

3-6-2014

A bound on the number of edges in graphs without an even cycle

Boris Bukh

Carnegie Mellon University, bbukh@math.cmu.edu

Zilin Jiang

Carnegie Mellon University

Follow this and additional works at: <http://repository.cmu.edu/math>



Part of the [Mathematics Commons](#)

This Working Paper is brought to you for free and open access by the Mellon College of Science at Research Showcase @ CMU. It has been accepted for inclusion in Department of Mathematical Sciences by an authorized administrator of Research Showcase @ CMU. For more information, please contact research-showcase@andrew.cmu.edu.

A bound on the number of edges in graphs without an even cycle

Boris Bukh*

Zilin Jiang^{†,*}

Abstract

We show that, for each fixed k , an n -vertex graph not containing a cycle of length $2k$ has at most $80\sqrt{k \log k} \cdot n^{1+1/k} + O(n)$ edges.

Introduction

Let $\text{ex}(n, F)$ be the largest number of edges in an n -vertex graph that contains no copy of a fixed graph F . The first systematic study of $\text{ex}(n, F)$ was started by Turán [16], and now it is a central problem in extremal graph theory (see surveys [14, 9]).

The function $\text{ex}(n, F)$ exhibits a dichotomy: if F is not bipartite, then $\text{ex}(n, F)$ grows quadratically in n , and is fairly well understood. If F is bipartite, $\text{ex}(n, F)$ is subquadratic, and for very few F the order of magnitude is known. The two simplest classes of bipartite graphs are complete bipartite graphs, and cycles of even length. Most of the study of $\text{ex}(n, F)$ for bipartite F has been concentrated on these two classes. In this paper, we address the even cycles. For an overview of the status of $\text{ex}(n, F)$ for complete bipartite graphs see [2]. For a thorough survey on bipartite Turán problems see [8].

The first bound on the problem is due to Erdős[5] who showed that $\text{ex}(n, C_4) = \Theta(n^{3/2})$. Thanks to the works of Erdős and Rényi [6], Brown [4, Section 3], and Kövari, Sós and Turán [10] it is now known that

$$\text{ex}(n, C_4) = (1/2 + o(1))n^{3/2}.$$

The best current bound for $\text{ex}(n, C_6)$ for large values of n is

$$0.5338n^{4/3} < \text{ex}(n, C_6) \leq 0.6272n^{4/3}$$

due to Füredi, Naor and Verstraëte [7].

A general bound of $\text{ex}(n, C_{2k}) \leq \gamma_k n^{1+1/k}$, for some unspecified constant γ_k , was asserted by Erdős. The first proof was by Bondy and Simonovits [3], who showed that $\text{ex}(n, C_{2k}) \leq 20kn^{1+1/k}$ for all sufficiently large n . This was improved by Verstraëte [17] to $8(k-1)n^{1+1/k}$ and by Pikhurko [13] to $(k-1)n^{1+1/k} + O(n)$. The principal result of the present paper is an improvement of these bounds:

*Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA 15213. Supported in part by U.S. taxpayers through NSF grant DMS-1301548.

[†]Supported in part by U.S. taxpayers through NSF grant DMS-1201380.

Main Theorem. *Suppose G is n -vertex graph that contains no C_{2k} , and $n \geq (2k)^{8k^2}$ then*

$$\text{ex}(n, C_{2k}) \leq 80\sqrt{k \log k} \cdot n^{1+1/k} + 10k^2n.$$

It is our duty to point out that the improvement offered by the Main Theorem is of uncertain value because we still do not know if $\Theta(n^{1+1/k})$ is the correct order of magnitude for $\text{ex}(n, C_{2k})$. Only for $k = 2, 3, 5$ constructions of C_{2k} -free graphs with $\Omega(n^{1+1/k})$ edges are known [1, 18, 11, 12]. The first author believes it to be likely that $\text{ex}(n, C_{2k}) = o(n^{1+1/k})$ for all large k . We stress again that the situation is completely different for odd cycles, where the value of $\text{ex}(n, C_{2k+1})$ is known exactly for all large n [15].

Proof method and organization of the paper Our proof is inspired by that of Pikhurko [13]. Apart from a couple of lemmas that we quote from [13], the proof is self-contained. However, we advise the reader to at least skim [13] to see the main idea in a simpler setting.

Pikhurko's proof builds a breadth-first search tree, and then argues that a pair of adjacent levels of the tree cannot contain a Θ -graph¹. It is then deduced that each level must be at least $\delta/(k-1)$ times larger than the previous, where δ is the (minimum) degree. The bound on $\text{ex}(n, C_{2k})$ then follows. The estimate of $\delta/(k-1)$ is sharp when one restricts one's attention to a pair of levels.

