde{v}(T,x,w) = -e^{-\alpha w}$. Notice that in the exponential function in the last term of (71), we have w instead of $w-\pi$ because there is no loss at default.

We assume that $\tilde{v}$ takes the form

$$\tilde{v}(t,x,w) = -e^{-\alpha w}e^{-\alpha\tilde{G}(t,x)},$$

and substitute it back into (71) to have

$$\tilde{G}_t + \frac{(cc^\top)^{ij}}{2}D_{ij}\tilde{G} - \frac{\alpha}{2}(cc^\top)^{ij}D_i\tilde{G}D_j\tilde{G} + \max_{\pi}\Big[(b^i - \alpha\sigma\pi c^{ij}\rho^j)D_i\tilde{G} - \frac{\alpha}{2}\sigma^2\pi^2 + \tilde{\mu}\pi + \frac{\gamma}{\alpha}\left(1 - e^{\alpha\tilde{G}}\right)\Big] = 0, \quad (72)$$

with the terminal condition $\tilde{G}(T,x) = 0$.

The maximum of the above is attained at

$$\tilde{\pi} \triangleq \frac{1}{\alpha\sigma^2}\left(\tilde{\mu} - \alpha\sigma c^{ij}\rho^j D_i\tilde{G}\right). \quad (73)$$

Thus evaluating (72) at $\pi = \tilde{\pi}$ gives,

$$\tilde{G}_t + \frac{(cc^\top)^{ij}}{2}D_{ij}\tilde{G} - \frac{\alpha}{2}(cc^\top)^{ij}D_i\tilde{G}D_j\tilde{G} + b^iD_i\tilde{G} + \frac{1}{2\alpha\sigma^2}\left(\tilde{\mu} - \alpha\sigma c^{ij}\rho^j D_i\tilde{G}\right)^2 + \frac{\gamma}{\alpha}\left(1 - e^{\alpha\tilde{G}}\right) = 0. \quad (74)$$

Because we assume that the investor is indifferent between the two strategies anytime before the time of default, we impose that

$$G(t,x) = \tilde{G}(t,x); \quad \forall (t,x) \in [0,T]\times\mathbb{R}^n,$$

which is possible when G and $\tilde{G}$ satisfy the same PDE, i.e., (25) and (74) are the same.

Replacing $\tilde{G}$ by G in (74) and equating it with (25), we obtain

$$\begin{aligned} b^i D_i G &+ \frac{1}{2\alpha\sigma^2}\left(\tilde{\mu} - \alpha\sigma c^{ij}\rho^j D_i G\right)^2 + \frac{\gamma}{\alpha}(1 - e^{\alpha G}) \\ &= \frac{\alpha}{2}(c^{ij}\rho^j D_i G)^2 + (b^i - \frac{\mu}{\sigma}c^{ij}\rho^j)D_i G - \frac{\sigma^2}{2\alpha}(\theta^2 + 2\theta) + \frac{1}{\alpha}\left(\gamma + \frac{\mu^2}{2\sigma^2}\right), \end{aligned}$$

or,

$$\left(\tilde{\mu} - \alpha\sigma c^{ij}\rho^j D_i G\right)^2 = \left(\mu - \alpha\sigma c^{ij}\rho^j D_i G\right)^2 + 2\sigma^2\gamma e^{\alpha G} - \sigma^4(\theta^2 + 2\theta). \tag{75}$$

Above recall that $\theta = \theta(x, G, DG)$ where $\theta(x, z, p)$ is defined in (24). The right-hand side of (75) is non-negative because Corollary 58 with

$$x = \frac{\gamma}{\sigma^2}e^{\alpha G}, \quad y = \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}c^{ij}\rho^j D_i G$$

proves the inequality

$$\theta^2 + 2\theta \le \left[\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}c^{ij}\rho^j D_i G\right]^2 + 2\frac{\gamma}{\sigma^2}e^{\alpha G}.$$

Thus we can solve for $\tilde{\mu}$ in (75) to get

$$\tilde{\mu} = \alpha\sigma c^{ij}\rho^j D_i G \pm \sqrt{(\mu - \alpha\sigma c^{ij}\rho^j D_i G)^2 + 2\sigma^2\gamma e^{\alpha G} - \sigma^4(\theta^2 + 2\theta)}.$$

The sign above is undetermined at this point, but if we substitute it back into (73), we have

$$\tilde{\pi} = \pm\frac{1}{\alpha}\sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma}c^{ij}\rho^j D_i G\right)^2 + 2\frac{\gamma}{\sigma^2}e^{\alpha G} - (\theta^2 + 2\theta)}.$$

In light of the definition of $\tilde{\pi}$, (73), we choose the positive sign in $\tilde{\pi}$ (and the corresponding positive sign in $\tilde{\mu}$) to make $\tilde{\pi}$ non-negative, because we are considering the investor who is long the stock, seeking to purchase the insurance to protect herself from the default loss.

Thus, by (70), we obtain the buyer's indifference coupon process $f_t = f(t, X_t)$ where

$$\begin{aligned} f(t,x) &\triangleq \mu(x) - \alpha\sigma(x)c^{ij}\rho^j(x)D_iG(t,x) \\ &\quad + \sqrt{\left(\mu(x) - \alpha\sigma(x)c^{ij}\rho^j(x)D_iG(t,x)\right)^2 + 2\sigma^2(x)\gamma(x)e^{\alpha G(t,x)} - \sigma^4(x)(\theta^2+2\theta)}, \end{aligned} \tag{76}$$

and the optimal investment in Strategy 2, $\tilde{\pi}_t = \tilde{\pi}(t, X_t)$, where

$$\tilde{\pi}(t,x) \triangleq \frac{1}{\alpha}\sqrt{\left[\frac{\mu(x)}{\sigma^2(x)} - \frac{\alpha}{\sigma(x)}c^{ij}(x)\rho^j(x)D_iG(t,x)\right]^2 + 2\frac{\gamma(x)}{\sigma^2(x)}e^{\alpha G(t,x)} - (\theta^2+2\theta)}, \tag{77}$$

where θ is evaluated at $(x, G(t,x), DG(t,x))$.

5.2. Verification Results for Strategy 2

Now that we obtain the candidate indifference coupon process f, we state and prove the precise optimal investment result for Strategy 2 in this section. Note that to verify the optimal investment result for Strategy 1, which is already done in Theorem 7, we require Assumption 5. It turns out that Assumption 5 is also sufficient to verify the optimal investment result for Strategy 2. Therefore, we enforce Assumption 5 in this section.

We introduce a fictitious stock process $\tilde{S}$ by

$$\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = \mathbb{1}_{t\le\tau}\left(\tilde{\mu}_t dt + \sigma(X_t)dB_t\right),$$

and define those classes in a similar way as in Section 2.2:

$$\begin{aligned} \tilde{\mathcal{M}}_e &\triangleq \left\{Q \sim P \text{ on } \mathcal{G}_T : \tilde{S} \text{ is a local martingale under } Q\right\}, \\ \tilde{\mathcal{M}}_{e,f} &\triangleq \left\{Q \in \tilde{\mathcal{M}}_e \ : \ H(Q|P) < \infty\right\}, \\ \tilde{\mathcal{A}}_{\mathbb{G}} &\triangleq \left\{\pi \ : \ \tilde{\mathcal{W}}^\pi \text{ is a } Q \text{ super-martingale for all } Q \in \tilde{\mathcal{M}}_{e,f}\right\}. \end{aligned}$$

Then we have,

Theorem 44. *Under Assumptions 5, for any $t \in [0,T]$ with initial wealth $\mathcal{W}_t = \tilde{\mathcal{W}}_t = w \in \mathbb{R}$, we have*

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[\mathbb{1}_{\tau>t} U(\mathcal{W}_T^\pi)|\mathcal{G}_t\right] = \operatorname*{ess\,sup}_{\pi \in \tilde{\mathcal{A}}_{\mathbb{G}}} \mathbb{E}\left[\mathbb{1}_{\tau>t} U(\tilde{\mathcal{W}}_T^\pi)|\mathcal{G}_t\right] = \mathbb{1}_{\tau>t} U(w + G(t, X_t)), \tag{78}$$

where each supremum is attained by the optimal control in $\mathcal{A}_{\mathbb{G}}$ and in $\tilde{\mathcal{A}}_{\mathbb{G}}$, respectively, and $G \in C^{1,2}([0,T] \times \mathbb{R}^n)$ is a pre-default certainty equivalent for the investor satisfying the PDE (25) *with the terminal condition $G(T,x) = 0$ for $x \in \mathbb{R}^n$. Thus the investor is indifferent between Strategy 1 and Strategy 2 at any time before $\tau \wedge T$.*

Before the proof of Theorem 44, we provide here a couple of auxiliary lemmas.

