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DEFINABLE CARDINALS JUST BEYOND \mathbb{R}/\mathbb{Q}

CLINTON T. CONLEY AND BENJAMIN D. MILLER

ABSTRACT. We establish the inexistence of countable bases for the family of definable cardinals associated with countable Borel equivalence relations which are not measure reducible to \mathbb{E}_0 , thereby ruling out natural generalizations of the Glimm-Effros dichotomy. We also push the primary known results concerning the abstract structure of the Borel cardinal hierarchy nearly to its base, using arguments substantially simpler than those previously employed. Our main tool is a strong notion of separability, which holds of orbit equivalence relations induced by group actions satisfying an appropriate measureless local rigidity property.

Introduction

The usual notion of *cardinality* entails that $|X| \leq |Y|$ if and only if there is an injection of X into Y . Over the last few decades, a finer notion of *definable cardinality* has emerged, whereby one requires that the injection is suitably definable. This refined notion has been used to identify obstacles of definability inherent in classification problems throughout mathematics. While there are far too many such applications to provide an exhaustive list here, a few notable examples include the classifications of torsion-free abelian groups [Hjo99, AK00, Tho03, Tho06], ergodic measure-preserving transformations [Hjo01, FW04, FRW06, FRW11], separable Banach spaces [FLR09, Ros11], and separable C^* -algebras [FTT13a, FTT13b, Sab13]. In order to better understand such results, one must obtain deep insight into the abstract structure of the definable cardinal hierarchy. Unfortunately, the elucidation of this structure has turned out to be a very difficult task.

The first of the two main lines of research into the abstract structure of the definable cardinality hierarchy concerns its base. The first

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substantial difference between the usual notion of cardinality and its definable counterpart appeared in [Sil80], where it was shown that the definable cardinality of \mathbb{R} is the immediate successor of that of \mathbb{N} , among definable cardinals associated with co-analytic equivalence relations. Building upon this and earlier operator-algebraic work in [Gli61, Eff65], it was shown in [HKL90, DJK94] that the definable cardinality of \mathbb{R}/\mathbb{Q} is the immediate successor of that of \mathbb{R} , among definable cardinals associated with Borel equivalence relations. Unfortunately, work in this direction stalled shortly thereafter, with [KL97, Theorem 2] ruling out further analogs of the continuum hypothesis for definable cardinals associated with Borel equivalence relations. However, such questions remained open for countable Borel equivalence relations.

The first of the two main goals of this paper is to show that even in the latter context, every basis for the definable cardinals beyond \mathbb{R}/\mathbb{Q} under measure reducibility has cardinality at least 2^{\aleph_0} , in particular ruling out the kinds of single-element bases provided by the aforementioned results.

The second of the two main lines of research into the abstract structure of the definable cardinality hierarchy concerns the fact that if one goes sufficiently far beyond its base, then the properties of cardinality and its definable counterpart diverge wildly. The first such result, due originally to Woodin and later refined in [LV94], was the existence of uncountable families of Borel equivalence relations whose associated definable cardinals are pairwise incomparable. However, the underlying arguments depended heavily on Baire category techniques, and [HK96, Theorem 6.2] implies that such an approach cannot yield incomparability of definable cardinals associated with countable Borel equivalence relations.

This difficulty was eventually overcome in [AK00], establishing the existence of uncountable families of countable Borel equivalence relations whose associated definable cardinals are pairwise incomparable, in addition to myriad further results concerning the complexity of the definable cardinal hierarchy. The arguments behind these theorems marked a sharp departure from earlier approaches, relying upon sophisticated superrigidity machinery for actions of linear algebraic groups.

Soon thereafter, similar techniques were used in [Ada02, Tho02] to obtain many striking new properties of definable cardinals, such as the existence of infinite definable cardinals κ lying strictly below 2κ . While many of the underlying arguments were later simplified in [HK05], even these refinements depended upon complex rigidity phenomena. And while the still simpler arguments of [Hjo12] gave rise to treeable countable Borel equivalence relations whose associated definable cardinals

are incomparable, it still gave little sense of how far one must travel beyond the base of the definable cardinal hierarchy before encountering such extraordinary behavior.

The second of the two main goals of this paper is to show that under measure reducibility, such definable cardinals appear just beyond \mathbb{R}/\mathbb{Q} .

We obtain our results by introducing a measureless notion of rigidity, which we establish directly for the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{T}^2 . In the presence of a measure, this yields strong separability properties of the induced orbit equivalence relation. Many of our results follow rather easily from the latter, while others require an additional graph-theoretic stratification theorem, also established via elementary methods.

BASIC NOTIONS

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is *Borel* if it is in the σ -algebra generated by the underlying topology.

A *standard Borel space* is a set X equipped with the family of Borel sets associated with a Polish topology on X . Every subset of a standard Borel space inherits the σ -algebra consisting of its intersection with each Borel subset of the original space; this restriction is again standard Borel exactly when the subset in question is Borel. The *product* of standard Borel spaces X and Y is the set $X \times Y$, equipped with the σ -algebra generated by the family of all sets of the form $A \times B$, where $A \subseteq X$ and $B \subseteq Y$ are Borel.

A function between standard Borel spaces is *Borel* if pre-images of Borel sets are Borel. We say that a sequence $(x_i)_{i \in I}$ of points of X is *Borel* if $\{(i, x_i) \mid i \in I\}$ is Borel, and more generally, a sequence $(B_i)_{i \in I}$ of subsets of X is *Borel* if $\{(i, x) \in I \times X \mid x \in B_i\}$ is Borel.

Suppose that E and F are equivalence relations on X and Y . A *homomorphism* from E to F is a function $\phi: X \rightarrow Y$ sending E -equivalent points to F -equivalent points, a *reduction* of E to F is a homomorphism sending E -inequivalent points to F -inequivalent points, and an *embedding* of E into F is an injective reduction.

Every reduction of E to F induces an injection of X/E into Y/F . The existence of a Borel reduction of E to F yields the most prevalent notion of definable cardinality among such quotients.

A *Borel measure* on X is a function $\mu: \mathcal{B} \rightarrow [0, \infty]$, where \mathcal{B} is the family of Borel subsets of X , with the property that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$ whenever $(B_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of X . We say that μ is σ -finite if there are Borel sets $B_n \subseteq X$ such that $X = \bigcup_{n \in \mathbb{N}} B_n$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$, μ is

finite if $\mu(X) < \infty$, and μ is a *Borel probability measure* if $\mu(X) = 1$. We use $P(X)$ to denote the set of all Borel probability measures on X , equipped with the smallest standard Borel structure making the functions $\Lambda_B: P(X) \rightarrow \mathbb{R}$ given by $\Lambda_B(\mu) = \mu(B)$ Borel, where $B \subseteq X$ varies over all Borel sets.

A Borel set $B \subseteq X$ is μ -*null* if $\mu(B) = 0$, μ -*positive* if $\mu(B) > 0$, and μ -*conull* if $\sim B$ is μ -null. We say that μ is *E-ergodic* if every E -invariant Borel set is μ -null or μ -conull. We say that μ is *E-quasi-invariant* if $\mu(B) = 0 \iff \mu(T(B)) = 0$ for every Borel set $B \subseteq X$ and every Borel automorphism $T: X \rightarrow X$ whose graph is contained in E , and μ is *E-invariant* if $\mu(B) = \mu(T(B))$ for every Borel set $B \subseteq X$ and every Borel automorphism $T: X \rightarrow X$ whose graph is contained in E .

We say that E is μ -*nowhere reducible* to F if there is no μ -positive Borel set on which E is Borel reducible to F , E is μ -*somewhere reducible* to F if there is a μ -positive Borel set on which E is Borel reducible to F , E is μ -*reducible* to F if there is a μ -conull Borel set on which E is Borel reducible to F , and E is *measure reducible* to F if E is μ -reducible to F for every Borel probability measure μ on X . Intuitively, the latter means that it is impossible to rule out Borel reducibility of E to F using measure-theoretic techniques. Moreover, all notions of definable reducibility lie between measure reducibility and Borel reducibility. A useful intermediate notion is that of *invariant-measure reducibility*, in which one merely asks that $E \upharpoonright B$ is μ -reducible to F for every Borel set $B \subseteq X$ and every $(E \upharpoonright B)$ -invariant Borel probability measure μ on B . The corresponding notions of embeddability are defined analogously.

Following the standard abuse of language, we say that an equivalence relation is *countable* if all of its classes are countable. A Borel equivalence relation is *smooth* if it is Borel reducible to equality on a standard Borel space, and *treeable* if its classes coincide with the connected components of an acyclic Borel graph.

We use \mathbb{E}_0 to denote the countable Borel equivalence relation on $2^{\mathbb{N}}$ given by $x \mathbb{E}_0 y \iff \exists n \in \mathbb{N} \forall m \geq n \ x(m) = y(m)$. This relation is Borel bi-embeddable with the orbit equivalence relation induced by the action of \mathbb{Q} on \mathbb{R} by addition, thus the corresponding quotients have the same definable cardinalities.

BASES

A *quasi-order* on a set Q is a binary relation \leq on Q such that $\forall q \in Q \ q \leq q$ and $\forall p, q, r \in Q \ (p \leq q \leq r \implies p \leq r)$. A *basis* for a set $P \subseteq Q$ under \leq is a set $B \subseteq Q$ such that $\forall p \in P \exists b \in B \ b \leq p$.

In this paper, we seek to elucidate the extent to which measure theory can shed light on the structure of the Borel reducibility hierarchy just beyond \mathbb{E}_0 . But given our limited knowledge of the structure of the hierarchy, the appropriate meaning of *just beyond* is not entirely clear.

Let \mathcal{E} denote the class of countable Borel equivalence relations, on standard Borel spaces, which are not measure reducible to \mathbb{E}_0 . We will focus primarily on those properties which hold of some relation in every basis $\mathcal{B} \subseteq \mathcal{E}$ for \mathcal{E} under measure reducibility.

Of course, one should first strive to understand the structure of such bases, the original motivation for this paper.

Theorem A. *Suppose that $\mathcal{B} \subseteq \mathcal{E}$ is a basis for \mathcal{E} under measure reducibility. Then $|\mathcal{B}| \geq 2^{\aleph_0}$.*

SEPARABILITY

Although we will later give a somewhat different definition, for the sake of the introduction we will say that a countable Borel equivalence relation F on a standard Borel space Y is *projectively separable* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is a Borel probability measure on X for which E is μ -nowhere reducible to \mathbb{E}_0 , there is a Borel set $R \subseteq X \times Y$, whose vertical sections are countable, such that $\mu(\{x \in B \mid \neg x R \phi(x)\}) = 0$ for every Borel set $B \subseteq X$ and every countable-to-one Borel homomorphism $\phi: B \rightarrow Y$ from $E \upharpoonright B$ to F . Intuitively, this means that modulo μ -null sets, each E -class maps to only countably many F -classes under countable-to-one Borel partial homomorphisms from E to F . The primary tool of this paper is the existence of treeable countable Borel equivalence relations which are not measure reducible to \mathbb{E}_0 , yet are nevertheless projectively separable.

Theorem B. *The orbit equivalence relation induced by the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on \mathbb{T}^2 is projectively separable.*

Suppose that E is a projectively separable treeable countable Borel equivalence relation which is not measure reducible to \mathbb{E}_0 . In order to establish Theorem A, we establish the stronger result that if $\mathcal{B} \subseteq \mathcal{E}$ is merely a basis for the set of Borel subequivalence relations of E in \mathcal{E} , then $|\mathcal{B}| \geq 2^{\aleph_0}$.

Ultimately, one would like to have the even stronger result in which \mathcal{B} is a basis for the family of relations in \mathcal{E} which are measure reducible to E . We show that E is a counterexample to this strengthening if and only if it is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility. While the existence of such successors

remains open, we show that if there are any at all, then there is a Borel sequence of 2^{\aleph_0} -many pairwise non-measure-reducible such relations.

One final comment concerning projective separability is that it is closed downward under Borel reducibility. As a consequence, if $\mathcal{B} \subseteq \mathcal{E}$ is a basis for \mathcal{E} under measure reducibility, then it necessarily contains a relation whose restriction to some Borel set is not measure reducible to \mathbb{E}_0 , but is both projectively separable and treeable. In particular, if we wish to prove that some relation in every such basis satisfies a given property, then it is sufficient to show that the property holds of every projectively separable treeable countable Borel equivalence relation which is not measure reducible to \mathbb{E}_0 .

ANTICHAINS

In light of the above observations, the following allows us to push the existence of large antichains to the base of the reducibility hierarchy, simultaneously giving a simple new proof of the main result of [Hjo12].

Theorem C. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is a Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E .*

As with our anti-basis theorem, one would like to have the strengthening of this result in which the relations E_r are measure reducible to E . We show that E is a counterexample to this strengthening if and only if there is a finite family \mathcal{F} of successors of \mathbb{E}_0 under measure reducibility for which E is a countable disjoint union of Borel equivalence relations which are measure bi-reducible with those in \mathcal{F} .

In particular, it follows that the existence of a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible countable Borel equivalence relations which are measure reducible to E is equivalent to the existence of a Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible countable equivalence relations which are measure reducible to E . Moreover, the inexistence of such sequences implies the stronger fact that every sequence $(E_n)_{n \in \mathbb{N}}$ of countable Borel equivalence relations which are measure reducible to E contains an infinite subsequence which is increasing under measure reducibility.

COMPLEXITY

In [AK00], the existence of perfect families of pairwise incomparable countable Borel equivalence relations with distinguished ergodic Borel probability measures is used to establish a host of complexity results. We obtain simple new proofs of these results, while simultaneously pushing them to the base of the reducibility hierarchy, by establishing the following strengthening of Theorem C.

Theorem D. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There are Borel sequences $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E and $(\mu_r)_{r \in \mathbb{R}}$ of Borel probability measures on X such that:*
 - (a) *Each μ_r is E_r -quasi-invariant and E_r -ergodic.*
 - (b) *For all distinct $r, s \in \mathbb{R}$, the relation E_r is μ_r -nowhere reducible to the relation E_s .*

While this result is somewhat technical, the complexity results of [AK00] are all obtained as abstract consequences of condition (2).

Again, one would like to have the strengthening in which the relations E_r are measure reducible to E . We show that E is a counterexample to this strengthening if and only if it is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.

PRODUCTS

An elementary fact concerning the usual notion of cardinality is that $\kappa\lambda = \max(\kappa, \lambda)$ whenever κ and λ are infinite cardinals. However, the analogous statement for definable cardinality fails just above \mathbb{R}/\mathbb{Q} . Let $\Delta(X)$ denote the diagonal on $X \times X$, and identify $E \times F$ with the equivalence relation on the product of the underlying spaces given by $(x_1, y_1) (E \times F) (x_2, y_2) \iff (x_1 E x_2 \text{ and } y_1 F y_2)$.

Theorem E. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then the following are equivalent:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation $E \times \Delta(\mathbb{R})$ is measure reducible to E .*

It follows that no projectively separable treeable countable Borel equivalence relation measure reduces every treeable countable Borel equivalence relation, yielding a simple new proof of [Hjo08, Theorem 1.6], asserting the existence of treeable countable Borel equivalence

relations which are neither Borel reducible to \mathbb{E}_0 nor Borel reduce every treeable countable Borel equivalence relation. In fact, it yields the generalization in which the class of treeable countable Borel equivalence relations is replaced with any class closed under products with $\Delta(\mathbb{R})$, and containing a projectively separable countable Borel equivalence relation which is not measure reducible to \mathbb{E}_0 .

In [Tho02, Theorem 3.3a], the rigidity results behind [AK00] are used to establish the existence of countable Borel equivalence relations E with the property that for no $n \in \mathbb{N}$ is $E \times \Delta(n+1)$ measure reducible to $E \times \Delta(n)$. While there are projectively separable countable Borel equivalence relations which are neither measure reducible to \mathbb{E}_0 nor have this property, in light of our observations on bases, the following yields a simple new proof of this result, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem F. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is a Borel set $B \subseteq X$ with the property that for no $n \in \mathbb{N}$ is $(E \upharpoonright B) \times \Delta(n+1)$ measure reducible to $(E \upharpoonright B) \times \Delta(n)$.*

CONTAINMENT VERSUS REDUCIBILITY

In [Ada02], the rigidity results behind [AK00] are used to establish the existence of countable Borel equivalence relations $E_1 \subseteq E_2$ such that E_1 is not measure reducible to E_2 . This was strengthened by the main result of [Hjo12], which actually provided an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible treeable countable equivalence relations. In light of our observations on bases, the following provides a simple new proof of this fact, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem G. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E .*

EMBEDDABILITY VERSUS REDUCIBILITY

A Borel equivalence relation is *aperiodic* if all of its classes are infinite. In [Tho02, Theorem 3.3b], the rigidity results behind [AK00] are used to establish the existence of aperiodic countable Borel equivalence relations E and F for which E is Borel reducible to F , but E is not measure embeddable into F . In fact, such examples were produced with $E = F \times I(2)$, where $I(X) = X \times X$.

If E is invariant-measure reducible to \mathbb{E}_0 and F is aperiodic, then E is measure reducible to F if and only if E is measure embeddable into F . In particular, if E is aperiodic but invariant-measure reducible to \mathbb{E}_0 , then $E \times I(\mathbb{N})$ is measure embeddable into E . On the other hand, we show the following, which in light of our observations on bases, yields simple new proofs of the aforementioned result, while simultaneously pushing it to the base of the reducibility hierarchy.

Theorem H. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *Every Borel subequivalence relation of E is invariant-measure reducible to \mathbb{E}_0 .*
- (2) *There is an aperiodic Borel subequivalence relation F of E with the property that for no $n \in \mathbb{N}$ is $F \times I(n+1)$ measure embeddable into $F \times I(n)$.*

REFINEMENTS

We have taken great care to state our results in forms which make both the theorems and the underlying arguments as clear as possible. Nevertheless, by utilizing several additional ideas, one can obtain many generalizations and strengthenings.

In particular, by establishing analogs of our results for orbit equivalence relations induced by free Borel actions of countable discrete non-abelian free groups, one can rule out strong dynamical forms of the von Neumann conjecture, while simultaneously providing an elementary proof of the existence of continuum-many pairwise incomparable such relations, as found, for example, in [GP05]. Moreover, as the notion of comparison we consider is far weaker than those typically appearing in ergodic theory, our results are correspondingly stronger.

One can also obtain similar results for substantial weakenings of measure reducibility, as well as for broader classes of equivalence relations. We plan to explore such developments in a future paper.

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Part I. Tools

In this first part of the paper, we introduce the new ideas underlying our arguments. In §1, we show that $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ satisfies a measureless strengthening of amenability. In §2, we use this to prove that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ satisfies a measureless local rigidity property. In §3, we establish a strong separability property for orbit equivalence relations induced by such actions. In §4, we show that the latter yields countability of an appropriate auxiliary equivalence relation on the underlying space of ergodic, quasi-invariant Borel probability measures

witnessing the failure of hyperfiniteness, and we derive several consequences of this countability. In §5, we provide a general stratification theorem for treeable countable Borel equivalence relations.

1. PRODUCTIVE HYPERFINITENESS

Suppose that Γ is a countable discrete group. The *diagonal product* of actions $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright Y$ is the action $\Gamma \curvearrowright X \times Y$ given by $\gamma \cdot (x, y) = (\gamma \cdot x, \gamma \cdot y)$. We say that a Borel action $\Gamma \curvearrowright X$ on a standard Borel space is *productively hyperfinite* if whenever $\Gamma \curvearrowright Y$ is a Borel action on a standard Borel space, the orbit equivalence relation induced by the diagonal product action $\Gamma \curvearrowright X \times Y$ is hyperfinite.

Proposition 1.1. *Suppose that Γ is a countable discrete group, X is a standard Borel space, and $\Gamma \curvearrowright X$ is a Borel action such that:*

- (1) *The induced orbit equivalence relation is hyperfinite.*
- (2) *The stabilizer of every point is hyperfinite.*
- (3) *Only countably many points have infinite stabilizers.*

Then $\Gamma \curvearrowright X$ is productively hyperfinite.

Proof. Let C denote the Borel set of points whose stabilizers are infinite, and fix an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations whose union is E_Γ^X .

Suppose now that Y is a standard Borel space and $\Gamma \curvearrowright Y$ is a Borel action. For each $n \in \mathbb{N}$, let F_n denote the equivalence relation on $(X \setminus C) \times Y$ for which two $E_\Gamma^{(X \setminus C) \times Y}$ -equivalent pairs (x, y) and (x', y') are related exactly when $x E_n x'$. As each F_n is finite and their union is $E_\Gamma^{(X \setminus C) \times Y}$, the latter equivalence relation is hyperfinite.

It only remains to show that $E_\Gamma^{C \times Y}$ is hyperfinite. As C is countable and Proposition F.1 ensures that the family of Borel sets on which a countable Borel equivalence relation is hyperfinite forms a σ -ideal, we need only show that $E_\Gamma^{X \times Y}$ is hyperfinite on $\{x\} \times Y$, for all $x \in C$. But this follows from the fact that each is the orbit equivalence relation induced by a Borel action of the stabilizer of x . \square

To apply this to $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, we must first consider its stabilizers.

Proposition 1.2. *Suppose that $\theta \in \mathbb{T}$. Then the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is either trivial or infinite cyclic.*

Proof. We consider first the case that $\theta \cap \mathbb{Z}^2 \neq \emptyset$. Let v denote the unique element of $\theta \cap \mathbb{Z}^2$ of minimal length. Note that the stabilizers of θ and v are one and the same, for if A is in the stabilizer of θ , then v is an eigenvector of A , so minimality ensures that $Av = v$. Minimality also ensures that the coordinates of v are relatively prime, so there

exists $a \in \mathbb{Z}^2$ such that $a \cdot v = 1$, in which case $B = \begin{pmatrix} a_1 & a_2 \\ -v_2 & v_1 \end{pmatrix}$ is a matrix in $\mathrm{SL}_2(\mathbb{Z})$ for which $Bv = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, thus conjugation by B yields an isomorphism of the stabilizer of v with that of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and the latter is the infinite cyclic group $\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z}\}$.