In our proof, we use three adjacent levels. We find a Θ -graph satisfying an extra technical condition that permits an extension of Pikhurko's argument. Annoyingly, this extension requires a bound on the *maximum degree*. To achieve such a bound we use a modification of breadth-first search that avoids the high-degree vertices.

What we really prove in this paper is the following:

Theorem 1. *Suppose $k \geq 4$, and suppose G is a bipartite n -vertex graph of minimum degree at least $2d + 5k^2$, where*

$$d \geq \max(20\sqrt{k \log k} \cdot n^{1/k}, (2k)^{8k}), \tag{1}$$

then G contains C_{2k} .

The Main Theorem follows from Theorem 1 and two well-known facts: every graph contains a bipartite subgraph with half of the edges, and every graph of average degree d_{avg} contains a subgraph of minimum degree at least $d_{\text{avg}}/2$.

The rest of the paper is organized as follows. We present our modification of breadth-first search in Section 1. In Section 2, which is the heart of the paper, we explain how to find Θ -graphs in triples of consecutive levels. Finally, in Section 3 we assemble the pieces of the proof.

1 Graph exploration

Our aim is to have vertices of degree at most Δd for some $k \ll \Delta \ll d^{1/k}$. The particular choice is fairly flexible; we choose to use

$$\Delta \stackrel{\text{def}}{=} k^3.$$

¹We recall the definition of a Θ -graph in Section 2

Let G be a graph, and let x be any vertex of G . We start our exploration with the set $V_0 = \{x\}$, and mark the vertex x as explored. Suppose V_0, V_1, \dots, V_{i-1} are the sets explored in the 0th, 1st, \dots , $(i-1)$ st steps respectively. We then define V_i as follows:

1. Let V'_i consist of those neighbors of V_{i-1} that have not yet been explored. Let Bg_i be the set of those vertices in V'_i that have more than Δd unexplored neighbors, and let $Sm_i = V'_i \setminus Bg_i$.
2. Define

$$V_i = \begin{cases} V'_i & \text{if } |Bg_i| > \frac{1}{k+1}|V'_i|, \\ Sm_i & \text{if } |Bg_i| \leq \frac{1}{k+1}|V'_i|. \end{cases}$$

The vertices of V_i are then marked as explored.

We call sets V_0, V_1, \dots *levels* of G . A level V_i is *big* if $|Bg_i| > \frac{1}{k+1}|V'_i|$, and is *normal* otherwise.

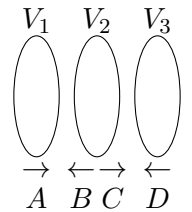
Lemma 2. *If $\delta \leq \Delta d$, and G is a bipartite graph of minimum degree δ , then each $v \in V_{i+1}$ has at least δ neighbors in $V_i \cup V'_{i+2}$.*

Proof. Fix a vertex $v \in V(G)$. We will show, by induction on i , that if $v \notin V_1 \cup \dots \cup V_i$, then v has at least δ neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$. The base case $i = 1$ is clear. Suppose $i > 1$. If $v \in Bg_i$, then v has $\Delta d \geq \delta$ neighbors in the required set. Otherwise, v is not in V'_i and hence has no neighbors in V_{i-1} . Hence, v has as many neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$ as in $V(G) \setminus (V_1 \cup \dots \cup V_{i-2})$, and our claim follows from the induction hypothesis.

If $v \in V_{i+1}$, then the neighbors of v are a subset of $V_1 \cup \dots \cup V_i \cup V'_{i+2}$. Hence, at least δ of these neighbors lie in $V_i \cup V'_{i+2}$. \square

Trilayered graphs A *trilayered graph* with layers V_1, V_2, V_3 is a graph G on a vertex set V_1, V_2, V_3 such that the only edges in G are between V_1 and V_2 , and between V_2 and V_3 . If $V'_1 \subset V_1$, $V'_2 \subset V_2$ and $V'_3 \subset V_3$, then we denote by $G[V'_1, V'_2, V'_3]$ the trilayered subgraph induced by three sets V'_1, V'_2, V'_3 . Any three sets V_{i-1}, V_i, V'_{i+1} from the exploration process naturally form a trilayered graph; these graphs and their subgraphs are the only trilayered graphs that appear in this paper.

We say that a trilayered graph has *minimum degree* at least $[A : B, C : D]$ if each vertex in V_1 has at least A neighbors in V_2 , each vertex in V_2 has at least B neighbors in V_1 , each vertex in V_2 has at least C neighbors in V_3 , and each vertex in V_3 has at least D neighbors in V_2 . A schematic drawing of such a graph is on the right.



