

An improvement of the Beck–Fiala theorem

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Abstract

In 1981 Beck and Fiala proved an upper bound for the discrepancy of a set system of degree d that is independent of the size of the ground set. In the intervening years the bound has been decreased from $2d - 2$ to $2d - 4$. We improve the bound to $2d - \log^* d$.

MSC CLASSES: 05D05, 11K38, 05C15

1 Introduction

Let X be a finite set, and let \mathcal{F} be a family of subsets of X . A *two-coloring* of X is a function $\chi: X \rightarrow \{-1, +1\}$. For $S \subset X$ we define $\chi(S) \stackrel{\text{def}}{=} \sum_{x \in S} \chi(x)$. The *discrepancy* of a coloring χ is

$$\text{disc } \chi \stackrel{\text{def}}{=} \max_{S \in \mathcal{F}} |\chi(S)|.$$

The discrepancy of \mathcal{F} is then defined as the discrepancy of an optimal coloring,

$$\text{disc } \mathcal{F} \stackrel{\text{def}}{=} \min_{\chi} \text{disc } \chi.$$

The *degree* of $x \in X$ in the family \mathcal{F} is the number of sets that contain x . Over 30 years ago Beck and Fiala [2] proved that if the maximum degree of vertices in \mathcal{F} is d , then $\text{disc } \mathcal{F} \leq 2d - 2$. The remarkable feature of the result is that it depends neither on the number of sets in \mathcal{F} nor on the size of X . If dependence on these quantities is permitted, the best result is due to Banaszczyk [1, Theorem 2], and asserts that $\text{disc } \mathcal{F} \leq c\sqrt{d \log |X|}$. Let $f(d) \stackrel{\text{def}}{=} \max \text{disc } \mathcal{F}$, where the maximum is taken over all set families of degree at most d . In [2] Beck and Fiala conjectured that $f(d) = O(\sqrt{d})$. If true, the conjecture would be a strengthening of Spencer’s six deviation theorem [8]. A related, but stronger conjecture was made by János Komlós [8, p. 680]. A relaxation of Komlós’s conjecture to vector-valued colorings has been established by Nikolov [7]. For a general overview of discrepancy theory see books [6, 4].

The original Beck–Fiala bound has been improved twice. First, Bednarchak and Helm [3] proved that $f(d) \leq 2d - 3$ for $d \geq 3$. Then Helm [5] claimed¹ that $f(d) \leq 2d - 4$ for all sufficiently large d . In this paper, we improve the bound by a function growing to infinity with d :

Theorem. *For all sufficiently large d we have*

$$f(d) \leq 2d - \log^* d.$$

Here $\log^* x \stackrel{\text{def}}{=} \min\{t : \log^{(t)} x \leq 1\}$, where $\log^{(1)} x = \log x$ and $\log^{(t+1)} x = \log^{(t)} \log x$, and the logarithms are to the base 2.

¹The author has been unable to understand Helm’s proof, or to reach Martin Helm for clarification.

2 Proof ideas

Like the proofs in [2, 3, 5], our proof uses the method of floating colors. In this section we present the method informally, and explain the main difficulty in its application. We present the three main ideas of our proof, and how these ideas address the difficulty. In following sections we give the proof in full.

A *floating coloring* is a function $\chi: X \rightarrow [-1, +1]$. The value $\chi(x)$ is the “color” of x according to χ . If $-1 < \chi(x) < +1$, we consider the color of x “floating”, whereas $\chi(x) \in \{-1, +1\}$ means that the color of x is “frozen”. Once frozen, the elements never change their color again. All the floating elements eventually turn frozen, giving a genuine two-coloring of X . Our goal is to ensure that the discrepancy of that coloring does not exceed $2d - \Delta$, where $2d - \Delta$ is the desired bound on $f(d)$.

The purpose of evolving coloring χ gradually is to focus on the dangerous sets. A set $S \in \mathcal{F}$ is *dangerous* if there is a way to freeze the floating elements to make the discrepancy exceed $2d - \Delta$. Only dangerous sets matter in the subsequent evolution of χ . In the argument of Beck and Fiala, an invariant is maintained: for each dangerous set $S \in \mathcal{F}$ we have $\chi(S) = 0$. *Size of a dangerous set* is the number of floating elements in the set. It is easily shown that if the average size of dangerous sets exceeds d , then the number of floating colors exceeds the number of dangerous sets; hence, there is a way to perturb the floating colors, in a manner that preserves the invariant. The coloring is perturbed until one of the floating elements becomes frozen, and the process repeats.

The dangerous sets of size at most d thus pose a natural problem. If not for them, the process would never stop, and the result is a coloring of discrepancy at most $2d - \Delta$. Let us call these particularly troublesome dangerous sets *nasty*. The sum of the floating elements of a nasty set is nearly $\pm d$, for otherwise the floating elements do not have enough “room” to change much. Hence, most elements of a nasty set are close to ± 1 . The *first idea* thus is to forcibly round elements x such that $|\chi(x)| > 1 - \alpha$, where α is a small number. Forcible rounding of only $O(\Delta)$ elements in a nasty set is enough to render the set benign.

Forcible rounding introduces an “error” of at most α into the invariant $\chi(S) = 0$. Therefore, forcible rounding is tolerable only if the rounded element is not contained in any very large set. The *second idea* consists in noting that if there are nasty sets, but no elements in them can be forcibly rounded, then the nasty sets must be highly overlapping. Indeed, it is possible to perturb χ if the average size of sets exceeds d , and the large sets that prevent forcible rounding contribute a lot to this average. Making this idea precise requires a charging argument, whose details are in section 5.

We take advantage of the overlap in the nasty sets by selecting an element b that is common to many nasty sets, and singling out the nasty sets containing b into a separate *cohort*. We call b the cohort’s *banner*. Since perturbation of χ might get stuck more than once, we might create several cohorts over the course of our algorithm. We shall treat each cohort as a fully autonomous set system. Thus it will be subject to its own invariants, and will impose its own linear conditions in the perturbation step.

Since cohorts consist of nasty sets, the average set size in a cohort is less than d , and so for the perturbation step to be possible, a cohort cannot impose as many linear conditions as there are sets in a cohort. The *third idea* is to use the few available linear conditions to render some sets in a cohort benign. The benign sets pose no threat, but still contribute elements towards the average set size in a cohort because they contain the banner. (At this point the term “average set size” becomes a

misnomer since in the formula $\frac{\text{number of elements}}{\text{number of sets}}$ the benign sets contribute to the numerator, but not to the denominator. However we will keep on using the term.)

To understand how linear conditions in a cohort work, we imagine that sets in a cohort engage in an elimination tournament. When two nasty sets S and S' are matched against one another instead of two invariants $\chi(S) = C$ and $\chi(S') = C'$, there is just one invariant $\chi(S) + \chi(S') = C + C'$. The match is declared finished when the total size of S and S' drops below d . The loser is the smaller of S and S' , for it can be shown that it became benign. The winner, on the other hand, might have even larger error in $\chi(S)$ after the end of the match. Fortunately for us, the winner also gets a virtual trophy — the banner of the loser. This means that for purposes of computing the average set size we can count the winner’s banner element twice. In general, a winner of D matches will have 2^D virtual banner elements.

The final obstacle is the possibility that the banner of a cohort might get frozen. In that case the defeated sets cease to contribute to the average set size, and the argument collapses. The rescue comes from the fact that when a cohort was formed the banner was an element satisfying $|\chi(b)| > 1 - \alpha$. So, by replacing the invariant $\chi(S) = C$ by $\chi(S) + \beta\chi(b) = \hat{C}$, where β is very large, we can ensure that b can be frozen only to the value that we originally attempted to round it to. So, the banner can get frozen only in the favourable direction. If that happens, the sets in the cohort become a bit less dangerous, but not yet benign; so we dissolve the cohort, and return the sets to the general pool, where they can become parts of new cohorts. After being in a cohort Δ times, a set is guaranteed to be benign, and so the error in χ cannot explode.