It remains to consider the case that $\theta \cap \mathbb{Z}^2 = \emptyset$. Fix $v \in \theta$. An elementary calculation reveals that the stabilizer of v is trivial. Let Λ denote the set of eigenvalues of matrices in the stabilizer of θ , noting that Λ forms a group under multiplication.

Lemma 1.3. *The group Λ is cyclic.*

Proof. It is sufficient to show that 1 is isolated in $\Lambda \cap [1, \infty)$. Towards this end, suppose that A is in the stabilizer of θ and v is an eigenvector of A with eigenvalue $\lambda > 1$. If μ is the other eigenvalue of A , then $\lambda\mu = \det(A) = 1$, so $\mathrm{tr}(A) = \lambda + \mu = \lambda + 1/\lambda$. As $\mathrm{tr}(A) \in \mathbb{Z}$, another elementary calculation reveals that $\lambda \geq (3 + \sqrt{5})/2$. \square

By Lemma 1.3, there is a matrix A in the stabilizer of θ which has an eigenvalue λ generating Λ . Note that if B is any matrix in the stabilizer of θ , then there exists $n \in \mathbb{Z}$ for which v is an eigenvector of B with eigenvalue λ^n , in which case $A^n B^{-1}$ is in the stabilizer of v , so $B = A^n$, thus A generates the stabilizer of θ , hence the latter is cyclic.

Observe finally that if A is a non-identity matrix fixing θ , then any two distinct powers of A are themselves distinct, since the eigenvalues corresponding to v are distinct. In particular, it follows that if the stabilizer of θ is non-trivial, then it is infinite. \square

As a consequence, we can now obtain the main result of this section.

Proposition 1.4. *The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is productively hyperfinite.*

Proof. As Proposition K.1 ensures that the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is hyperfinite, Proposition 1.2 ensures that the non-trivial stabilizers of $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are infinite cyclic, and Proposition F.5 ensures that infinite cyclic groups are hyperfinite, it is sufficient to show that only countably many $\theta \in \mathbb{T}$ have non-trivial stabilizers, by Proposition 1.1. As every such θ is the equivalence class of an eigenvector of some non-trivial matrix in $\mathrm{SL}_2(\mathbb{Z})$, and every such matrix admits at most two such classes of eigenvectors, this follows from the countability of $\mathrm{SL}_2(\mathbb{Z})$. \square

2. PROJECTIVE RIGIDITY

Given $R \subseteq X \times X$, $\Delta \curvearrowright Y$, and $\rho: R \rightarrow \Delta$, we say that a function $\phi: X \rightarrow Y$ is ρ -invariant if $x_1 R x_2 \implies \phi(x_1) = \rho(x_1, x_2) \cdot \phi(x_2)$

for all $x_1, x_2 \in X$. The *difference set* associated with two functions $\phi: A \subseteq X \rightarrow Y$ and $\psi: B \subseteq X \rightarrow Y$ is given by

$$D(\phi, \psi) = \{x \in A \cap B \mid \phi(x) \neq \psi(x)\} \cup (A \Delta B).$$

We say that $\Delta \curvearrowright Y$ is *projectively rigid* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X , and $\rho: E \rightarrow \Delta$ is a Borel function, there is essentially at most one countable-to-one ρ -invariant Borel function, in the sense that for any two such functions ϕ and ψ , the relation $E \upharpoonright D(\phi, \psi)$ is hyperfinite.

Theorem 2.1. *The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is projectively rigid.*

Proof. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $\rho: E \rightarrow \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, $\phi: X \rightarrow \mathbb{R}^2$ is a countable-to-one ρ -invariant Borel function, and $\psi: X \rightarrow \mathbb{R}^2$ is a ρ -invariant Borel function.

Define $\pi: D(\phi, \psi) \rightarrow \mathbb{T}$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\phi(x) - \psi(x))$, and define $\sigma: E \upharpoonright D(\phi, \psi) \rightarrow \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x_1, x_2) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}(\rho(x_1, x_2))$.

Lemma 2.2. *The function π is σ -invariant.*

Proof. Simply observe that if $x_1 (E \upharpoonright D(\phi, \psi)) x_2$, then

$$\begin{aligned} \pi(x_1) &= \mathrm{proj}_{\mathbb{T}}(\phi(x_1) - \psi(x_1)) \\ &= \mathrm{proj}_{\mathbb{T}}(\rho(x_1, x_2) \cdot \phi(x_2) - \rho(x_1, x_2) \cdot \psi(x_2)) \\ &= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot \phi(x_2) - \sigma(x_1, x_2) \cdot \psi(x_2)) \\ &= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot (\phi(x_2) - \psi(x_2))) \\ &= \sigma(x_1, x_2) \cdot \mathrm{proj}_{\mathbb{T}}(\phi(x_2) - \psi(x_2)) \\ &= \sigma(x_1, x_2) \cdot \pi(x_2), \end{aligned}$$

thus π is σ -invariant. \(\boxtimes\)

As $(\mathrm{proj}_{\mathbb{T}^2} \circ \phi) \upharpoonright D(\phi, \psi)$ is also σ -invariant, it follows that the product $\pi \times (\mathrm{proj}_{\mathbb{T}^2} \circ \phi)$ is a countable-to-one homomorphism from $E \upharpoonright D(\phi, \psi)$ to the orbit equivalence relation induced by the diagonal product action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T} \times \mathbb{T}^2$. As Proposition 1.4 ensures that $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is productively hyperfinite, it follows that the latter relation is hyperfinite. As Proposition F.2 ensures that the family of hyperfinite Borel equivalence relations is closed downward under countable-to-one Borel homomorphism, it follows that the former relation is also hyperfinite. \(\boxtimes\)

3. PROJECTIVE SEPARABILITY

Let $L(X, \mu, Y)$ denote the set of Borel functions $\phi: B \rightarrow Y$, where B varies over Borel subsets of X . We equip $L(X, \mu, Y)$ with the pseudometric d_μ given by $d_\mu(\phi, \psi) = \mu(D(\phi, \psi))$.

Proposition 3.1. *Suppose that X and Y are standard Borel spaces, μ is a finite Borel measure on X , and $\mathcal{L} \subseteq L(X, \mu, Y)$. Then the following are equivalent:*

- (1) *The space \mathcal{L} is separable.*
- (2) *There is a Borel set $R \subseteq X \times Y$, whose vertical sections are countable, with the property that*

$$\forall \phi \in \mathcal{L} \quad \mu(\{x \in \text{dom}(\phi) \mid \neg x R \phi(x)\}) = 0.$$

Proof. To see (1) \implies (2), note that if \mathcal{D} is a countable dense subset of \mathcal{L} , then the set $R = \bigcup_{\phi \in \mathcal{D}} \text{graph}(\phi)$ is as desired, since Proposition A.2 ensures that graphs of Borel functions are Borel. To see (2) \implies (1), it is sufficient to show that if condition (2) holds, then there is a countable subset of $L(X, \mu, Y)$ whose closure contains \mathcal{L} . As the vertical sections of R are countable, Theorem C.3 yields a countable family \mathcal{F} of Borel partial functions, the union of whose graphs is R . By Proposition H.2, there is a countable algebra \mathcal{B} of Borel subsets of X , containing the domain of every $\phi \in \mathcal{F}$, such that for all Borel sets $A \subseteq X$ and all $\epsilon > 0$, there exists $B \in \mathcal{B}$ with $\mu(A \triangle B) \leq \epsilon$. We then obtain the desired countable dense family by considering those $\psi: B \rightarrow Y$, where B ranges over \mathcal{B} , for which there is a finite partition $\mathcal{A} \subseteq \mathcal{B}$ of B with the property that $\forall A \in \mathcal{A} \exists \phi \in \mathcal{F} \quad \phi \upharpoonright A = \psi \upharpoonright A$. \square

We say that a function $\phi: Y \rightarrow Y'$ is a *homomorphism* from a set $\mathcal{L} \subseteq L(X, \mu, Y)$ to a set $\mathcal{L}' \subseteq L(X, \mu, Y')$ if $\forall \psi \in \mathcal{L} \quad \phi \circ \psi \in \mathcal{L}'$.

Proposition 3.2. *Suppose that X , Y , and Y' are standard Borel spaces, μ is a Borel probability measure on X , $\mathcal{L} \subseteq L(X, \mu, Y)$ and $\mathcal{L}' \subseteq L(X, \mu, Y')$, there is a countable-to-one Borel homomorphism $\phi: Y \rightarrow Y'$ from \mathcal{L} to \mathcal{L}' , and \mathcal{L}' is separable. Then \mathcal{L} is separable.*

Proof. Fix a Borel set $R' \subseteq X \times Y'$ satisfying the analog of condition (2) of Proposition 3.1 for \mathcal{L}' , and observe that the set $R = (\text{id} \times \phi)^{-1}(R')$ satisfies condition (2) of Proposition 3.1 for \mathcal{L} . \square

Let $\text{Hom}(E, \mu, F)$ denote the subspace of $L(X, \mu, Y)$ consisting of all countable-to-one partial homomorphisms $\phi \in L(X, \mu, Y)$ from E to F .

Proposition 3.3. *Suppose that X , Y , and Y' are standard Borel spaces, E , F , and F' are countable Borel equivalence relations on X , Y ,*

and Y' , μ is a Borel probability measure on X , there is a countable-to-one Borel homomorphism $\phi: Y \rightarrow Y'$ from F to F' , and $\text{Hom}(E, \mu, F')$ is separable. Then $\text{Hom}(E, \mu, F)$ is separable.

Proof. As the function ϕ is also a homomorphism from $\text{Hom}(E, \mu, F)$ to $\text{Hom}(E, \mu, F')$, the desired conclusion follows from Proposition 3.2. \square

We say that F is *projectively separable* if whenever X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is a Borel probability measure on X with respect to which E is μ -nowhere hyperfinite, the space $\text{Hom}(E, \mu, F)$ is separable.

Proposition 3.4. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , there is a countable-to-one Borel homomorphism from E to F , and F is projectively separable. Then E is projectively separable.*

Proof. This is a direct consequence of Proposition 3.3. \square

We next establish the connection between projective rigidity and projective separability.

Theorem 3.5. *Suppose that Δ is a countable discrete group, Y is a standard Borel space, and $\Delta \curvearrowright Y$ is a projectively rigid Borel action. Then the induced orbit equivalence relation E_Δ^Y is projectively separable.*

Proof. Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is a Borel probability measure on X with respect to which E is μ -nowhere hyperfinite. Let μ_c denote the counting measure on X , and fix a sequence of positive real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. By Proposition H.2, there is a countable family \mathcal{B} of Borel subsets of X such that for all Borel sets $A \subseteq X$ and all real numbers $\epsilon > 0$, there exists $B \in \mathcal{B}$ with $\mu(A \triangle B) \leq \epsilon$.

As the vertical sections of E are countable, Theorem C.3 yields an increasing sequence $(R_n)_{n \in \mathbb{N}}$ of Borel subsets of $X \times X$ such that $E = \bigcup_{n \in \mathbb{N}} R_n$ and every vertical section of every R_n has cardinality at most n . As each of the measures $\nu_n = (\mu \times \mu_c) \upharpoonright R_n$ is finite and Δ is countable, Proposition 3.1 yields countable dense sets $\mathcal{D}_n \subseteq L(R_n, \nu_n, \Delta)$. For each set $B \in \mathcal{B}$, rational number $\epsilon > 0$, natural number $n \in \mathbb{N}$, and function $\rho \in \mathcal{D}_n$ for which it is possible, fix a Borel set $A_{B, \epsilon, n, \rho} \subseteq X$ with $\mu(A_{B, \epsilon, n, \rho} \triangle B) \leq \epsilon$, a Borel function $\sigma_{B, \epsilon, n, \rho}: E \upharpoonright A_{B, \epsilon, n, \rho} \rightarrow \Delta$ with $d_{\nu_n}(\rho, \sigma_{B, \epsilon, n, \rho} \upharpoonright R_n) \leq \epsilon$, and a $\sigma_{B, \epsilon, n, \rho}$ -invariant Borel function $\phi_{B, \epsilon, n, \rho}: A_{B, \epsilon, n, \rho} \rightarrow Y$. It only remains to check that the set of functions of the form $\phi_{B, \epsilon, n, \rho}$ is dense in $\text{Hom}(E, \mu, F)$.

Towards this end, suppose that $\phi \in \text{Hom}(E, \mu, F)$ and $\epsilon > 0$ is rational, and fix a Borel function $\sigma: E \upharpoonright \text{dom}(\phi) \rightarrow \Delta$ for which ϕ

is σ -invariant. For each $n \in \mathbb{N}$, fix $B_n \in \mathcal{B}$ and $\rho_n \in \mathcal{D}_n$ such that $\mu(B_n \Delta \text{dom}(\phi)) \leq \epsilon_n$ and $d_{\nu_n}(\rho_n, \sigma \upharpoonright R_n) \leq \epsilon_n$. Then the sets $A_n = A_{B_n, \epsilon_n, n, \rho_n}$ are well-defined, as are the maps $\sigma_n = \sigma_{B_n, \epsilon_n, n, \rho_n}$ and $\phi_n = \phi_{B_n, \epsilon_n, n, \rho_n}$. Let S_n denote the binary relation on $\text{dom}(\phi)$ given by $S_n = (R_n \setminus D(\sigma, \sigma_n)) \upharpoonright \text{dom}(\phi)$.

Lemma 3.6. *There is a $(\mu \upharpoonright \text{dom}(\phi))$ -conull Borel set $C \subseteq \text{dom}(\phi)$ such that $E \cap (C \times \text{dom}(\phi)) \subseteq \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} S_m$.*

Proof. Set $C_n = \{x \in \text{dom}(\phi) \mid \exists y \in \text{dom}(\phi) \ x \ R_n \ y\}$. As Theorem H.11 ensures that

$$\mu(C_n) \leq d_{\nu_n}(\sigma \upharpoonright R_n, \sigma_n) \leq d_{\nu_n}(\sigma \upharpoonright R_n, \rho_n) + d_{\nu_n}(\rho_n, \sigma_n) \leq 2\epsilon_n,$$

it follows that the set $B = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} C_m$ is μ -null, and therefore the set $C = \text{dom}(\phi) \setminus B$ is $(\mu \upharpoonright \text{dom}(\phi))$ -conull. Moreover, if (x, y) is in $E \cap (C \times \text{dom}(\phi))$, then there exists $n \in \mathbb{N}$ for which $x \ R_n \ y$, in which case the fact that $x \in C$ ensures that if $m \geq n$ is sufficiently large, then $x \ S_m \ y$. \square

By Lemma 3.6, there is a $(\mu \upharpoonright \text{dom}(\phi))$ -conull Borel set $C \subseteq \text{dom}(\phi)$ for which $E \upharpoonright C = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} S_m \upharpoonright C$. For each $n \in \mathbb{N}$, let E_n denote the equivalence relation on C generated by $S_n \upharpoonright C$.

Lemma 3.7. *Suppose that $n \in \mathbb{N}$. Then there is a Borel function $\sigma'_n: E_n \rightarrow \Delta$ such that every $(\sigma \upharpoonright (S_n \upharpoonright C))$ -invariant function is σ'_n -invariant.*

Proof. Note that if $x \ E_n \ y$, then there are only countably many $\ell \in \mathbb{N}$ and $(z_0, \dots, z_\ell) \in C^{\ell+1}$ such that $x = z_0$, $\forall i < \ell \ z_i \ S_n \ z_{i+1}$, and $y = z_n$, so Theorem C.3 yields Borel functions $\ell: E_n \rightarrow \mathbb{N}$ and $f: E_n \rightarrow C^{<\mathbb{N}}$ with the property that

$$x = f_0(x, y) \ S_n \ f_1(x, y) \ S_n \ \cdots \ S_n \ f_{\ell(x, y)}(x, y) = y$$

whenever $x \ E_n \ y$. Set $\sigma'_n(x, y) = \prod_{i < \ell} \sigma(f_i(x, y), f_{i+1}(x, y))$. \square

As $\phi \upharpoonright C$ and $\phi_n \upharpoonright C$ are both $\sigma \upharpoonright (S_n \upharpoonright C)$ -invariant, it follows that they are both σ'_n -invariant.

As Theorem J.2 ensures that the class of μ -hyperfinite Borel equivalence relations is closed under increasing unions, it follows that for $n \in \mathbb{N}$ sufficiently large, there is a Borel set $D \subseteq C$, on which $\bigcap_{m \geq n} E_m$ is μ -nowhere hyperfinite, such that $\mu(C \setminus D) \leq \epsilon/3$. By increasing n , we can assume that $\epsilon_n \leq \epsilon/3$.

As the projective rigidity of $\Delta \curvearrowright Y$ ensures that $\phi \upharpoonright D = \phi_n \upharpoonright D$ off of a μ -null set, it follows that

$$\begin{aligned} d_\mu(\phi, \phi_n) &\leq \mu(\text{dom}(\phi) \triangle \text{dom}(\phi_n)) + \mu(C \setminus D) \\ &\leq \mu(\text{dom}(\phi) \triangle B_n) + \mu(B_n \triangle \text{dom}(\phi_n)) + \mu(C \setminus D), \end{aligned}$$

and the latter sum is bounded above by ϵ . \square

In particular, we can now establish the existence of non-trivial projectively separable countable Borel equivalence relations.

Theorem 3.8. *The orbit equivalence relation induced by $\text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ is not measure hyperfinite, but is projectively separable and treeable.*

Proof. Propositions K.2 and K.3 ensure that the orbit equivalence relation induced by $\text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ is not measure hyperfinite, but is treeable. To see that it is projectively separable, note that it is Borel reducible to the orbit equivalence relation induced by $\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$. As Theorem 2.1 ensures that the latter action is projectively rigid, its induced orbit equivalence relation is projectively separable by Theorem 3.5. As Proposition 3.4 ensures that the class of projectively separable countable Borel equivalence relations is closed downward under Borel reducibility, the desired result follows. \square

4. THE SPACE OF MEASURES

Theorem I.7 ensures that the set \mathcal{EQ}_E of E -ergodic E -quasi-invariant Borel probability measures on X is Borel, and Theorem J.8 implies that the set \mathcal{H}_E of Borel probability measures μ on X with respect to which E is μ -hyperfinite is also Borel. Proposition A.1 therefore ensures that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ inherits a standard Borel structure from $P(X)$.

Proposition 4.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and the set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a single measure-equivalence class. Then E is a successor of \mathbb{E}_0 under measure reducibility.*

Proof. Suppose that Y is a standard Borel space and F is a countable Borel equivalence relation on Y which is measure reducible to E , but not to \mathbb{E}_0 . We must show that E is measure reducible to F .

By Theorem J.11, there is a Borel probability measure $\nu \in \mathcal{EQ}_F \setminus \mathcal{H}_F$. By Proposition I.1, there is a ν -null Borel set $N \subseteq Y$ on which F is non-smooth. As F is countable, Theorem C.3 ensures that $[N]_F$ is Borel, so by replacing N with $[N]_F$, we can assume that N is F -invariant. Fix a ν -conull Borel set $C \subseteq \sim N$ for which there is a Borel reduction $\phi: C \rightarrow X$ of $F \upharpoonright C$ to E . As E and F are countable, Theorem C.3

ensures that the set $B = [\phi(C)]_E$ is Borel, and that there is a Borel function $\psi: B \rightarrow C$ such that $\text{graph}(\phi \circ \psi) \subseteq E$. In particular, it follows that ψ is a Borel reduction of $E \upharpoonright B$ to $F \upharpoonright C$.

Suppose now that μ is a Borel probability measure on X . As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, it follows that the push-forward ν' of $\nu \upharpoonright C$ through ϕ is not in \mathcal{H}_E . By Proposition I.5, there is an E -quasi-invariant Borel probability measure ν'' on X such that $\nu' \ll \nu''$ and the two measures have the same E -invariant null Borel sets. Then $\nu'' \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, so $E \upharpoonright \sim B$ is measure hyperfinite, thus there is a Borel set $A \subseteq \sim B$ such that $E \upharpoonright A$ is hyperfinite and $\mu(A \cup B) = 1$. As Theorem F.8 ensures that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth Borel equivalence relation, it follows that there is a Borel reduction $\psi': A \rightarrow N$ of $E \upharpoonright A$ to $F \upharpoonright N$. As $\psi \cup \psi'$ is a reduction of $E \upharpoonright (A \cup B)$ to F , it follows that E is μ -reducible to F , thus E is measure reducible to F . \square

As a corollary, we obtain the following similar result.

Proposition 4.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes. Then E is a countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.*

Proof. Suppose that N is a non-empty countable set and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is the disjoint union of the measure-equivalence classes of Borel probability measures μ_n on X , for $n \in N$. Fix a partition $(B_n)_{n \in N}$ of X into E -invariant Borel sets with the property that $\mu_n(B_n) = 1$ for all $n \in N$, and observe that Proposition 4.1 ensures that each $E \upharpoonright B_n$ is a successor of \mathbb{E}_0 under measure reducibility. \square

On the other hand, we have the following.

Proposition 4.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes. Then there are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E -invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. As Proposition H.10 ensures that measure equivalence is Borel, Theorem D.1 yields a Borel sequence $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of pairwise orthogonal Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$. Theorem H.12 then implies that by thinning down $(\mu_c)_{c \in 2^{\mathbb{N}}}$, we can ensure the existence of

a Borel sequence $(A_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint Borel subsets of X such that $\mu_c(A_c) = 1$ for all $c \in 2^{\mathbb{N}}$. Define $B_c = \{x \in X \mid [x]_E \subseteq A_c\}$. \square

Combining the previous two results yields the following.

Proposition 4.4. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then at least one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation E is a countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.*
- (3) *There are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E -invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. This follows from Propositions 4.2 and 4.3. \square

Let $\ll_{E,F}$ denote the set of all $(\mu, \nu) \in (\mathcal{EQ}_E \setminus \mathcal{H}_E) \times (\mathcal{EQ}_F \setminus \mathcal{H}_F)$ for which there is a μ -conull Borel set $C \subseteq X$ and a Borel reduction $\phi: C \rightarrow Y$ of $E \upharpoonright C$ to F such that $\phi_*(\mu \upharpoonright C) \ll \nu$. When $E = F$, we simply write \ll_E . In spite of our adherence to the usual measure-theoretic abuse of notation, it is not difficult to show that \ll_E is actually an equivalence relation.