2 Θ -graphs

A Θ -graph is a cycle of length at least $2k$ with a chord. We shall use several lemmas from the previous works.

Lemma 3 (Lemma 2.1 in [13], also Lemma 2 in [17]). *Let F be a Θ -graph and $1 \leq l \leq |V(F)| - 1$. Let $V(F) = W \cup Z$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in F of length l that begins in W necessarily ends in W . Then F is bipartite with parts W and Z .*

Lemma 4 (Lemma 2.2 in [13]). *Let $k \geq 3$. Any bipartite graph H of minimum degree at least k contains a Θ -graph.*

Corollary 5. *Let $k \geq 3$. Any bipartite graph H of average degree at least $2k$ contains a Θ -graph.*

For a graph G and a set $Y \subset V(G)$ let $G[Y]$ denote the graph induced on Y . For disjoint $Y, Z \subset V(G)$ let $G[Y, Z]$ denote the bipartite subgraph of G that is induced by the bipartition $Y \cup Z$.

Suppose G is a trilayered graph with layers V_1, V_2, V_3 . We say that a Θ -graph $F \subset G$ is *well-placed* if each vertex of $V(F) \cap V_2$ is adjacent to some vertex in $V_1 \setminus V(F)$.

Lemma 6. *Suppose G is a trilayered graph with layers V_1, V_2, V_3 such that the degree of every vertex in V_2 is between $2d + 5k^2$ and Δd . Suppose t is a nonnegative integer, and let $F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|}$. Assume that*

$$\begin{aligned}
a) \quad & F \geq 2, \\
b) \quad & e(V_1, V_2) \geq 2kF|V_1|, \\
c) \quad & e(V_1, V_2) \geq 8k(t+1)^2(2\Delta k)^{2k-1}|V_1|, \\
d) \quad & e(V_1, V_2) \geq 8(et/F)^t k|V_2|, \\
e) \quad & e(V_1, V_2) \geq 20(t+1)^2|V_2|.
\end{aligned} \tag{2}$$

Then at least one of the following holds:

- I) *There is a Θ -graph in $G[V_1, V_2]$.*
- II) *There is a well-placed Θ -graph in $G[V_1, V_2, V_3]$.*

The proof of Lemma 6 is in two parts: finding trilayered subgraph of large minimum degree (Lemmas 7 and 8), and finding a well-placed Θ -graph inside that trilayered graph (Lemma 9).

Finding a trilayered subgraph of large minimum degree The disjoint union of two bipartite graphs shows that a trilayered graph with many edges need not contain a trilayered subgraph of large minimum degree. We show that, in contrast, if a trilayered graph contains no Θ -graph between two of its levels, then it must contain a subgraph of large minimum degree:

Lemma 7. *Let a, A, B, C, D be positive real numbers. Suppose G is a trilayered graph with layers V_1, V_2, V_3 and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Assume also that*

$$a \cdot e(V_1, V_2) \geq (A + k + 1)|V_1| + B|V_2|. \tag{3}$$

Then one of the following holds:

- I) *There is a Θ -graph in $G[V_1, V_2]$.*
- II) *There exist non-empty subsets $V'_1 \subset V_1, V'_2 \subset V_2, V'_3 \subset V_3$ such that the induced trilayered subgraph $G[V'_1, V'_2, V'_3]$ has minimum degree at least $[A : B, C : D]$.*
- III) *There is a subset $\tilde{V}_2 \subset V_2$ such that $e(V_1, \tilde{V}_2) \geq (1 - a)e(V_1, V_2)$, and $|\tilde{V}_2| \leq D|V_3|/d$.*

Proof. We suppose that alternative (I) does not hold. Then, by Corollary 5, the average degree of every subgraph of $G[V_1, V_2]$ is at most $2k$.

Consider the process that aims to construct a subgraph satisfying (II). The process starts with $V'_1 = V_1, V'_2 = V_2$ and $V'_3 = V_3$, and at each step removes one of the vertices that violate the minimum degree condition on $G[V'_1, V'_2, V'_3]$. The process stops when either no vertices are left, or the minimum degree of $G[V'_1, V'_2, V'_3]$ is at least $[A : B, C : D]$. Since in the latter case we are done, we assume that this process eventually removes every vertex of G .

Let R be the vertices of V_2 that were removed because at the time of removal they had fewer than C neighbors in V'_3 . Put

$$E' \stackrel{\text{def}}{=} \{uv \in E(G) : u \in V_2, v \in V_3, \text{ and } v \text{ was removed before } u\},$$

$$S \stackrel{\text{def}}{=} \{v \in V_2 : v \text{ has at least } 4k^2 \text{ neighbors in } V_1\}.$$

Note that $|E'| \leq D|V_3|$. We cannot have $|S| \geq |V_1|/k$, for otherwise the average degree of the bipartite graph $G[V_1, S]$ would be at least $\frac{4k}{1+1/k} \geq 2k$. So $|S| \leq |V_1|/k$.