The concrete description of $\tilde{\mathcal{M}}_e$ similar to Lemma 36 is given by

Lemma 45. *For $Q \sim P$, we have*

$$Q \in \tilde{\mathcal{M}}_e \quad \text{if and only if}$$

$$\tilde{\mu}_t + \sigma_t \left(\sum\nolimits_{j=1}^n \rho_t^j \beta_t^j + \sqrt{1 - |\rho_t|^2} \beta_t^0 \right) = 0 \quad \text{for } t \le \tau \wedge T, \tag{79}$$

where the processes β and κ are as defined in Theorem 35.

Proof. Fix $Q \sim P$. Under Q, the dynamics $\tilde{S}$ follows

$$\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = \mathbb{1}_{t \le \tau} \left[\left(\tilde{\mu}_t + \sigma_t \left(\sum\nolimits_{j=1}^n \rho_t^j \beta_t^j + \sqrt{1 - |\rho_t|^2} \beta_t^0 \right) \right) dt + \sigma_t dB_t^Q \right].$$

where B^Q is a $(\mathbb{G}, Q)$-Brownian motion defined in (58). Thus $\tilde{S}$ is a $(\mathbb{G}, Q)$-local martingale if and only if the finite variation part vanishes, which is the statement of the lemma. □

For the martingale measure, $\tilde{Q}$, we assume

$$\frac{d\tilde{Q}}{dP} \triangleq \exp\left(-\alpha (\tilde{\mathcal{W}}_T^{\tilde{\pi}} - G(0,x)) \right),$$

which satisfies the first-order conditions for the duality.

As in (59), we assume the density process for $\tilde{Q}$ of the form

$$Z_t = e^{-\alpha(\tilde{\mathcal{W}}_t^{\tilde{\pi}} - G(0,x))} \cdot \Big(H_t + (1-H_t)e^{-\alpha G(t,X_t)}\Big);$$

$$= \begin{cases} \exp\left[-\alpha(\tilde{\mathcal{W}}_t^{\tilde{\pi}} - G(0,x) + G(t,X_t))\right]; & t < \tau, \\ \exp\left[-\alpha(\tilde{\mathcal{W}}_t^{\tilde{\pi}} - G(0,x))\right]; & \tau \le t, \end{cases}$$

whose differential is (see (60)),

$$dZ_t = Z_{t-}\left[\mathbb{1}_{t\le\tau}(-\alpha\tilde{\pi}_t\sigma_t\rho_t^j - \alpha D_i G c^{ij})dW_t^j - \mathbb{1}_{t\le\tau}\alpha\tilde{\pi}_t\sigma_t\sqrt{1-|\rho_t|^2}dW_t^0 + \left(e^{\alpha G}-1\right)dM_t\right].$$

With regards to the above differential, let us also define

$$\begin{cases} \tilde{\beta}_t^j & \triangleq \mathbb{1}_{t\le\tau}\tilde{\beta}^j(t,X_t); \\ & \triangleq \mathbb{1}_{t\le\tau}\left(-\alpha\tilde{\pi}(t,X_t)\sigma(X_t)\rho^j(X_t) - \alpha D_i G(t,X_t)c^{ij}\right); \quad 1\le j\le n, \\ \tilde{\beta}_t^0 & \triangleq \mathbb{1}_{t\le\tau}\tilde{\beta}^0(t,X_t); \\ & \triangleq -\mathbb{1}_{t\le\tau}\alpha\tilde{\pi}(t,X_t)\sigma(X_t)\sqrt{1-|\rho(X_t)|^2}, \\ \tilde{\kappa}_t & \triangleq e^{\alpha G(t,X_t)} - 1. \end{cases} \tag{80}$$

We provide here linear bound results for $\tilde{\pi}$ and $\tilde{\beta}$ similar to Lemma 40.

Lemma 46. *Under Assumption 5, $\tilde{\pi}(t,x)$ and $\tilde{\beta}(t,x)$ defined in (77) and (80), respectively, satisfy the linear growth condition with respect to $|x|$, i.e., we have positive constants, C_1, and C_2 such that*

$$|\tilde{\pi}(t,x)| \le C_1(1+|x|),$$

$$|\tilde{\beta}(t,x)| \le C_2(1+|x|).$$

for $(t,x) \in [0,T]\times\mathbb{R}^n$.

Proof. Observe from (77),

$$0 \le \tilde{\pi}(t,x) \le \frac{1}{\alpha}\left[\left|\frac{\mu(x)}{\sigma^2(x)} - \frac{\alpha}{\sigma(x)}c^{ij}\rho^j(x)D_iG(t,x)\right| + \sqrt{2\frac{\gamma(x)}{\sigma^2(x)}e^{\alpha G(t,x)}}\right].$$

Under Assumption 5, G is bounded in $\mathbb{R}^n$ by Remark 33, and DG grows linearly with $|x|$ in $\mathbb{R}^n$ by Proposition 39. Note also that under the assumption, we have $\gamma/\sigma^2, \mu/\sigma^2$, and c^{ij}/σ bounded in $\mathbb{R}^n$, thus $\tilde{\pi}$ indeed grows linearly with $|x|$. The linear growth condition of $\tilde{\beta}$ is now straightforward from its definition (80). □

Finally we are ready for the proof of Theorem 44.

Proof of Theorem 44. We prove the statement for $t = 0$ without loss of generality. Under Assumptions 5 and $\varphi \equiv 0$, the optimal control problem in the first term of (78) is already solved in Theorem 7, and in particular, the existence of such $G \in C^{1,2}([0,T]\times\mathbb{R}^n)$ is guaranteed. We therefore focus on the optimal control problem for Strategy 2 in the second term of (78).

We by and large follow Section 4.6.

To show the optimal control problem for Strategy 2, we again use the duality result to verify (i.e. Theorem 34):

(i) $\tilde{Q} \in \tilde{\mathcal{M}}_{e,f}$,
(ii) $(\tilde{\mathcal{W}}_t^{\tilde{\pi}})_{t\le T}$ is a true $\tilde{Q}$-martingale and a Q-supermartingale for all $Q \in \tilde{\mathcal{M}}_{e,f}$.

For condition (i), observe first that, since the processes $\overline{\beta}$ in (62) and $\tilde{\beta}$ in (80) have the same form, the proof that $(Z_t)_{t\le T}$ is a P-martingale in Section 4.6 carries over here verbatim, together with Lemma 46, to show that $\tilde{Q}$ is indeed a probability measure equivalent to P. Moreover, we have for $t \leq \tau \wedge T$,

$$\begin{aligned}
&\tilde{\mu}_t + \sigma_t \left(\sum\nolimits_{j=1}^n \rho_t^j \tilde{\beta}_t^j + \sqrt{1 - |\rho_t|^2} \tilde{\beta}_t^0 \right); \\
=&\tilde{\mu}_t + \sigma_t \left(-\alpha \tilde{\pi}_t \sigma_t |\rho_t|^2 - \alpha c^{ij} \rho_t^j D_i G - \alpha \tilde{\pi}_t \sigma_t (1 - |\rho_t|^2) \right); \\
=&\tilde{\mu}_t - \alpha \sigma_t^2 \tilde{\pi}_t - \alpha \sigma_t c^{ij} \rho_t^j D_i G; \\
=&0,
\end{aligned}$$

where the last equality is by (73). Thus Lemma 45 yields $\tilde{Q} \in \tilde{\mathcal{M}}_e$. To prove $\tilde{Q} \in \tilde{\mathcal{M}}_{e,f}$, we further need to show

$$H(\tilde{Q}|P) = \mathbb{E}^{\tilde{Q}}[-\alpha(\tilde{\mathcal{W}}_T^{\tilde{\pi}} - G(0,x))]$$

is finite. Since G is bounded, it suffices to show that $\mathcal{W}^{\tilde{\pi}}$ is a $\tilde{Q}$-martingale (which is actually the first half of condition (ii)). Observe that the dynamics of X under $\tilde{Q}$ is the same as the dynamics of X under $\overline{Q}$ in (66), with $\overline{\beta}$ replaced by $\tilde{\beta}$, which grows linearly with $|x|$ by Lemma 46. Therefore by Lemma 41 we have

$$\sup_{t \leq T} \mathbb{E}^{\tilde{Q}} \left[\exp(\varepsilon |X_t|^2) \right] < \infty,$$

for some $\varepsilon > 0$. In particular, $(X_t)_{t \leq T}$ has uniformly bounded moments of any order under $\tilde{Q}$, i.e.,

$$\sup_{t \leq T} \mathbb{E}^{\tilde{Q}} |X_t|^n < \infty; \quad n = 1, 2, \dots. \tag{81}$$