The following fact provides a partial converse to Proposition 4.1.

Proposition 4.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and some vertical section of \ll_E is a countable union of measure-equivalence classes. Then the following are equivalent:*

- (1) *The set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a single measure-equivalence class.*
- (2) *The relation E is a successor of \mathbb{E}_0 under measure reducibility.*

Proof. By Proposition 4.1, it is sufficient to show that if $\mathcal{EQ}_E \setminus \mathcal{H}_E$ contains multiple measure-equivalence classes, then E is not a successor of \mathbb{E}_0 under measure reducibility. Towards this end, fix a Borel probability measure μ on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ for which the corresponding vertical section of \ll_E is a countable union of measure-equivalence classes, as well as a Borel probability measure ν on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ for which $\mu \not\sim \nu$. Fix an E -invariant ν -conull Borel set $D \subseteq X$ which is null with respect to every measure in the μ^{th} vertical section of \ll_E which is not measure equivalent to ν .

Lemma 4.6. *Suppose that $C \subseteq X \setminus D$ is a μ -conull Borel set. Then there is no Borel reduction $\phi: C \cup D \rightarrow D$ of $E \upharpoonright (C \cup D)$ to $E \upharpoonright D$.*

Proof. Suppose that ϕ is such a reduction. Then our choice of D ensures that $\phi_*(\mu \upharpoonright C) \ll \nu$, so $\mu \ll_E \nu$. As \ll_E is transitive, our choice of D also ensures that $\phi_*(\nu \upharpoonright D) \ll \nu$. But then there exist $x \in C$ and $y \in D$ for which $\phi(x) E \phi(y)$, contradicting the fact that ϕ is a reduction. \square

In particular, it follows that E is not measure reducible to $E \upharpoonright D$, and therefore cannot be a successor of \mathbb{E}_0 under measure reducibility. \square

The following provides a partial converse to Proposition 4.2.

Proposition 4.7. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and every vertical section of \ll_E is a countable union of measure-equivalence classes. Then the following are equivalent:*

- (1) *The set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes.*
- (2) *The relation E is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.*

Proof. By Proposition 4.2, it is sufficient to show that if N is a non-empty countable set and $(B_n)_{n \in N}$ is a partition of X into E -invariant Borel sets on which E is a successor of \mathbb{E}_0 under measure reducibility, then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a countable union of measure-equivalence classes. Towards this end, note that for all $n \in N$, every vertical section of $\ll_{E \upharpoonright B_n}$ is a countable union of measure-equivalence classes, so Proposition 4.5 ensures that $\mathcal{EQ}_{E \upharpoonright B_n} \setminus \mathcal{H}_{E \upharpoonright B_n}$ is the measure-equivalence class of some Borel probability measure μ_n on B_n . Identifying μ_n with the corresponding Borel probability measure on X , it follows that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is the union of the measure-equivalence classes of μ_n , for $n \in N$. \square

Summarizing these results, we obtain the following.

Theorem 4.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and every vertical section of \ll_E is a countable union of measure-equivalence classes. Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation E is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.*
- (3) *There are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E -invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. If $\mathcal{EQ}_E \subseteq \mathcal{H}_E$, then Theorem J.11 ensures that condition (1) holds, in which case conditions (2) and conditions (3) trivially fail.

If $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is a non-empty countable union of measure-equivalence classes, then condition (1) trivially fails, Proposition 4.7 ensures that condition (2) holds, and its proof implies that condition (3) fails. If $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes, then condition (1) trivially fails, Proposition 4.7 ensures that condition (2) fails, and Proposition 4.3 implies that condition (3) holds. \square

In light of our earlier results, the following yields a criterion for ensuring that Borel subequivalence relations of successors of \mathbb{E}_0 under measure reducibility are again successors of \mathbb{E}_0 under measure reducibility.

Proposition 4.9. *Suppose that X is a standard Borel space, $E \subseteq F$ are countable Borel equivalence relations on X , μ is an E -ergodic F -quasi-invariant Borel probability measure on X , and $\mathcal{EQ}_F \setminus \mathcal{H}_F$ is contained in the measure-equivalence class of μ . Then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is also contained in the measure-equivalence class of μ .*

Proof. Suppose that $\nu \in \mathcal{EQ}_E$ but $\mu \not\sim \nu$. Then there is an E -invariant μ -null ν -conull Borel set $C \subseteq X$, in which case Proposition I.5 yields an F -quasi-invariant Borel probability measure ν' on X such that $\nu \ll \nu'$ and the two measures have the same F -invariant null Borel sets. As the F -quasi-invariance of μ ensures that $[C]_F$ is μ -null, it follows that $\nu' \in \mathcal{H}_F$. As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that $\nu' \in \mathcal{H}_E$, thus $\nu \in \mathcal{H}_E$. \square

We also have the following criterion for ensuring strong ergodicity.

Proposition 4.10. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , F is hyperfinite, and $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$ is not (E, F) -ergodic. Then $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes.*

Proof. Fix a μ -null-to-one Borel homomorphism $\phi: X \rightarrow Y$ from E to F . By appealing to Theorem I.8, we obtain a Borel disintegration $(\mu_y)_{y \in Y}$ of μ through ϕ .

Then the set $C = \{y \in Y \mid E \text{ is not } \mu_y\text{-hyperfinite}\}$ is Borel by Theorem J.8. As E is μ -nowhere hyperfinite, Proposition J.13 ensures that C is $(\phi_*\mu)$ -conull.

In particular, as ϕ is μ -null-to-one, it follows that C is uncountable, in which case Theorem D.1 yields an uncountable partial transversal $P \subseteq C$ of F . Theorem J.11 then yields Borel probability measures ν_y on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $[\phi^{-1}(y)]_E$ is ν_y -conull, for all $y \in P$. As the latter sets are pairwise disjoint, it follows that $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is not a countable union of measure-equivalence classes. \square

We next compute a bound on the complexity of the relation $\ll_{E,F}$.

Proposition 4.11. *Suppose that X and Y are standard Borel spaces and E and F are countable Borel equivalence relations on X and Y . Then $\ll_{E,F}$ is analytic.*

Proof. Note that $\mu \ll_{E,F} \nu$ if and only if there is a code c for a measurable function from X to Y such that $(\phi_c)_*(\mu \upharpoonright \text{dom}(\phi_c)) \ll \nu$ and ϕ_c is a reduction of E to F on a μ -conull set. Propositions H.7 and H.15 ensure that the former relation is Borel, and Proposition I.15 implies that the latter relation is analytic. \square

We close this section by noting that our hypothesis on \ll_E holds of all projectively separable countable Borel equivalence relations.

Proposition 4.12. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , and F is projectively separable. Then the vertical sections of $\ll_{E,F}$ are countable unions of measure-equivalence classes.*

Proof. Suppose that $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and let A denote the vertical section of $\ll_{E,F}$ corresponding to μ . As Proposition 4.11 ensures that $\ll_{E,F}$ is analytic, so too is A . As Proposition H.10 ensures that measure equivalence is Borel, Theorem D.1 implies that if A is not a union of countably many measure-equivalence classes, then there is a Borel sequence $(\nu_c)_{c \in 2^{\mathbb{N}}}$ of pairwise orthogonal Borel probability measures on Y in A . Theorem H.12 then ensures that by passing to an appropriate subsequence, we can ensure that there is a Borel sequence $(D_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint subsets of Y such that $\nu_c(D_c) = 1$ for all $c \in 2^{\mathbb{N}}$. But for each $c \in 2^{\mathbb{N}}$, there is a μ -conull Borel set $C_c \subseteq X$ for which there is a Borel reduction $\phi_c: C_c \rightarrow D_c$ from $E \upharpoonright C_c$ to $F \upharpoonright D_c$, contradicting the projective separability of F . \square

5. STRATIFICATION

Proposition F.3 ensures that every aperiodic countable Borel equivalence relation has an aperiodic hyperfinite Borel subequivalence relation. This is the special case of the following fact, in which G is the difference of E and equality, and ρ is the constant cocycle.

Proposition 5.1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , G is a Borel graphing of E , and $\rho: E \rightarrow \mathbb{R}^+$ is an aperiodic Borel cocycle. Then there is a Borel subgraph H of G generating a hyperfinite Borel equivalence relation on which ρ is aperiodic.*

Proof. As Proposition A.2 ensures that graphs of Borel functions are themselves Borel, the following observation implies that it is sufficient to establish the proposition on an E -complete Borel set.

Lemma 5.2. *Suppose that $B \subseteq X$ is an E -complete Borel set and H is a Borel subgraph of $G \upharpoonright B$ generating a hyperfinite Borel equivalence relation on which ρ is aperiodic. Then there is a Borel function $f: \sim B \rightarrow X$ such that $\text{graph}(f^{\pm 1}) \cup H$ is a subgraph of G generating a hyperfinite Borel equivalence relation on which ρ is aperiodic.*

Proof. As the vertical sections of G are countable, Theorem C.3 yields Borel sets $B_n \subseteq X$ and Borel functions $f_n: B_n \rightarrow X$ with the property that $G = \bigcup_{n \in \mathbb{N}} \text{graph}(f_n)$. Let $d_G(x, B)$ denote the length of the shortest G -path from x to an element of B . Observe that this function is Borel, as it can also be expressed, for $x \notin B$, as the least $n \in \mathbb{N}$ for which there exist $k_1, \dots, k_n \in \mathbb{N}$ such that $f_{k_1} \circ \dots \circ f_{k_n}(x) \in B$. Noting that for each $x \in X$, the set of $y \in G_x$ with the property that $d_G(y, B) = d_G(x, B) - 1$ is countable, one more application of Theorem C.3 yields a Borel function $f: \sim B \rightarrow X$, whose graph is contained in G , such that $d_G(f(x), B) = d_G(x, B) - 1$ for all $x \in \sim B$. As every connected component of $\text{graph}(f^{\pm 1}) \cup H$ contains a connected component of H , it follows that ρ is aperiodic on the equivalence relation generated by $\text{graph}(f^{\pm 1}) \cup H$. As the function sending x to $f^{d_G(x, B)}(x)$ is a Borel reduction of the latter equivalence relation to that generated by H , and Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed under Borel reducibility, it follows that the equivalence relation generated by $\text{graph}(f^{\pm 1}) \cup H$ is hyperfinite. \square

We will now recursively construct an increasing sequence $(H_n)_{n \in \mathbb{N}}$ of approximations to the desired graph, beginning with $H_0 = \emptyset$. Given H_n , let E_n denote the equivalence relation induced by H_n , and let B_n denote the set of all $x \in X$ for which ρ is finite on $E_n \upharpoonright [x]_{E_n}$. As H_n and E are countable, Theorem C.3 ensures that these sets are Borel. As Proposition I.2 implies that countable Borel equivalence relations admitting finite Borel cocycles to \mathbb{R} are smooth, it follows that $E_n \upharpoonright B_n$ is smooth. Remark E.2 therefore yields a Borel transversal $A_n \subseteq B_n$ of $E_n \upharpoonright B_n$. Let R_n denote the relation consisting of all $(x, (y, (x', y'))) \in A_n \times (A_n \times (B_n \times B_n))$ with $x E_n x' (G \setminus E_n) y' E_n y$ and $\rho([x]_{E_n}, [y]_{E_n}) \leq 1$. As the vertical sections of R_n are countable, Theorem C.3 ensures that the set $A'_n = \text{proj}_{A_n}(R_n)$ is Borel, there is a Borel uniformization $f'_n: A'_n \rightarrow A_n \times (B_n \times B_n)$ of R_n , and both of the sets $S_n = f'_n(A'_n)$ and $H'_n = \text{proj}_{B_n \times B_n}(S_n)^{\pm 1}$ are Borel.

Set $H_{n+1} = H_n \cup H'_n$. To see that the equivalence relation E_{n+1} generated by H_{n+1} is hyperfinite, we consider the function $f_n: A_n \rightarrow A_n$ given by $f_n = (\text{proj}_{A_n} \circ f'_n) \cup (\text{id} \upharpoonright (A_n \setminus A'_n))$. As Theorem F.7 ensures that $E_t(f_n)$ is hypersmooth, Theorem F.6 implies that it is hyperfinite. As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, and the unique function $\phi_n: B_n \rightarrow A_n$ such that $\forall x \in B_n \ x E_n \phi_n(x)$ is a Borel reduction of $E_{n+1} \upharpoonright B_n$ to $E_t(f_n)$, it follows that E_{n+1} is hyperfinite. This completes the recursive construction.

As every equivalence class of $E \upharpoonright \sim B_n$ contains a ρ -infinite equivalence class of E_n , it follows from Lemma 5.2 that we can construct the desired graph off of the set $B_\infty = \bigcap_{n \in \mathbb{N}} B_n$. In order to construct the desired graph on B_∞ , set $H_\infty = \bigcup_{n \in \mathbb{N}} H_n$ and let E_∞ denote the equivalence relation generated by H_∞ . As $E_\infty = \bigcup_{n \in \mathbb{N}} E_n$, it follows that $E_\infty \upharpoonright B_\infty$ is hypersmooth, so Theorem F.6 ensures that it is hyperfinite. By one more application of Lemma 5.2, it is therefore sufficient to observe that there do not exist $(G \setminus E_\infty)$ -related points $x, y \in B_\infty$ for which the corresponding equivalence classes $[x]_{E_\infty}, [y]_{E_\infty}$ are ρ -finite.

Suppose, towards a contradiction, that there are such points. Then there exists $n \in \mathbb{N}$ such that $\rho([x]_{E_\infty}, [x]_{E_n}), \rho([y]_{E_\infty}, [y]_{E_n}) < 2$. As $\rho([x]_{E_n}, [y]_{E_n}) \leq 1$ or $\rho([y]_{E_n}, [x]_{E_n}) \leq 1$, it follows that $\phi_n(x) \in A'_n$ or $\phi_n(y) \in A'_n$, so $\rho([x]_{E_{n+1}}, [x]_{E_n}) \geq 2$ or $\rho([y]_{E_{n+1}}, [y]_{E_n}) \geq 2$, thus $\rho([x]_{E_\infty}, [x]_{E_{n+1}}) < 1$ or $\rho([y]_{E_\infty}, [y]_{E_{n+1}}) < 1$, which is impossible. \square

In particular, we obtain the following measure-theoretic corollary.

Proposition 5.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , μ is an E -quasi-invariant Borel probability measure on X for which E is μ -nowhere smooth, and G is a Borel graphing of E . Then there is a Borel subgraph H of G whose induced equivalence relation is μ -nowhere smooth but hyperfinite.*

Proof. By Proposition I.3, there is a Borel cocycle $\rho: E \rightarrow \mathbb{R}^+$ with respect to which μ is invariant. As Proposition I.2 ensures that countable Borel equivalence relations admitting finite Borel cocycles to \mathbb{R} are smooth, by throwing away an E -invariant μ -null Borel set on which E is smooth, we can assume that ρ is aperiodic. Proposition 5.1 then yields a Borel subgraph H of G generating a hyperfinite equivalence relation on which ρ is aperiodic. As Proposition I.4 ensures that every such relation is μ -nowhere smooth, the result follows. \square

The following observation allows us to find disjoint Borel sets which, in the measure-theoretic setting, are complete with respect to different equivalence relations.

Proposition 5.4. *Suppose that X is a standard Borel space, E and F are aperiodic countable Borel equivalence relations on X , and μ and ν are Borel probability measures on X . Then there are disjoint Borel sets $A, B \subseteq X$ such that $\mu([A]_E) = \nu([B]_F) = 1$.*

Proof. By two applications of Proposition E.5, there are decreasing sequences $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ of Borel subsets of X such that each A_n is E -complete, each B_n is F -complete, and $\bigcap_{n \in \mathbb{N}} A_n = \bigcap_{n \in \mathbb{N}} B_n = \emptyset$. Fix real numbers $\epsilon_n > 0$ such that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and recursively construct strictly increasing sequences $(i_n)_{n \in \mathbb{N}}$ and $(j_n)_{n \in \mathbb{N}}$ of natural numbers by setting $i_0 = 0$, and given $n \in \mathbb{N}$ and $i_n \in \mathbb{N}$, choosing $j_n > \max_{m < n} j_m$ sufficiently large that $\mu([A_{i_n} \setminus B_{j_n}]_E) \geq 1 - \epsilon_n$, as well as $i_{n+1} > i_n$ sufficiently large that $\nu([B_{j_n} \setminus A_{i_{n+1}}]_F) \geq 1 - \epsilon_n$. Define $A = \bigcup_{n \in \mathbb{N}} (A_{i_n} \setminus B_{j_n})$ and $B = \bigcup_{n \in \mathbb{N}} (B_{j_n} \setminus A_{i_{n+1}})$. \square

A *directed graph* on X is an irreflexive subset G of $X \times X$. The *domain* of such a relation is the set of x for which G_x is non-empty. An *oriented graph* on X is an irreflexive antisymmetric subset H of $X \times X$. An *orientation* of G is an oriented graph H such that $G = H^{\pm 1}$.

Proposition 5.5. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X , G is a locally countable Borel graph on X , and μ is an E -quasi-invariant Borel probability measure on X for which E is μ -nowhere smooth and the domain of G has μ -conull E -saturation. Then there is a Borel orientation H of G whose domain has μ -conull E -saturation.*

Proof. For each Borel set $B \subseteq X$, put $X_B = \{x \in B \mid [x]_{E \upharpoonright B} \text{ is finite}\}$. As E is countable, Theorem C.3 ensures that such sets are Borel, as are E -saturations of Borel sets.

Lemma 5.6. *Suppose that $B \subseteq X$ is Borel. Then $[X_B]_E$ is μ -null.*

Proof. As E is countable, Theorem C.3 ensures that there is a Borel reduction of $E \upharpoonright [X_B]_E$ to $E \upharpoonright X_B$. As Proposition E.1 ensures that $E \upharpoonright X_B$ is smooth, so too is $E \upharpoonright [X_B]_E$. As E is μ -nowhere smooth, it follows that $[X_B]_E$ is μ -null. \square

We consider now the special case that G is of the form $\text{graph}(I)$, where $A \subseteq X$ is a Borel set and $I: A \rightarrow A$ is a Borel involution. Proposition E.1 and Remark E.2 yield a Borel transversal $B \subseteq A$ of the equivalence relation generated by G . Lemma 5.6 ensures that the set $C = [A]_E \setminus [X_B \cup X_{A \setminus B}]_E$ is μ -conull.

We use E_B , $E_{A \setminus B}$, μ_B , and $\mu_{A \setminus B}$ to denote the restrictions of E , $(I \times I)^{-1}(E)$, μ , and $I_*\mu$ to $B \cap C$. As E_B and $E_{A \setminus B}$ are aperiodic,

Proposition 5.4 yields a Borel set $B' \subseteq B$, an E_B -invariant μ_B -null Borel set $N_B \subseteq C$, and an $E_{A \setminus B}$ -invariant $\mu_{A \setminus B}$ -null Borel set $N_{A \setminus B} \subseteq C$ such that $B' \cup N_B$ is E_B -complete and $(B \setminus B') \cup N_{A \setminus B}$ is $E_{A \setminus B}$ -complete. As μ is E -quasi-invariant, the set $D = C \setminus [N_B \cup N_{A \setminus B}]_E$ is μ -conull. Let H denote the graph of the restriction of I to $B' \cup I(B \setminus B')$.

The fact that B is a transversal of the equivalence relation generated by I ensures that H is an oriented graph. To see that H is an orientation of G , note that if $x G y$, then $x \in B$ or $y \in B$, from which it follows that $(x \in B'$ or $y \in I(B \setminus B'))$ or $(y \in B'$ or $x \in I(B \setminus B'))$, so $(x H y$ or $y H x)$ or $(y H x$ or $x H y)$, thus $x H y$ or $y H x$. To see that the E -saturation of the domain of H is μ -conull, it is enough to show that the domain of H intersects the E -class of every $x \in D$. Towards this end, note that $A \cap [x]_E$ is non-empty, thus so too is either $B \cap [x]_E$ or $(A \setminus B) \cap [x]_E$, in which case either $B' \cap [x]_E$ or $I(B \setminus B') \cap [x]_E$ is non-empty as well, thus the domain of H intersects $[x]_E$.

We now consider the general case. As G is locally countable, Theorem C.3 yields Borel sets $A_n \subseteq X$ and Borel involutions $I_n: A_n \rightarrow A_n$, with pairwise disjoint graphs, such that $G = \bigcup_{n \in \mathbb{N}} \text{graph}(I_n)$. Setting $G_n = \text{graph}(I_n)$, $X_n = [A_n]_E$, and $\mu_n = \mu \upharpoonright X_n$, the above special case yields Borel orientations H_n of G_n whose domains have μ_n -conull E -saturation. Then $H = \bigcup_{n \in \mathbb{N}} H_n$ is a Borel orientation of G whose domain has μ -conull E -saturation. \square

A μ -stratification of E is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E whose union is E and which is strictly increasing on every μ -positive Borel set.

Theorem 5.7. *Suppose that X is a standard Borel space, E is a treeable countable Borel equivalence relation on X , and μ is an E -quasi-invariant Borel probability measure on X for which E is μ -nowhere hyperfinite. Then there is a Borel μ -stratification of E .*

Proof. Fix a Borel treeing G of E . By Proposition 5.3, we can assume that there is Borel subgraph H of G whose induced equivalence relation F is μ -nowhere smooth but hyperfinite. As E is μ -nowhere hyperfinite, the F -saturation of the domain of $G \setminus H$ is μ -conull. As F is μ -nowhere smooth, Proposition 5.5 ensures that, by throwing out a μ -null Borel set, we can assume that there is a Borel orientation K of $G \setminus H$ whose domain intersects every F -class. As μ is E -quasi-invariant, we can ensure that the set we throw out is E -invariant. As F is μ -nowhere smooth and Proposition I.2 ensures that F is smooth on the finite part of $\rho \upharpoonright (F \upharpoonright \text{dom}(K))$, by throwing out another μ -null Borel set, we can assume that $F \upharpoonright \text{dom}(K)$ is $(\rho \upharpoonright (F \upharpoonright \text{dom}(K)))$ -aperiodic,

and therefore aperiodic. The E -quasi-invariance of μ again allows us to ensure that the set we throw out is E -invariant. Proposition E.6 then yields a partition of the domain of K into a sequence $(B_q)_{q \in \mathbb{Q}}$ of pairwise disjoint F -complete Borel sets. Set $K_r = K \upharpoonright (\bigcup_{q < r} B_q \times X)$ and define $G_r = H \cup K_r^{\pm 1}$. As G_r is locally countable, Theorem C.3 ensures that the equivalence relation E_r induced by G_r is Borel.