The average degree condition on $G[V_1, S]$ implies that

$$e(V_1, S) \leq k(|V_1| + |S|) \leq (k+1)|V_1|.$$

Let u be any vertex in $R \setminus S$. Since it is connected to at least $d+C$ vertices of V_3 , it must be adjacent to at least d edges of E' . Thus,

$$|R \setminus S| \leq |E'|/d \leq D|V_3|/d.$$

Assume that the conclusion (III) does not hold with $\tilde{V}_2 = R \setminus S$. Then $e(V_1, R \setminus S) < (1-a)e(V_1, V_2)$. Since the total number of edges between V_1 and V_2 that were removed due to the minimal degree conditions on V_1 and V_2 is at most $A|V_1|$ and $B|V_2|$ respectively, we conclude that

$$\begin{aligned} e(V_1, V_2) &\leq e(V_1, S) + e(V_1, R \setminus S) + A|V_1| + B|V_2| \\ &< (k+1)|V_1| + (1-a)e(V_1, V_2) + A|V_1| + B|V_2|, \\ a \cdot e(V_1, V_2) &< (A+k+1)|V_1| + B|V_2|. \end{aligned}$$

The contradiction completes the proof. □

Remark. The preceding lemma by itself is sufficient to prove the estimate $\text{ex}(n, C_{2k}) = O(k^{2/3}n^{1+1/k})$. For that, one chooses approximately $B = k^{2/3}, D = k^{1/3}$ and $a = 1/2$. One can then show that when applied to trilayered graphs arising from the exploration process the alternative (III) leads to a subgraph of average degree $2k$. The two remaining alternatives are dealt by Corollary 5 and Lemma 9. However, it is possible to obtain a better bound by iterating the preceding lemma.

Lemma 8. *Let C be a positive real number. Suppose G is a trilayered graph with layers V_1, V_2, V_3 , and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Let $F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|}$, and assume that F and $e(V_1, V_2)$ satisfy (2). Then one of the following holds:*

I) *There is a Θ -graph in $G[V_1, V_2]$.*

II) There exist numbers A, B, D and non-empty subsets $V'_1 \subset V_1, V'_2 \subset V_2, V'_3 \subset V_3$ such that the induced trilayered subgraph $G[V'_1, V'_2, V'_3]$ has minimum degree at least $[A : B, C : D]$, with the following inequalities that bind A, B , and D :

$$\begin{aligned} B &\geq 5, & (B-4)D &\geq 2k, \\ A &\geq 2k(\Delta D)^{D-1}. \end{aligned} \tag{4}$$

Proof. Assume, for the sake of contradiction, that neither (I) nor (II) hold. With hindsight, set $a_j = \frac{1}{t-j+1}$ for $j = 0, \dots, t-1$. We shall define a sequence of sets $V_2 = V_2^{(0)} \supseteq V_2^{(1)} \supseteq \dots \supseteq V_2^{(t)}$ inductively. We denote by

$$d_i \stackrel{\text{def}}{=} e(V_1, V_2^{(i)})/|V_2^{(i)}|$$

the average degree from $V_2^{(i)}$ into V_1 . The sequence $V_2^{(0)}, V_2^{(1)}, \dots, V_2^{(t)}$ will be constructed so as to satisfy

$$e(V_1, V_2^{(i+1)}) \geq (1 - a_i)e(V_1, V_2^{(i)}), \tag{5}$$

$$d_{i+1} \geq d_i \cdot \prod_{j=0}^i (1 - a_j). \tag{6}$$

Note that (5) and the choice of a_0, \dots, a_i imply that

$$e(V_1, V_2^{(i)}) \geq \frac{1}{t+1}e(V_1, V_2). \tag{7}$$

The sequence starts with $V_2^{(0)} = V_2$. Assume $V_2^{(i)}$ has been defined. We proceed to define $V_2^{(i+1)}$. Put

$$\begin{aligned} A &= a_i e(V_1, V_2^{(i)})/2|V_1| - k - 1, \\ B &= a_i d_i/4 + 5, \\ D &= \min(2k, 8k/a_i d_i). \end{aligned}$$

With help of (7) and (2c) it is easy to check that the inequalities (4) hold for this choice of constants.