Note by Lemma 45 that, under any $Q \in \tilde{\mathcal{M}}_e$, the dynamics of $\tilde{\mathcal{W}}^{\tilde{\pi}}$ becomes

$$d\tilde{\mathcal{W}}_t^{\tilde{\pi}} = \tilde{\pi}_t \mathbb{1}_{t \leq \tau} \sigma_t dB_t^Q.$$

Thus the expectation of the quadratic variation of $(\tilde{\mathcal{W}}^{\tilde{\pi}})_{t\leq T}$ under $\tilde{Q}$ can be computed as

$$\begin{aligned}\mathbb{E}^{\tilde{Q}}[\tilde{\mathcal{W}}^{\tilde{\pi}}]_T &\leq \mathbb{E}^{\tilde{Q}}\left[\int_0^T \sigma_s^2(\tilde{\pi}_s)^2 ds\right];\\ &\leq \sup_{\mathbb{R}^n}\sigma^2(x)\int_0^T \mathbb{E}^{\tilde{Q}}(\tilde{\pi}_s)^2 ds;\\ &\leq C_1^2 \sup_{\mathbb{R}^n}\sigma^2(x)\int_0^T \mathbb{E}^{\tilde{Q}}(1+|X_s|)^2 ds;\\ &< \infty,\end{aligned}$$

where the third inequality is by Lemma 46 and the last one by (81). Thus we show that $H(\tilde{Q}|P) < \infty$, i.e., $\tilde{Q} \in \mathcal{M}_{e,f}$.

For condition (ii), we are going to show that $\tilde{\mathcal{W}}^{\tilde{\pi}}$ is a true Q-martingale for all $Q \in \tilde{\mathcal{M}}_{e,f}$. As in (68), we estimate the expectation of the quadratic variation of $(\tilde{\mathcal{W}}^{\tilde{\pi}})_{t\leq T}$ by

$$\begin{aligned}\mathbb{E}^{Q}[\tilde{\mathcal{W}}^{\tilde{\pi}}]_T &\leq \mathbb{E}^{Q}\left[\int_0^{\tau\wedge T} \sigma_s^2(\tilde{\pi}_s)^2 ds\right];\\ &\leq 2C_1^2 \sup_{\mathbb{R}^n}\sigma^2(x)\int_0^T \mathbb{E}^{Q}(1+|X_s|^2) ds,\end{aligned}$$

which is finite because of (69) and Lemma 41. Thus (ii) is proved. □

5.3. Simple Case

If we consider the constant coefficients case as in Chapter 3, we can rather explicitly compute the buyer's indifference coupon rate process $(f(t))_{t\leq T}$ as a deterministic function of time.

Since the certainty equivalent G only depends on t for the simple case, f in (76) reduces to

$$\begin{aligned}f(t) &= \mu - \sqrt{\mu^2 + 2\sigma^2\gamma e^{\alpha G(t)} - \sigma^4(\theta^2(G(t)) + 2\theta(G(t)))};\\ &= \mu - \sqrt{\mu^2 + \sigma^4\left(2e^{-\frac{\mu}{\sigma^2}}p(T-t)e^{p(T-t)} - p^2(T-t) - 2p(T-t)\right)},\end{aligned}$$

where in the first line $G(t)$ is defined in (18) and $\theta(z)$ in (14), and in the second line we use the relation (15). Since the ODE for p, (17), does not involve the absolute risk aversion parameter α, f does not depend on α either.

For example, when $\mu = .04, \sigma = .3, \gamma = .03$, and $T = 1$, $(f(t))_{t\leq T}$ looks as in Figure 1. We can see how the investor estimates the default risk and is willing pay more than γ to protect against the default in this market.

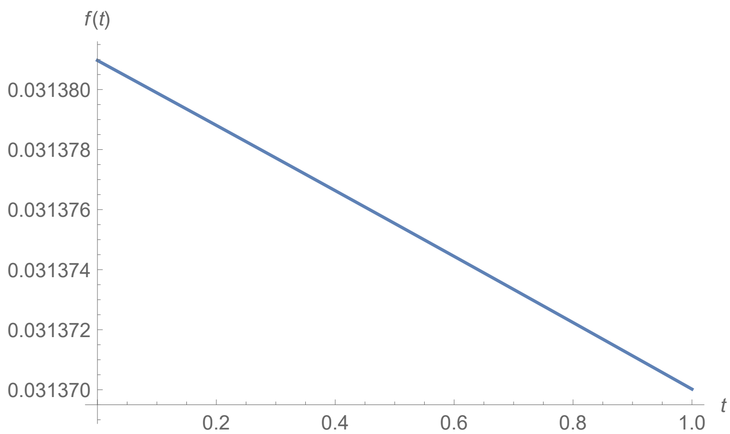

FIGURE 1. Graph of $(f(t))_{t\leq T}$ when $\mu = .04, \sigma = .3, \gamma = .03, T = 1$.

Remark 47. In the long-time-horizon limit, we have

$$\begin{aligned}
\lim_{T\to\infty} f(0) &= \lim_{T\to\infty}\left(\mu - \sqrt{\mu^2 + \sigma^4\left(2e^{-\frac{\mu}{\sigma^2}}p(T)e^{p(T)} - p^2(T) - 2p(T)\right)}\right)\\
&= \mu - \sqrt{\mu^2 + \sigma^4\left(2e^{-\frac{\mu}{\sigma^2}}p_+e^{p_+} - p_+^2 - 2p_+\right)}\\
&= \mu - \sqrt{2\sigma^2\gamma\left(\frac{\left(-1+\sqrt{1+2\frac{\gamma}{\sigma^2}+\frac{\mu^2}{\sigma^4}}\right)e^{-1+\sqrt{1+2\frac{\gamma}{\sigma^2}+\frac{\mu^2}{\sigma^4}}}}{\frac{\gamma}{\sigma^2}e^{\frac{\mu}{\sigma^2}}} - 1\right)}.
\end{aligned}$$

For the given parameters above, this is .0317116.

APPENDIX A

Estimations for Proposition 29

Assumptions 20, 21, and 22 are in effect throughout this chapter. Moreover, we assume the result of Proposition 28, and thus have the uniform bounds of local solutions, u_0 and u_1, introduced in Proposition 28. We say $f(x,z,p) \in O(|p|^i)$ for $i = 0, 1, 2, \dots$, if

$$\limsup_{|p|\to\infty} \frac{1}{|p|^i} \sup_{\Omega_m \times [u_0,u_1]} |f(x,z,p)| < \infty \quad \text{for all } m = 1, 2, \dots\ .$$

Before we start, readers should be reminded of (30):

$$a(x,z,p) = -\frac{\alpha}{2} p^\top \overline{a}(x) p + p^\top \overline{b}(x) - \frac{\sigma^2(x)}{2\alpha} \left(\theta^2(x,z,p) + 2\theta(x,z,p)\right) + \overline{\gamma}(x),$$

where $\overline{a}, \overline{b}$, and $\overline{\gamma}$ are differentiable in D, and θ is in $O(|p|^1)$ from Corollary 56 below.

Lemma 48 (Estimation of B_m). *$B_m(x,z,p)$ defined in (42) satisfies*

$$\limsup_{|p|\to\infty} \sup_{\Omega_m \times [u_0,u_1]} B_m(x,z,p) = C(m) < \infty.$$

Proof. From (42), we have

$$\begin{aligned} B_m(x,z,p) &= \frac{1}{\mathcal{E}(x,p)} (\delta(p)\mathcal{E}(x,p) + (\overline{\delta}(p) - 1) a(x,z,p)); \\ &= \frac{1}{\mathcal{E}(x,p)} \left(|p|^{-2} p \cdot D_x \mathcal{E}(x,p) + (p \cdot D_p - 1) a(x,z,p)\right); \\ &= \frac{1}{\mathcal{E}(x,p)} \left(|p|^{-2} p_i p_j p_k D_k a^{ij}(x) + (p \cdot D_p - 1) a(x,z,p)\right). \end{aligned} \qquad (82)$$

Since $|p|^{-2} p_i p_j p_k D_k a^{ij} = O(|p|)$, the first term in (82) disappears in the limit as $|p| \to \infty$. We will focus on the second term.