Suppose now that $B \subseteq X$ is a Borel set for which there are real numbers $r < s$ with $E_r \upharpoonright B = E_s \upharpoonright B$. Then $B \cap [x]_{E_s} \subseteq [x]_{E_r}$ for all $x \in B$. As $G_r \subseteq G_s$ and the latter graph is acyclic, it follows that for all $y \in [x]_{E_s} \setminus [x]_{E_r}$, there is a unique point of $[y]_{E_r}$ of minimal distance to $[x]_{E_r}$ with respect to the graph metric associated with G_s . Let $\phi: [B]_{E_s} \setminus [B]_{E_r} \rightarrow [B]_{E_s} \setminus [B]_{E_r}$ be the function sending each point of its domain to the corresponding point of its E_r -class. As E is countable, Theorem C.3 ensures that $[B]_{E_r}$, $[B]_{E_s}$, and ϕ are Borel. As ϕ is a selector for the restriction of E_r to $[B]_{E_s} \setminus [B]_{E_r}$, it follows that this restriction is smooth. As F is μ -nowhere smooth and Proposition E.3 ensures that the class of smooth countable Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that E_r is also μ -nowhere smooth. In particular, this means that the set $[B]_{E_s} \setminus [B]_{E_r}$ is μ -null, and since μ is E_s -quasi-invariant, so too is the set $[B]_{E_s}$, thus B is μ -null as well, hence $(E_r)_{r \in \mathbb{R}}$ is indeed a μ -stratification of E . \square

Part II. Applications

In this second part of the paper, we obtain our main results. While our theorems were listed in the introduction in order of importance, here we proceed according to the amount of new machinery behind the arguments, with those requiring the least appearing first. In §6, we use the countability of the vertical sections of \ll_E to establish our results on products. In §7, we combine the countability of the vertical sections of \ll_E with facts about compressibility and costs of equivalence relations to obtain our results on the distinction between embeddability and reducibility. In §8, we combine projective separability, facts about \ll_E , and the existence of stratifications to obtain our results on antichains and the distinction between containment and reducibility. In §9, we use these tools to obtain our anti-basis theorems. And in §10, we combine these tools with [AK00] to obtain our complexity results.

6. PRODUCTS

We begin this section with an observation concerning measurable reducibility of products.

Proposition 6.1. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , m is a continuous Borel probability measure on \mathbb{R} , $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, and the μ^{th} vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes. Then $E \times \Delta(\mathbb{R})$ is $(\mu \times m)$ -nowhere reducible to F .*

Proof. Suppose, towards a contradiction, that there is a $(\mu \times m)$ -positive Borel set $B \subseteq X \times \mathbb{R}$ on which there is a Borel reduction $\phi: B \rightarrow Y$ of $E \times \Delta(\mathbb{R})$ to F . As E is countable, Theorem C.3 ensures that $[B]_{E \times \Delta(\mathbb{R})}$ is Borel, in addition to yielding a Borel reduction of $(E \times \Delta(\mathbb{R})) \upharpoonright [B]_{E \times \Delta(\mathbb{R})}$ to $(E \times \Delta(\mathbb{R})) \upharpoonright B$. By replacing B with its $(E \times \Delta(\mathbb{R}))$ -saturation, we can therefore assume that B is $(E \times \Delta(\mathbb{R}))$ -invariant. Note that the set $R = \{r \in \mathbb{R} \mid \mu(B^r) > 0\}$ is m -positive, by Theorem H.11. As m is continuous, it follows that R is uncountable. For each $r \in R$, Proposition I.5 yields an F -quasi-invariant Borel probability measure ν_r on Y such that $(\phi^r)_*(\mu \upharpoonright B^r) \ll \nu_r$, but the two measures have the same E -invariant null Borel sets. Then the measures ν_r are pairwise orthogonal elements of the vertical section of $\ll_{E,F}$ corresponding to μ , the desired contradiction. \square

This has the following consequence for measure reducibility.

Proposition 6.2. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , E is a countable Borel equivalence relation on X , and F is a non-smooth projectively separable countable Borel equivalence relation on Y . Then the following are equivalent:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation $E \times \Delta(\mathbb{R})$ is measure reducible to F .*

Proof. To see (1) \implies (2), note that, as Theorem F.9 implies that every countable Borel equivalence relation which is Borel reducible to \mathbb{E}_0 is hyperfinite, it follows that E , and therefore $E \times \Delta(\mathbb{R})$, are measure hyperfinite. As Theorem F.8 implies that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth Borel equivalence relation, it follows that $E \times \Delta(\mathbb{R})$ is measure reducible to F .

To see $\neg(1) \implies \neg(2)$, appeal to Theorem J.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$. Proposition 4.12 then implies that the μ^{th} vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes. Fix a continuous Borel probability measure m on \mathbb{R} . As Proposition 6.1 ensures that $E \times \Delta(\mathbb{R})$ is $(\mu \times m)$ -nowhere reducible to F , the former is not measure reducible to the latter. \square

In particular, we obtain the following corollaries.

Proposition 6.3. *Suppose that X is an uncountable standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then the following are equivalent:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation $E \times \Delta(\mathbb{R})$ is measure reducible to E .*

Proof. Note first that if E is smooth, then it is Borel reducible to \mathbb{E}_0 (this follows from Theorem D.1, although in this case one can also give a simple direct argument). Moreover, in this case $E \times \Delta(\mathbb{R})$ is also smooth, thus Borel reducible to equality on \mathbb{R} . As X is uncountable, Theorem D.1 ensures that equality on \mathbb{R} is Borel reducible to E , thus $E \times \Delta(\mathbb{R})$ is Borel reducible to E .

It only remains to observe that when E is non-smooth, the desired result is the special case of Proposition 6.2 when $E = F$. \square

Proposition 6.4. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X , and \mathcal{F} is a class of countable Borel equivalence relations which is closed under products with equality on \mathbb{R} and contains a countable Borel equivalence relation which is not measure reducible to \mathbb{E}_0 . Then there is a relation in \mathcal{F} which is not measure reducible to E .*

Proof. This is a direct consequence of Proposition 6.2. \square

This last observation yields a new proof of [Hjo08, Theorem 1.6].

Theorem 6.5 (Hjorth). *There is a treeable countable Borel equivalence relation which is not measure reducible to \mathbb{E}_0 , and to which some treeable countable Borel equivalence relation is not measure reducible.*

Proof. Proposition 6.2 ensures that every projectively separable treeable countable Borel equivalence relation, which is not measure reducible to \mathbb{E}_0 , has the desired property. \square

In light of Theorem 3.8, the orbit equivalence relation induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ provides a new, very explicit example of a treeable countable Borel equivalence relation satisfying the conclusion of the above theorem.

We next turn our attention to another observation concerning measurable reducibility, but this time for shorter products.

Proposition 6.6. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , m is a strictly positive probability measure on 2 , $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$, $\nu \in \mathcal{EQ}_F \setminus \mathcal{H}_F$, and $\mu \ll_{E,F} \nu$. If the μ^{th} vertical section of $\ll_{E,F}$ is a countable union*

of measure-equivalence classes, then there is an F -invariant ν -conull Borel set $C \subseteq Y$ for which $E \times \Delta(2)$ is not $(\mu \times m)$ -reducible to $F \upharpoonright C$.

Proof. Fix an F -invariant ν -conull Borel set $C \subseteq Y$ which is ν' -null for every measure ν' in the vertical section of $\ll_{E,F}$ corresponding to μ , other than those which are measure equivalent to ν . Suppose, towards a contradiction, that there is a $(\mu \times m)$ -positive Borel set $B \subseteq X \times 2$ on which there is a Borel reduction $\phi: B \rightarrow Y$ of $E \times \Delta(2)$ to $F \upharpoonright C$. As E is countable, Theorem C.3 ensures that $[B]_{E \times \Delta(2)}$ is Borel, in addition to yielding a Borel reduction of $(E \times \Delta(2)) \upharpoonright [B]_{E \times \Delta(2)}$ to $(E \times \Delta(2)) \upharpoonright B$. By replacing B with its $(E \times \Delta(2))$ -saturation, we can therefore assume that B is $(E \times \Delta(2))$ -invariant. Proposition I.5 then yields $(F \upharpoonright C)$ -quasi-invariant Borel probability measures ν_i on C with the property that $(\phi^i)_*(\mu \upharpoonright C_i) \ll \nu_i$ but the two measures have the same E -invariant null Borel sets, for all $i < 2$. As ν_0 and ν_1 are orthogonal elements of the vertical section of $\ll_{E,F \upharpoonright C}$, this contradicts our choice of C . \square

This has the following consequences for measure reducibility.

Proposition 6.7. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , E is a countable Borel equivalence relation on X , and F is a projectively separable countable Borel equivalence relation on Y which is not measure reducible to \mathbb{E}_0 . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is a Borel set $B \subseteq Y$ such that $E \times \Delta(2)$ is not measure reducible to $F \upharpoonright B$, and $F \upharpoonright B$ is not measure reducible to \mathbb{E}_0 .*

Proof. To see (1) \implies $\neg(2)$, note that, as Theorem F.9 implies that a countable Borel equivalence relation is Borel reducible to \mathbb{E}_0 if and only if it is hyperfinite, it follows that E is measure hyperfinite, thus so too is $E \times \Delta(2)$. Suppose now that $B \subseteq Y$ is a Borel set for which $F \upharpoonright B$ is not measure reducible to \mathbb{E}_0 . One more appeal to Theorem F.9 then ensures that $F \upharpoonright B$ is not measure hyperfinite, and therefore non-smooth. As Theorem F.8 implies that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth Borel equivalence relation, it follows that $E \times \Delta(2)$ is Borel reducible to F .

To see $\neg(1) \implies (2)$, appeal to Theorem J.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$. Proposition 4.12 then implies that the μ^{th} vertical section of $\ll_{E,F}$ is a countable union of measure-equivalence classes. Clearly we can assume that E is measure reducible to F , so there is a μ -conull Borel set $C \subseteq X$ on which there is a Borel

reduction $\phi: C \rightarrow Y$ of E to F . By Proposition I.5, there is an F -quasi-invariant Borel probability measure ν on Y such that $\phi_*(\mu \upharpoonright C) \ll \nu$ but the two measures have the same E -invariant null Borel sets. Fix a strictly positive probability measure m on 2 . As Proposition 6.6 yields a ν -conull Borel set $D \subseteq Y$ for which $E \times \Delta(2)$ is not $(\mu \times m)$ -reducible to $F \upharpoonright D$, the former is not measure reducible to the latter. \square

Proposition 6.8. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and F is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is a Borel set $B \subseteq X$ such that $(E \upharpoonright B) \times \Delta(2)$ is not measure reducible to $E \upharpoonright B$, and $E \upharpoonright B$ is not measure reducible to \mathbb{E}_0 .*

Proof. To see (2) $\implies \neg(1)$, note that if there is a Borel set $B \subseteq X$ on which E is not measure reducible to \mathbb{E}_0 , then E itself cannot be measure reducible to \mathbb{E}_0 .

To see $\neg(1) \implies (2)$, appeal to Theorem J.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{H}_E$. Proposition 4.12 then implies that the μ^{th} vertical section of \ll_E is a countable union of measure-equivalence classes. Fix a strictly positive probability measure m on 2 . As Proposition 6.6 yields a μ -conull Borel set $C \subseteq X$ for which $E \times \Delta(2)$ is not $(\mu \times m)$ -reducible to $E \upharpoonright C$, it follows that $(E \upharpoonright C) \times \Delta(2)$ is not measure reducible to $E \upharpoonright C$. \square

7. REDUCIBILITY WITHOUT EMBEDDABILITY

We begin this section with an observation concerning the relationship between measurable reducibility and measurable embeddability.

Proposition 7.1. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X which is invariant-measure reducible to \mathbb{E}_0 , F is an aperiodic countable Borel equivalence relation on Y , and μ is a Borel probability measure on X . Then E is μ -reducible to F if and only if E is μ -embeddable into F .*

Proof. Suppose that E is μ -reducible to F , and fix a μ -conull Borel set $C \subseteq X$ on which there is a Borel reduction $\phi: C \rightarrow Y$ of E to F . As E is countable, Theorem C.3 ensures that $[C]_E$ is Borel, and there is a Borel reduction of $E \upharpoonright [C]_E$ to $E \upharpoonright C$. By replacing ϕ with its composition with such a function, we can therefore assume that C is itself E -invariant. Proposition I.5 ensures that there is an E -quasi-invariant Borel probability measure on X , with respect to which μ is

absolutely continuous, which agrees with μ on all E -invariant Borel sets. By replacing μ with such a measure, we can therefore assume that μ is E -quasi-invariant.

We handle first the case that F is smooth. Then $E \upharpoonright C$ is also smooth. As E is countable, Remark E.2 yields partitions $(C_n)_{n \in \mathbb{N}}$ of C into Borel partial transversals of E , and $(D_n)_{n \in \mathbb{N}}$ of Y into Borel transversals of F . One then obtains an embedding $\pi: C \rightarrow Y$ of $E \upharpoonright C$ into F by setting

$$\pi(x) = y \iff \exists n \in \mathbb{N} (x \in C_n, y \in D_n, \text{ and } \phi(x) F y).$$

As Proposition A.1 ensures that C inherits a standard Borel structure from X , and Proposition A.2 ensures that a function between standard Borel spaces is Borel if and only if its graph is Borel, it follows that π is Borel.

We next turn to the case that F is non-smooth. As Proposition F.11 ensures that every non-smooth Borel equivalence relation can be Borel embedded into its restriction to an invariant Borel set off of which it is non-smooth, by composing ϕ with such an embedding, we can assume that the restriction of F to the set $Z = \sim[\phi(X)]_E$ is non-smooth. As ϕ is countable-to-one, Theorem C.3 yields an $(E \upharpoonright C)$ -complete Borel set $B \subseteq C$ on which ϕ is injective.

Fix a Borel set $A \subseteq B$ of maximal μ -measure on which E is compressible. As E is countable, Theorem C.3 ensures that $[A]_E$ is Borel. As Proposition I.10 ensures that countable Borel equivalence relations can be Borel embedded into their restrictions to complete compressible Borel sets, there is a Borel injection $\psi: [A]_E \rightarrow A$ whose graph is contained in E . Then the function $\pi = \phi \circ \psi$ is a Borel embedding of $E \upharpoonright [A]_E$ into $F \upharpoonright \phi(C)$.

If $\mu([A]_E) = 1$, then it follows that E is μ -embeddable into F . Otherwise, Theorem I.11 ensures that $\mu \upharpoonright (B \setminus [A]_E)$ is equivalent to an $E \upharpoonright (B \setminus [A]_E)$ -invariant Borel probability measure ν on $B \setminus [A]_E$. As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed under Borel reducibility, it follows that E is invariant-measure hyperfinite, in which case there is a ν -conull Borel set $A' \subseteq B \setminus [A]_E$ on which E is hyperfinite. As E is countable, Theorem C.3 ensures that $[A']_E$ is Borel and there is a Borel reduction of $E \upharpoonright [A']_E$ to $E \upharpoonright A$. As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed downward under Borel reducibility, it follows that $E \upharpoonright [A']_E$ is also hyperfinite. As Theorem F.8 ensures that every hyperfinite Borel equivalence relation is Borel embeddable into every non-smooth Borel equivalence relation, there is a Borel embedding $\pi': [A']_E \rightarrow Z$ of $E \upharpoonright [A']_E$ into $F \upharpoonright Z$. As

$\mu([A \cup A']_E) = 1$ and $\pi \cup \pi'$ is an embedding of $E \upharpoonright [A \cup A']_E$ into F , the proposition follows. \square

This has the following consequence for the relationship between measure embeddability and measure reducibility.

Proposition 7.2. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , E is invariant-measure reducible to \mathbb{E}_0 , and F is aperiodic. Then E is measure reducible to F if and only if E is measure embeddable into F .*

Proof. This is a direct consequence of Proposition 7.1. \square

In particular, we obtain the following.

Proposition 7.3. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X which is invariant-measure reducible to \mathbb{E}_0 . Then $E \times I(\mathbb{N})$ is measure embeddable into E .*

Proof. This is a direct consequence of Proposition 7.2. \square

We next record a natural obstacle to measurable embeddability.

Proposition 7.4. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , $\mu \in \mathcal{E}\mathcal{I}_E \setminus \mathcal{H}_E$, $\nu \in \mathcal{E}\mathcal{I}_F \setminus \mathcal{H}_F$, $C_\mu(E) < C_\nu(F)$, and the μ^{th} vertical section of $\ll_{E,F}$ is the measure-equivalence class of ν . Then E is not μ -embeddable into F .*

Proof. Suppose, towards a contradiction, that there is a μ -conull Borel set $C \subseteq X$ on which there is a Borel embedding $\pi: C \rightarrow Y$ of E into F . Then $\pi_*\mu \ll \nu$, since otherwise Proposition I.5 would yield an F -ergodic F -quasi-invariant Borel probability measure ν' on Y with the same F -invariant null Borel sets as $\pi_*\mu$, in which case the E -ergodicity of μ would ensure that ν' is F -ergodic, and the downward closure of the family of hyperfinite Borel equivalence relations under Borel embeddability (see Proposition F.2) would imply that E is ν' -nowhere hyperfinite, despite the fact that ν and ν' are orthogonal. Let ν_B be the Borel probability measure on B given by $\nu_B(D) = \nu(D)/\nu(B)$. As $\pi_*\mu \ll \nu_B$ and both measures are $(F \upharpoonright B)$ -ergodic and $(F \upharpoonright B)$ -invariant, Proposition I.5 implies that $\pi_*\mu = \nu_B$. The formula for the cost of Borel restrictions given by Proposition I.12 then ensures that $C_\nu(F) \leq C_{\nu_D}(F \upharpoonright D) = C_\mu(E)$, a contradiction. \square

As a special case, we obtain the following.

Proposition 7.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $\mu \in \mathcal{E}\mathcal{I}_E \setminus \mathcal{H}_E$, $1 < C_\mu(E) < \infty$, and the μ^{th} vertical section of \ll_E is the measure-equivalence class of μ . Then for no $n \in \mathbb{N}$ is it the case that the relation $(E \upharpoonright C) \times I(n+1)$ is μ -embeddable into $(E \upharpoonright C) \times I(n)$.*

Proof. Let m_n denote the uniform probability measure on n . Then the formula for the cost of Borel restrictions given by Proposition I.12 ensures that $C_{\mu \times m_{n+1}}(E \times I(n+1)) < C_{\mu \times m_n}(E \times I(n))$ for all $n \in \mathbb{N}$, so Proposition 7.4 implies that $E \times I(n+1)$ is not $(\mu \times m_{n+1})$ -embeddable into $E \times I(n)$. \square

Putting these observations together, we obtain the following.

Proposition 7.6. *Suppose that X is a standard Borel space and E is an aperiodic projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *Every Borel subequivalence relation of E is invariant-measure reducible to \mathbb{E}_0 .*
- (2) *There is an aperiodic Borel subequivalence relation F of E such that for no $n \in \mathbb{N}$ is $F \times I(n+1)$ measure embeddable into $F \times I(n)$.*

Proof. Proposition 7.3 immediately yields (1) \implies \neg (2). To see \neg (1) \implies (2), suppose that F is a Borel subequivalence relation of E which is not invariant-measure reducible to \mathbb{E}_0 . As Theorem F.9 ensures that countable Borel equivalence relations Borel reducible to \mathbb{E}_0 are hyperfinite, it follows that F is not invariant-measure hyperfinite. Fix a Borel set $B \subseteq X$ as well as an $(F \upharpoonright B)$ -invariant Borel probability measure μ on B such that $F \upharpoonright B$ is not μ -hyperfinite. Fix a Borel graphing G of $F \upharpoonright B$. As G is locally countable, Theorem C.3 yields an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of Borel subgraphs of G of bounded vertex degree whose union is G . As Theorem J.2 ensures that the increasing union of μ -hyperfinite Borel equivalence relations is μ -hyperfinite, there exists $n \in \mathbb{N}$ sufficiently large for which the equivalence relation F' generated by G_n is not μ -hyperfinite. Note that $C_\nu(F') < \infty$ for every F' -invariant Borel probability measure ν on B . By Proposition J.10, there exists $\nu \in \mathcal{E}\mathcal{I}_{F'} \setminus \mathcal{H}_{F'}$. As Proposition 3.4 ensures that the class of projectively separable countable Borel equivalence relations is closed downward under Borel restrictions and Borel subequivalence relations, it follows that F' is projectively separable. As Proposition 4.12 ensures that the vertical sections of $\ll_{F'}$ are countable unions of measure-equivalence classes, there is a ν -conull Borel set $C \subseteq B$ which is null with respect to every measure in $\ll_{F'}$,

with the exception of those in the measure-equivalence class of ν . By removing a ν -null Borel subset of C , we can assume that the relation $F'' = F' \upharpoonright C$ is aperiodic. As Proposition G.1 ensures that the family of treeable countable Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that F'' is treeable, so $1 < C_\nu(F'') < \infty$ by Proposition I.13, thus Proposition 7.5 implies that for no $n \in \mathbb{N}$ is it the case that $F'' \times I(n+1)$ is μ -embeddable into $F'' \times I(n)$. As Proposition F.3 ensures that every aperiodic countable Borel equivalence relation has an aperiodic hyperfinite Borel subequivalence relation, there is an aperiodic hyperfinite subequivalence relation F''' of $E \upharpoonright \sim C$. Then $F'' \cup F'''$ is as desired. \boxtimes

8. ANTICHAINS

In this section, we produce perfect sequences of pairwise non-measure reducible Borel subequivalence relations of a given projectively separable treeable countable Borel equivalence relation.