In addition,

$$\begin{aligned} (A + k + 1)|V_1| + B|V_2^{(i)}| &= \frac{3}{4}a_i e(V_1, V_2^{(i)}) + 5|V_2^{(i)}| \\ &\stackrel{(2c)}{\leq} \frac{3}{4}a_i e(V_1, V_2^{(i)}) + \frac{1}{4(t+1)^2}e(V_1, V_2) \\ &\stackrel{(7)}{\leq} a_i e(V_1, V_2^{(i)}). \end{aligned}$$

So, the condition (3) of Lemma 7 is satisfied for the graph $G[V_1, V_2^{(i)}, V_3]$. By Lemma 7 there is a subset $V_2^{(i+1)} \subset V_2^{(i)}$ satisfying (5) and

$$|V_2^{(i+1)}| \leq D|V_3|/d.$$

Next we show that the set $V_2^{(i+1)}$ satisfies inequality (6). Indeed, we have

$$\begin{aligned} d_{i+1} &= \frac{e(V_1, V_2^{(i+1)})}{|V_2|} \geq \frac{(1-a_i)e(V_1, V_2^{(i)})}{D|V_3|/d} = (1-a_i)a_i d_i \frac{d}{8k|V_3|} e(V_1, V_2^{(i)}) \\ &\geq (1-a_i)a_i d_i \frac{d \cdot e(V_1, V_2)}{8k|V_3|} \prod_{j=0}^{i-1} (1-a_j) = d_i \cdot F a_i \prod_{j=0}^i (1-a_j). \end{aligned}$$

Iterative application of (6) implies

$$d_t \geq d_0 F^t \prod_{j=0}^{t-1} a_j (1-a_j)^{t-j} \geq d_0 F^t \prod_{j=0}^{t-1} \frac{e^{-1}}{t-j+1} = d_0 \frac{(F/e)^t}{(t+1)!}. \quad (8)$$

If we have $|V_2^{(t)}| < |V_1|$, then the average degree of induced subgraph $G[V_1, V_2^{(t)}]$ is greater than $e(V_1, V_2^{(t)})/|V_1| \stackrel{(7)}{\geq} e(V_1, V_2)/(t+1)|V_1| \stackrel{(2c)}{\geq} 2k$, which by Corollary 5 leads to outcome (I).

If $|V_2^{(t)}| \geq |V_1|$ and $d_t \geq 4k$, then the average degree of $G[V_1, V_2^{(t)}]$ is at least $d_t/2 \geq 2k$, again leading to the outcome (I). So, we may assume that $d_t < 4k$. Since $(t+1)! \leq 2t^t$ we deduce from (8) that

$$d_0 \leq 4k(t+1)!(e/F)^t \leq 8k(et/F)^t.$$

This contradicts (2d), and so the proof is complete. \square

Locating well-placed Θ -graphs in trilayered graphs We come to the central argument of the paper. It shows how to embed well-placed Θ -graphs into trilayered graphs of large minimum degree. Or rather, it shows how to embed well-placed Θ -graphs into regular trilayered graphs; the contortions of the previous two lemmas, and the factor of $\sqrt{\log k}$ in the final bound, come from authors' inability to deal with irregular graphs.

Lemma 9. *Let A, B, D be positive real numbers. Let G be a trilayered graph with layers V_1, V_2, V_3 of minimum degree at least $[A, B, d+k, D]$. Suppose that no vertex in V_2 has more than Δd neighbors in V_3 . Assume also that*

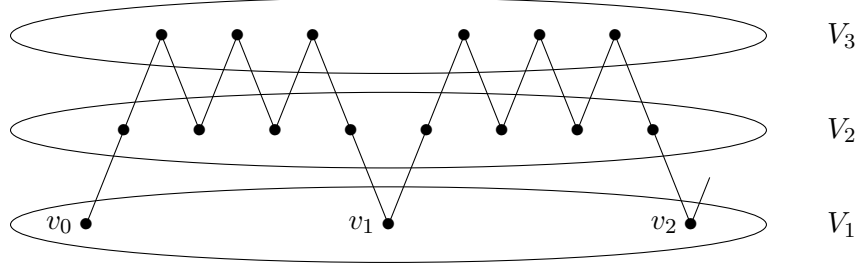
$$B \geq 5 \quad (9)$$

$$(B-4)D \geq 2k-2 \quad (10)$$

$$A \geq 2k(\Delta D)^{D-1}. \quad (11)$$

Then G contains a well-placed Θ -graph.

Proof. Assume, for the sake of contradiction, that G contains no well-placed Θ -graphs. Leaning on this assumption we shall build an arbitrary long path P of the form



where, for each i , vertices v_i and v_{i+1} are joined by a path of length $2D$ that alternates between V_2 and V_3 . Since the graph is finite, this would be a contradiction.