Applying Lemma 54 to $\theta(x,z,p)$ defined in (24), with

$$f(x,z,p)=\frac{\gamma(x)}{\sigma^2(x)}\exp\left(\frac{\mu(x)}{\sigma^2(x)}+\alpha z-\frac{\alpha}{\sigma(x)}c^{ij}(x)\rho_j(x)p_i\right), \tag{83}$$

and varying p_i while fixing other arguments, we have

$$\begin{aligned}D_{p_i}\theta&=\frac{\theta}{1+\theta}\frac{D_{p_i}f}{f};\\&=-\frac{\theta}{1+\theta}\frac{\alpha}{\sigma}c^{ij}\rho_j.\end{aligned}$$

Thus

$$\begin{aligned}D_{p_i}a&=-\frac{\alpha}{2}\left[2(cc^\top)^{ij}p_j-2(c^{ij}\rho_j)^2p_i\right]+\left(b^i-\frac{\mu}{\sigma}c^{ij}\rho_j\right)-\frac{\sigma^2}{2\alpha}2(\theta+1)D_{p_i}\theta;\\&=-\frac{\alpha}{2}\left[2(cc^\top)^{ij}p_j-2(c^{ij}\rho_j)^2p_i\right]+\overline{b}^i+\sigma\theta c^{ij}\rho_j.\end{aligned}\tag{84}$$

The second term in (82) is

$$\begin{aligned}(p\cdot D_p-1)a&=-\frac{\alpha}{2}\left[(cc^\top)^{ij}p_ip_j-(c^{ij}\rho_jp_i)^2\right]+\sigma\theta c^{ij}\rho_jp_i+\frac{\sigma^2}{2\alpha}(\theta^2+2\theta)-\overline{\gamma};\\&=-\frac{\alpha}{2}p^\top\overline{a}p+\sigma\theta p^\top c\rho+\frac{\sigma^2}{2\alpha}(\theta^2+2\theta)-\overline{\gamma}.\end{aligned}$$

Thus $(p\cdot D_p-1)a=O(|p|^2)$ because $\sigma,c,\rho,\bar{a}$, and $\bar{\gamma}$ only depend on x while $\theta=O(|p|^1)$. Finally we obtain

$$\limsup_{|p|\to\infty}\sup_{\Omega_m\times[u_0,u_1]}B_m(x,z,p)=C(m)<\infty.$$

$\square$

Lemma 49 (Estimation of C_m). *$C_m(x,z,p)$ defined in (43) satisfies*

$$\limsup_{|p|\to\infty}\sup_{\Omega_m\times[u_0,u_1]}C_m(x,z,p)=0.$$

Proof. From (43), we have

$$\begin{aligned} C_m(x,z,p) &= \frac{1}{\mathcal{E}(x,p)}\left(\frac{|p|^2}{2\lambda_m}\sum\nolimits_{i,j}(\delta(p)a^{ij}(x))^2 + \delta(p)a(x,z,p)\right); \\ &= \frac{1}{\mathcal{E}(x,p)}\left(\frac{1}{2\lambda_m|p|^2}\sum\nolimits_{i,j}\left(\sum\nolimits_k p_k D_k a^{ij}(x)\right)^2 + \delta(p)a(x,z,p)\right). \end{aligned}$$

The first term in the outer parenthesis is $O(|p|^0)$. Thus we focus on the second term.

Using Lemma 54 to $D_z\theta$, we have

$$D_z\theta = \frac{\alpha\theta}{1+\theta}.$$

Thus

$$\begin{aligned} \delta a &= \left(D_z + |p|^{-2}p\cdot D_x\right)a; \\ &= -\frac{\sigma^2}{2\alpha}2(\theta+1)D_z\theta + |p|^{-2}p_kD_ka; \\ &= -\sigma^2\theta + |p|^{-2}p_kD_ka. \end{aligned}$$

D_ka in the last term is computed as

$$\begin{aligned} D_ka &= -\frac{\alpha}{2}p^\top(D_k\overline{a})p + p^\top(D_k\overline{b}) - \frac{1}{2\alpha}D_k\left(\sigma^2(\theta^2+2\theta)\right) + D_k\overline{\gamma}; \\ &= -\frac{\alpha}{2}p^\top\left(D_k\overline{a}\right)p + p^\top\left(D_k\overline{b}\right) - \frac{\sigma}{\alpha}(\theta^2+2\theta)D_k\sigma - \frac{\sigma^2}{\alpha}(\theta+1)D_k\theta + D_k\overline{\gamma}. \end{aligned}$$

Applying Lemma 54 to $D_k\theta$ with f defined in (83), we have

$$\begin{aligned} D_k\theta &= \frac{\theta}{1+\theta}\frac{D_kf}{f}; \\ &= \frac{\theta}{1+\theta}\left[\frac{D_k(\gamma/\sigma^2)}{\gamma/\sigma^2} + D_k\left(\frac{\mu}{\sigma^2}\right) - \alpha p_iD_k\left(\frac{c^{ij}\rho_j}{\sigma}\right)\right]. \end{aligned}$$

Therefore,

$$D_k a = -\frac{\alpha}{2} p^\top (D_k \overline{a}) p + p^\top \left(D_k \overline{b}\right) - \frac{\sigma}{\alpha}(\theta^2 + 2\theta) D_k \sigma$$
$$- \frac{\sigma^2}{\alpha}\theta \left[\frac{D_k(\gamma/\sigma^2)}{\gamma/\sigma^2} + D_k\left(\frac{\mu}{\sigma^2}\right) - \alpha p_i D_k \left(\frac{c^{ij}\rho_j}{\sigma}\right)\right] + D_k \overline{\gamma},$$

which is $O(|p|^2)$. This implies $\delta a = O(|p|)$. Thus

$$\limsup_{|p|\to\infty} \sup_{\Omega_m \times [u_0, u_1]} C_m = 0.$$

□

Lemma 50 (Estimation of D_m). *$D_m(x, z, p)$ defined in (45) satisfies*

$$\limsup_{|p|\to\infty} \sup_{\Omega_m \times [u_0, u_1]} D_m(x, z, p) = C(m) < \infty.$$

PROOF.

$$D_m(x, z, p) = \frac{1}{\mathcal{E}(x, p)} \left(|p|^2 \Lambda_m + |p|(|\mathcal{E}_p(x, p)| + |a_p(x, z, p)|)\right);$$
$$= \frac{1}{\mathcal{E}(x, p)} \left(|p|^2 \Lambda_m + |p|(|(cc^\top)^{ij}(x) p_j| + |a_p(x, z, p)|)\right).$$

From (84), a_p is proved to be $O(|p|)$. Thus it is straightforward to see

$$\limsup_{|p|\to\infty} \sup_{\Omega_m \times [u_0, u_1]} D_m(x, z, p) < \infty.$$

□

APPENDIX B

Miscellaneous Proofs

B.1. Proof of Proposition 39

In this proof, we revisit the result [**Lie96**, Theorem 11.3, (b)] used in Section 4.3.2, only this time we explicitly show that the gradient grows linearly with respect to $|x|$ in an unbounded domain $\mathbb{R}^n$. Unfortunately, we cannot just cite [**Lie96**, Theorem 11.3, (b)], because the desired result requires a careful evaluation of the constant appearing in [**Lie96**, Theorem 11.3, (b)], which is only mentioned there as being dependent on "the limit behavior" as $|p| \to \infty$ of a couple of quantities, including $A(x, z, p), B(x, z, p)$, and $C(x, z, p)$ introduced in Section 4.3.2. We instead need to go through the argument leading to [**Lie96**, Theorem 11.3, (b)] with more careful estimations of large p limits as in Lemma 51 below.

The main tool in the proof is the maximum principle applied to the PDE satisfied by $|Du|^2$ where u is the solution to (28). To be more precise, we first establish the linear PDE (92) below for the squared gradient of $\psi_0(u)$, $\overline{v}$, where the auxiliary function ψ_0 is to be determined in Lemma 52 below. Since we have to deal with the local gradient estimate in the unbounded domain, we further need to multiply $\overline{v}$ by a truncating function η, so that we can apply the maximum principle to $w = \eta\overline{v}$ in the cylindrical domain of a fixed radius. The PDE for w turns out to be a little more complex than that for $\overline{v}$, requiring extra estimations before applying the maximum principle.

Let us stress that Assumptions 5 and 6 prevail throughout this section. Thus Remark 33 is valid: we freely speak of $C^{1,2}([0, T] \times \mathbb{R}^n)$-solution u for (32) with $u_0 \leq u \leq u_1$.

Under Assumption 5, our domain is

$$\Omega = (0,T) \times D = (0,T) \times \mathbb{R}^n,$$

and we set the localized domains to be

$$\Omega_m = (0,T) \times D_m = (0,T) \times B(0, mR) \quad \text{for } m = 1, 2, \ldots,$$

where R is a fixed positive constant.