We begin by noting that hyperfiniteness rules out such sequences.

Proposition 8.1. *Suppose that X is a standard Borel space, E is a hyperfinite Borel equivalence relation on X , and E_1 and E_2 are Borel subequivalence relations of E . Then E_1 and E_2 are comparable under Borel reducibility.*

Proof. As Proposition F.2 ensures that the family of hyperfinite Borel equivalence relations is closed downward under Borel subequivalence relations, it follows that E_1 and E_2 are themselves hyperfinite. But Theorem F.10 implies that any two hyperfinite Borel equivalence relations are comparable under Borel reducibility. \boxtimes

We next turn our attention to very special sorts of antichains.

Proposition 8.2. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation E is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.*
- (3) *There are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint E -invariant subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ with the property that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$, and for no distinct $c, d \in 2^{\mathbb{N}}$ is it the case that $E \upharpoonright B_c$ is μ_c -reducible to $E \upharpoonright B_d$.*

Proof. By Theorem 4.8, it is sufficient to show that if $(B_c)_{c \in 2^{\mathbb{N}}}$ is a Borel sequence of pairwise disjoint E -invariant sets and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ is a Borel sequence of Borel probability measures on X in $\mathcal{EQ}_E \setminus \mathcal{H}_E$ such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$, then by passing to a perfect subsequence, one can ensure that for no distinct $c, d \in 2^{\mathbb{N}}$ is it the case that $E \upharpoonright B_c$ is μ_c -reducible to $E \upharpoonright B_d$. Towards this end, let R denote the binary relation on $2^{\mathbb{N}}$ in which two sequences $c, d \in 2^{\mathbb{N}}$ are R -related if $E \upharpoonright B_c$ is μ_c -reducible to $E \upharpoonright B_d$. Then Proposition I.15 ensures that R is analytic, so Proposition B.7 implies that R has the Baire property. As the projective separability of E ensures that the vertical sections of R are countable, it follows that the vertical sections of R are meager, so Theorem B.3 ensures that R is itself meager, in which case Theorem B.6 yields the desired perfect subsequence. \square

In particular, this allows us to characterize the circumstances under which there is a perfect sequence of pairwise non-measure reducible countable Borel equivalence relations which are measure reducible to a given projectively separable countable Borel equivalence relation.

Proposition 8.3. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is a finite family \mathcal{F} of successors of \mathbb{E}_0 under measure reducibility for which E is a non-empty countable disjoint union of Borel equivalence relations which are measure bi-reducible with those in \mathcal{F} .*
- (3) *There is a Borel sequence $(E_c)_{c \in 2^{\mathbb{N}}}$ of pairwise non-measure-reducible countable equivalence relations measure reducible to E .*

Proof. In light of Proposition 8.2, we can assume that E is a non-empty countable disjoint union of successors of \mathbb{E}_0 under measure reducibility.

Suppose first that there is a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible such successors, and fix a Borel sequence $(N_c)_{c \in 2^{\mathbb{N}}}$ of subsets of \mathbb{N} such that $N_c \not\subseteq N_d$ for all distinct $c, d \in 2^{\mathbb{N}}$. Then the equivalence relations $E_c = \bigcup_{n \in N_c} E_n$ are pairwise non-measure-reducible.

On the other hand, if there is no such sequence $(E_n)_{n \in \mathbb{N}}$, then there is a finite family \mathcal{F} of successors of \mathbb{E}_0 under measure reducibility for which E is a non-empty countable disjoint union of Borel equivalence relations which are measure bi-reducible with those in \mathcal{F} . Set $n = |\mathcal{F}|$. Then the restriction of measure reducibility to the class of countable Borel equivalence relations which are not measure reducible to \mathbb{E}_0 but are measure reducible to E is itself reducible to the partial ordering of

domination on \mathbb{N}^n , which is given by $s \leq t \iff \forall i < n \ s(i) \leq t(i)$. As \mathbb{N}^n is countable, this completes the proof of the proposition. \square

As corollaries, we obtain the following.

Proposition 8.4. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then the following are equivalent:*

- (1) *There is a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible countable Borel equivalence relations measure reducible to E .*
- (2) *There is a Borel sequence $(E_c)_{c \in 2^{\mathbb{N}}}$ of pairwise non-measure-reducible countable equivalence relations measure reducible to E .*

Proof. As in the proof of Proposition 8.3, it is sufficient to note that for no $n \in \mathbb{N}$ is there an infinite sequence of elements of \mathbb{N}^n which forms an antichain with respect to domination on \mathbb{N}^n . \square

Proposition 8.5. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *There is a sequence $(E_n)_{n \in \mathbb{N}}$ of pairwise non-measure-reducible countable Borel equivalence relations measure reducible to E .*
- (2) *Every sequence $(E_n)_{n \in \mathbb{N}}$ of countable Borel equivalence relations measure reducible to E has an infinite subsequence which is increasing with respect to measure reducibility.*

Proof. As in the proof of Proposition 8.3, it is sufficient to note that for all $n \in \mathbb{N}$, every infinite sequence of elements of \mathbb{N}^n has an infinite subsequence which is increasing with respect to domination on \mathbb{N}^n . \square

We next turn our attention to subequivalence relations. The main additional tool we will need is the following observation concerning the power of μ -stratifications in the presence of projective separability.

Proposition 8.6. *Suppose that X is a standard Borel space, E is a projectively separable countable Borel equivalence relation on X , μ is a Borel probability measure on X , $(B_n)_{n \in \mathbb{N}}$ is a sequence of μ -positive Borel subsets of X , and $(E_{n,r})_{r \in \mathbb{R}}$ is a Borel $(\mu \upharpoonright B_n)$ -stratification of $E \upharpoonright B_n$ such that $\bigcap_{r \in \mathbb{R}} E_{n,r}$ is $(\mu \upharpoonright B_n)$ -nowhere hyperfinite, for all $n \in \mathbb{N}$. Then there is a Borel embedding $\pi: \mathbb{R} \rightarrow \mathbb{R}$ of the usual ordering of \mathbb{R} into itself such that $E_{m,\pi(r)}$ is $(\mu \upharpoonright B_m)$ -nowhere reducible to $E_{n,\pi(s)}$ for all distinct $(m,r), (n,s) \in \mathbb{N} \times \mathbb{R}$.*

Proof. Let $R_{m,n}$ denote the relation on \mathbb{R} in which two real numbers r and s are related if $E_{m,r}$ is $(\mu \upharpoonright B_m)$ -somewhere reducible to $E_{n,s}$.

Lemma 8.7. *Every horizontal section of every $R_{m,n}$ is countable.*

Proof. Suppose, towards a contradiction, that there exist $m, n \in \mathbb{N}$ and $t \in \mathbb{R}$ for which $R_{m,n}^t$ is uncountable. For each $r \in R_{m,n}^t$, fix a μ -positive Borel set $B_{m,r} \subseteq B_m$ on which there is a Borel reduction $\phi_r: B_{m,r} \rightarrow B_n$ of $E_{m,r}$ to $E_{n,t}$. Then there exists $\epsilon > 0$ such that $\mu(B_{m,r}) \geq \epsilon$ for uncountably many $r \in R_{m,n}^t$. As each ϕ_r is a homomorphism from $(\bigcap_{r \in \mathbb{R}} E_{m,r}) \upharpoonright B_{m,r}$ to E , the $(\mu \upharpoonright B_m)$ -nowhere hyperfiniteness of $(\bigcap_{r \in \mathbb{R}} E_{m,r}) \upharpoonright B_m$ coupled with the projective separability of E ensures the existence of distinct $r, s \in R_{m,n}^t$ for which $\mu(B_{m,r}), \mu(B_{m,s}) \geq \epsilon$ and $d_\mu(\phi_r, \phi_s) < \epsilon$. Then $\{x \in B_{m,r} \cap B_{m,s} \mid \phi_r(x) = \phi_s(x)\}$ is a μ -positive Borel set on which $E_{m,r}$ and $E_{m,s}$ coincide, a contradiction. \square

Proposition I.15 ensures that each $R_{m,n}$ is analytic, so Proposition B.7 implies that each $R_{m,n}$ has the Baire property. As the horizontal sections of each $R_{m,n}$ are countable and therefore meager, Theorem B.3 ensures that each $R_{m,n}$ is meager, thus so too is their union R , in which case Theorem B.5 yields a continuous embedding $\phi: 2^\mathbb{N} \rightarrow \mathbb{R}$ of the lexicographical ordering of $2^\mathbb{N}$ into the usual ordering of \mathbb{R} with respect to which pairs of distinct sequences in $2^\mathbb{N}$ are mapped to R -unrelated pairs of real numbers. Fix a Borel embedding $\psi: \mathbb{R} \rightarrow 2^\mathbb{N}$ of the usual ordering of \mathbb{R} into the lexicographical ordering of $2^\mathbb{N}$, and observe that the function $\pi = \phi \circ \psi$ is as desired. \square

In particular, this yields the following measure-theoretic result.

Theorem 8.8. *Suppose that X is a standard Borel space, E is a projectively separable treeable countable Borel equivalence relation on X , and μ is a Borel probability measure on X . Then the following are equivalent:*

- (1) *The relation E is μ -nowhere reducible to \mathbb{E}_0 .*
- (2) *There is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise μ -nowhere reducible subequivalence relations of E .*

Proof. To see $\neg(1) \implies \neg(2)$, suppose that $B \subseteq X$ is a μ -positive Borel set on which E is Borel reducible to \mathbb{E}_0 . As Theorem F.9 ensures that countable Borel equivalence relations Borel reducible to \mathbb{E}_0 are hyperfinite, it follows that E is hyperfinite, so Proposition 8.1 implies that if E_1 and E_2 are Borel subequivalence relations of E , then $E_1 \upharpoonright B$ and $E_2 \upharpoonright B$ are comparable under Borel reducibility.

To see (1) \implies (2), note first that Theorem 5.7 yields a Borel μ -stratification $(F_r)_{r \in \mathbb{R}}$ of E . As Theorem J.2 ensures that the family of μ -hyperfinite countable Borel equivalence relations is closed under increasing unions, there is a partition $(B_n)_{n \in \mathbb{N}}$ of X into μ -positive

Borel sets, as well as a sequence $(r_n)_{n \in \mathbb{N}}$ of real numbers, such that $F_{r_n} \upharpoonright B_n$ is $(\mu \upharpoonright B_n)$ -nowhere hyperfinite for all $n \in \mathbb{N}$. Fix order-preserving Borel injections $\phi_n: \mathbb{R} \rightarrow (r_n, \infty)$, and appeal to Proposition 8.6 to obtain a Borel embedding $\phi: \mathbb{R} \rightarrow \mathbb{R}$ of the usual ordering of \mathbb{R} into itself such that $F_{(\phi_m \circ \phi)(r)} \upharpoonright B_m$ is $(\mu \upharpoonright B_m)$ -nowhere reducible to $F_{(\phi_n \circ \phi)(r)}$ for all distinct $(m, r), (n, s) \in \mathbb{N} \times \mathbb{R}$. Then the relations $E_r = \bigcup_{n \in \mathbb{N}} (F_{(\phi_n \circ \phi)(r)} \upharpoonright B_n)$ are as desired. \square

In the special case that the equivalence relation in question is a successor of \mathbb{E}_0 under measure reducibility, we can ensure that the same holds of the subequivalence relations.

Theorem 8.9. *Suppose that X is a standard Borel space, E is a projectively separable treeable countable Borel equivalence relation on X which is a successor of \mathbb{E}_0 under measure reducibility, and μ is a Borel probability measure on X . Then the following are equivalent:*

- (1) *The relation E is μ -nowhere reducible to \mathbb{E}_0 .*
- (2) *There is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise μ -nowhere reducible subequivalence relations of E consisting of successors of \mathbb{E}_0 under measure reducibility with the property that μ is $(\bigcap_{r \in \mathbb{R}} E_r)$ -ergodic.*

Proof. By Theorem 8.8, it is enough to show (1) \implies (2). As Theorem F.9 ensures that hyperfinite Borel equivalence relations are Borel reducible to \mathbb{E}_0 , it follows that E is μ -nowhere hyperfinite. Proposition I.5 yields an E -quasi-invariant Borel probability measure for which $\mu \ll \nu$ but the two measures agree on all E -invariant Borel sets. By Proposition I.3, there is a Borel cocycle $\rho: E \rightarrow \mathbb{R}^+$ with respect to which ν is invariant. As Theorem I.9 ensures the existence of an ergodic decomposition of ρ , Proposition 4.5 implies that ν is E -ergodic, in which case Proposition 4.10 implies that ν is (E, \mathbb{E}_0) -ergodic.

Theorem 5.7 yields a Borel μ -stratification $(F_r)_{r \in \mathbb{R}}$ of E . Proposition I.14 ensures that not every F_r is μ -hyperfinite, so by passing to a Borel subsequence, we can assume that there is a μ -positive Borel set $B \subseteq X$ on which $\bigcap_{r \in \mathbb{R}} F_r$ is μ -nowhere hyperfinite. Proposition I.14 implies that by passing to a further subsequence, we can also assume that $\mu \upharpoonright B$ is $(\bigcap_{r \in \mathbb{R}} F_r \upharpoonright B)$ -ergodic. Proposition 8.6 therefore yields a Borel embedding $\phi: \mathbb{R} \rightarrow \mathbb{R}$ of the usual ordering of \mathbb{R} into itself such that $F_{\phi(r)} \upharpoonright B$ is $(\mu \upharpoonright B)$ -nowhere reducible to $F_{\phi(s)}$ for all distinct $r, s \in \mathbb{R}$. As the equivalence relation E is countable, Theorem C.3 ensures that the set $[B]_E$ is Borel, and that there is an extension of the identity function on B to a Borel function $\psi: [B]_E \rightarrow B$ whose graph is contained in E . Let E_r denote the equivalence relation given by

$x E_r y \iff \psi(x) F_{\phi(r)} \psi(y)$ on $[B]_E$, and which is trivial off of $[B]_E$. Then Propositions 4.9 and 4.10 ensure that $(E_r)_{r \in \mathbb{R}}$ is as desired. \square

We close this section with the Borel analogs of these results.

Theorem 8.10. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E .*

Proof. To see (1) \implies \neg (2), suppose that $E_1 \subseteq E_2$ are Borel subequivalence relations of E , and μ is a Borel probability measure on X . Clearly we can assume that X is uncountable, since otherwise E_2 is trivially Borel reducible to E_1 . Fix a μ -conull Borel set $C \subseteq X$ on which E is Borel reducible to \mathbb{E}_0 . As Theorem F.9 ensures that countable Borel equivalence relations Borel reducible to \mathbb{E}_0 are hyperfinite, it follows that $E \upharpoonright C$ is hyperfinite. As Proposition F.2 ensures that the class of hyperfinite Borel equivalence relations is closed under Borel subequivalence relations, it follows that $E_1 \upharpoonright C$ and $E_2 \upharpoonright C$ are also hyperfinite. If $E_1 \upharpoonright C$ is smooth, then Theorem D.1 and our assumption that X is uncountable yields a Borel reduction of $E_1 \upharpoonright C$ to E_2 . If $E_1 \upharpoonright C$ is not smooth, then Proposition E.3 ensures that neither is E_2 , and since Theorem F.8 implies that there is a Borel embedding of every hyperfinite Borel equivalence relation into every non-smooth Borel equivalence relation, it follows that there is a Borel embedding of $E_1 \upharpoonright C$ into E_2 , thus E_1 is measure reducible to E_2 .

To see \neg (1) \implies (2), appeal to Theorem J.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{HE}$, and apply Theorem 8.8. \square

Theorem 8.11. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X which is a successor of \mathbb{E}_0 under measure reducibility. Then there is an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E which are themselves successors of \mathbb{E}_0 under measure reducibility.*

Proof. Appeal to Theorem J.11 to obtain a Borel probability measure $\mu \in \mathcal{EQ}_E \setminus \mathcal{HE}$, and apply Theorem 8.9. \square

9. BASES

In this section, we establish the inexistence of small bases $\mathcal{B} \subseteq \mathcal{E}$ for \mathcal{E} under measure reducibility.

Theorem 9.1. *Suppose that X is a standard Borel space and E is a projectively separable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The relation E is a non-empty countable disjoint union of successors of E under measure reducibility.*
- (3) *For every basis $\mathcal{B} \subseteq \mathcal{E}$ for the family of relations in \mathcal{E} measure reducible to E , the set \mathbb{R} is a union of $|\mathcal{B}|$ -many countable sets.*

Proof. By Theorem 4.8, we can assume that conditions (1) and (2) fail, and moreover, that there is a Borel sequence $(B_r)_{r \in \mathbb{R}}$ of pairwise disjoint subsets of X on which the corresponding restrictions of E are non-measure-hyperfinite. As Theorem J.11 ensures the existence of Borel probability measures $\nu_F \in \mathcal{EQ}_F \setminus \mathcal{HF}_F$ for each $F \in \mathcal{B}$, the projective separability of E ensures that for each $F \in \mathcal{B}$, the set R_F of $r \in \mathbb{R}$ for which F is measure reducible to $E \upharpoonright B_r$ is countable, so it only remains to note that $\mathbb{R} = \bigcup_{F \in \mathcal{B}} R_F$. \square

We next consider successors of \mathbb{E}_0 under measure reducibility.

Theorem 9.2. *Suppose that X is a standard Borel space, E is a projectively separable treeable countable Borel equivalence relation on X which is a successor of \mathbb{E}_0 under measure reducibility, and $\mathcal{B} \subseteq \mathcal{E}$ is a basis for the family of Borel subequivalence relations of E in \mathcal{E} under measure reducibility. Then $|\mathcal{B}| \geq 2^{\aleph_0}$.*

Proof. Note first that if E is a successor of \mathbb{E}_0 under measure reducibility, then Theorem 8.11 yields an increasing Borel sequence $(E_r)_{r \in \mathbb{R}}$ of pairwise non-measure-reducible subequivalence relations of E , which are also successors of \mathbb{E}_0 under measure reducibility. Then each element of \mathcal{B} is measure reducible to at most one E_r , thus $|\mathcal{B}| \geq 2^{\aleph_0}$. \square

Putting these results together, we obtain the following.

Theorem 9.3. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *For every basis $\mathcal{B} \subseteq \mathcal{E}$ for the family of Borel subequivalence relations of E in \mathcal{E} , the set \mathbb{R} is a union of $|\mathcal{B}|$ -many countable sets.*

Proof. It is sufficient to show $\neg(1) \implies (2)$. To handle the case that E is not a countable disjoint union of successors of \mathbb{E}_0 under measure reducibility, appeal to Theorem 9.1. To handle the case that E is such a union, appeal to Theorem 9.2. \square

Remark 9.4. Suppose that \mathbb{R} is a union of $|X|$ -many countable sets. Under $\text{ZF} + \text{AC}_{\aleph_0}$, it follows that $|X| > \aleph_0$. Under ZFC, it follows that $|X| \geq 2^{\aleph_0}$. By Theorem B.4, the latter also holds under $\text{ZF} + \text{DC} +$ all subsets of \mathbb{R} have the Baire property + there is an injection of \mathbb{R} into every non-well-orderable set, which is itself known to hold in natural models of AD^+ (see [CK11]).

10. COMPLEXITY

In this section, we establish a technical strengthening of Theorem 8.8 which gives rise to our complexity results.

Theorem 10.1. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *There are Borel sequences $(E_r)_{r \in \mathbb{R}}$ of subequivalence relations of E and $(\mu_r)_{r \in \mathbb{R}}$ of Borel probability measures on X such that:*
 - (a) *Each μ_r is E_r -quasi-invariant and E_r -ergodic.*
 - (b) *The relation E_r is μ_r -nowhere reducible to the relation E_s , for all distinct $r, s \in \mathbb{R}$.*

Proof. By Theorem 8.10, it is sufficient to establish $\neg(1) \implies (2)$. Towards this end, note that if E is not a countable disjoint union of successors of \mathbb{E}_0 under measure reducibility, then Theorem 4.8 yields the desired result. On the other hand, if E is a countable disjoint union of successors of \mathbb{E}_0 under measure reducibility, then there is an E -invariant Borel set $B \subseteq X$ on which E is a successor of \mathbb{E}_0 under measure reducibility. Proposition 4.5 then yields a Borel probability measure μ on B for which $E \upharpoonright B$ is μ -nowhere hyperfinite, in which case one obtains the desired equivalence relations by trivially extending those given by Theorem 8.9 from B to X . \square

As a consequence, we obtain the following.

Theorem 10.2. *Suppose that X is a standard Borel space and E is a projectively separable treeable countable Borel equivalence relation on X which is not measure reducible to \mathbb{E}_0 . Then the following hold:*

- (a) *There is an embedding of containment on Borel subsets of \mathbb{R} into Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E (in the codes).*
- (b) *Borel bi-reducibility and reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E are both Σ_2^1 -complete (in the codes).*

- (c) *Every Borel quasi-order is Borel reducible to Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E .*
- (d) *Borel and $\sigma(\Sigma_1^1)$ -measurable reducibility do not agree on the countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E .*

Proof. This follows from Theorems 10.1 and L.1. \(\square\)

Part III. Appendices

In this final part of the paper, we provide the standard definitions and previously known results utilized throughout. Although we mainly give references to the relevant arguments, we do provide proofs when they are either particularly short or difficult to find in the literature.

APPENDIX A. STANDARD BOREL SPACES

A *Polish space* is a separable topological space admitting a compatible complete metric. A subset of such a space is *Borel* if it is in the σ -algebra generated by the underlying topology. A *standard Borel space* is a set X equipped with the family of Borel sets generated by some Polish topology on X .

Proposition A.1. *Every Borel subspace of a standard Borel space is standard Borel.*

Proof. See, for example, [Kec95, Corollary 13.4]. \(\square\)

A function between standard Borel spaces is *Borel* if pre-images of Borel sets are Borel. We say that a sequence $(x_i)_{i \in I}$ of points of X is *Borel* if $\{(i, x_i) \mid i \in I\}$ is Borel, and more generally, a sequence $(B_i)_{i \in I}$ of subsets of X is *Borel* if $\{(i, x) \in I \times X \mid x \in B_i\}$ is Borel.