Despite its length, the sketch above misses a couple of crucial technical moments that can be seen only from details. Most crucially, it does not explain why the final bound, $\log^* d$, is pitifully tiny. It is because the winners not only accumulate virtual banners, but also lose some of their own elements to freezing.

It turns out that the banner accumulation happens faster, but the reason is subtle: Let R_D be the maximal possible error in χ in a set that defeated D sets. Crudely, we may think of R_D as the number of elements that a set lost due to freezing. One can show that it satisfies an approximate recurrence $R_{D+1} \approx 2R_D - 2^D$, where the term 2^D reflects 2^D virtual banners of a loser. Though R_D eventually becomes negative, it grows exponentially at first. In particular, a cohort might get disbanded when χ is exponential in R_0 . For any single set, this might happen up to Δ times, and so the error might grow to be a tower of height Δ .

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3 Two-coloring algorithm

Data. The only data that the Beck–Fiala algorithm remembers is the floating coloring $\chi: X \rightarrow [-1, +1]$. The data that our algorithm uses is more elaborate:

Data	Informal meaning
Function $\chi: X \rightarrow [-1, 1]$	Floating coloring
Partition of the family \mathcal{F} as $\mathcal{F} = \mathcal{B} \cup \mathcal{G} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$	\mathcal{B} contains benign sets \mathcal{G} is a general pool $\mathcal{C}_1, \dots, \mathcal{C}_m$ are cohorts
Elements $b_1, \dots, b_m \in X$	Banners for the cohorts
Signs $\varepsilon_1, \dots, \varepsilon_m \in \{-1, +1\}$	The color of banner b_i is close to ε_i
Integers $r_1, \dots, r_m \in \{0, \dots, \Delta - 1\}$	The larger r_i is, the less dangerous \mathcal{C}_i is
Set families M_1, \dots, M_m	M_i is a current matching in a cohort \mathcal{C}_i
Integer $D[S] \in \{0, 1, \dots\}$ for each $S \in \mathcal{C}_i$	Set S has defeated $D[S]$ sets in its cohort

Notation. To describe the algorithm, and the invariants that data satisfies, we need notation which we introduce now. An element $x \in X$ such that $\chi(x) \in \{-1, +1\}$ is called *frozen*; otherwise x is *floating*. With this in mind, here is our notation:

$\text{Sz}(S) \stackrel{\text{def}}{=} \sum_{x \in S \text{ floating}} 1$	number of floating elements in S ,
$\chi(S) \stackrel{\text{def}}{=} \sum_{x \in S} \chi(x)$	current color (discrepancy) of S ,
$\text{Fr}(S) \stackrel{\text{def}}{=} \sum_{x \in S \text{ frozen}} \chi(x)$	total color of frozen elements in S ,
$\text{Fl}(S) \stackrel{\text{def}}{=} \sum_{x \in S \text{ floating}} \chi(x)$	total color of floating elements in S ,
$\text{Th}_-(S) \stackrel{\text{def}}{=} \text{Sz}(S) - \text{Fr}(S)$	threat of negative discrepancy,
$\text{Th}_+(S) \stackrel{\text{def}}{=} \text{Sz}(S) + \text{Fr}(S)$	threat of positive discrepancy,
$\text{Th}(S) \stackrel{\text{def}}{=} \max(\text{Th}_-(S), \text{Th}_+(S))$	threat of discrepancy.

For later use we record two identities,

$$\text{Th}_-(S) = \text{Sz}(S) + \text{Fl}(S) - \chi(S), \quad (1)$$

$$\text{Th}_+(S) = \text{Sz}(S) - \text{Fl}(S) + \chi(S). \quad (2)$$

The algorithm proceeds in stages. To refer to the data at a particular stage we use superscripts. For example, $\text{Sz}^{(n)}(S)$ and $D^{(n)}[S]$ denote the values of $\text{Sz}(S)$ and $D[S]$ respectively at the n 'th stage of the algorithm. The notation for other quantities follows the same pattern.

Constants. In the proof we use several constants, which we introduce now. Their informal meanings appear to the right of their definitions.

$\Delta \stackrel{\text{def}}{=} \log^* d$	A set S is benign if $\text{Th}(S) \leq 2d - \Delta$,
$\alpha \stackrel{\text{def}}{=} \frac{1}{4}$	The threshold for rounding is $1 - \alpha$,
$\text{Tw}_r \stackrel{\text{def}}{=} \begin{cases} \Delta & \text{if } r \in \{0, 1\} \\ 2^{8\text{Tw}_{r-2}} & \text{otherwise} \end{cases}$	Tower function controlling blowup of Th ,
$\beta_r \stackrel{\text{def}}{=} 4\text{Tw}_r$	Clamping factor for cohorts with $r_i = r$,
$W \stackrel{\text{def}}{=} d/(64\Delta^2 \text{Tw}_{\Delta-1})$	Size of the newly-created cohorts,
$R_D \stackrel{\text{def}}{=} (D - 2)2^D - (2^D - 1)\Delta + 2$	This term controls $\chi(S)$ in a cohort. (3)

Of these constants, the most important is R_D . The whole proof is based on the fact that R_D is eventually larger than $C2^D$ for any constant C .

In what follows we assume that d is so large that $\Delta > 10$ holds.

The choice of constants is fairly flexible, and the chosen constants are far from unique or optimal. For example, any choice for α from Tw_Δ/d to a constant less than 1 would have worked (with minor changes in the other constants). To avoid burdening the main exposition, we record the needed inequalities between constants here (valid for all $r < \Delta$):

$$\begin{array}{ll}
\text{a) } \Delta \leq \text{Tw}_r & \text{b) } \text{Tw}_r + \Delta < \beta_r(1 - \alpha) \\
\text{c) } (\beta_r + 1)\alpha + \text{Tw}_r + \Delta \leq 4\text{Tw}_r & \text{d) } 16\text{Tw}_r \leq 2^{16\text{Tw}_{r-2}} \\
\text{e) } \text{Tw}_{\Delta-1} \leq \log \log d & \text{f) } 2\text{Tw}_{\Delta-1} + (2 - \alpha)\Delta \leq \alpha d \\
\text{g) } d \geq W\Delta(\Delta + 2\text{Tw}_{\Delta-1} + 2)(8 + 4\Delta) & \text{h) } 2^{8\text{Tw}_r + 6}\text{Tw}_r < W \\
\text{i) } (1 - \frac{1}{2}\alpha)(2d - \Delta) \geq (1 - \frac{1}{2}\alpha)d + 2^{4\text{Tw}_r + 1}\beta_r\alpha + 2^{4\text{Tw}_r + 1}\text{Tw}_r + 2^{4\text{Tw}_r}\Delta & \\
\text{j) } \text{Tw}_{r+2} \geq 2^{4\text{Tw}_r + 1}\beta_r\alpha + 2^{4\text{Tw}_r + 1}\text{Tw}_r + 2^{4\text{Tw}_r}\Delta &
\end{array} \tag{4}$$

Proofs: (a) is proved by induction on r ; (b) and (c) are implied by (a); (d) follows from $1 \leq \Delta$ and (a); (e) is true because $\log^*(16\text{Tw}_r) \leq r/2 + \log^*(16\Delta)$ by induction on r , and because $\Delta/2 + \log^*(16\Delta) \leq \Delta - 1$ for $\Delta > 10$; (f) follows from (e); (g) follows from (e) and the definition of W ; (h) is implied by (a) and (e); (i) follows from (a) and three uses of (e) to bound each of the summands on the right; (j) is a consequence of (a) and the definition of Tw_r . \square

Invariants. The data satisfies the following fourteen invariants. Note that the invariants are symmetric with respect to flipping the coloring, i.e., with respect to the inversion $\chi \mapsto -\chi$, $\varepsilon_i \mapsto -\varepsilon_i$. Our algorithm is also symmetric in this sense. We suggest that the reader examine invariants 1 to 7, and refer to other invariants later.