While building the path we maintain the following property:

$$\text{Every } v \in P \cap V_2 \text{ has at least one neighbor in } V_1 \setminus P. \quad (\star)$$

We call a path satisfying (\star) *good*.

We construct the path inductively. We begin by picking v_0 arbitrarily from V_1 . Suppose a good path $P = v_0 \leftrightarrow v_1 \leftrightarrow \dots \leftrightarrow v_{l-1}$ has been constructed, and we wish to find a path extension $v_0 \leftrightarrow v_1 \leftrightarrow \dots \leftrightarrow v_{l-1} \leftrightarrow v_l$.

For each $i = 1, 2, \dots, 2D - 1$ we shall define a family \mathcal{Q}_i of good paths that satisfy

1. Each path in \mathcal{Q}_i is of the form $v_0 \leftrightarrow v_1 \leftrightarrow \dots \leftrightarrow v_{l-1} \leftrightarrow u$, where $v_{l-1} \leftrightarrow u$ is a path of length i that alternates between V_2 and V_3 . The vertex u is called a *terminal* of the path. The set of terminals of the paths in \mathcal{Q}_i is denoted by $T(\mathcal{Q}_i)$.
2. For each i , the paths in \mathcal{Q}_i have distinct terminals.
3. For odd-numbered indices, we have the inequality

$$|\mathcal{Q}_{2i+1}| \geq -3k + A \left(\frac{1}{\Delta} \right)^i \prod_{j \leq i} \left(1 - \frac{j}{D} \right).$$

4. For even-numbered indices, we have the inequality

$$e(T(\mathcal{Q}_{2i}), V_2) \geq d|\mathcal{Q}_{2i-1}|.$$

Let

$$t \stackrel{\text{def}}{=} \lceil B/2 \rceil.$$

We will repeatedly use the following straightforward fact, which we call the *small-degree argument*: whenever Q is a good path and $u \in V_2$ is adjacent to the terminal of Q , then the path Qu is adjacent to fewer than t vertices in $V_1 \cap Q$. Indeed, if vertex u were adjacent to $v_{j_1}, v_{j_2}, \dots, v_{j_t} \in V_1 \cap Q$, then $v_{j_2} \leftrightarrow u$ (along path Q) and the edge uv_{j_2} would form a cycle of total length at least $2D(t-2) + 2 \stackrel{(10)}{\geq} 2k$. As uv_{j_3} is a chord of the cycle, and u is adjacent to v_{j_1} that is not on the cycle, that would contradict the assumption that G contains no well-placed Θ -graph.

The set \mathcal{Q}_1 consists of all paths of the form Pu for $u \in V_2 \setminus P$. Let us check that the preceding conditions hold for \mathcal{Q}_1 . Vertex v_{l-1} cannot be adjacent to k or more vertices in $P \cap V_2$, for otherwise G would contain a well-placed Θ -graph with a chord through v_{l-1} . So, $|\mathcal{Q}_1| \geq A - k$. Next, consider any $u \in V_2 \setminus P$ that is a neighbor of v_{l-1} . By the small-degree argument vertex u cannot be adjacent to t or more vertices of $P \cap V_1$, and Pu is good.

Suppose \mathcal{Q}_{2i-1} has been defined, and we wish to define \mathcal{Q}_{2i} . Consider an arbitrary path $Q = v_0 \rightsquigarrow v_1 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow u \in \mathcal{Q}_{2i-1}$. Vertex u cannot have k or more neighbors in $Q \cap V_3$, for otherwise G would contain a well-placed Θ -graph with a chord through u . Hence, there are at least d edges of the form uw , where $w \in V_3 \setminus Q$. As we vary u we obtain a family of at least $d|\mathcal{Q}_{2i-1}|$ paths eligible for inclusion into \mathcal{Q}_{2i} . We let \mathcal{Q}_{2i} consist of any maximal set of such paths with distinct terminals.

Suppose \mathcal{Q}_{2i} has been defined, and we wish to define \mathcal{Q}_{2i+1} . Consider an arbitrary path $Q = v_0 \rightsquigarrow v_1 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow u \in \mathcal{Q}_{2i}$. An edge uw is called *long* if $w \in P$, and w is at a distance exceeding $2k$ from u along path Q . If uw is a long edge, then from u to Q there is only one edge, namely the edges to the predecessor of u on Q , for otherwise there is a well-placed Θ -graph. Also, at most i neighbors of u lie on the path $v_{l-1} \rightsquigarrow u$. Since $\deg u \geq D$, it follows that at least $(1 - i/D)\deg u$ short edges from u that miss $v_{l-1} \rightsquigarrow u$. Thus there is a set \mathcal{W} of at least $(1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)$ walks (not necessarily paths!) of the form $v_0 \rightsquigarrow v_1 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow uw$ such that $v_{l-1} \rightsquigarrow uw$ is a path and w occurs only among the last $2k$ vertices of the walk.