Proposition A.2. *Suppose that X and Y are standard Borel spaces and $\phi: X \rightarrow Y$. Then ϕ is Borel if and only if $\text{graph}(\phi)$ is Borel.*

Proof. See, for example, [Kec95, Proposition 14.12]. \(\square\)

The following result shows that, up to Borel isomorphism, standard Borel spaces are determined by their cardinalities.

Theorem A.3. *Suppose that X and Y are standard Borel spaces. Then there is a Borel embedding of X into Y if and only if $|X| \leq |Y|$, and there is a Borel isomorphism of X and Y if and only if $|X| = |Y|$.*

Proof. See, for example, the proof of [Kec95, Theorem 15.6]. \(\square\)

Given a compact Polish space X and a Polish space Y , we use $C(X, Y)$ to denote the set of continuous functions $f: X \rightarrow Y$, with the topology generated by the metric $d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$, where d_Y is a compatible complete metric on Y .

Proposition A.4. *Suppose that X is a compact Polish space and Y is a Polish space. Then $C(X, Y)$ is Polish.*

Proof. See, for example, [Kec95, Theorem 4.19]. □

A subset of a standard Borel space is *analytic* if it is the image of a Borel subset of a standard Borel space under a Borel function, *co-analytic* if its complement is analytic, and $\sigma(\Sigma_1^1)$ if it is in the σ -algebra generated by the analytic sets. A function between standard Borel spaces is $\sigma(\Sigma_1^1)$ -*measurable* if pre-images of Borel sets are $\sigma(\Sigma_1^1)$.

APPENDIX B. BAIRE CATEGORY

A subset of a topological space is *nowhere dense* if its closure has empty interior, *meager* if it is the union of countably many nowhere dense sets, and *comeager* if its complement is meager.

Theorem B.1 (Baire). *Suppose that X is a complete metric space. Then every comeager subset of X is dense.*

Proof. See, for example, [Kec95, Theorem 8.4]. □

A subset of a topological space has the *Baire property* if its symmetric difference with some open set is meager, and a function between topological spaces is *Baire measurable* if pre-images of open sets have the Baire property. A *Baire space* is a topological space in which every comeager set is dense.

Proposition B.2. *Suppose that X is a Baire space and $B \subseteq X$ has the Baire property. Then exactly one of the following holds:*

- (1) *The set B is meager.*
- (2) *There is a non-empty open set $U \subseteq X$ in which B is comeager.*

Proof. As B has the Baire property, its symmetric difference with some open set $U \subseteq X$ is meager. If U is empty, then B is meager. Otherwise B is comeager in U . □

Given a set $R \subseteq X \times Y$ and a point $y \in Y$, the y^{th} *horizontal section* of R is given by $R^y = \{x \in X \mid x R y\}$, and given a point $x \in X$, the x^{th} *vertical section* of R is given by $R_x = \{y \in Y \mid x R y\}$.

Similarly, given a function $\phi: X \times Y \rightarrow Z$ and a point $y \in Y$, we define $\phi^y: X \rightarrow Z$ by $\phi^y(x) = \phi(x, y)$, and given a point $x \in X$, we define $\phi_x: Y \rightarrow Z$ by $\phi_x(y) = \phi(x, y)$.

Theorem B.3 (Kuratowski-Ulam). *Suppose that X and Y are Baire spaces, Y is second countable, and $R \subseteq X \times Y$ has the Baire property.*

- (1) *The set $\{x \in X \mid R_x \text{ has the Baire property}\}$ is comeager.*
- (2) *The set R is comeager if and only if $\{x \in X \mid R_x \text{ is comeager}\}$ is comeager.*

Proof. See, for example, [Kec95, Theorem 8.41]. \(\square\)

A binary relation R on a set X is a *quasi-order* if it is reflexive and transitive. We say that R is *well-founded* if for every non-empty set $Y \subseteq X$, there exists $y \in Y$ such that $\forall z \in Y (z R y \implies y R z)$.

Theorem B.4. *Suppose that X is a Polish space and R is a well-founded quasi-order on X which has the Baire property and is non-meager in every non-empty open square. Then some vertical section of $R \cap R^{-1}$ has the Baire property and is non-meager.*

Proof. By Proposition B.2, there is a non-empty basic open set in which R is comeager, in which case Theorem B.3 ensures that some horizontal section of R has the Baire property and is non-meager. Fix $x \in X$ with the property that R^x has the Baire property and is non-meager, and $y R x \implies x R y$ for all $y \in X$ with the property that R^y has the Baire property and is non-meager. To see that $(R \setminus R^{-1})^x$ is meager, note that otherwise this argument could be repeated to produce a point $y \in X$ such that R^y has the Baire property and is non-meager, $y R x$, and $\neg x R y$, a contradiction. \(\square\)

The *lexicographical ordering* of $2^{\mathbb{N}}$ is the binary relation on $2^{\mathbb{N}}$ given by $x <_{\text{lex}} y \iff \exists n \in \mathbb{N} (x \upharpoonright n = y \upharpoonright n \text{ and } x(n) < y(n))$.

Theorem B.5 (Galvin). *Suppose that X is a perfect Polish space, R is a binary relation on X with the Baire property, and for no non-empty open set $U \subseteq X$ is $(U \times U) \setminus R$ meager. Then there is a continuous function $\phi: 2^{\mathbb{N}} \rightarrow X$ such that $\forall x, y \in 2^{\mathbb{N}} (x <_{\text{lex}} y \implies \neg x R y)$.*

Proof. See, for example, the proof of [Kec95, Theorem 19.7]. \(\square\)

In particular, this yields the following perfect set theorem.

Theorem B.6 (Mycielski). *Suppose that X is a perfect Polish space and $R \subseteq X \times X$ is meager. Then there is a continuous injection $\phi: 2^{\mathbb{N}} \rightarrow X$ sending distinct sequences to R -inequivalent points.*

Proof. This follows from an application of Theorem B.5 to $R^{\pm 1}$. \(\square\)

It is straightforward to check that the family of sets with the Baire property forms a σ -algebra, and therefore contains the Borel sets. Moreover, we have the following.

Proposition B.7. *Suppose that X is a Polish space and $A \subseteq X$ is analytic. Then A has the Baire property.*

Proof. See, for example, [Kec95, Theorem 21.6]. ☒

APPENDIX C. UNIFORMIZATION

Given sets X and Y , we define $\text{proj}_X: X \times Y \rightarrow X$ by setting $\text{proj}_X(x, y) = x$. A *uniformization* of a set $R \subseteq X \times Y$ is a function $f: \text{proj}_X(R) \rightarrow Y$ whose graph is contained in R .

Theorem C.1 (Jankov, von Neumann). *Suppose that X and Y are standard Borel spaces and $R \subseteq X \times Y$ is analytic. Then there is a $\sigma(\Sigma_1^1)$ -measurable uniformization of R .*

Proof. See, for example, [Kec95, Theorem 18.1]. ☒

A function $f: X \rightarrow Y$ is *countable-to-one* if pre-images of singletons are countable.

Theorem C.2. *Suppose that X and Y are standard Borel spaces, and $f: X \rightarrow Y$ is a countable-to-one Borel function. Then $f(B)$ is Borel.*

Proof. See, for example, [Kec95, Lemma 18.12]. ☒

A *partial uniformization* of a set $R \subseteq X \times Y$ is a function $f: B \rightarrow Y$, where $B \subseteq X$, such that $\text{graph}(f) \subseteq R$.

Theorem C.3 (Lusin-Novikov). *Suppose that X and Y are standard Borel spaces and $R \subseteq X \times Y$ is a Borel set whose vertical sections are countable. Then R is the union of countably many graphs of Borel partial uniformizations.*

Proof. See, for example, [Kec95, Theorem 18.10]. ☒

APPENDIX D. BOREL EQUIVALENCE RELATIONS

A *partial transversal* of an equivalence relation is a set which intersects every equivalence class in at most one point.

Theorem D.1 (Silver). *Suppose that X is a Polish space and E is a co-analytic equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E has only countably many classes.*
- (2) *There is a continuous injection of $2^{\mathbb{N}}$ into a partial transversal of E .*

Proof. See, for example, [Sil80]. ☒

Suppose that E and F are equivalence relations on X and Y . A *homomorphism* from E to F is a function $\phi: X \rightarrow Y$ sending E -equivalent points to F -equivalent points, a *reduction* of E to F is a homomorphism sending E -inequivalent points to F -inequivalent points, and an *embedding* of E into F is an injective reduction.

A Borel equivalence relation E on a Polish space X is *generically ergodic* if every E -invariant set with the Baire property is meager or comeager, or equivalently, if for every standard Borel space Y and Baire measurable homomorphism $\phi: X \rightarrow Y$ from E to equality, there exists $y \in Y$ such that $\phi^{-1}(y)$ is comeager.

We use \mathbb{E}_0 to denote the equivalence relation on $2^{\mathbb{N}}$ given by

$$x \mathbb{E}_0 y \iff \exists n \in \mathbb{N} \forall m \geq n \ x(m) = y(m).$$

Proposition D.2. *The equivalence relation \mathbb{E}_0 is generically ergodic.*

Proof. Suppose that $B \subseteq 2^{\mathbb{N}}$ is an \mathbb{E}_0 -invariant non-meager set with the Baire property. By Proposition B.2, there exist $n \in \mathbb{N}$ and $s \in 2^n$ such that B is comeager in the open set $\mathcal{N}_s = \{x \in 2^{\mathbb{N}} \mid s \sqsubseteq x\}$. Then, by \mathbb{E}_0 -invariance, B is comeager in \mathcal{N}_t for all $t \in 2^n$, so B is comeager. \square

A Borel equivalence relation E is *smooth* if it is Borel reducible to equality on a standard Borel space.

Proposition D.3. *Suppose that $B \subseteq 2^{\mathbb{N}}$ is a non-meager Borel set. Then $\mathbb{E}_0 \upharpoonright B$ is not smooth.*

Proof. As \mathbb{E}_0 is countable, Theorem C.3 ensures that the set $C = [B]_{\mathbb{E}_0}$ is Borel, and that there is a Borel reduction of $\mathbb{E}_0 \upharpoonright C$ to $\mathbb{E}_0 \upharpoonright B$, thus the former is smooth as well. But Proposition D.2 implies that \mathbb{E}_0 is generically ergodic, so C is comeager, thus $\mathbb{E}_0 \upharpoonright C$ is not smooth. \square

The following fact shows that \mathbb{E}_0 is the minimal non-smooth Borel equivalence relation.

Theorem D.4 (Harrington-Kechris-Louveau). *Suppose that X is a Polish space and E is a Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is smooth.*
- (2) *There is a continuous embedding of \mathbb{E}_0 into E .*

Proof. See, for example, [HKL90, Theorem 1.1]. \square

APPENDIX E. COUNTABLE BOREL EQUIVALENCE RELATIONS

Following the standard abuse of language, we say that a Borel equivalence relation is *aperiodic* if all of its equivalence classes are infinite,

countable if all of its equivalence classes are countable, and *finite* if all of its equivalence classes are finite.

Proposition E.1. *Suppose that X is a standard Borel space and F is a finite Borel equivalence relation on X . Then F is smooth.*

Proof. Fix a Borel linear ordering \leq of X . This can be accomplished, for example, by first fixing an enumeration $(U_n)_{n \in \mathbb{N}}$ of an open basis for X , and then pulling back the lexicographical ordering of $2^{\mathbb{N}}$ through the function $\phi: X \rightarrow 2^{\mathbb{N}}$ given by $\phi(x)(n) = \chi_{U_n}(x)$. Theorem C.3 ensures that the function $\psi: X \rightarrow X$, given by $\psi(x) = \min_{\leq}[x]_F$, is a Borel reduction of F to equality on X . \square

Remark E.2. A *selector* for an equivalence relation F on X is a reduction of F to equality on X , whose graph is contained in F . Although the above argument actually gives the apparently stronger fact that every finite Borel equivalence relation has a Borel selector, Theorem C.3 implies that among countable Borel equivalence relations, smoothness is equivalent to the existence of a Borel selector. A *complete set* for an equivalence relation is a set which intersects every equivalence class, and such a set is a *transversal* if it is also a partial transversal. Theorem C.3 implies that the existence of a Borel selector is equivalent to the existence of a Borel transversal, and that the existence of a Borel transversal is equivalent to the existence of a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel partial transversals. Moreover, in the special case that E is aperiodic, this is equivalent to the existence of a partition $(B_n)_{n \in \mathbb{N}}$ of X into Borel transversals.

Proposition E.3. *Suppose that X is a standard Borel space and E is a smooth countable Borel equivalence relation on X . Then every Borel subequivalence relation of E is smooth.*

Proof. By Remark E.2, there is a partition of X into countably many Borel partial transversals of E . As every Borel partial transversal of E is a Borel partial transversal of all of its subequivalence relations, one more application of Remark E.2 yields the desired result. \square

A function $I: X \rightarrow X$ is an *involution* if $I^2 = \text{id}$.

Theorem E.4 (Feldman-Moore). *Suppose that X is a standard Borel space and $R \subseteq X \times X$ is a symmetric Borel set whose vertical sections are all countable. Then there are Borel involutions $I_n: X \rightarrow X$ with the property that $R = \bigcup_{n \in \mathbb{N}} \text{graph}(I_n)$.*

Proof. This follows from the proof of [FM77, Theorem 1]. \square

The following fact can be viewed as a very general form of Rokhlin's Lemma in ergodic theory.

Proposition E.5 (Slaman-Steel). *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X . Then there is a decreasing sequence $(B_n)_{n \in \mathbb{N}}$ of E -complete Borel subsets of X with empty intersection.*

Proof. This follows from the proof of [SS88, Lemma 1]. \square

As an immediate corollary, we obtain the following similar result.

Proposition E.6. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X . Then there is a sequence $(B_n)_{n \in \mathbb{N}}$ of pairwise disjoint E -complete Borel subsets of X .*

Proof. By Proposition E.5, there is a decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of E -complete Borel subsets of X with empty intersection. Recursively define functions $k_n: X \rightarrow \mathbb{N}$ by first setting $k_0(x) = 0$, and then defining $k_{n+1}(x) = \min\{k \in \mathbb{N} \mid (A_{k_n(x)} \setminus A_k) \cap [x]_E \neq \emptyset\}$. As E is countable, Theorem C.3 ensures that these functions are Borel. It follows that the sets $B_n = \{x \in X \mid x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}\}$ are as desired. \square

APPENDIX F. HYPERFINITENESS

A Borel equivalence relation is *hyperfinite* if it is the union of an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite Borel subequivalence relations.

Proposition F.1. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then the family of Borel sets on which E is hyperfinite is closed under countable unions.*

Proof. See, for example, [DJK94, Proposition 5.2]. \square

Proposition F.2. *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , there is a countable-to-one Borel homomorphism from E to F , and F is hyperfinite. Then E is hyperfinite.*

Proof. This follows, for example, from [DJK94, Proposition 5.2]. \square

Proposition F.3 (Jackson-Kechris-Louveau). *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X . Then there is an aperiodic hyperfinite Borel subequivalence relation F of E .*

Proof. See, for example, [JKL02, Lemma 3.25]. \square

The orbit equivalence relation induced by $\Gamma \curvearrowright X$ is given by

$$x E_{\Gamma}^X y \iff \exists \gamma \in \Gamma \gamma \cdot x = y.$$

Theorem F.4 (Slaman-Steel, Weiss). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then the following are equivalent:*

- (1) *The relation E is hyperfinite.*
- (2) *The relation E is the orbit equivalence relation induced by a Borel \mathbb{Z} -action.*

Proof. See, for example, [SS88, Lemma 1]. \(\boxtimes\)

We say that a countable discrete group Γ is *hyperfinite* if whenever X is a standard Borel space and $\Gamma \curvearrowright X$ is a Borel action, the induced orbit equivalence relation E_{Γ}^X is hyperfinite.

Proposition F.5 (Slaman-Steel, Weiss). *The group \mathbb{Z} is hyperfinite.*

Proof. This is a rephrasing of (2) \implies (1) from Theorem F.4. \(\boxtimes\)

A Borel equivalence relation is *hypersmooth* if it is the union of an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of smooth Borel subequivalence relations.

Theorem F.6 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and E is a hypersmooth countable Borel equivalence relation on X . Then E is hyperfinite.*

Proof. See, for example, the beginning of [DJK94, §8]. \(\boxtimes\)

The tail equivalence relation induced by a Borel function $T: X \rightarrow X$ is given by

$$x E_t(T) y \iff \exists m, n \in \mathbb{N} T^m(x) = T^n(y).$$

Theorem F.7 (Dougherty-Jackson-Kechris). *Suppose that X is a standard Borel space and $T: X \rightarrow X$ is Borel. Then $E_t(T)$ is hypersmooth.*

Proof. See, for example, [DJK94, Theorem 8.1]. \(\boxtimes\)

We next turn our attention to comparability of hyperfinite Borel equivalence relations.

Theorem F.8 (Dougherty-Jackson-Kechris). *Suppose that X and Y are standard Borel spaces, E and F are countable Borel equivalence relations on X and Y , E is hyperfinite, and F is non-smooth. Then there is a Borel embedding $\phi: X \rightarrow Y$ of E into F .*

Proof. This follows from Theorem D.4 and [DJK94, Theorem 1]. \(\boxtimes\)

In particular, this yields the following corollaries.

Theorem F.9 (Dougherty-Jackson-Kechris). *Suppose that X is a Polish space and E is a countable Borel equivalence relation on X . Then the following are equivalent:*

- (1) *The relation E is hyperfinite.*
- (2) *The relation E is Borel reducible to \mathbb{E}_0 .*

Proof. To see (1) \implies (2), note that \mathbb{E}_0 is non-smooth by Proposition D.3, and appeal to the fact that every hyperfinite Borel equivalence relation is Borel embeddable into every non-smooth Borel equivalence relation, by Theorem F.8. To see (2) \implies (1), note that \mathbb{E}_0 is trivially hyperfinite, and by Proposition F.2, the family of hyperfinite Borel equivalence relations is closed downward under Borel reducibility. \square

Theorem F.10 (Dougherty-Jackson-Kechris). *All hyperfinite Borel equivalence relations are comparable under Borel reducibility.*

Proof. Theorem A.3 ensures that all standard Borel spaces are comparable under Borel embeddability, Remark E.2 implies that smooth countable Borel equivalence relations have Borel transversals, and Theorem F.8 ensures that every hyperfinite Borel equivalence relation is Borel reducible to every non-smooth Borel equivalence relation. \square

We say that a Borel set $B \subseteq X$ is *E -smooth* if $E \upharpoonright B$ is smooth.

Proposition F.11. *Suppose that X is a standard Borel space and E is a non-smooth countable Borel equivalence relation on X . Then there is a Borel embedding $\pi: X \rightarrow X$ of E into E such that the complement of the set $[\pi(X)]_E$ is not E -smooth.*

Proof. By Theorem D.4, it is sufficient to establish the proposition for \mathbb{E}_0 . Towards this end, observe that the function $\pi: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ given by

$$\pi(x)(n) = \begin{cases} x(m) & \text{if } n = 2m, \text{ and} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

is as desired. \square

APPENDIX G. TREEABILITY

A *graph* on a set X is an irreflexive symmetric subset of $X \times X$. A path through such a graph is a sequence $(x_i)_{i < n}$ with the property that $\forall i < n \ x_i G x_{i+1}$, in which case n is the *length* of the path. A graph is *acyclic* if there is at most one injective path between any two distinct vertices. A Borel equivalence relation is *treeable* if its classes coincide with the connected components of an acyclic Borel graph.

Proposition G.1 (Jackson-Kechris-Louveau). *The family of treeable countable Borel equivalence relations is closed downward under countable-to-one Borel homomorphism.*

Proof. See [JKL02, Proposition 3.3]. ⊠

APPENDIX H. MEASURES

A Borel measure on X is a function $\mu: \mathcal{B} \rightarrow [0, \infty]$, where \mathcal{B} is the family of Borel subsets of X , with the property that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} \mu(B_n)$ whenever $(B_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel subsets of X . We say that μ is σ -finite if there are Borel sets $B_n \subseteq X$ such that $X = \bigcup_{n \in \mathbb{N}} B_n$ and $\mu(B_n) < \infty$ for all $n \in \mathbb{N}$, μ is finite if $\mu(X) < \infty$, and μ is a Borel probability measure if $\mu(X) = 1$.

Proposition H.1. *Suppose that X is a Polish space and μ is a Borel probability measure on X . Then for every Borel set $B \subseteq X$ and real number $\epsilon > 0$, there is an open set $U \subseteq X$ such that $B \subseteq U$ and $\mu(U \setminus B) \leq \epsilon$.*

Proof. See, for example, [Kec95, Theorem 17.10]. ⊠

Proposition H.2. *Suppose that X is a standard Borel space and μ is a Borel probability measure on X . Then there is a countable family \mathcal{B} of Borel subsets of X with the property that for all Borel sets $A \subseteq X$ and real numbers $\epsilon > 0$, there exists $B \in \mathcal{B}$ for which $\mu(A \triangle B) \leq \epsilon$.*

Proof. This is a direct consequence of Proposition H.1. ⊠

Suppose that X is a Polish space. We endow the set $P(X)$ of all Borel probability measures on X with the smallest topology making the functions $\Lambda_f(\mu) = \int f d\mu$ continuous, where $f: X \rightarrow \mathbb{R}$ varies over all bounded continuous functions.

Proposition H.3. *Suppose that X is a Polish space. Then so is $P(X)$.*

Proof. See, for example, [Kec95, Theorem 17.23]. ⊠

We will be interested primarily in the following description of the corresponding standard Borel structure.