Invariant 1: If $x \in X$ is frozen in coloring $\chi^{(i)}$, then it is also frozen in $\chi^{(j)}$ for all $j > i$.

Invariant 2: Every set $S \in \mathcal{B}$ satisfies $\text{Th}(S) \leq 2d - \Delta$.

Invariant 3: Every set $S \in \mathcal{F}$ satisfying $\text{Sz}(S) \leq d$ also satisfies $\text{Th}(S) \leq 2d$.

Invariant 4: Element b_i is common to all the sets in \mathcal{C}_i .

Invariant 5: Family M_i is a matching on \mathcal{C}_i , i.e., M_i consists of pairs of sets from \mathcal{C}_i , and the pairs in M_i are disjoint. Note that the matching is not perfect: some sets in \mathcal{C}_i might be unmatched.

Invariant 6: If $\{S, S'\} \in M_i$, then $D[S] = D[S']$.

Invariant 7: If $S \in \mathcal{C}_i$, then $\text{Sz}(S) \leq d + 1 - 2^{D[S]}$.

Invariant 8⁺: If $S \in \mathcal{C}_i$, and $\varepsilon_i = +1$ then $\text{Th}_+(S) \leq \Delta + 2 - 2^{D[S]+1}$.

Invariant 8⁻: If $S \in \mathcal{C}_i$, and $\varepsilon_i = -1$ then $\text{Th}_-(S) \leq \Delta + 2 - 2^{D[S]+1}$.

Invariant 9: If element $b \in X$ is a banner of k cohorts $\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_k}$, then b is contained in at least $\sum_{j=1}^k (W - |\mathcal{C}_{i_j}|)$ sets of \mathcal{B} .

Invariant 10: For each cohort \mathcal{C}_i we have $\sum_{S \in \mathcal{C}_i} 2^{D[S]} \leq W$.

Invariant 11⁺: If $S \in \mathcal{G}$ and $\text{Th}_+(S) = 2d - r$, then

$$\begin{aligned} \chi(S) &\leq 0 && \text{if } r < \Delta \text{ and } \text{Sz}(S) \geq d, \\ \chi(S) &\leq \text{Tw}_r - \frac{1}{2}\alpha \text{Th}_-(S) && \text{if } r < \Delta \text{ and } \text{Sz}(S) \leq d. \end{aligned}$$

Invariant 11⁻: If $S \in \mathcal{G}$ and $\text{Th}_-(S) = 2d - r$, then

$$\begin{aligned} -\chi(S) &\leq 0 && \text{if } r < \Delta \text{ and } \text{Sz}(S) \geq d, \\ -\chi(S) &\leq \text{Tw}_r - \frac{1}{2}\alpha \text{Th}_+(S) && \text{if } r < \Delta \text{ and } \text{Sz}(S) \leq d. \end{aligned}$$

Invariant 12⁻: If $\varepsilon_i = -1$, and a set $S \in \mathcal{C}_i$ is not in any edge of M_i , and $D = D[S]$ then

$$\chi(S) + 2^D \beta_{r_i} (\chi(b_i) + 1 - \alpha) \leq 2^D \text{Tw}_{r_i} - \frac{1}{2}\alpha \text{Th}_-(S) - R_D.$$

Invariant 12⁺: If $\varepsilon_i = +1$, and a set $S \in \mathcal{C}_i$ is not in any edge of M_i , and $D = D[S]$ then

$$-\chi(S) - 2^D \beta_{r_i} (\chi(b_i) - 1 + \alpha) \leq 2^D \text{Tw}_{r_i} - \frac{1}{2}\alpha \text{Th}_+(S) - R_D.$$

Invariant 13⁻: If $\varepsilon_i = -1$, and $\{S, S'\} \in M_i$, and $D = D[S] = D[S']$ then

$$\chi(S) + \chi(S') + 2^{D+1} \beta_{r_i} (\chi(b_i) + 1 - \alpha) \leq 2^{D+1} \text{Tw}_{r_i} - \frac{1}{2}\alpha (\text{Th}_-(S) + \text{Th}_-(S')) - 2R_D.$$

Invariant 13⁺: If $\varepsilon_i = +1$, and $\{S, S'\} \in M_i$, and $D = D[S] = D[S']$ then

$$-\chi(S) - \chi(S') - 2^{D+1} \beta_{r_i} (\chi(b_i) - 1 + \alpha) \leq 2^{D+1} \text{Tw}_{r_i} - \frac{1}{2}\alpha (\text{Th}_+(S) + \text{Th}_+(S')) - 2R_D.$$

Invariant 14: If $S \in \mathcal{C}_i$, then $D[S] \leq 4 \text{Tw}_{r_i}$.

Initialization of the algorithm. At start, we set $\chi = 0$, $\mathcal{B} = \emptyset$, $\mathcal{G} = \mathcal{F}$, $\mathcal{C} = \emptyset$ and $m = 0$.

Steps of the algorithm. The algorithm of Beck–Fiala makes progress in two ways: by using linear perturbation of the current floating coloring, and by discarding the benign sets. Our algorithm uses nine ways to make progress. We refer to these ways as *steps*. For each step, there is a condition that must hold for it to be executed. The steps are ordered, and the choice of a step to be executed is greedy: we always execute the first step whose condition is satisfied. For example, step 5 is executed only if steps 1 through 4 cannot be executed.

Step 1 (Moving benign sets from \mathcal{G} to \mathcal{B}):

EXECUTION CONDITION: There is an $S \in \mathcal{G}$ such that $\text{Th}(S) \leq 2d - \Delta$.

STEP DESCRIPTION: Move S from \mathcal{G} to \mathcal{B} .

Step 2 (Removing empty cohorts):

EXECUTION CONDITION: There is a cohort \mathcal{C}_i such that $\mathcal{C}_i = \emptyset$.

STEP DESCRIPTION: Remove cohort \mathcal{C}_i by deleting \mathcal{C}_i , b_i , ε_i , r_i and M_i and renumbering the remaining cohorts appropriately.

Step 3 (Rounding elements that are safe to round):

EXECUTION CONDITION: One of the following two conditions holds:

- +) There is an $x \in X$ such that $\chi(x) > 1 - \alpha$ and there exists no $S \in \mathcal{G}$ containing x that satisfies $\text{Sz}(S) \geq d + 1$ and $\text{Th}_+(S) > 2d - \Delta$.
-) There is an $x \in X$ such that $-\chi(x) > 1 - \alpha$ and there exists no $S \in \mathcal{G}$ containing x that satisfies $\text{Sz}(S) \geq d + 1$ and $\text{Th}_-(S) > 2d - \Delta$.

STEP DESCRIPTION: In case (+), set $\chi(x)$ equal to $+1$. In case (-), set $\chi(x)$ equal to -1 .

Step 4 (Moving benign cohort sets to \mathcal{B}):

EXECUTION CONDITION: There is an $S \in \mathcal{C}_i$ that is not a part of M_i and such that $\text{Th}(S) \leq 2d - \Delta$.

STEP DESCRIPTION: Move S to \mathcal{B} .

Step 5 (Disbanding cohorts whose banners were frozen):

EXECUTION CONDITION: For some i we have $\chi(b_i) \in \{-1, +1\}$.

STEP DESCRIPTION: Disband cohort \mathcal{C}_i by moving all sets in \mathcal{C}_i to \mathcal{G} , and then removing cohort as in step 2.

Step 6 (Declaring some matches finished):

EXECUTION CONDITION: There is an edge $\{S, S'\} \in M_i$ with $D = D[S] = D[S']$ such that

$$\text{Sz}(S) + \text{Sz}(S') + 2^{D+1} - 2 \leq d. \quad (5)$$

STEP DESCRIPTION: Without loss of generality, $\text{Sz}(S) \geq \text{Sz}(S')$. Perform the next three actions:

- 1) Remove edge $\{S, S'\}$ from M_i .
- 2) Move S' to \mathcal{B} .
- 3) If $\text{Th}(S) \leq 2d - \Delta$, move S to \mathcal{B} as well. Otherwise, increment $D[S]$.