The fact that a^{ij} is constant under Assumption 5 simplifies most of the quantities already introduced in Section 4.3.2. For example, the Bernstein $\mathcal{E}$ (39) becomes

$$\mathcal{E}(p) = a^{ij} p_i p_j.$$

The quantities A, B, and C defined in (41) – (43) become much simpler, too. For $(x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, we have

$$\begin{aligned} A(p) &= \frac{1}{\mathcal{E}(p)} (\overline{\delta}(p) - 1) \mathcal{E}(p); \\ &= \frac{1}{\mathcal{E}(p)} (p \cdot D_p - 1) a^{ij} p_i p_j; \\ &= 1, \qquad\qquad (85) \\ B(x, z, p) &= \frac{1}{\mathcal{E}(p)} (\overline{\delta}(p) - 1) a(x, z, p), \\ C(x, z, p) &= \frac{1}{\mathcal{E}(p)} \delta(p) a(x, z, p). \end{aligned}$$

We first establish the PDE for $\overline{v} = |D\overline{u}|^2$, where

$$\overline{u}(t, x) \triangleq \psi_0(u(t, x)),$$

and ψ_0 is a strictly increasing function defined on $[u_0, u_1]$. Set ψ to be the inverse of ψ_0, defined on the interval $[\overline{u}_0, \overline{u}_1] = [\psi_0(u_0), \psi_0(u_1)]$. We further assume ψ and ψ_0 are C^3 on their domains. Thus

$$\begin{aligned} u(t,x) &= \psi(\overline{u}(t,x)), \\ u_t = \psi'(\overline{u})\overline{u}_t, &\quad D_i u = \psi'(\overline{u}) D_i \overline{u}, \\ D_{ij} u &= \psi'(\overline{u}) D_{ij}\overline{u} + \psi''(\overline{u}) D_i\overline{u} D_j \overline{u}, \\ \overline{v} &= |D\overline{u}|^2 = \frac{|Du|^2}{(\psi'(\overline{u}))^2}. \end{aligned} \tag{86}$$

(Here and in all that follows, we drop natural function arguments (t,x) or (x).)

We also define

$$\omega(\overline{z}) = \frac{\psi''(\overline{z})}{(\psi'(\overline{z}))^2}; \quad \overline{z} \in [\overline{u}_0, \overline{u}_1]. \tag{87}$$

(Whenever $\overline{z}$ appears in this section, it should be understood to take values in $[\overline{u}_0, \overline{u}_1]$.)

Substituting (86) into (28), we have

$$0 = -\overline{u}_t + a^{ij} D_{ij}\overline{u} + \frac{1}{\psi'(\overline{u})}(a(x,u,Du) + \omega(\overline{u})\mathcal{E}(Du)), \tag{88}$$

where we explicitly denoted the function arguments of ψ', a, ω, and $\mathcal{E}$. Adopting the notation where the superscript r denotes differentiation with respect to p_r, the chain-rule gives

$$\begin{aligned} D_k a &= a_k + a_z D_k u + a^r D_{rk} u, \\ D_k \mathcal{E} &= \mathcal{E}^r D_{rk} u. \end{aligned}$$

Thus we first differentiate (88) with respect to x_k to have[1],

$$0 = -D_k\overline{u}_t + a^{ij} D_{ijk}\overline{u} - (a+\omega\mathcal{E})\frac{\psi''}{(\psi')^2} D_k\overline{u} + \frac{1}{\psi'}\left(a_k + a_z D_k u + a^r D_{rk} u + \omega' D_k\overline{u}\mathcal{E} + \omega\mathcal{E}^r D_{rk} u\right).$$

[1]This operation requires $Du \in C^{1,2}$, which is more than $u \in C^{1,2}$ that we obtain in Theorem 26. This is not an issue because we can always rewrite (92) or (96) below in the weak form, so that in the end we can apply the strong maximum principle for weak subsolution [**Lie96**, Theorem 6.25]. See the remark that follows [**Lie96**, (11.8)] for the explicit weak form.

Multiply the above equation by $D_k\overline{u}$ and sum on k to have,

$$0 = -\frac{1}{2}\overline{v}_t + a^{ij}D_{ijk}\overline{u}D_k\overline{u} + \frac{1}{\psi'}\left(a^r D_{rk}uD_k\overline{u} + \psi' a_z\overline{v} + a_k D_k\overline{u} + \omega\mathcal{E}^r D_{rk}uD_k\overline{u}\right) + \left(\frac{\omega'}{\psi'}\mathcal{E} - \omega a - \omega^2\mathcal{E}\right)\overline{v}. \quad (89)$$

The PDE above can be transformed into the linear parabolic PDE of $\overline{v}$ as follows. For the second term of (89), notice the relation

$$\frac{1}{2}a^{ij}D_{ij}\overline{v} = \frac{1}{2}a^{ij}D_{ij}(D_k\overline{u}D_k\overline{u}) = a^{ij}D_i(D_{jk}\overline{u}D_k\overline{u}) = a^{ij}(D_{ijk}\overline{u}D_k\overline{u} + D_{ik}\overline{u}D_{jk}\overline{u}).$$

Also, since

$$D_{rk}u = D_r(D_k u) = D_r(\psi' D_k\overline{u}) = \psi'' D_r\overline{u}D_k\overline{u} + \psi' D_{rk}\overline{u} = \omega D_r u D_k u + \psi' D_{rk}\overline{u},$$

the third and sixth terms of (89) become

$$\begin{aligned}\frac{1}{\psi'}a^r D_{rk}uD_k\overline{u} &= \frac{1}{\psi'}a^r(\omega D_r u D_k u + \psi' D_{rk}\overline{u})D_k\overline{u} = \omega\overline{\delta}a\overline{v} + \frac{1}{2}a^r D_r\overline{v},\\ \frac{1}{\psi'}\omega\mathcal{E}^r D_{rk}uD_k\overline{u} &= \frac{1}{\psi'}\omega\mathcal{E}^r(\omega D_r u D_k u + \psi' D_{rk}\overline{u})D_k\overline{u} = \omega^2\overline{\delta}\mathcal{E}\overline{v} + \frac{1}{2}\omega\mathcal{E}^r D_r\overline{v},\end{aligned}$$

where $\overline{\delta}$, defined in (40), is evaluated at $p = Du$. Therefore, using

$$b^r \triangleq a^r(x, u, Du) + \omega(\overline{u})\mathcal{E}^r(Du),$$

(89) becomes

$$\begin{aligned}0 &= -\frac{1}{2}\overline{v}_t + \frac{1}{2}a^{ij}D_{ij}\overline{v} + \frac{1}{2}b^r D_r\overline{v} - a^{ij}D_{ik}\overline{u}D_{jk}\overline{u} + \left[\frac{\omega'}{\psi'}\mathcal{E} + \omega^2(\overline{\delta} - 1)\mathcal{E} + \omega(\overline{\delta} - 1)a + \delta a\right]\overline{v};\\ &= -\frac{1}{2}\overline{v}_t + \frac{1}{2}a^{ij}D_{ij}\overline{v} + \frac{1}{2}b^r D_r\overline{v} - a^{ij}D_{ik}\overline{u}D_{jk}\overline{u} + \left(\frac{\omega'}{\psi'} + \omega^2 + B\omega + C\right)\mathcal{E}\overline{v}, \qquad (90)\end{aligned}$$

where in the last line, we substituted the relations in (85), and both B and C are evaluated at (x, u, Du). Finally, noticing from Assumption 1 that

$$a^{ij} D_{ik}\overline{u} D_{jk}\overline{u} \geq \lambda \sum_{i,j} |D_{ij}\overline{u}|^2 \geq 0, \tag{91}$$

we have

$$-\overline{v}_t + a^{ij} D_{ij}\overline{v} + b^r D_r\overline{v} + 2c^0 \mathcal{E}\overline{v} \geq 0, \tag{92}$$

with

$$c^0(t,x) \triangleq \frac{\omega'(\overline{u})}{\psi'(\overline{u})} + \omega^2(\overline{u}) + B(x,u,Du)\omega(\overline{u}) + C(x,u,Du). \tag{93}$$

Regarding (93), we construct ψ (therefore ψ_0 as its inverse) in Lemma 52 such that, for some $\varepsilon > 0$ and $p_1 > 0$,

$$c^0(t,x) \leq -\varepsilon \quad \text{on } \{(t,x) \in \Omega : |Du(t,x)| > p_1\}. \tag{94}$$

The estimation so far results in [**Lie96**, Theorem 11.1], but now we turn to the (spacial) local bound on gradient, [**Lie96**, Section 11.3, (b)].

Pick a point x_0 in $D_{m-1} = B(0,(m-1)R)$ and let Q be a cylinder $(0,T) \times B(x_0,R)$ so that Q is contained in Ω_m. For the local estimation, we are going to apply the maximum principle to the PDE for

$$w(t,x) \triangleq \eta(x)\overline{v}(t,x),$$

in Q, where[2]

$$\eta(x) \triangleq \zeta^2(x), \quad \zeta(x) \triangleq \left(1 - \frac{|x-x_0|^2}{R^2}\right)^+. \tag{95}$$

[2]The choice of functions $\eta(x)$ and $\zeta(x)$ corresponds to setting $\theta = 1$ defined in [**Lie96**, Section 11.3].