Proposition H.4. *Suppose that X is a Polish space. Then the family of Borel subsets of $P(X)$ is the smallest σ -algebra making the functions $\Lambda_B(\mu) = \mu(B)$ Borel, where $B \subseteq X$ varies over all Borel sets.*

Proof. See, for example, [Kec95, Theorem 17.24]. ⊠

Proposition H.5. *Suppose that X and Y are standard Borel spaces and $B \subseteq X \times Y$ is Borel. Then the function $\phi: X \times P(Y) \rightarrow [0, 1]$ given by $\phi(x, \mu) = \mu(B_x)$ is Borel.*

Proof. See, for example, [Kec95, Theorem 17.25]. \square

A Borel set $B \subseteq X$ is μ -null if $\mu(B) = 0$, and μ -conull if its complement is μ -null. A Borel measure μ is *absolutely continuous* with respect to a Borel measure ν , or $\mu \ll \nu$, if every ν -null Borel set is also μ -null.

Theorem H.6 (Radon-Nikodým). *Suppose that X is a standard Borel space and $\mu \ll \nu$ are Borel probability measures on X . Then there is a non-negative Borel function $\phi: X \rightarrow \mathbb{R}$ such that $\mu(B) = \int_B \phi \, d\nu$ for all Borel sets $B \subseteq X$.*

Proof. See, for example, [Fre03, Corollary 232F]. \square

Proposition H.7. *Suppose that X is a standard Borel space. Then the absolute continuity relation on $P(X)$ is Borel.*

Proof. This is a consequence of the observation that if \mathcal{U} is a basis for a Polish topology compatible with the Borel structure of X , then $\mu \ll \nu \iff \forall \delta > 0 \exists \epsilon > 0 \forall U \in \mathcal{U} (\nu(U) < \epsilon \implies \mu(U) < \delta)$. \square

More generally, a set $Y \subseteq X$ is μ -null if it is contained in a μ -null Borel set. A set $Y \subseteq X$ is μ -measurable if there is a Borel set $B \subseteq Y$ such that $Y \setminus B$ is μ -null.

Proposition H.8 (Lusin). *Suppose that X is a standard Borel space, $A \subseteq X$ is an analytic set, and μ is a Borel probability measure on X . Then A is μ -measurable.*

Proof. See, for example, [Kec95, Theorem 21.10]. \square

A function between standard Borel spaces is μ -measurable if pre-images of Borel sets are μ -measurable.

Proposition H.9 (Lusin). *Suppose that X and Y are standard Borel spaces, μ is a Borel probability measure on X , and $\phi: X \rightarrow Y$ is μ -measurable. Then there is a μ -conull Borel set $C \subseteq X$ with the property that $\phi \upharpoonright C$ is Borel.*

Proof. Fix a Polish topology on Y generating its Borel structure, as well as a countable basis \mathcal{B} for Y . Then for each $B \in \mathcal{B}$, there is a Borel set $A_B \subseteq X$ and a μ -null Borel set $N_B \subseteq X$ with the property that $A_B \subseteq \phi^{-1}(B) \subseteq A_B \cup N_B$. Set $C = X \setminus \bigcup_{B \in \mathcal{B}} N_B$. \square

A Borel measure μ is *measure equivalent* to a Borel measure ν , or $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$.

Proposition H.10. *Suppose that X is a standard Borel space. Then measure equivalence on $P(X)$ is Borel.*

Proof. This is an immediate consequence of Proposition H.7. \square

The *product* of Borel probability measures μ and ν on X and Y is the unique Borel probability measure $\mu \times \nu$ on $X \times Y$ with the property that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for all Borel sets $A \subseteq X$ and $B \subseteq Y$.

Theorem H.11 (Fubini). *Suppose that X and Y are standard Borel spaces, μ and ν are Borel probability measures on X and Y , and $R \subseteq X \times Y$ is $(\mu \times \nu)$ -measurable. Then R is $(\mu \times \nu)$ -conull if and only if μ -almost every vertical section of R is ν -conull.*

Proof. See, for example, [Hal50, Theorem 36.A]. \square

Two Borel measures μ and ν are *orthogonal* if there is a μ -null, ν -conull Borel set.

Theorem H.12 (Burgess-Mauldin). *Suppose that X is a standard Borel space and $B \subseteq P(X)$ is an uncountable Borel set of pairwise orthogonal measures. Then there are Borel sequences $(B_c)_{c \in 2^{\mathbb{N}}}$ of pairwise disjoint Borel subsets of X and $(\mu_c)_{c \in 2^{\mathbb{N}}}$ of Borel probability measures on X in B such that $\mu_c(B_c) = 1$ for all $c \in 2^{\mathbb{N}}$.*

Proof. By Theorem A.3, we can assume that $X = 2^{\mathbb{N}}$. Let \mathcal{A} denote the algebra of clopen subsets of $2^{\mathbb{N}}$. By Theorem D.1, there is a continuous injection $\pi: 2^{\mathbb{N}} \rightarrow B$. Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and recursively construct $k_n \in \mathbb{N}$, $\phi_n: 2^n \rightarrow 2^{k_n}$, and $A_n: 2^n \rightarrow \mathcal{A}$ which satisfy the following conditions:

- (1) $\forall n \in \mathbb{N} \forall s \in 2^n \phi_{n+1}(s^\frown(0)) \neq \phi_{n+1}(s^\frown(1))$.
- (2) $\forall i < 2 \forall n \in \mathbb{N} \forall s \in 2^n \phi_n(s) \sqsubseteq \phi_{n+1}(s^\frown(i))$.
- (3) $\forall n \in \mathbb{N} \forall s, t \in 2^n (s = t \iff A_n(s) \cap A_n(t) \neq \emptyset)$.
- (4) $\forall n \in \mathbb{N} \forall s \in 2^n \forall \mu \in \pi(\mathcal{N}_{\phi_n(s)}) \mu(A_n(s)) \geq 1 - \epsilon_n$.

Define $\phi: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by $\phi(c) = \bigcup_{n \in \mathbb{N}} \phi_n(c \upharpoonright n)$, and for each $c \in 2^{\mathbb{N}}$, define $B_c = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} A_m(c \upharpoonright m)$ and $\mu_c = (\pi \circ \phi)(c)$. \square

It is sometimes convenient to have a means of coding Borel functions, modulo Borel probability measures, which is uniform in both the function and the measure in question. In order to keep our coding as transparent as possible, it will be useful to assume that every continuous partial function from X to Y has a continuous total extension, and that the spaces $C(X, Y)$ and $C(Y, X)$ are themselves Polish. This holds, for example, when $X = Y = 2^{\mathbb{N}}$, which we can assume by Theorem A.3.

Proposition H.13. *Suppose that X is a compact Polish space and Y is a Polish space. Then the function $\pi: C(X, Y) \times X \rightarrow Y$ given by $\pi(f, x) = f(x)$ is continuous.*

Proof. It is sufficient to show that if $U \subseteq Y$ is open and $\pi(f, x) \in U$, then there are open neighborhoods V and W of f and x such that $\pi(V \times W) \subseteq U$. Towards this end, fix a Polish metric d on Y compatible with its underlying topology. As U is open, there exists $\epsilon > 0$ such that $\mathcal{B}(f(x), \epsilon) \subseteq U$. As f is continuous, there is an open neighborhood W of x such that $f(W) \subseteq \mathcal{B}(f(x), \epsilon/2)$. Fix an open neighborhood V of f such that $\forall g \in V \forall x \in X \ d(f(x), g(x)) \leq \epsilon/2$. It only remains to note that if $(g, y) \in V \times W$, then $d(f(x), g(y)) \leq d(f(x), f(y)) + \epsilon/2 < \epsilon$, thus $g(y) \in U$. \square

We refer to elements c of $C(X, Y)^\mathbb{N}$ as *codes* for measurable functions. Proposition H.13 ensures that the sets

$$D_n = \{(c, x) \in C(X, Y)^\mathbb{N} \times X \mid \forall m \geq n \ c(x)(m) = c(x)(n)\}$$

and $D = \bigcup_{n \in \mathbb{N}} D_n$ are Borel. We associate with each $c \in C(X, Y)^\mathbb{N}$ the function $\phi_c: D_c \rightarrow Y$ for which $\phi_c(x)$ is the eventual value of $(c_n(x))_{n \in \mathbb{N}}$.

Proposition H.14. *Suppose that X and Y are standard Borel spaces. Then the function $\pi: D \rightarrow Y$ given by $\pi(c, x) = \phi_c(x)$ is Borel.*

Proof. Note that $\pi(c, x) = y \iff \exists n \in \mathbb{N} \forall m \geq n \ c(x)(m) = y$, thus the graph of π is Borel. As Proposition A.2 ensures that π is Borel if and only if its graph is Borel, the desired result follows. \square

The *push-forward* of a Borel measure μ on X through a Borel function $\phi: X \rightarrow Y$ is given by $(\phi_*\mu)(B) = \mu(\phi^{-1}(B))$.

Proposition H.15. *Suppose that X and Y are standard Borel spaces. Then the function $\pi: \{(c, \mu) \in C(X, Y)^\mathbb{N} \times P(X) \mid \mu(D_c) = 1\} \rightarrow P(Y)$ given by $\pi(c, \mu) = (\phi_c)_*\mu$ is Borel.*

Proof. It is sufficient to show that if $B \subseteq Y$ is Borel and $F \subseteq \mathbb{R}$ is of the form $(a, b]$, where $a < b$ are in \mathbb{R} , then the intersection of the sets

$$R = \{(c, \mu) \in C(X, Y)^\mathbb{N} \times P(X) \mid \mu(D_c) = 1\}$$

and

$$S = \{(c, \mu) \in C(X, Y)^\mathbb{N} \times P(X) \mid (\phi_c)_*\mu(B) \in F\}$$

is Borel. Proposition H.5 ensures that R is Borel. As $(c, \mu) \in S$ if and only if $\exists n \in \mathbb{N} \forall m \geq n \ \mu(c(m)^{-1}(B) \cap (D_n)_c) \in F$, one more application of Proposition H.5 ensures that S is Borel. \square

APPENDIX I. MEASURED EQUIVALENCE RELATIONS

We begin this section with an observation concerning non-smoothness in the presence of measures.

Proposition I.1. *Suppose that X is a standard Borel space, E is a non-smooth Borel equivalence relation on X , and μ is a Borel probability measure on X . Then there is a μ -null Borel set on which E is non-smooth.*

Proof. By Theorem D.4, there is a continuous embedding $\pi: 2^{\mathbb{N}} \rightarrow X$ of \mathbb{E}_0 into E . Note that for each $c \in 2^{\mathbb{N}}$, the function $\pi_c: 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ given by

$$\pi_c(d)(n) = \begin{cases} c(n) & \text{if } n \text{ is even, and} \\ d(n) & \text{if } n \text{ is odd} \end{cases}$$

is a continuous embedding of \mathbb{E}_0 into \mathbb{E}_0 . As the sets of the form $\pi_c(2^{\mathbb{N}})$ are pairwise disjoint, it follows that for all but countably many $c \in 2^{\mathbb{N}}$, the function $\pi_c \circ \pi$ is as desired. \square

Let \mathbb{R}^+ denote the group of strictly positive real numbers under multiplication, and suppose that $\rho: E \rightarrow \mathbb{R}^+$ is a Borel cocycle, meaning that $\rho(x, z) = \rho(x, y)\rho(y, z)$ whenever $x E y E z$. For each set $Y \subseteq [x]_E$, define $\rho(Y, x) = \sum_{y \in Y} \rho(y, x)$. We say that Y is ρ -finite or ρ -infinite according to whether $\rho(Y, x)$ is finite or infinite. Our assumption that ρ is a cocycle ensures that the ρ -finiteness of Y does not depend on the choice of $x \in [Y]_E$. We say that ρ is *finite* if every equivalence class of E is ρ -finite, and *aperiodic* if every equivalence class of E is ρ -infinite. Given $Y, Z \subseteq [x]_E$, define $\rho(Y, Z) = \rho(Y, x)/\rho(Z, x)$. Again, our assumption that ρ is a cocycle ensures that $\rho(Y, Z)$ does not depend on the choice of $x \in [Y]_E$.

Proposition I.2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and there is a finite Borel cocycle $\rho: E \rightarrow \mathbb{R}^+$. Then E is smooth.*

Proof. See, for example, [Mil08, Proposition 2.1]. \square

A measure μ is E -quasi-invariant if the family of all μ -null Borel sets is closed under E -saturation, and E -invariant if $\mu(\phi(B)) = \mu(B)$ for all Borel sets $B \subseteq X$ and Borel injections $\phi: X \rightarrow X$ whose graphs are contained in E . More generally, we say that a measure μ on X is ρ -invariant if

$$\mu(\phi(B)) = \int_B \rho(\phi(x), x) d\mu(x),$$

for all Borel sets $B \subseteq X$ and Borel injections $\phi: X \rightarrow X$ whose graphs are contained in E .

We use \mathcal{Q}_E to denote the family of all E -quasi-invariant Borel probability measures on X , \mathcal{I}_E to denote the family of all E -invariant Borel probability measures on X , and \mathcal{I}_ρ to denote the family of all ρ -invariant Borel probability measures on X .

Proposition I.3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is an E -quasi-invariant Borel probability measure on X . Then there is a Borel cocycle $\rho: E \rightarrow \mathbb{R}^+$ with respect to which μ is invariant.*

Proof. See, for example, [KM04, §8]. □

We say that E is μ -nowhere smooth if there is no μ -positive Borel set on which E is smooth.

Proposition I.4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $\rho: E \rightarrow \mathbb{R}^+$ is an aperiodic Borel cocycle, and μ is a ρ -invariant Borel probability measure on X . Then E is μ -nowhere smooth.*

Proof. See, for example, [Mil08, Proposition 2.1]. □

The following fact usually allows us to assume quasi-invariance.

Proposition I.5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is a Borel probability measure on X . Then there is an E -quasi-invariant Borel probability measure ν on X such that $\mu \ll \nu$ and the two measures take the same values on all E -invariant Borel sets.*

Proof. By Theorem E.4, there is a countable group $\Gamma = \{\gamma_n \mid n \in \mathbb{N}\}$ of Borel automorphisms of X whose induced orbit equivalence relation is E . Define $\nu = \sum_{n \in \mathbb{N}} (\gamma_n)_* \mu / 2^{n+1}$.

To see that ν is a Borel measure, simply note that $\nu(\emptyset) = 0$ and

$$\begin{aligned} \sum_{k \in \mathbb{N}} \nu(B_k) &= \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} ((\gamma_n)_* \mu)(B_k) / 2^{n+1} \\ &= \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} ((\gamma_n)_* \mu)(B_k) / 2^{n+1} \\ &= \sum_{n \in \mathbb{N}} ((\gamma_n)_* \mu)(\bigcup_{k \in \mathbb{N}} B_k) / 2^{n+1} \\ &= \nu(\bigcup_{k \in \mathbb{N}} B_k) \end{aligned}$$

whenever $(B_k)_{k \in \mathbb{N}}$ is a sequence of pairwise disjoint Borel sets. Observe now that if $B \subseteq X$ is an E -invariant Borel set, then $\mu(B) = (\gamma_* \mu)(B)$ for all $\gamma \in \Gamma$, thus $\nu(B) = \sum_{n \in \mathbb{N}} \mu(B) / 2^{n+1} = \mu(B)$. It follows that ν is a Borel probability measure and μ and ν take the same values on all

E -invariant Borel sets. It only remains to observe that if $N \subseteq X$ is a ν -null Borel set, then $\mu(N) \leq \sum_{n \in \mathbb{N}} (\gamma_n)_* \mu(N) = 0$, thus $\mu \ll \nu$. \square

A measure μ is E -ergodic if every E -invariant Borel set is μ -null or μ -conull. We use \mathcal{E}_E to denote the family of all E -ergodic Borel probability measures on X , \mathcal{EQ}_E to denote $\mathcal{E}_E \cap \mathcal{Q}_E$, \mathcal{EI}_E to denote $\mathcal{E}_E \cap \mathcal{I}_E$, and \mathcal{EI}_ρ to denote $\mathcal{E}_E \cap \mathcal{I}_\rho$. Note that any two measures in \mathcal{EQ}_E are either orthogonal or equivalent; the following gives a sufficient condition to strengthen equivalence to equality.

Proposition I.6. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $\rho: E \rightarrow \mathbb{R}^+$ is a Borel cocycle, and $\mu \ll \nu$ are E -ergodic ρ -invariant Borel probability measures on X . Then $\mu = \nu$.*

Proof. By Theorem H.6, there is a non-negative Borel map $\phi: X \rightarrow \mathbb{R}$ such that $\mu(B) = \int \phi(x) d\nu(x)$ for all Borel sets $B \subseteq X$. Noting that $\mu(X) = \nu(X) = 1$, to see that $\mu = \nu$, it is sufficient to show that ϕ is constant on a μ -conull Borel set. Suppose, towards a contradiction, that there are μ -positive Borel sets $A, B \subseteq X$ with the property that $\forall x \in A \forall y \in B \phi(x) < \phi(y)$. As E is countable, Theorem E.4 yields a countable group Γ of Borel automorphisms of X whose induced orbit equivalence relation is E . As μ is E -ergodic, there exists $\gamma \in \Gamma$ such that the set $A' = A \cap \gamma^{-1}(B)$ is μ -positive, in which case

$$\mu(\gamma(A')) = \int_{A'} \rho(\gamma \cdot x, x) d\mu(x) = \int_{A'} \phi(x) \rho(\gamma \cdot x, x) d\nu(x)$$

and

$$\mu(\gamma(A')) = \int_{\gamma(A')} \phi(x) d\nu(x) = \int_{A'} \phi(\gamma \cdot x) \rho(\gamma \cdot x, x) d\nu(x),$$

the desired contradiction. \square

Theorem I.7 (Ditzen). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and $\rho: E \rightarrow \mathbb{R}^+$ is a Borel cocycle. Then the set of E -ergodic, ρ -invariant Borel probability measures on X is Borel.*

Proof. See [Dit92, Theorem 2 of Chapter 2]. \square

A *disintegration* of a Borel probability measure μ on X through a Borel function $\phi: X \rightarrow Y$ is a sequence $(\mu_y)_{y \in Y}$ with the property that $\mu = \int \mu_y d\mu(y)$ and $\mu_y(\phi^{-1}(y)) = 1$ for all $y \in Y$.

Theorem I.8. *Suppose that X and Y are standard Borel spaces, μ is a Borel probability measure on X , and $\phi: X \rightarrow Y$ is Borel. Then there is a Borel disintegration of μ through ϕ .*

Proof. See, for example, [Kec95, Exercise 17.35]. \square

An *ergodic decomposition* of a Borel cocycle $\rho: E \rightarrow \mathbb{R}^+$ is a sequence $(\mu_x)_{x \in X}$ of Borel probability measures on X such that $\mu_x = \mu_y$ for all $(x, y) \in E$, $\mu(\{x \in X \mid \mu = \mu_x\}) = 1$ for all $\mu \in \mathcal{E}\mathcal{I}_\rho$, and $\mu = \int \mu_x d\mu(x)$ for all $\mu \in \mathcal{I}_\rho$.

Theorem I.9 (Ditzen). *Suppose that X is a standard Borel space, E is a Borel equivalence relation on X , and $\rho: E \rightarrow \mathbb{R}^+$ is a Borel cocycle. Then there is a Borel ergodic decomposition of ρ .*

Proof. See [Dit92, Theorem 6 of Chapter 2]. \square

An equivalence relation E is *compressible* if there is a Borel injection $\phi: X \rightarrow X$ such that $\text{graph}(\phi) \subseteq E$ and $\sim\phi(X)$ is E -complete.

Proposition I.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and $B \subseteq X$ is a Borel E -complete set for which $E \upharpoonright B$ is compressible. Then there is a Borel injection $\pi: X \rightarrow B$ whose graph is contained in E .*

Proof. Fix a Borel compression $\phi: B \rightarrow B$ of $E \upharpoonright B$. By Theorem C.3, there is a Borel function $\psi: X \rightarrow B \setminus \phi(B)$ whose graph is contained in E , as well as a Borel function $\xi: X \rightarrow \mathbb{N}$ such that the function $\psi \times \xi$ is injective. Then the function $\pi(x) = \phi^{\xi(x)} \circ \psi(x)$ is as desired. \square

We say that E is *μ -nowhere compressible* if it is not compressible on any μ -positive Borel set.

Theorem I.11 (Hopf). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , and μ is an E -quasi-invariant Borel σ -finite measure on X . If E is μ -nowhere compressible, then there is an E -invariant Borel probability measure $\nu \sim \mu$.*

Proof. See, for example, [Nad98, §10]. \square

The *μ -cost* of a Borel graph G on a standard Borel space X with respect to a Borel probability measure μ on X is given by

$$C_\mu(G) = \frac{1}{2} \int |G_x| d\mu(x).$$

The *μ -cost* of a countable Borel equivalence relation E with respect to an E -invariant Borel probability measure on the underlying space is the infimum of the costs of its Borel graphings.

Proposition I.12 (Gaboriau). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , μ is an E -invariant Borel probability measure on X , $B \subseteq X$ is an E -complete*

Borel set, and μ_B is the Borel probability measure on B given by $\mu_B(D) = \mu(D)/\mu(B)$. Then $C_\mu(E) - 1 = \mu(B)(C_{\mu_B}(E \upharpoonright B) - 1)$. In particular, it follows that $C_\mu(E) \leq C_{\mu_B}(E \upharpoonright B)$.

Proof. See, for example, [KM04, Theorem 21.1]. \square

Proposition I.13 (Gaboriau). *Suppose that X is a standard Borel space, E is an aperiodic treeable countable Borel equivalence relation on X , and μ is an E -invariant Borel probability measure on X for which E is not μ -hyperfinite. Then $C_\mu(E) > 1$.*

Proof. See, for example, [KM04, Corollary 27.12]. \square

An E -ergodic measure μ is (E, F) -ergodic if there is no μ -null-to-one Borel homomorphism from E to F .