Step 7 (Matching unmatched sets in a cohort):

EXECUTION CONDITION: There are sets $S, S' \in \mathcal{C}_i$ that are not part of M_i and such that $D[S] = D[S']$.

STEP DESCRIPTION: Add edge $\{S, S'\}$ to M_i .

Step 8 (Linear perturbation):

EXECUTION CONDITION: Let $N = |\mathcal{G}| + \sum_{i=1}^m (|\mathcal{C}_i| - |M_i|)$. Execute this step only if the number of floating elements exceeds N .

STEP DESCRIPTION: We will generate a set E of N linear equations in values of an unknown function $\tau: X \rightarrow \mathbb{R}$. The equations will have the property that the current floating coloring χ satisfies them all.

- Each $S \in \mathcal{G}$ generates the equation $\tau(S) = \chi(S)$.
- Each $S \in \mathcal{C}_i$ that is not in M_i generates the equation

$$\tau(S) + 2^D \beta_{r_i} \tau(b_i) = \chi(S) + 2^D \beta_{r_i} \chi(b_i).$$

- Each edge $\{S, S'\} \in M_i$ generates the equation

$$\tau(S) + \tau(S') + 2^{D+1} \beta_{r_i} \tau(b_i) = \chi(S) + \chi(S') + 2^{D+1} \beta_{r_i} \chi(b_i).$$

Let E be the resulting set of equations. Note that $|E| = N$. Let

$$A \stackrel{\text{def}}{=} \{ \tau : \tau \text{ satisfies all of } E, \text{ and } \tau(x) = \chi(x) \text{ whenever } \chi(x) \in \{-1, +1\} \}$$

Since the number of floating elements exceeds N , set A is an affine space of positive dimension. Since A contains χ , it also must contain a point $\tau \in [-1, +1]^X$ in which more elements are frozen than in the current value of χ . Set χ to that τ .

Step 9 (Creating a new cohort):

EXECUTION CONDITION: Not all sets are in \mathcal{B} , i.e., $\mathcal{F} \neq \mathcal{B}$.

STEP DESCRIPTION: In section 5 we will show that this step is executed only if there is a $b \in X$, and a family $\mathcal{D} \subset \mathcal{G}$ of size $|\mathcal{D}| = W$ and a number $r < \Delta$ such that one of the following holds.

- +) We have $\chi(b) > 1 - \alpha$, and each $S \in \mathcal{D}$ satisfies $b \in S$, $\text{Th}_-(S) = 2d - r$ and $\text{Sz}(S) \leq d$.
-) We have $-\chi(b) > 1 - \alpha$, and each $S \in \mathcal{D}$ satisfies $b \in S$, $\text{Th}_+(S) = 2d - r$ and $\text{Sz}(S) \leq d$.

Create a new empty cohort \mathcal{C}_{m+1} , move all sets in \mathcal{D} from \mathcal{G} to \mathcal{C}_{m+1} , and set $b_{m+1} = b$, $\varepsilon_{m+1} = \text{sign } \chi(b)$, $r_{m+1} = r$, $M_{m+1} = \emptyset$ and $D[S] = 0$ for all $S \in \mathcal{C}_{m+1}$.

4 Proof of the algorithm's correctness

In this section we show that each step of the algorithm preserves all the invariants enumerated in the previous section. We also show that the algorithm terminates, and that its termination implies that the discrepancy of the set family \mathcal{F} is at most $2d - \Delta$.

In the proofs that follow we use several consequences of the invariants that we stated. We record these consequences now.

Lemma 1. *If invariant 1 holds, then $\text{Th}_+^{(j)}(S)$ and $\text{Th}_-^{(j)}(S)$ are non-increasing functions of j .*

Proof. Suppose $j < k$. Let $r = \text{Sz}^{(j)}(S) - \text{Sz}^{(k)}(S)$ be the number of elements frozen between stages j and k . Then $\text{Fr}^{(k)} \leq \text{Fr}^{(j)} + r$. So $\text{Th}_+^{(k)}(S) = \text{Sz}^{(k)}(S) + \text{Fr}^{(k)}(S) \leq (\text{Sz}^{(j)}(S) - r) + (\text{Fr}^{(j)}(S) + r)$. The proof for Th_- is similar. \square

Lemma 2. *If $S \in \mathcal{F}$ satisfying $\text{Sz}(S) \leq d$ also satisfies $\text{Th}_+(S) > 2d - \Delta$, then $\text{Th}_-(S) < \Delta$. Conversely, if $\text{Th}_-(S) > 2d - \Delta$, then $\text{Th}_+(S) < \Delta$.*

Proof. This follows from $\text{Th}_+(S) + \text{Th}_-(S) = 2\text{Sz}(S)$. \square

Lemma 3. *If invariant 1 holds, then $\text{Th}_+(S) \leq 2d - \Delta$ whenever $S \in \mathcal{C}_i$ and $\varepsilon_i = +1$. Similarly, if invariant 1 holds, then $\text{Th}_-(S) \leq 2d - \Delta$ whenever $S \in \mathcal{C}_i$ and $\varepsilon_i = -1$.*

Proof. By symmetry, we may assume $\varepsilon_i = +1$. In view of lemma 1, it suffices to consider only the stage when S is added to a cohort at step 9. At that stage, lemma 2 applies. \square

Lemma 4. *Suppose invariants 8^+ and 8^- hold. If $S \in \mathcal{C}_i$, then*

$$\varepsilon_i \cdot \chi(S) - \gamma \text{Th}_{\varepsilon_i}(S) \leq \Delta + 2 - 2^{D[S]+1}$$

for all $0 \leq \gamma \leq 1$.

Proof. Assume $\varepsilon_i = +1$ by symmetry. We have

$$\begin{aligned} \chi(S) - \gamma \text{Th}_+(S) &= (1 - \gamma)\chi(S) - \gamma(\text{Sz}(S) - \text{Fl}(S)) && \text{using identity (2)} \\ &\leq (1 - \gamma)\text{Th}_+(S) && \text{due to } \chi \leq \text{Th}_+ \text{ and } \text{Fl} \leq \text{Sz} \\ &\leq (1 - \gamma)(\Delta + 2 - 2^{D[S]+1}) && \text{because of invariant } 8^+. \end{aligned} \quad \square$$

Lemma 5. *If all the invariants hold and cohort \mathcal{C}_i is non-empty, then $\text{sign } \chi(b_i) = \varepsilon_i$.*

Proof. Consider the case $\varepsilon_i = +1$, the other case is analogous. If M_i is non-empty, let $\{S, S'\}$ be any edge in M_i . If M_i is empty, let S be any set in \mathcal{C}_i , and put $S' = S$. In either case, invariants 12^+ and 13^+ imply that

$$-\chi(S) - \chi(S') - 2^{D+1}\beta_{r_i}(\chi(b_i) - 1 + \alpha) \leq 2^{D+1}\text{Tw}_{r_i} - \frac{1}{2}\alpha(\text{Th}_+(S) + \text{Th}_+(S')) - 2R_D.$$

By the preceding lemma with $\gamma = \frac{1}{2}\alpha$ and a bit of algebra, where nearly all terms cancel, it follows that

$$-\beta_{r_i}(\chi(b_i) - 1 + \alpha) \leq \text{Tw}_{r_i} - D + \Delta,$$

and the lemma follows since $\text{Tw}_{r_i} - D + \Delta < \beta_{r_i}(1 - \alpha)$ by inequality (4b). \square

Lemma 6. *If invariants 7, 12^- and 12^+ hold, then for every $S \in \mathcal{C}_i$ that is not in M_i and satisfying $\text{Th}(S) > 2d - \Delta$, invariant 14 holds.*