Observe

$$
\begin{aligned}
w_t &= \eta \overline{v}_t, \\
D_i w &= \eta D_i \overline{v} + \overline{v} D_i \eta, \\
D_{ij} w &= \eta D_{ij} \overline{v} + D_i \eta D_j \overline{v} + D_j \eta D_i \overline{v} + \overline{v} D_{ij} \eta; \\
&= \eta D_{ij} \overline{v} + \frac{D_i \eta}{\eta}(D_j w - \overline{v} D_j \eta) + \frac{D_j \eta}{\eta}(D_i w - \overline{v} D_i \eta) + \overline{v} D_{ij} \eta.
\end{aligned}
$$

Multiplying (90) by η, we have,

$$
\begin{aligned}
0 &= -\frac{1}{2} w_t + \frac{1}{2} a^{ij} \eta D_{ij} \overline{v} + \frac{1}{2} b^i \eta D_i \overline{v} - \eta a^{ij} D_{ik} \overline{u} D_{jk} \overline{u} + \left(\frac{\omega'}{\psi'} + \omega^2 + B\omega + C \right) \mathcal{E} w \\
&= -\frac{1}{2} w_t + \frac{1}{2} a^{ij} \left[D_{ij} w - \frac{D_i \eta}{\eta}(D_j w - \overline{v} D_j \eta) - \frac{D_j \eta}{\eta}(D_i w - \overline{v} D_i \eta) - \overline{v} D_{ij} \eta \right] \\
&+ \frac{1}{2} b^i (D_i w - \overline{v} D_i \eta) - \eta a^{ij} D_{ik} \overline{u} D_{jk} \overline{u} + \left(\frac{\omega'}{\psi'} + \omega^2 + B\omega + C \right) \mathcal{E} w \\
&= -\frac{1}{2} w_t + \frac{1}{2} a^{ij} D_{ij} w + \frac{1}{2} (b^i - \frac{2}{\eta} a^{ij} D_j \eta) D_i w + \left(\frac{\omega'}{\psi'} + \omega^2 + B\omega + C \right) \mathcal{E} w \\
&+ \left[\frac{1}{\eta^2} a^{ij} D_i \eta D_j \eta - \frac{1}{2\eta} a^{ij} D_{ij} \eta - \frac{1}{2\eta} b^i D_i \eta \right] w.
\end{aligned}
$$

Therefore, using (91), we see that the PDE satisfies by w is similar to that of $\overline{v}$, (92), i.e.,

$$
-w_t + a^{ij} D_{ij} w + b_2^i D_i w + 2(c^0 + B_0 \omega + C_0) \mathcal{E} w \geq 0, \tag{96}
$$

where

$$
b_2^i \triangleq b^i - \frac{2}{\eta} a^{ij} D_j \eta = \omega(\overline{u}) \mathcal{E}^i(Du) + a^i(x, u, Du) - \frac{2}{\eta} a^{ij} D_j \eta,
$$

and

$$
\begin{aligned}
B_0 &\triangleq -\frac{1}{\mathcal{E}(Du)} \frac{D_i \eta}{2\eta} \mathcal{E}^i(Du), \\
C_0 &\triangleq \frac{1}{\mathcal{E}(Du)} \left[\frac{1}{\eta^2} a^{ij} D_i \eta D_j \eta - \frac{1}{2\eta} a^{ij} D_{ij} \eta - \frac{1}{2\eta} a^i(x, u, Du) D_i \eta \right].
\end{aligned} \tag{97}
$$

which are defined right before (11.17) in [**Lie96**][3].

In order to apply the maximum principle to (96), we seek to have bounds on B_0 and C_0 that go to 0 as $|p| \to \infty$, or similarly, as $w \to \infty$, so that $c^0 + B_0\omega + C_0$ becomes negative for w large enough. Note that ψ is already determined in Lemma 52. Thus from (87), ω is a fixed function on $[\overline{u}_0, \overline{u}_1]$. In particular, we have by Lemma 53,

$$c^0 + B\omega + C_0 \leq 0 \quad \text{on } \{(t,x) \in \Omega_m : w > C_3 m^2\}, \tag{98}$$

for some positive constant C_3.

Applying the maximum principle to (96) in the region Q, with (98) yields,

$$\sup_Q w \leq^4 \max(C_3 m^2, \sup_{\mathcal{P}Q} w) \leq \max(C_3 m^2, \frac{\sup_{B(x_0,R)} |D\varphi|^2}{\min \psi'^2})$$

where the last inequality is from (106) below. Hence w is bounded by the right-most quantity in Q, and in particular at (t, x_0),

$$\frac{|Du(t,x_0)|^2}{\max \psi'^2} \leq \overline{v}(t,x_0) = w(t,x_0) \leq \max(C_3 m^2, \frac{\sup_{B(x_0,R)} |D\varphi|^2}{\min \psi'^2}).$$

Finally, evaluating this inequality at any point with $x_0 \in D_{m-1}$ gives

$$\sup_{\Omega_{m-1}} |Du| \leq \max \psi' \cdot \max(C_3^{1/2} m, \frac{\sup_{D_m} |D\varphi|}{\min \psi'}).$$

Thus, in $\Omega_{m-1}/\Omega_{m-2}$, we have

$$\sup_{\Omega_{m-1}/\Omega_{m-2}} \frac{|Du|}{|x|} \leq \frac{\max \psi'}{(m-2)R} \cdot \max(C_3^{1/2} m, \frac{\sup_{D_m} |D\varphi|}{\min \psi'}).$$

[3]The definition of C_0 in [**Lie96**, Section 11.3] misses the term $\frac{1}{2\eta} a^i D_i \eta$.

[4]Suppose w takes the maximum in $\overline{Q}$ at (t', x'). We can assume $(t', x') \in \overline{Q}/\mathcal{P}Q$ and $w(t', x') > C_3 m^2$, because otherwise the maximum principle result is trivial. By the strong maximum principle for weak subsolutions [**Lie96**, Theorem 6.25], w takes the value $w(t', x')$ for on the cylinder $(t - l^2, t) \times B(x, l)$ for any $l > 0$ as long as $(t' - l^2, t') \times B(x', l) \subset \{(t,x) \in Q : w > C_3 m^2\}$. However, this is a contradiction because it means we can pick points on which w takes value $w(t', x')$, arbitrarily close to $\mathcal{P}Q \cup \{w = C_3 m^2\}$, but w is continuous.

Since the right-hand side is bounded by a constant not depending on m, Proposition 39 is proved.

Below we list the lemmas used in the proof of Proposition 39.

Lemma 51. *Under Assumption 5,*

$$B^\infty \triangleq \limsup_{|p|\to\infty} \sup_{\Omega\times[u_0,u_1]} B, \quad C^\infty \triangleq \limsup_{|p|\to\infty} \sup_{\Omega\times[u_0,u_1]} C \tag{99}$$

are both finite with $C^\infty = 0$, *and*

$$\sup_{\Omega\times[u_0,u_1]} \frac{1}{\mathcal{E}} \left(|p|^2\Lambda + |p||\mathcal{E}_p|\right) \leq C_1 < \infty, \tag{100}$$

$$\sup_m \sup_{|p|>mp_2} \sup_{\Omega_m\times[u_0,u_1]} \frac{1}{\mathcal{E}}|p||a_p| \leq C_2 < \infty, \tag{101}$$

where, in (101), p_2 *is some positive constant.*

Proof. Estimations in (99) are carried out in a similar manner as (44), where the only difference is that all the estimations localized to some Ω_m are now valid for the entire domain Ω. As such, Lemma 49, and Lemma 48 prove $C^\infty = 0$, and the finiteness of B^∞, respectively, where each appearance of λ_m should be replaced by λ during the proofs.

Noting $\mathcal{E} = a^{ij}p_ip_j \geq \lambda|p|^2$, we obtain (100) as a straightforward consequence of the positive definiteness of the constant matrix a^{ij}.