From the maximum degree condition on V_2 it follows that walks in \mathcal{W} have at least $(1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d$ distinct terminals. A walk fails to be a path only if the terminal vertex lies on P . However, since the edge uw is short, this can happen for at most $2k$ possible terminals. Hence, there is a $\mathcal{Q}_{2i+1} \subset \mathcal{W}$ of size $|\mathcal{Q}_{2i+1}| \geq (1 - i/D)e(T(\mathcal{Q}_{2i}), V_2)/\Delta d - 2k$ that consists of paths with distinct terminals. It remains to check that every path in \mathcal{Q}_{2i+1} is good. The only way that $Q = v_0 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow uw \in \mathcal{Q}_{2i+1}$ may fail to be good is if w has no neighbors in $V_1 \setminus Q$. By the small-degree argument w has fewer than t neighbors in V_1 . Since w has at least B neighbors in V_1 and $B \geq t + 2$, we conclude that w has at least *two* neighbors in V_1 outside the path. Of course, the same is true for every terminal of a path in \mathcal{Q}_{2i+1} .

Note that \mathcal{Q}_{2D-1} is non-empty. Let $Q = v_0 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow u \in \mathcal{Q}_{2D-1}$ be an arbitrary path. Note that since $2D - 1$ is odd, $u \in V_2$. By the property of terminals of V_i (odd i) that we noted in the previous paragraph, there are two vertices in $V_1 \setminus Q$ that are neighbors of u . Let v_l be any of them, and let the new path be $Qv_l = v_0 \rightsquigarrow \dots \rightsquigarrow v_{l-1} \rightsquigarrow uv_l$. This path can fail to be good if there is a vertex w on the path Q that is good in Q , but is bad in Qv_l . By the small-degree argument, w is adjacent to fewer than t vertices in $Q \cap V_1$ that precede w in Q . The same argument applied to the reversal of the path Qv_l shows that w is adjacent to fewer than t vertices in $Q \cap V_1$ that succeed w in Q . Since $2t - 2 < B$, the path Qv_l is good.

Hence, it is possible to build an arbitrarily long path in G . This contradicts the finiteness of G . \square

Lemma 6 follows from Lemmas 8 and 9 by setting $C = d + k$, in view of inequality $4k^2 + k \leq 5k^2$.

3 Proof of Theorem 1

Suppose G has minimum degree of at least $2d + 4k^2 + k$ and contains no C_{2k} . Pick a root vertex x arbitrarily, and let V_0, V_1, \dots, V_{k-1} be the levels obtained from the exploration process in Section 1.

Lemma 10. *For $1 \leq i \leq k-1$, the graph $G[V_{i-1}, V_i, V_{i+1}]$ contains no well-placed Θ -graph.*

Proof. The following proof is almost an exact repetition of the proof of Claim 3.1 from [13] (which is also reproduced as Lemma 11 below).

Suppose, for the sake of contradiction, that a well-placed Θ -graph $F \subset G[V_{i-1}, V_i, V_{i+1}]$ exists. Let $Y = V_i \cap V(F)$. Since F is well-placed, for every vertex of Y there is a path of length i to the vertex x . The union of these paths forms a tree T with x as a root. Let y be the vertex farthest from x such that every vertex of Y is a T -descendant of y . Paths that connect y to Y branch at y . Pick one such branch, and let $W \subset Y$ be the set of all the T -descendants of that branch. Let $Z = V(F) \setminus W$. From $Y \neq V_i \cap V(F)$ it follows that Z is not an independent set of F , and so $W \cup Z$ is not a bipartition of F .

Let ℓ be the distance between x and y . We have $\ell < i$ and $2k - 2i + 2\ell < 2k \leq |V(F)|$. By Lemma 3 in F there is a path P of length $2k - 2i + 2\ell$ that starts at some $w \in W$ and ends in $z \in Z$. Since the length of P is even, $z \in Y$. Let P_w and P_z be unique paths in T that connect y to respectively w and z . They intersect only at y . Each of P_w and P_z has length $i - \ell$. The union of paths P, P_w, P_z forms a $2k$ -cycle in G . \square

The same argument (with a different Y) also proves the next lemma.