Proof. Say $\varepsilon_i = +1$, the other case being symmetric. For brevity write $D = D[S]$ and $r = r_i$. Lemma 3 asserts that $\text{Th}_+(S) \leq 2d - \Delta$, and so $\text{Th}(S) > 2d - \Delta$ implies that

$$\text{Th}_-(S) > 2d - \Delta. \quad (6)$$

We thus have

$$\begin{aligned}
0 &\leq 2^D \beta_r \alpha + 2^D \text{Tw}_r - R_D + \chi(S) - \frac{1}{2} \alpha \text{Th}_+(S) && \text{from invariant 12}^+ \\
&= 2^D \beta_r \alpha + 2^D \text{Tw}_r - R_D + (1 - \alpha) \text{Sz}(S) + \text{Fl}(S) - (1 - \frac{1}{2} \alpha) \text{Th}_-(S) && \text{using identities (1) and (2)} \\
&\leq 2^D \beta_r \alpha + 2^D \text{Tw}_r - R_D + (2 - \alpha) \text{Sz}(S) - (1 - \frac{1}{2} \alpha)(2d - \Delta) && \text{from } \text{Fl}(S) \leq \text{Sz}(S) \text{ and (6)} \\
&\leq 2^D \beta_r \alpha + 2^D \text{Tw}_r - R_D + (2 - \alpha)(d + 1 - 2^D) - (1 - \frac{1}{2} \alpha)(2d - \Delta) && \text{from invariant 7} \\
&= 2^D (\beta_r \alpha + \text{Tw}_r + \Delta + \alpha - D) - \frac{1}{2} \alpha \Delta - \alpha && \text{from (3) and some algebra}
\end{aligned}$$

and so $D \leq 4 \text{Tw}_r$ by inequality (4c). Hence invariant 14 holds. \square

It is easy to check that the algorithm satisfies all the invariants at the initialization stage. Only invariants 11⁺ and 11⁻ require an invocation of lemma 2 and of inequality (4a); the other invariants are immediate.

In what follows we assume that the algorithm satisfies all the invariants at stage $n - 1$, and that our goal is to show that the algorithm satisfies them at stage n . For brevity, we write $\text{Sz}(S)$, $\text{Fr}(S)$ etc in place of $S^{(n-1)}(S)$, $\text{Fr}^{(n-1)}(S)$ etc. We still write $\text{Sz}^{(n)}(S)$ etc in full.

Verification of invariant 1: The only steps that modify χ are steps 3 and 8. They do not unfreeze any elements.

Verification of invariant 2: The only steps that move sets into \mathcal{B} are steps 1, 4 and 6. The moves in steps 1 and 4 are preconditioned on $\text{Th}(S) \leq 2d - \Delta$, and so invariant 2 holds trivially. The only potential problem is the movement of S' to \mathcal{B} in step 6, which we now tackle.

Assume that $\varepsilon_i = +1$, the other case is symmetric. Since $\text{Sz}(S') \leq \text{Sz}(S)$, it follows from the execution condition of step 6 that $\text{Sz}(S') \leq d/2$. By lemma 3, $\text{Th}_+(S') \leq 2d - \Delta$, so it remains to prove that $\text{Th}_-(S') \leq 2d - \Delta$.

Use of the identity (1), inequality $\text{Fl}(S') \leq \text{Sz}(S')$, and lemma 4 in that order, yields

$$\text{Th}_-(S') \leq 2 \text{Sz}(S') - \chi(S') \leq 2 \text{Sz}(S') - \chi(S') - \chi(S) + \frac{1}{2} \alpha \text{Th}_+(S) + \Delta + 1 - 2^D.$$

Applying invariant 13⁺ and using $\text{Th}_+(S') + \text{Th}_-(S') = 2 \text{Sz}(S')$ gives

$$\begin{aligned}
\text{Th}_-(S') &\leq 2 \text{Sz}(S') - \frac{1}{2} \alpha \text{Th}_+(S') + 2^{D+1} \beta_{r_i} \alpha + 2^{D+1} \text{Tw}_{r_i} - 2R_D + \Delta \\
&= (2 - \alpha) \text{Sz}(S') + \frac{1}{2} \alpha \text{Th}_-(S') + 2^{D+1} \beta_{r_i} \alpha + 2^{D+1} \text{Tw}_{r_i} - 2R_D + \Delta.
\end{aligned}$$

The invariant 2 then follows by using $\text{Sz}(S') \leq d/2$, invariant 14 and inequality (4i).

Verification of invariant 3: Since $\text{Th}(S)$ is non-increasing by lemma 1, only the steps that might decrease $\text{Sz}(S)$ need to be examined. These are steps 3 and 8. So, let us consider a set S such that $\text{Sz}(S) > d$ and $\text{Sz}^{(n)}(S) \leq d$.

We will prove that $\text{Th}_+^{(n)}(S) \leq 2d$ (the other case is similar). We may assume that $\text{Th}_+(S) > 2d$, for else we are done by lemma 1. By invariant 11⁺, $\chi(S) \leq 0$. Furthermore, for step 8 we have $\chi^{(n)}(S) = \chi(S)$, whereas for step 3 we have $\chi^{(n)}(S) \leq \chi(S) + \alpha$. Hence, identity (2) implies

$$\text{Th}_+^{(n)}(S) = \text{Sz}^{(n)}(S) - \text{Fl}^{(n)}(S) + \chi^{(n)}(S) \leq 2 \text{Sz}^{(n)}(S) + \alpha \leq 2d + \alpha.$$

Since $\text{Th}_+^{(n)}(S)$ is an integer, we conclude that $\text{Th}_+^{(n)}(S) \leq 2d$ as desired.

Verification of invariant 4: The only step that adds sets to a cohort, or modifies a cohort's banner is step 9. However, that step clearly respects invariant 4.

Verification of invariant 5: The only step that adds edges to M_i is step 7. It adds edges only between unmatched sets.

Verification of invariant 6: The only step that adds edges to M_i is step 7. It adds edges only between sets with equal value of D . The only steps that change value of $D[S]$ are steps 6 and 9. Step 6 changes $D[S]$ only after having removed the edge that contains S from M_i . Step 9 is not a problem either, as the cohort that it creates has empty matching.

Verification of invariant 7: In view of invariant 1, $\text{Sz}(S)$ can only decrease. Thus, it suffices to verify only the steps that either increase the value of $D[S]$ or add sets to \mathcal{C}_i . These are steps 6 and 9 respectively.

Step 9 adds sets with $\text{Sz}(S) \leq d$ and assigns $D[S] = 0$ for them, satisfying invariant 7.

Consider step 6. Since it is that step which is executed, and not step 5, it follows that b_i is not frozen, and so $\text{Sz}(S') \geq 1$. Hence, from the inequality (5) we have

$$\text{Sz}(S) \leq d + 1 - 2^{D[S]+1} = d + 1 - 2^{D^{(n)}[S]}.$$

Verification of invariants 8^+ and 8^- : By lemma 1 it suffices to check only the steps that move sets to \mathcal{C}_i or increase the value of $D[S]$. These are steps 6 and 9.

By symmetry it suffices to treat only invariant 8^+ . Note that both in step 6 and in step 9, the sets S for which we need to establish invariant 8^+ satisfy $\text{Th}_-^{(n)}(S) > 2d - \Delta$. Hence,

$$\text{Th}_+^{(n)}(S) = \text{Sz}^{(n)}(S) + \text{Fr}^{(n)}(S) = 2\text{Sz}^{(n)}(S) - \text{Th}_-^{(n)}(S) < 2\text{Sz}^{(n)}(S) - 2d + \Delta,$$

and invariant 8^+ follows from the validity of invariant 7 at stage n , which was proved immediately above.