For (101), recall that our a_p has the form (84):

$$D_{p_i}a = -\frac{\alpha}{2}\left[2(cc^\top)^{ij}p_j - 2(c^{ij}\rho_j)^2p_i\right] + \left(b^i - \frac{\mu}{\sigma}c^{ij}\rho_j\right) + \sigma\theta c^{ij}\rho_j.$$

Under Assumption 5, we see from Corollary 56 below that θ grows linearly with respect to p in Ω. Thus all the terms in $D_{p_i}a$ grow linearly with respect to p in Ω except the term b^i. Because b is in C^1_b, we can pick a positive constant b_1 such that $|b| \leq b_1 m$ in Ω_m for all $m \in \mathbb{N}$. Fix any positive constant p_2. Then for each $m \in \mathbb{N}$, we have for $x \in D_m$ and

$|p| > mp_2$,

$$\frac{1}{\mathcal{E}}|p||b| \leq \frac{1}{\lambda|p|}b_1 m \leq \frac{b_1}{\lambda p_2},$$

i.e., bounded by a constant not depending on m. Thus (101) is proved. $\square$

Lemma 52. *We can choose a strictly increasing function ψ_0 defined on $[u_0, u_1]$ such that both ψ_0 and its inverse ψ are C^3 on their domains. Moreover, we can pick $\varepsilon > 0$ and $p_1 > 0$ such that (94) holds.*

Proof. We follow the argument of Section 11.1 leading to [**Lie96**, Theorem 11.1].

Let

$$\chi(z) \triangleq (\log\phi(z))' \tag{102}$$

with

$$\phi(z) \triangleq \psi'(\psi^{-1}(z)). \tag{103}$$

Note that, with $z = \psi(\overline{z})$, straightforward calculus gives

$$\chi(z) = \frac{\phi'(z)}{\phi(z)} = \frac{1}{\phi(z)}\frac{\psi''(\psi^{-1}(z))}{\psi'(\psi^{-1}(z))} = \frac{\psi''(\overline{z})}{(\psi'(\overline{z}))^2} = \omega(\overline{z}),$$
$$\chi'(z) = \big(\omega(\psi^{-1}(z))\big)' = \frac{\omega'(\psi^{-1}(z))}{\psi'(\psi^{-1}(z))} = \frac{\omega'(\overline{z})}{\psi'(\overline{z})}.$$

Thus we can rewrite c^0 in (93) as $c^0(t,x) = \chi'(u) + \chi^2(u) + B(x,u,Du)\chi(u) + C(x,u,Du)$.

We first seek $\chi(z)$ that satisfies

$$\chi'(z) + \chi^2(z) + B^\infty\chi(z) \leq -2\varepsilon; \quad z \in [u_0, u_1],$$

for some positive ε. As such, we take

$$\chi(z) = \exp((2 + B_\infty)(u_0 - z)),$$

and $\varepsilon = \chi(u_1)/2$. Indeed, noticing $0 < \chi(u_1) \leq \chi(z) \leq 1$ for $z \in [u_0, u_1]$, we have

$$\begin{aligned} &\max_{z\in[u_0,u_1]} (\chi'(z) + \chi^2(z) + B^\infty \chi(z)) \\ &= \max_{z\in[u_0,u_1]} \chi(z)[-(2 + B^\infty) + \chi(z) + B_\infty]; \\ &= \max_{z\in[u_0,u_1]} \chi(z)[\chi(z) - 2]; \\ &\leq \max_{z\in[u_0,u_1]} -\chi(z) \leq -\chi(u_1) = 2\varepsilon. \end{aligned}$$

By (99), we can take p_1 large enough so that

$$\chi'(z) + \chi^2(z) + B(x, z, p)\chi(z) + C(x, z, p) \leq \chi'(z) + \chi^2(z) + B^\infty \chi(z) + \varepsilon;$$
$$\text{for all } x \in \mathbb{R}^n, z \in [u_0, u_1], |p| > p_1.$$

Thus we finally obtain

$$\begin{aligned} c^0(t, x) &= \chi'(u) + \chi^2(u) + B(x, u, Du)\chi(u) + C(x, u, Du); \\ &\leq \chi'(u) + \chi^2(u) + B^\infty \chi(u) + \varepsilon; \\ &\leq -\varepsilon, \end{aligned}$$

for all $(t, x) \in \Omega$, $|Du(t, x)| > p_1$.

Once we have $\chi(z)$, we can integrate it to obtain $\phi(z)$ by (102). From (103), we have

$$\psi'(\overline{z}) = \phi(\psi(\overline{z})).$$

Thus solving this ODE on $[\overline{u}_0, \overline{u}_1]$ gives us $\psi(\overline{z})$. Finally we get ψ_0 by inverting ψ. $\square$

Lemma 53. *In* (96)*, we can take* L_m *so that*

$$c^0 + B_0\omega + C_0 \leq 0 \quad \textit{on } \{(t, x) \in \Omega_m : w(t, x) > L_m\},$$

for $m = 1, 2, \ldots$.

Moreover, L_m *is quadratic in* m*, i.e.,* $L_m = C_3 m^2$*.*

Proof. To evaluate B_0 and C_0 defined in (97), we first evaluate the terms with η. From (95), straightforward calculus gives,

$$\begin{aligned} D_i\eta &= 2\zeta D_i\zeta \\ &= -2\zeta \cdot \frac{2(x-x_0)_i}{R^2}, \\ D_{ij}\eta &= -\frac{4}{R^2}[(x-x_0)_i D_j\zeta + \zeta\delta_{ij}] \\ &= -\frac{4}{R^2}\left[-2\frac{(x-x_0)_i(x-x_0)_j}{R^2} + \zeta\delta_{ij}\right], \end{aligned}$$

thus the bounds

$$|D\eta| \leq \frac{4\zeta}{R}, \quad |D_{xx}\eta| \leq \frac{12}{R^2}, \tag{104}$$

where $|D_{xx}\eta|$ is understood as the operator norm for the $n \times n-$matrix $D_{xx}\eta$.

Furthermore, from the defining relation $u(t,x) = \psi(\overline{u}(t,x))$ and $0 \leq \zeta \leq 1$, we have the inequality

$$|Du| = \psi'(\overline{u})\overline{v}^{1/2} \geq (\min\psi')\overline{v}^{1/2} = (\min\psi')\frac{w^{1/2}}{\zeta} \tag{105}$$

$$\geq (\min\psi')w^{1/2}, \tag{106}$$

where $\min_{\overline{z}\in[\overline{u}_0,\overline{u}_1]}\psi'(\overline{z}) = \min_{z\in[u_0,u_1]}\phi(z)$ is a fixed positive constant (see (103)).

Combining (104) and (105), we have

$$\frac{|D\eta|}{\eta|Du|} \leq \frac{4}{\min\psi'}\frac{1}{Rw^{1/2}}, \quad \frac{|D_{xx}\eta|}{\eta|Du|^2} \leq \frac{12}{\min\psi'^2}\frac{1}{R^2 w}. \tag{107}$$

Therefore, from (97), (100), and (107), we have in Ω,

$$|B_0| \leq \frac{|D\eta||\mathcal{E}_p|}{2\eta\mathcal{E}} = \frac{1}{2}\frac{|D\eta|}{\eta|Du|}\frac{|Du||\mathcal{E}_p|}{\mathcal{E}} \leq \frac{2C_1}{\min\psi'}\frac{1}{Rw^{1/2}}. \tag{108}$$

Similarly, from (97), (100), (101), and (107), we have in Ω_m and $|Du| > mp_2$[5],

$$\begin{aligned}|C_0| &\le \frac{1}{\mathcal{E}}\left(\frac{1}{\eta^2}\Lambda|D\eta|^2 + \frac{1}{2\eta}n\Lambda|D_{xx}\eta| + \frac{1}{2\eta}|a_p||D\eta|\right)\\ &= \frac{|Du|^2\Lambda}{\mathcal{E}}\left(\frac{|D\eta|}{\eta|Du|}\right)^2 + \frac{n}{2}\frac{|Du|^2\Lambda}{\mathcal{E}}\frac{|D_{xx}\eta|}{\eta|Du|^2} + \frac{1}{2}\frac{|Du||a_p|}{\mathcal{E}}\frac{|D\eta|}{\eta|Du|}\\ &\le \frac{16C_1}{\min\psi'^2}\frac{1}{R^2w} + \frac{6nC_1}{\min\psi'^2}\frac{1}{R^2w} + \frac{2C_2}{\min\psi'}\frac{1}{Rw^{1/2}}. \end{aligned} \tag{109}$$

Thus, by (108) and (109), we can pick a positive constant w_0, that only depends on $n, \min\psi', C_1, C_2, R, \varepsilon$ and $\max|\omega|$, such that

$$B_0\omega + C_0 \le \varepsilon \quad \text{on } \{(t,x) \in \Omega_m : |Du(t,x)| > mp_2,\ w(t,x) > w_0\}. \tag{110}$$

Observing the inequality (106), we combine the results (94) and (110) to have,

$$c^0 + B_0\omega + C_0 \le 0 \quad \text{on } \{(t,x) \in \Omega_m : w > L_m\}$$

where

$$L_m = \max\left(\frac{p_1^2}{\min\psi'^2}, \frac{m^2p_2^2}{\min\psi'^2}, w_0\right),$$

which is the statement of Lemma 53. □

B.2. Proof of Lemma 41

We follow the idea described in [**Pha02**, Remark 2.2].