Proposition I.14. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , μ is an E -ergodic (E, \mathbb{E}_0) -ergodic E -invariant Borel probability measure on X , and $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of countable Borel equivalence relations on X whose union is E . Then for all $\epsilon > 0$, there is a Borel set $B \subseteq X$ of μ -measure at least $1 - \epsilon$ on which μ is E_n -ergodic, for all sufficiently large $n \in \mathbb{N}$.*

Proof. See, for example, [Mil12, Proposition 2.2]. \square

Proposition I.15. *Suppose that X and Y are compact Polish spaces, and E and F are countable Borel equivalence relations on X and Y . Then the set of pairs $(c, \mu) \in C(X, Y)^\mathbb{N} \times P(X)$ for which ϕ_c is a reduction of E to F on an E -invariant μ -conull Borel set is analytic.*

Proof. By Theorem E.4, there are countable groups Γ and Δ of Borel automorphisms of X and Y whose induced orbit equivalence relations are E and F . Then (c, μ) has the desired property if and only if there exists $d \in C(Y, X)^\mathbb{N}$ such that the following conditions hold:

- (1) $\mu(D_c) = 1$.
- (2) $\forall_\mu^* x \in X \forall \gamma \in \Gamma \phi_c(x) F \phi_c(\gamma \cdot x)$.
- (3) $(\phi_c)_* \mu(D_d) = 1$.
- (4) $\forall_\mu^* x \in X \forall \delta \in \Delta (\delta \cdot \phi_c(x) \in D_d \implies x E \phi_d(\delta \cdot \phi_c(x)))$.

Proposition H.5 ensures that condition (1) is Borel, Propositions H.5 and H.14 ensure that condition (2) is Borel, Propositions H.5 and H.15 ensure that condition (3) is Borel, and Propositions H.5 and H.14 ensure that condition (4) is Borel. \square

We say that E is μ -nowhere reducible to F if there is no μ -positive Borel set on which E is Borel reducible to F , E is μ -somewhere reducible

to F if there is a μ -positive Borel set on which E is Borel reducible to F , E is μ -reducible to F if there is a μ -conull Borel set on which E is Borel reducible to F , and E is *measure reducible* to F if E is μ -reducible to F for every Borel probability measure μ on X . Intuitively, the latter means that it is impossible to rule out Borel reducibility of E to F using measure-theoretic techniques. Moreover, all notions of definable reducibility lie between measure reducibility and Borel reducibility. A useful intermediate notion is that of *invariant-measure reducibility*, in which one merely asks that $E \upharpoonright B$ is μ -reducible to F for every Borel set $B \subseteq X$ and every $(E \upharpoonright B)$ -invariant Borel probability measure μ on B . The corresponding notions of embeddability are defined analogously.

APPENDIX J. MEASURE HYPERFINITENESS

An equivalence relation is μ -nowhere hyperfinite if there is no μ -positive Borel set on which it is hyperfinite, and μ -hyperfinite if there is a μ -conull Borel set on which it is hyperfinite.

Proposition J.1. *Suppose that Γ is a countable discrete non-amenable group, X is a standard Borel space, $\Gamma \curvearrowright X$ is a free Borel action, and μ is an E_Γ^X -invariant Borel probability measure on X . Then the induced orbit equivalence relation E_Γ^X is not μ -hyperfinite.*

Proof. See, for example, [JKL02, Proposition 2.5]. \(\square\)

Theorem J.2 (Dye, Krieger). *Suppose that X is a standard Borel space, μ is a Borel probability measure on X , and $(E_n)_{n \in \mathbb{N}}$ is an increasing sequence of μ -hyperfinite Borel equivalence relations on X . Then the equivalence relation $E = \bigcup_{n \in \mathbb{N}} E_n$ is also μ -hyperfinite.*

Proof. See, for example, [KM04, Proposition 6.11]. \(\square\)

Suppose that X is a standard Borel space. Given a Borel measure μ on X and Borel equivalence relations E and F on X , define

$$e_\mu(E, F) = \mu(\{x \in X \mid [x]_E \neq [x]_F\}).$$

Proposition J.3. *Suppose that X is a standard Borel space and μ is a Borel probability measure on X . Then e_μ is a complete pseudo-metric.*

Proof. To see that e_μ is a pseudo-metric, it is sufficient to check the triangle inequality. Towards this end, suppose that E_1 , E_2 , and E_3 are

Borel equivalence relations on X , and observe that

$$\begin{aligned}
e_\mu(E_1, E_3) &= 1 - \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_3}\}) \\
&\leq 1 - \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\} \cap \{x \in X \mid [x]_{E_2} = [x]_{E_3}\}) \\
&= 1 + \mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\} \cup \{x \in X \mid [x]_{E_2} = [x]_{E_3}\}) - \\
&\quad (\mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\}) + \mu(\{x \in X \mid [x]_{E_2} = [x]_{E_3}\})) \\
&\leq 2 - (\mu(\{x \in X \mid [x]_{E_1} = [x]_{E_2}\}) + \mu(\{x \in X \mid [x]_{E_2} = [x]_{E_3}\})) \\
&= e_\mu(E_1, E_2) + e_\mu(E_2, E_3).
\end{aligned}$$

To see that e_μ is complete, suppose that $(E_n)_{n \in \mathbb{N}}$ is an e_μ -Cauchy sequence, fix a sequence of real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, and fix a strictly increasing sequence of natural numbers k_n such that

$$\forall n \in \mathbb{N} \forall i, j \geq k_n \quad e_\mu(E_i, E_j) \leq \epsilon_n.$$

Note that for all $n \in \mathbb{N}$, the set $Y_n = \{x \in X \mid \forall m \geq n \ [x]_{E_{k_m}} = [x]_{E_{k_n}}\}$ has μ -measure at least $1 - \sum_{m \geq n} \epsilon_m$. In particular, it follows that the set $Y = \bigcup_{n \in \mathbb{N}} Y_n$ is μ -conull. Letting E denote the union of the diagonal on X with the equivalence relations of the form $E_{k_n} \upharpoonright Y_n$ for $n \in \mathbb{N}$, it follows that $E_{k_n} \rightarrow_{e_\mu} E$ as $n \rightarrow \infty$, thus e_μ is indeed complete. \square

It is not difficult to see that e_μ is not separable, even when restricted to the family of Borel equivalence relations on X whose classes are all of cardinality two. In contrast, we have the following.

Proposition J.4. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then there is a countable family \mathcal{F} of finite Borel subequivalence relations of E such that for all Borel probability measures μ on X , the family \mathcal{F} is e_μ -dense in the set of all finite Borel subequivalence relations of E .*

Proof. Fix an enumeration $(U_n)_{n \in \mathbb{N}}$ of a basis for a Polish topology generating the Borel structure of X . By Theorem E.4, there is a sequence $(f_n)_{n \in \mathbb{N}}$ of Borel automorphisms of X such that $E = \bigcup_{n \in \mathbb{N}} \text{graph}(f_n)$.

For each $n \in \mathbb{N}$ and $s \in \mathbb{N}^n$, let X_s denote the Borel set of $x \in X$ with the property that whenever $i, j, k < n$, $x \in U_{s(i)} \cap U_{s(j)}$, $y \in U_{s(k)}$, and $f_i(x) = f_k(y)$, there exists $\ell < n$ such that $y \in U_{s(\ell)}$ and $f_j(x) = f_\ell(y)$. Let F_s denote the reflexive Borel relation on X in which distinct points x and y related if there exist $i, j < n$ and $z \in U_{s(i)} \cap U_{s(j)} \cap X_s$ such that $x = f_i(z)$ and $y = f_j(z)$.

Lemma J.5. *Each F_s is an equivalence relation.*

Proof. As F_s is clearly reflexive and symmetric, it is sufficient to show that it is transitive. Towards this end, observe that if $x F_s y F_s z$ are pairwise distinct, then there exist $i, j < n$ and $v \in U_{s(i)} \cap U_{s(j)} \cap X_s$ with the property that $x = f_i(v)$ and $y = f_j(v)$, as well as $k, \ell < n$ and $w \in U_{s(k)} \cap U_{s(\ell)} \cap X_s$ with the property that $y = f_k(w)$ and $z = f_\ell(w)$. As $v \in X_s$, there exists $m < n$ with $w \in U_{s(m)}$ and $x = f_i(v) = f_m(w)$, in which case the definition of F_s ensures that $x F_s z$. \square

To see that the family $\mathcal{F} = \{F_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ is as desired, suppose that $\epsilon > 0$, F is a finite Borel subequivalence relation of E , and μ is a Borel probability measure on X . Fix $n \in \mathbb{N}$ sufficiently large that the μ -measure of the set $Y = \{x \in X \mid \forall y, z \in [x]_F \exists i < n f^i(y) = z\}$ is strictly greater than $1 - \epsilon$. Set $\delta = \mu(Y) - (1 - \epsilon)$, and define $Y_k = \{x \in X \mid x F f_k(x)\}$ for all $k < n$. By Proposition H.1, there exists $s \in \mathbb{N}^n$ with the property that μ -measure of the set

$$Z_{i,j,k} = \{x \in X \mid (f_i^{-1} \circ f_j)(x) \in U_{s(k)} \iff (f_i^{-1} \circ f_j)(x) \in Y_k\}$$

is at least $1 - \delta/n^3$, for all $i, j, k < n$.

Lemma J.6. *The set $Z = Y \cap \bigcap_{i,j,k < n} Z_{i,j,k}$ is contained in X_s .*

Proof. We must show that if $i, j, k < n$, $z \in U_{s(i)} \cap U_{s(j)} \cap Z$, $y \in U_{s(k)}$, and $f_i(z) = f_k(y)$, then there exists $\ell < n$ such that $y \in U_{s(\ell)}$ and $f_j(z) = f_\ell(y)$. Towards this end, note that $y = (f_k^{-1} \circ f_i)(z)$, so the fact that $z \in Z$ ensures that $z \in Y_i \cap Y_j$ and $y \in Y_k$. In particular, it follows that $f_j(z) F z F f_i(z) = f_k(y) F y$. The fact that $z \in Y$ then yields $\ell < n$ such that $f_j(z) = f_\ell(y)$. As $y \in Y_\ell$, one more appeal to the fact that $z \in Z$ ensures that $y \in U_{s(\ell)}$. \square

Lemma J.7. *Suppose that $z \in Z$. Then $[z]_F = [z]_{F_s}$.*

Proof. Suppose first that $x \in [z]_F$. As $z \in Y$, there exist $i, j < n$ such that $x = f_i(z)$ and $z = f_j(z)$. Then $z \in Y_i \cap Y_j$, so the fact that $z \in Z$ ensures that $z \in U_{s(i)} \cap U_{s(j)}$. As Lemma J.6 implies that $z \in X_s$, the definition of F_s ensures that $x \in [z]_{F_s}$.

Suppose now that $x \in [z]_{F_s}$. The definition of F_s then yields $i, j < n$ and $w \in U_{s(i)} \cap U_{s(j)} \cap X_s$ such that $x = f_i(w)$ and $z = f_j(w)$. As $z \in Y$, there exists $\ell < n$ such that $z = f_\ell(z)$. As $z \in Y_\ell$, the fact that $z \in Z$ ensures that $z \in U_{s(\ell)}$, so the fact that $w \in X_s$ yields $k < n$ such that $z \in U_{s(k)}$ and $x = f_k(z)$. One more appeal to the fact that $z \in Z$ then ensures that $z \in Y_k$, in which case $x = f_k(z) \in [z]_F$. \square

As $\mu(Z) \geq 1 - \epsilon$, it follows that $e_\mu(F, F_s) \leq \epsilon$. \square

Let \mathcal{H}_E denote the space of Borel probability measures μ on X with respect to which E is μ -hyperfinite.

Theorem J.8 (Segal). *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then there is a Borel set $F \subseteq (\mathbb{N} \times (X \times X)) \times P(X)$ such that for all $\mu \in P(X)$, the following conditions hold:*

- (1) *The sets $(F^\mu)_n$ form an increasing sequence of finite Borel subequivalence relations of E .*
- (2) *The set $B^\mu = \{x \in X \mid [x]_E \neq \bigcup_{n \in \mathbb{N}} [x]_{(F^\mu)_n}\}$ does not contain a μ -positive Borel subset on which E is hyperfinite.*

In particular, it follows that \mathcal{H}_E is Borel.

Proof. Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$. By Proposition J.4, there is a family $\mathcal{E} = \{E_k \mid k \in \mathbb{N}\}$ of finite Borel subequivalence relations of E such that for all Borel probability measures μ on X , the family \mathcal{E} is e_μ -dense in the set of all finite Borel subequivalence relations of E . The characterization of the Borel structure of $P(X)$ provided by Proposition H.4 ensures that the functions $m_n: P(X) \rightarrow [0, 1]$ given by

$$m_n(\mu) = \sup_{k \in \mathbb{N}} \mu(\{x \in X \mid \forall i < n \ x \ E_k \ f_i(x)\})$$

are Borel, as are the functions $k_n: P(X) \rightarrow \mathbb{N}$ given by

$$k_n(\mu) = \min\{k \in \mathbb{N} \mid \mu(\{x \in X \mid \forall i < n \ x \ E_k \ f_i(x)\}) > m_n(\mu) - \epsilon_n\},$$

thus so too is the set $F \subseteq (\mathbb{N} \times (X \times X)) \times P(X)$ given by

$$x \ (F^\mu)_n \ y \iff \forall m \geq n \ x \ E_{k_m(\mu)} \ y.$$

To see that F is as desired, suppose that $\mu \in P(X)$. As the sets $(F^\mu)_n = \bigcap_{m \geq n} E_{k_m(\mu)}$ clearly form an increasing sequence of finite Borel subequivalence relations of E , it is enough to show that if $A \subseteq B^\mu$ is a Borel set on which E is hyperfinite, then $\mu(A) = 0$. As B^μ is E -invariant and E is countable, Theorem C.3 and Proposition F.2 allow us to assume that A is E -invariant as well.

Lemma J.9. *Suppose that $n \in \mathbb{N}$. Then*

$$\mu(\{x \in A \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}) \geq \mu(A) - \epsilon_n.$$

Proof. As $E_{k_n(\mu)}$ is finite, Remark E.2 ensures that it has a Borel transversal $C \subseteq X$ from which the quotient $X/E_{k_n(\mu)}$ inherits a standard Borel structure, and moreover, that the map associating each $E_{k_n(\mu)}$ -class with the unique point of C it contains is a Borel reduction of $E/E_{k_n(\mu)}$ to E . Proposition F.2 therefore implies that the restriction of $E/E_{k_n(\mu)}$ to $A/E_{k_n(\mu)}$ is hyperfinite.

Given $\epsilon > 0$, observe that all but finitely many relations E' along any sequence witnessing the hyperfiniteness of the restriction of $E/E_{k_n(\mu)}$

to $A/E_{k_n(\mu)}$, when viewed as equivalence relations on A , satisfy the condition that $\mu(\{x \in A \mid \forall i < n \ x \ E' \ f_i(x)\}) > \mu(A) - \epsilon$. The e_μ -density of \mathcal{E} therefore yields $k \in \mathbb{N}$ such that

$$\mu(A) - \mu(\{x \in A \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}) - \epsilon$$

is strictly less than

$$\mu(\{x \in X \mid \forall i < n \ x \ E_k \ f_i(x)\}) - \mu(\{x \in X \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}).$$

As the definition of $k_n(\mu)$ ensures that the latter quantity is itself strictly less than ϵ_n , it follows that

$$\mu(\{x \in A \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}) > \mu(A) - \epsilon_n - \epsilon,$$

thus $\mu(\{x \in A \mid \forall i < n \ x \ E_{k_n(\mu)} \ f_i(x)\}) \geq \mu(A) - \epsilon_n$, as the former inequality holds for all $\epsilon > 0$. \square

Set $A' = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} \{x \in A \mid \forall i < m \ x \ E_{k_m(\mu)} \ f_i(x)\}$, and note that $\mu(A) = \mu(A')$, since $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$, thus $\mu(A) = 0$, since $A' \cap B^\mu = \emptyset$.

Finally, as $\mathcal{H}_E = \{\mu \in P(X) \mid \mu(B^\mu) = 0\}$ and Theorem C.2 ensures that the set $B = \{(x, \mu) \in X \times P(X) \mid x \in B^\mu\}$ is Borel, Proposition H.5 implies that \mathcal{H}_E is Borel as well. \square

We now turn our attention to ergodic measures.

Proposition J.10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $\rho: E \rightarrow \mathbb{R}^+$ is a Borel cocycle, and there is a ρ -invariant Borel probability measure μ on X for which E is not μ -hyperfinite. Then there is such a measure which is also E -ergodic.*

Proof. This follows from Theorems I.9 and J.8. \square

Theorem J.11. *Suppose that X is a standard Borel space and E is a countable Borel equivalence relation on X . Then exactly one of the following holds:*

- (1) *The relation E is measure reducible to \mathbb{E}_0 .*
- (2) *The set $\mathcal{EQ}_E \setminus \mathcal{H}_E$ is non-empty.*

Proof. To see (1) $\implies \neg(2)$, note that if E is measure reducible to \mathbb{E}_0 , then E is measure hyperfinite, by Theorem F.9. To see $\neg(1) \implies (2)$, note that if E is not measure reducible to \mathbb{E}_0 , then E is not measure hyperfinite, since Proposition D.3 ensures that \mathbb{E}_0 is not smooth, and Theorem F.8 implies that every measure hyperfinite Borel equivalence relation is measure reducible to every non-smooth Borel equivalence relation. Proposition I.5 therefore yields an E -quasi-invariant

Borel probability measure μ on X with respect to which E is not μ -hyperfinite, and Propositions I.3 and J.10 give rise to an E -ergodic such measure. \square

We close this section by considering preservation of μ -hyperfiniteness under Borel homomorphisms.

Proposition J.12. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X , $\phi: X \rightarrow Y$ is a Borel homomorphism from E to equality, μ is a Borel probability measure on X , $(\mu_y)_{y \in Y}$ is a Borel disintegration of μ through ϕ , and $E \upharpoonright \phi^{-1}(y)$ is μ_y -hyperfinite for $(\phi_*\mu)$ -almost every $y \in Y$. Then E is μ -hyperfinite.*

Proof. By Theorem J.8, the set $D = \{y \in Y \mid E \text{ is } \mu_y\text{-hyperfinite}\}$ is Borel, and there is a hyperfinite Borel equivalence relation F on X for which there is a Borel set $C \subseteq X$ such that $\mu_y(C) = 1$ and $E \upharpoonright C = F \upharpoonright C$ for all $y \in D$. Then $\mu(C) = 1$, so E is μ -hyperfinite. \square

Proposition J.13. *Suppose that X and Y are standard Borel spaces, E is a countable Borel equivalence relation on X , F is a hyperfinite Borel equivalence relation on Y , $\phi: X \rightarrow Y$ is a Borel homomorphism from E to F , μ is a Borel probability measure on X , $(\mu_y)_{y \in Y}$ is a Borel disintegration of μ through ϕ , and $E \upharpoonright \phi^{-1}(y)$ is μ_y -hyperfinite for $(\phi_*\mu)$ -almost every $y \in Y$. Then E is μ -hyperfinite.*

Proof. Fix an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite Borel equivalence relations on Y whose union is F . Proposition J.12 then ensures that each of the equivalence relations $E_n = E \cap (\phi \times \phi)^{-1}(F_n)$ is μ -hyperfinite. As $E = \bigcup_{n \in \mathbb{N}} E_n$, Theorem J.2 implies that E is μ -hyperfinite. \square

APPENDIX K. ACTIONS OF $\mathrm{SL}_2(\mathbb{Z})$

Let \sim denote the equivalence relation on $\mathbb{R}^2 \setminus \{(0, 0)\}$ given by

$$v \sim w \iff \exists r \in \mathbb{R} (r > 0 \text{ and } rv = w),$$

and let \mathbb{T} denote the corresponding quotient. Define $\mathrm{proj}_{\mathbb{T}}: \mathbb{R}^2 \rightarrow \mathbb{T}$ by $\mathrm{proj}_{\mathbb{T}}(v) = [v]_{\sim}$, and let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Proposition K.1 (Jackson-Kechris-Louveau). *The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is hyperfinite.*

Proof. See the remark following the proof of [JKL02, Lemma 3.6]. \square

Let $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ denote the group of all transformations $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form $T(x) = Ax + b$, where $A \in \mathrm{SL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$, under composition. Define $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ and $\mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})}: \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z})$

by $\text{proj}_{\text{SL}_2(\mathbb{Z})}(Ax + b) = A$, and let $\text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ denote the action induced by $\text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Proposition K.2. *Suppose that μ is the Borel probability measure on \mathbb{T}^2 induced by Lebesgue measure on \mathbb{R}^2 . Then the orbit equivalence relation $E_{\text{SL}_2(\mathbb{Z})}^{\mathbb{T}^2}$ is not μ -hyperfinite.*

Proof. As $\text{SL}_2(\mathbb{Z})$ is not amenable and [JKL02, Lemma 3.6] ensures that $\text{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is free off of a μ -null set, this is a consequence of Proposition J.1. \square

Proposition K.3. *The orbit equivalence relation $E_{\text{SL}_2(\mathbb{Z})}^{\mathbb{T}^2}$ is treeable.*

Proof. See [JKL02, Proposition 3.5]. \square

APPENDIX L. COMPLEXITY

The following fact summarizes the main results of [AK00].

Theorem L.1 (Adams-Kechris). *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X , $(E_r)_{r \in \mathbb{R}}$ is a Borel sequence of subequivalence relations of E , and $(\mu_r)_{r \in \mathbb{R}}$ is a Borel sequence of Borel probability measures on X such that:*

- (1) *Each μ_r is E_r -quasi-invariant and E_r -ergodic.*
- (2) *The relation E_r is μ_r -nowhere reducible to the relation E_s , for all distinct $r, s \in \mathbb{R}$.*

Then the following hold:

- (a) *There is an embedding of containment on Borel subsets of \mathbb{R} into Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E (in the codes).*
- (b) *Borel bi-reducibility and reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E are both Σ_2^1 -complete (in the codes).*
- (c) *Every Borel quasi-order is Borel reducible to Borel reducibility of countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E .*
- (d) *Borel and $\sigma(\Sigma_1^1)$ -measurable reducibility do not agree on the countable Borel equivalence relations with smooth-to-one Borel homomorphisms to E .*

Proof. The proof of [AK00, Theorem 4.1] yields (a), the proof of [AK00, Theorem 5.1] yields (b), the final paragraph of [AK00, §7] yields (c), and the proof of [AK00, Theorem 5.5] yields (d). \square

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