Verification of invariant 9: We need to check only the steps that remove sets from \mathcal{C}_i , as well as creation of new cohorts in step 9. The latter is trivial since $|\mathcal{D}| = W$. The only steps that remove sets from \mathcal{C}_i are steps 4, 5 and 6. Of these, steps 4 and 6 move sets from \mathcal{C}_i to \mathcal{B} , thus preserving invariant 9. Step 5 removes sets to \mathcal{G} , but also disbands the cohort. The fact that the step preserves invariant 9 follows from $|\mathcal{C}_i| \leq W$, which is a consequence of invariant 10.

Verification of invariant 10: We need to check only the steps that either increase the value of $D[S]$ or add sets to \mathcal{C}_i . These are steps 6 and 9. For step 6 we note that $D[S] = D[S']$ by invariant 6, and so $2^{D^{(n)}(S)} = 2^{D[S]+1} = 2^{D[S]} + 2^{D[S']}$. For step 9 the invariant follows from $|\mathcal{D}| = W$ and $D[S] = 0$.

Verification of invariants 11^+ and 11^- : We only treat invariant 11^+ , for invariant 11^- is symmetric. We need to check only the steps that modify χ or move sets to \mathcal{G} . These are steps 3, 5 and 8. Let $S \in \mathcal{F}$ be arbitrary, and let us check that the invariant holds for S .

Step 8 does not change $\chi(S)$, and may only decrease $\text{Th}_+(S)$ and $\text{Th}_-(S)$, by lemma 1. So, if $\text{Sz}(S), \text{Sz}^{(n)}(S)$ are either both greater than d , or are both at most d , then the invariant holds at stage n because it held at stage $n - 1$. So, consider the case when $\text{Sz}(S) > d$ and $\text{Sz}^{(n)}(S) \leq d$. Note that since $\text{Sz}(S) > d$ and invariant 11^+ held at stage $n - 1$, it follows that $\chi(S) \leq 0$. If $\text{Th}_+^{(n)}(S) \leq 2d - \Delta$ then invariant 11^+ holds vacuously. Otherwise, $\text{Th}_-^{(n)}(S) < \Delta$ by lemma 2. Thus $\text{Tw}_r - \frac{1}{2}\alpha \text{Th}_-^{(n)}(S) \geq 0$ by inequality (4a). Since $\chi^{(n)}(S) = \chi(S) \leq 0$, invariant 11^+ holds.

Step 3 alters $\chi(S)$, $\text{Sz}(S)$, $\text{Th}_-(S)$ or $\text{Th}_+(S)$ only if it rounds an element x that is in S . So, assume $x \in S$. There are two cases according to the sign of $\chi(x)$:

- +) If $\chi(x) > 1 - \alpha$, then for the condition of step 3 to have triggered, we must have either $\text{Th}_+(S) \leq 2d - \Delta$ or $\text{Sz}(S) \leq d$. In the former case, $\text{Th}_+^{(n)}(S) \leq \text{Th}_+(S)$ by lemma 1, and so invariant 11^+ holds vacuously. In the latter case, $\chi^{(n)}(S) \leq \chi(S) + \alpha$, $\text{Th}_+^{(n)}(S) = \text{Th}_+(S)$ and $\text{Th}_-^{(n)}(S) = \text{Th}_-(S) - 2$, and the invariant is verified by substitution.
-) If $-\chi(x) > 1 - \alpha$, then $\chi^{(n)}(S) < \chi(S)$, and $\text{Th}_+^{(n)}(S) = \text{Th}_+(S) - 2$ and $\text{Th}_-^{(n)}(S) = \text{Th}_-(S)$. As in the treatment of step 8 above, the only case worthy of attention is $\text{Sz}(S) = d + 1$, $\text{Sz}^{(n)}(S) = d$, and the same argument as above disposes of it.

We treat step 5 next. Suppose cohort \mathcal{C}_i is being dissolved, and $S \in \mathcal{C}_i$ is an arbitrary set in it. If $\varepsilon_i = +1$, then by lemma 3 invariant 11^+ holds vacuously. So, assume $\varepsilon_i = -1$. If S is not a part of M_i , declare $S' = S$, otherwise let $\{S, S'\} \in M_i$ be the edge containing S . In both cases we conclude (either from invariant 12^- or invariant 13^-) that

$$\chi(S) + \chi(S') + 2^{D+1}\beta_{r_i}(\chi(b_i) + 1 - \alpha) \leq 2^{D+1}\text{Tw}_{r_i} - \frac{1}{2}\alpha(\text{Th}_-(S) + \text{Th}_-(S')) - 2R_D.$$

An application of lemma 4 to $-\chi(S') - \frac{1}{2}\alpha \text{Th}_-(S')$ gives

$$\begin{aligned} \chi(S) &\leq 2^{D+1}\beta_{r_i}\alpha + 2^{D+1}\text{Tw}_{r_i} + \Delta + 2 - 2^{D+1} - 2R_D - \frac{1}{2}\alpha \text{Th}_-(S) \\ &\leq \text{Tw}_{r_i+2} - \frac{1}{2}\alpha \text{Th}_-(S), \end{aligned} \tag{7}$$

with the last line holding because of inequality (4j) and invariant 14.

Let j be the stage when cohort \mathcal{C}_i was created. By the description of step 9 we had $\text{Th}_+^{(j)}(S) = 2d - r_i$ and banner b_i was a floating element. So, since $\chi(b_i) = -1$ by lemma 5, we conclude that $\text{Th}_+(S) \leq 2d - r_i - 2$. In view of inequality (7) above, this implies that invariant 11^+ holds.

Verification of invariants 12^- and 12^+ : We treat invariant 12^- , for invariant 12^+ is symmetric. We need to verify the steps that remove sets from M_i , modify $D[S]$ or χ , or add sets to \mathcal{C}_i . These are steps 3, 6, 8 and 9. We handle them in (reverse) order.

Validity of invariant 12^- after step 9 follows from $\chi(b_i) + 1 - \alpha \leq 0$, $R_0 = 0$, $\text{Sz}(S) \leq d$ and the validity of invariant 11^+ before the step.

Step 8 does not change the left-hand side of the inequality in invariant 12⁻, and might only increase the right-hand side (by decreasing Th₋(S)).

We treat step 6 next. We need to consider only the case Th(S) > 2d - Δ, for otherwise the invariant holds vacuously. Let D = D[S]. Our goal is to bound

$$Q \stackrel{\text{def}}{=} \chi^{(n)}(S) + 2^{D^{(n)}[S]} \beta_{r_i}(\chi(b_i) + 1 - \alpha) = \chi(S) + 2^{D+1} \beta_{r_i}(\chi(b_i) + 1 - \alpha).$$

Since invariant 13⁻ held at stage n - 1, we conclude that

$$Q \leq 2^{D+1} \text{Tw}_{r_i} - \frac{1}{2} \alpha (\text{Th}_-(S) + \text{Th}_-(S')) - \chi(S') - 2R_D \quad \text{by invariant 13}^-.$$

Hence, it suffices to prove that $-\frac{1}{2} \alpha \text{Th}_-(S') - \chi(S') \leq 2R_D - R_{D+1}$. This follows from lemma 4 and from $2R_D - R_{D+1} = \Delta + 2 - 2^{D+1}$. We are thus done with step 6.

Finally, we check step 3. We may assume that x, the element that is rounded, is in S. If $\chi(x) < 1 - \alpha$, then rounding decreases left-hand side of the inequality in invariant 12⁻, and does not affect the right-hand side at all. So, we may assume that $\chi(x) > 1 - \alpha$. By lemma 5 $\chi(b_i) < 0$, and so $x \neq b_i$. The rounding thus increases the left-hand side of the inequality in invariant 12⁻ by at most α. Since the right-hand side of the inequality increases by exactly α, we are done.

Verification of invariants 13⁻ and 13⁺: We need to check only steps 3, 7 and 8 as these are the only steps that either change χ or create an edge in M_i. The verification of steps 3 and 8 is an almost verbatim repetition of the verification of invariants 12⁻ and 12⁺ for those steps, and we omit it. The validity of invariants 13⁻ and 13⁺ after step 7 follows from the validity of invariants 12⁻ and 12⁺ respectively before the step.