Introducing a process $X' \triangleq c^{-1}X$, we see that

$$dX'_t = c^{-1}b(\omega, t, X_t)dt + dW_t = b'(\omega, t, X'_t) + dW_t,$$

where we define

$$b'(\omega, t, x) = c^{-1}b(\omega, t, cx)$$

[5]For the second term in C_0, note that $a^{ij}D_{ij}\eta = \mathrm{tr}(aD_{xx}\eta) \le |D_{xx}\eta|\,\mathrm{tr}(a) = |D_{xx}\eta|(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \le |D_{xx}\eta|n\Lambda$. where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are eigenvalues of the constant matrix a.

for $(t,x) \in \Omega \times [0,T] \times \mathbb{R}^n$.

It is straightforward to see that b' also satisfies the linear growth condition with respect to $|x|$, i.e.,

$$|b'(\omega,t,x)| \leq C'(1+|x|); \quad \forall(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^n.$$

We are going to show

$$\sup_{t\leq T} \mathbb{E}\left[\exp(k|X_t'|^2)\right] < \infty,$$

for some positive constant k. The statement of Lemma 41 follows immediately because $|X'|^2 = X^\top(cc^\top)^{-1}X \geq |X|^2/(2\Lambda)$ where Λ is defined in Assumption 1.

Since

$$\begin{aligned} |X_t'| &\leq |c^{-1}x| + \int_0^t |b'(\omega,t,X_u')|du + |W_t| \\ &\leq |c^{-1}x| + \int_0^t C'(1+|X_u'|)du + |W_t|; \quad 0 \leq t \leq T, \end{aligned}$$

by the Gronwall inequality (see [**KS12**, Problem 2.7]), we have

$$|X_t'| \leq C_1\left(1+|W_t| + \int_0^t |W_u|du\right); \quad 0 \leq t \leq T,$$

for some positive constant C_1.

Thus for any $t \in [0,T]$,

$$\begin{aligned} \mathbb{E}\left[\exp(k|X_t'|^2)\right] &\leq \mathbb{E}\left[\exp\left(9kC_1^2\left(1+|W_t|^2+\left(\int_0^t |W_u|du\right)^2\right)\right)\right] \\ &\leq \mathbb{E}\left[\exp\left(9kC_1^2\left(1+|W_t|^2+t\int_0^t |W_u|^2du\right)\right)\right] \\ &\leq e^{9kC_1^2}\mathbb{E}\left[e^{9kC_1^2|W_t|^2}\frac{1}{t}\int_0^t e^{9kC_1^2t^2|W_u|^2}du\right] \\ &\leq e^{9kC_1^2}\left(\mathbb{E}e^{18kC_1^2|W_t|^2}\right)^{\frac{1}{2}}\frac{1}{t}\int_0^t \left(\mathbb{E}e^{18kC_1^2t^2|W_u|^2}\right)^{\frac{1}{2}}du, \end{aligned} \tag{111}$$

where the second and fourth inequalities are by Cauchy-Schwarz and the third by Jensen. Note that in the last line above, we have bounds

$$\mathbb{E}e^{18kC_1^2|W_t|^2} = \mathbb{E}e^{18kC_1^2 t|W_1|^2} \leq \mathbb{E}e^{18kC_1^2 T|W_1|^2}, \tag{112}$$

and for $0 \leq u \leq t$,

$$\mathbb{E}e^{18kC_1^2 t^2|W_u|^2} = \mathbb{E}e^{18kC_1^2 t^2 u|W_1|^2} \leq \mathbb{E}e^{18kC_1^2 T^3|W_1|^2}. \tag{113}$$

We can choose $k > 0$, only depending on C_1 and T, that makes the right-most terms of (112) and (113) finite. For such choice of k, let

$$C_2 \triangleq \max\left(\mathbb{E}e^{18kC_1^2 T|W_1|^2}, \mathbb{E}e^{18kC_1^2 T^3|W_1|^2}\right) < \infty.$$

Then from (111), we have

$$\mathbb{E}\left[\exp(k|X_t'|^2)\right] \leq e^{9kC_1^2} C_2; \quad 0 \leq t \leq T.$$

Thus the lemma is proved.

B.3. Elementary Results

Lemma 54. *Suppose f is positive and differentiable at $x \in \mathbb{R}$. Then, for $y = W(f(x))$, we have*

$$y'(x) = \frac{y}{1+y}\frac{f'(x)}{f(x)}.$$

PROOF. By the definition of y, we have

$$ye^y = f(x).$$

Differentiating with respect to x and using the definition of y once again, we obtain $y'(x)$. □

Lemma 55. *For $x > 0, y \in \mathbb{R}$, we have*

$$W(xe^y) \leq W(x) + y^+.$$

Proof. Since the product-log function is a strictly increasing function, we only have to show the claim for $y > 0$. Fix $x > 0$ and let

$$f(y) \triangleq W(xe^y), \quad y > 0.$$

By Lemma 54, we have

$$f'(y) = \frac{f(y)}{1 + f(y)} \leq 1, \quad y > 0.$$

Thus for any $y > 0$, we see

$$f(y) \leq f(0) + y$$

holds by the mean value theorem. This is exactly the claim of the lamma for $y > 0$. □

Corollary 56. *For $\theta(x, z, p)$ defined by (24), we have the following bound (we drop the function arguments to simplify notations below)*

$$\begin{aligned}\theta &\leq \max\left(\frac{\gamma}{\sigma^2}, \frac{\mu}{\sigma^2}\right) + \left(\alpha z - \frac{\alpha}{\sigma} c^{ij} \rho_j p_i\right)^+ \\ &\leq \max\left(\frac{\gamma}{\sigma^2}, \frac{\mu}{\sigma^2}\right) + \alpha z^+ + \left(-\frac{\alpha}{\sigma} c^{ij} \rho_j p_i\right)^+.\end{aligned}$$

Proof. The last inequality follows from the sublinearity of the function $(\cdot)^+$, so we only need to prove the first inequality.

Using Lemma 55 with

$$\begin{aligned}x &= \frac{\gamma}{\sigma^2} \exp\left(\frac{\mu}{\sigma^2}\right), \\ y &= \alpha z - \frac{\alpha}{\sigma} c^{ij} \rho_j p_i,\end{aligned}$$

we obtain

$$\theta \leq W\left(\frac{\gamma}{\sigma^2} \exp\left(\frac{\mu}{\sigma^2}\right)\right) + \left(\alpha z - \frac{\alpha}{\sigma} c^{ij} \rho_j p_i\right)^+.$$

Since the product-log function is a strictly increasing function, it is straightforward to see

$$W\left(\frac{\gamma}{\sigma^2}\exp\left(\frac{\mu}{\sigma^2}\right)\right) \leq \max\left(\frac{\gamma}{\sigma^2}, \frac{\mu}{\sigma^2}\right).$$

Thus

$$\theta \leq \max\left(\frac{\gamma}{\sigma^2}, \frac{\mu}{\sigma^2}\right) + \left(\alpha z - \frac{\alpha}{\sigma} c^{ij} \rho_j p_i\right)^+.$$

□

Lemma 57. *For $x > 0, y \in \mathbb{R}$, we have*

$$xe^y \leq (-1 + \sqrt{1 + 2x + y^2}) \exp\left(-1 + \sqrt{1 + 2x + y^2}\right)$$

where the equality holds if and only if $x = y$.

Proof. Fix $y \geq 0$. Let

$$f(x) = (-1 + \sqrt{1 + 2x + y^2}) \exp\left(-1 + \sqrt{1 + 2x + y^2}\right) - xe^y.$$

Then, we see that

$$f'(x) = \exp\left(-1 + \sqrt{1 + 2x + y^2}\right) - e^y.$$

Note that $f'(x)$ is strictly negative for $x < y$, strictly positive for $x > y$, and zero for $x = y$ because $-1 + \sqrt{1 + 2x + y^2}$ lies strictly between x and y when $x \neq y$. Since $f(y) = 0$, we proved the claim for $y \geq 0$.

For $y < 0$, observe that for any $x \geq 0$,

$$xe^y \leq xe^{-y} \leq (-1 + \sqrt{1 + 2x + y^2}) \exp\left(-1 + \sqrt{1 + 2x + y^2}\right),$$

where the second inequality is from the case $y \geq 0$. □

Corollary 58. *For $x > 0, y \in \mathbb{R}$, let $\theta = W(xe^y)$. Then $\theta^2 + 2\theta \leq y^2 + 2x$ where the equality holds if and only if $x = y$.*

PROOF. Taking the product-log of the inequality in Lemma 57 yields $\theta \leq -1+\sqrt{1+2x+y^2}$, where the equality holds if and only if $x = y$. By rewriting this inequality, we have the claim. $\square$

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