Verification of invariant 14: We need to consider only step 6 as it is the only step that increases the value of D[S]. Since invariants 7, 12⁻ and 12⁺ have been proved above, the invariant 14 follows from lemma 6.

Proof that the algorithm terminates: Let F be the number of frozen elements, and consider the quantity

$$I \stackrel{\text{def}}{=} F + 4|\mathcal{B}| - m + \sum_{i=1}^m (|M_i| + |\mathcal{C}_i|).$$

We claim that each step other than step 5 increases I. Step 1 increases |B|; step 2 decreases m; step 3 increases F; step 4 increases 4|B| by 4 and decreases |C_i| by 1; step 6 increases 4|B| by at least 4, decreases |M_i| by 1, and decreases |C_i| by at most 2; step 7 increases |M_i|; step 8 increases F; step 9 increases $\sum_i |\mathcal{C}_i|$ by W and increases m by 1.

Step 5 is executed only if there are no empty cohorts (step 2 cannot be executed), and so $-m + \sum |\mathcal{C}_i| \geq 0$. Hence, after step 5 is executed, I is nonnegative. Since other steps increase I, and I is bounded by |X| + 4|F|, it follows that step 5 must be executed at least once every |X| + 4|F| steps.

After a cohort is disbanded in step 5, no new cohort with the same banner can be created, as banner is a floating element. Furthermore, since any cohort disbanded in step 5 is non-empty, that

step can be executed at most d times for any given value of the banner. So, step 5 can be executed total of at most $d|X|$ times. In particular, the algorithm terminates after at most $d|X|(|X| + 4|\mathcal{F}|)$ steps.

Since step 9 is executed unless $\mathcal{F} = \mathcal{B}$, when the algorithm terminates we have $\text{Th}(S) \leq 2d - \Delta$ for all $S \in \mathcal{F}$. While it does not mean that the final coloring χ takes only values -1 and $+1$, it does imply that no matter how we round the remaining floating elements, the resulting coloring will have discrepancy at most $2d - \Delta$.

5 Creation of a new cohort

In this section we prove a claim made in step 9, namely, that if that step can be executed, then there is a family $\mathcal{D} \subset \mathcal{G}$ all of whose sets contain a common element $b \in X$, and that satisfies the right conditions for making a new cohort. In what follows, we assume that none of the steps 1 through 8 can be executed, and that all the invariants hold.

Lemma 7. *For each cohort \mathcal{C}_i we have*

$$d(|\mathcal{C}_i| - |M_i|) < W + \sum_{S \in \mathcal{C}_i} (\text{Sz}(S) - 1). \quad (8)$$

Proof. By symmetry assume $\varepsilon_i = +1$.

Because $D[S] = D[S']$ for $\{S, S'\} \in M_i$ by invariant 6 and because step 6 cannot be executed, we have

$$(\text{Sz}(S) - 1) + (\text{Sz}(S') - 1) \geq d - 2^{D[S]} - 2^{D[S']} + 1 \quad \text{for all } \{S, S'\} \in M_i. \quad (9)$$

Let $S \in \mathcal{C}_i \setminus M_i$ be any unmatched cohort set, and let $D = D[S]$ for brevity. Because step 4 cannot be executed, we necessarily have $\text{Th}(S) > 2d - \Delta$, which by lemma 3 implies that $\text{Th}_-(S) > 2d - \Delta$, and so

$$\begin{aligned} 2\text{Sz}(S) &\geq \text{Sz}(S) + \text{Fl}(S) && \text{since } |\text{Fl}(S)| \leq \text{Sz}(S) \\ &= \text{Th}_-(S) + \chi(S) && \text{because of (1)} \\ &> 2d - \Delta + \chi(S) = 2d - \Delta - \chi(S) + 2\chi(S) \\ &\geq 2d - \Delta - \chi(S) + \alpha \text{Th}_+(S) - 2^{D+1}\beta_{r_i}\alpha - 2^{D+1}\text{Tw}_{r_i} + 2R_D && \text{by invariant 12}^+ \\ &\geq 2d - 2\Delta - 2 + 2^{D+1} - 2^{D+1}\beta_{r_i}\alpha - 2^{D+1}\text{Tw}_{r_i} + 2R_D && \text{by lemma 4.} \end{aligned}$$

Hence by inequality (4c)

$$\text{Sz}(S) - 1 \geq d - 2^{D[S]+3}\text{Tw}_{r_i} \quad \text{for all } S \in \mathcal{C}_i \setminus V(M_i).$$

Let $L = 4\text{Tw}_{r_i}$. Since step 7 cannot be executed, for each D there is at most one set $S \in \mathcal{C}_i$ satisfying $D[S] = D$ that is not in M_i . Also, $D[S] \leq L$ by invariant 14. Thus,

$$\sum_{S \in \mathcal{C}_i \setminus V(M_i)} (\text{Sz}(S) - 1) \geq \sum_{S \in \mathcal{C}_i \setminus V(M_i)} (d - 2^{D[S]+3}\text{Tw}_{r_i}) \geq d|\mathcal{C}_i \setminus V(M_i)| - 2^{L+4}\text{Tw}_{r_i}.$$

Combining this with (9) we obtain

$$\sum_{S \in \mathcal{C}_i} (\text{Sz}(S) - 1) \geq d(|\mathcal{C}_i| - |M_i|) - \sum_{S \in V(M_i)} 2^{D[S]} + |M_i| - 2^{L+4} \text{Tw}_{r_i}. \quad (10)$$

If $|M_i| > 2^{L+4} \text{Tw}_{r_i}$, then the lemma follows from invariant 10. Otherwise, $|M_i| \leq 2^{L+4} \text{Tw}_{r_i}$, hence $|V(M_i)| \leq 2^{L+5} \text{Tw}_{r_i}$, and so

$$\sum_{S \in V(M_i)} 2^{D[S]} + 2^{L+4} \text{Tw}_{r_i} \leq 2^L \cdot 2^{L+5} \text{Tw}_{r_i} + 2^{L+4} \text{Tw}_{r_i} \leq 2^{2L+6} \text{Tw}_{r_i} < W$$

by inequality (4h), and the lemma follows from (10). \square

Lemma 8. *The sets in \mathcal{G} satisfy*

$$\sum_{S \in \mathcal{G}} (\text{Sz}(S) - d) \leq 0, \quad (11)$$

and the equality is possible only if there are no cohorts.

Proof. Let F be the number of floating elements, and consider

$$dF \geq \sum_{S \in \mathcal{F}} \text{Sz}(S) = \sum_{S \in \mathcal{G}} \text{Sz}(S) + \sum_{S \in \mathcal{B}} \text{Sz}(S) + \sum_{i=1}^m \sum_{S \in \mathcal{C}_i} \text{Sz}(S) = \Sigma_1 + \Sigma_2 + \Sigma_3.$$

By invariant 9 we have $\Sigma_2 \geq \sum_i (W - |\mathcal{C}_i|)$. Bounding Σ_3 by the preceding lemma we obtain

$$dF \geq \sum_{S \in \mathcal{G}} (\text{Sz}(S) - d) + d|\mathcal{G}| + d \sum_{i=1}^m (|\mathcal{C}_i| - |M_i|). \quad (12)$$

Furthermore, since the inequality in the preceding lemma is strict, the equality in (12) can hold only if there are no cohorts.

Finally, since step 8 cannot be executed, $F \leq |\mathcal{G}| + \sum_i (|\mathcal{C}_i| - |M_i|)$ and so (11) holds. \square

Let

$$\begin{aligned} B^+ &= \{x \text{ floating} : \chi(x) \geq 1 - \alpha\}, \\ B^- &= \{x \text{ floating} : -\chi(x) \geq 1 - \alpha\}. \end{aligned}$$

The next lemma shows that nasty sets contain many nearly-frozen elements.

Lemma 9. *Suppose $S \in \mathcal{G}$ satisfies $\text{Sz}(S) \leq d$. Then the following hold:*

- +) *If $\text{Th}_+(S) > 2d - \Delta$, then $|B^- \cap S| \geq d/2$ and $\text{Sz}(S) \geq d - \Delta/2 - \text{Tw}_{\Delta-1}$*
-) *If $\text{Th}_-(S) > 2d - \Delta$, then $|B^+ \cap S| \geq d/2$ and $\text{Sz}(S) \geq d - \Delta/2 - \text{Tw}_{\Delta-1}$*

Proof. Consider the case $\text{Th}_-(S) > 2d - \Delta$. The other case is similar.

Define r by $\text{Th}_-(S) = 2d - r$. From (1) and invariant 11⁻ we deduce that

$$\frac{1}{2}\alpha \text{Th}_+(S) - \text{Tw}_r \leq \chi(S) = \text{Sz}(S) + \text{Fl}(S) - \text{Th}_-(S),$$

and since $\text{Th}_-(S) + \text{Th}_+(S) = 2\text{Sz}(S)$, this implies that

$$- \text{Tw}_r \leq (1 - \alpha)\text{Sz}(S) + \text{Fl}(S) - (1 - \frac{1}{2}\alpha) \text{Th}_-(S). \quad (13)$$

We then use bounds $\text{Fl}(S) \leq (1 - \alpha)|\text{Sz}(S)| + \alpha|B^+ \cap S|$, $\text{Th}_-(S) \geq 2d - \Delta$ and $\text{Sz}(S) \leq d$ to obtain

$$- \text{Tw}_r \leq -\alpha d + \alpha|B^+ \cap S| + (1 - \frac{1}{2}\alpha)\Delta$$

Hence $|B^+ \cap S| \geq d - \text{Tw}_r/\alpha \geq d/2$ by inequality (4f).

When combined with $\text{Fl}(S) \leq \text{Sz}(S)$, the inequality (13) also implies that

$$(2 - \alpha)\text{Sz}(S) \geq (1 - \frac{1}{2}\alpha)(2d - \Delta) - \text{Tw}_r,$$

and so $\text{Sz}(S) \geq d - \Delta/2 - \text{Tw}_r$. □

We are now ready to demonstrate the claim made in step 9.

Define a *charge* for a pair (x, S) where $x \in B^+ \cup B^-$ and $S \in \mathcal{G}$ by the following rule:

$$\text{Ch}(x, S) = \begin{cases} \frac{\text{Sz}(S) - d - 1}{|S \cap B^+|} & \text{if } x \in S \cap B^+ \text{ and } \text{Sz}(S) \leq d \text{ and } \text{Th}_-(S) > 2d - \Delta, \\ \frac{\text{Sz}(S) - d - 1}{|S \cap B^-|} & \text{if } x \in S \cap B^- \text{ and } \text{Sz}(S) \leq d \text{ and } \text{Th}_+(S) > 2d - \Delta, \\ \frac{\text{Sz}(S) - d}{4|S \cap B^+|} & \text{if } x \in S \cap B^+ \text{ and } \text{Sz}(S) \geq d + 1, \\ \frac{\text{Sz}(S) - d}{4|S \cap B^-|} & \text{if } x \in S \cap B^- \text{ and } \text{Sz}(S) \geq d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since step 1 cannot be executed, $\text{Sz}(S) \leq d$ implies that either $\text{Th}_-(S) > 2d - \Delta$ or $\text{Th}_+(S) > 2d - \Delta$. Consider any $S \in \mathcal{G}$ satisfying $\text{Sz}(S) \leq d$. By lemma 9 we have $S \cap (B_+ \cup B_-) \neq \emptyset$, and so

$$\sum_{x \in X} \text{Ch}(x, S) < \text{Sz}(S) - d. \quad (14)$$

Inequality (14) also clearly holds for $S \in \mathcal{G}$ satisfying $\text{Sz}(S) > d$. So (14) holds for all $S \in \mathcal{G}$.

If $\mathcal{G} = \emptyset$, then the left-hand side of inequality (11) would be zero, implying that there are no cohorts, and so $\mathcal{F} = \mathcal{B}$. Since $\mathcal{F} = \mathcal{B}$ contradicts the assumption that step 9 can be executed, we conclude that \mathcal{G} is non-empty. Hence the inequality (14) and lemma 8 imply

$$\sum_{\substack{x \in X \\ S \in \mathcal{G}}} \text{Ch}(x, S) < \sum_{S \in \mathcal{G}} (\text{Sz}(S) - d) \leq 0.$$

So, there is an element $b \in B^+ \cup B^-$ such that

$$\sum_{S \in \mathcal{G}} \text{Ch}(b, S) < 0. \quad (15)$$

Fix such a b , and assume that $b \in B^+$ (the case $b \in B^-$ is analogous). Define

$$\begin{aligned} \mathcal{N} &\stackrel{\text{def}}{=} \{S \in \mathcal{G} : b \in S, \text{Sz}(S) \leq d \text{ and } \text{Th}_-(S) > 2d - \Delta\}, \\ \mathcal{P} &\stackrel{\text{def}}{=} \{S \in \mathcal{G} : b \in S, \text{Sz}(S) \geq d + 1 \text{ and } \text{Th}_+(S) > 2d - \Delta\}. \end{aligned}$$

From (15) and the definition of $\text{Ch}(b, S)$ it follows that \mathcal{N} is non-empty.

Since step 3 is unable to round b , the set \mathcal{P} must be non-empty as well. We next show that the contribution of a set from \mathcal{P} to the sum (15) is very large, whereas the contribution of a set from \mathcal{N} is very small.

Proposition 10. *For each $S \in \mathcal{N}$ we have $\text{Ch}(b, S) \geq -\frac{\Delta+2\text{Tw}_{\Delta-1}+2}{d}$.*

Proposition 11. *For each $S \in \mathcal{P}$ we have $\text{Ch}(b, S) \geq \frac{1}{8+4\Delta}$.*

Proof of proposition 10. This follows from $\text{Ch}(b, S) = (\text{Sz}(S) - d - 1)/|S \cap B^+|$, and the bounds on $\text{Sz}(S)$ and on $|S \cap B^+|$ from lemma 9. \square

Proof of proposition 11. We show that $|S \cap B^+|$ is very small by bounding $\text{Fl}(S)$ both from above and from below:

$$\begin{aligned} \text{Fl}(S) &= \text{Sz}(S) + \chi(S) - \text{Th}_+(S) && \text{by identity (2)} \\ &\leq \text{Sz}(S) + \chi(S) - 2d + \Delta, && \text{since } S \in \mathcal{P} \\ &\leq \text{Sz}(S) - 2d + \Delta, && \text{by invariant 11}^+ \\ \text{Fl}(S) &\geq -\text{Sz}(S) + (2 - \alpha)|S \cap B^+|. \end{aligned}$$

Since $\alpha \leq 1$, these two inequalities imply

$$|S \cap B^+| \leq 2(\text{Sz}(S) - d) + \Delta,$$

$$\text{and so } \text{Ch}(b, S) \geq \frac{1}{8} \cdot \frac{\text{Sz}(S) - d}{\text{Sz}(S) - d + \Delta/2} \geq \frac{1}{8(1 + \Delta/2)}. \quad \square$$

From the two propositions and from (15) we conclude that $|\mathcal{N}| \geq \frac{d}{\Delta+2\text{Tw}_{\Delta-1}+2} \cdot \frac{1}{8+4\Delta} \geq \Delta W$ by inequality (4g). By invariant 3 we have $2d - \Delta < \text{Th}_-(S) \leq 2d$ for all $S \in \mathcal{N}$. Hence, by the pigeonhole principle there is an $0 \leq r < \Delta$ and a $\mathcal{D} \subset \mathcal{N}$ of size $|\mathcal{D}| = W$ such that $\text{Th}_+(S) = 2d - r$ for all $S \in \mathcal{D}$. This completes the proof of the claim made in step 9.

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