Lemma 11 (Claim 3.1 in [13]). *For $1 \leq i \leq k-1$, neither of $G[V_i]$ and $G[V_i, V_{i+1}]$ contains a bipartite Θ -graph.*

The next step is to show that the levels V_0, V_1, V_2, \dots increase in size. We shall show by induction on i that

$$e(V_i, V_{i+1}) \geq d|V_i|, \tag{12}$$

$$e(V_i, V_{i+1}) \leq 2k|V_{i+1}|, \tag{13}$$

$$e(V_i, V'_{i+1}) \leq 2k|V'_{i+1}|, \tag{14}$$

$$|V_{i+1}| \geq (2k)^{-1}d|V_i|, \tag{15}$$

$$|V_{i+1}| \geq \frac{d^2}{400k \log k} |V_{i-1}|. \tag{16}$$

Clearly, these hold for $i = 0$ since each vertex of V_1 sends only one edge to V_0 .

Proof of (12): By Lemma 2 the degree of every vertex in V_i is at least $d + 3k + 1$, and so

$$e(V_i, V'_{i+1}) \geq |V_i|(d + 3k + 1) - e(V_{i-1}, V_i) \stackrel{\text{induc.}}{\geq} (d + k + 1)|V_i|.$$

We next distinguish two cases depending on whether V_{i+1} is big (in the sense of the definition from Section 1). If V_{i+1} is big, then $e(V_i, V_{i+1}) = e(V_i, V'_{i+1})$, and (12) follows. If V_{i+1} is normal, then Corollary 5 implies that

$$e(V_i, \text{Bg}_{i+1}) \leq k(|V_i| + |\text{Bg}_{i+1}|) \leq (k + 1)|V_i|$$

and so

$$e(V_i, V_{i+1}) = e(V_i, V'_{i+1}) - e(V_i, \text{Bg}_{i+1}) \geq d|V_i|$$

implying (12). □

Proof of (13): Consider the graph $G[V_i, V_{i+1}]$. Inequality (12) asserts that the average degree of V_i is at least $d \geq 2k$. If (13) does not hold, then the average degree of V_{i+1} is at least $2k$ as well, contradicting Corollary 5. □

Proof for (14): The argument is the same as for (13) with $G[V_i, V'_{i+1}]$ in place of $G[V_i, V_{i+1}]$. □

Proof for (15): This follows from (13) and (12). □

Proof of (16) in the case V_i is a normal level: We assume that (16) does not hold and will derive a contradiction. Consider the trilayered graph $G[V_{i-1}, V_i, V'_{i+1}]$. Let $t = 2 \log k$. Suppose momentarily that the inequalities (2) in Lemma 6 hold. Then since V_i is normal, the degrees of vertices in V_i are bounded from above by Δd , and so Lemma 6 applies. However, the lemma's conclusion contradicts Lemmas 10 and 11. Hence, to prove (16) it suffices to verify inequalities (2a-d) with $F = d \cdot e(V_{i-1}, V_i) / 8k|V'_{i+1}|$.

We may assume that

$$F \geq 2e^2 \log k, \tag{17}$$

and in particular that (2a) holds. Indeed, if (17) were not true, then inequality (12) would imply $|V'_{i+1}| \geq (d^2/16e^2 k \log k)|V_{i-1}|$, and thus

$$|V_{i+1}| \geq (1 - \frac{1}{k})|V'_{i+1}| \geq (d^2/32e^2 k \log k)|V_{i-1}|,$$

and so (16) would follow in view of $32e^2 \leq 400$.

Inequality (2b) is implied by (15). Indeed,

$$e(V_{i-1}, V_i) = 8k|V_{i+1}|F/d \stackrel{(15)}{\geq} 4F|V_i| \stackrel{(15)}{\geq} 2k^{-1}dF|V_{i-1}|,$$

and $d \geq k^2$ by the definition of d from (1).

Inequality (2c) is implied by (1) and (12).

Next, suppose (2d) were not true. Since $F/t \geq e^2$ by (17), we would then conclude

$$\begin{aligned} |V_{i+1}| &\stackrel{(15)}{\geq} (2k)^{-1}d|V_i| \geq (16k^2)^{-1}(F/et)^t e(V_{i-1}, V_i) \\ &\geq (16k^2)^{-1}e^{2 \log k} e(V_{i-1}, V_i) \stackrel{(12)}{\geq} \frac{1}{16}d|V_{i-1}|, \end{aligned}$$

and so (16) would follow.

Finally, (2e) is a consequence of (12). □

Proof of (16) in the case V_i is a big level: We have

$$\begin{aligned} |V_{i+1}| &\geq \frac{1}{2}|V'_{i+1}| \stackrel{(14)}{\geq} (4k)^{-1}e(V_i, V'_{i+1}) \geq (4k)^{-1}e(\text{Bg}_i, V'_{i+1}) \geq (4k)^{-1}\Delta d|\text{Bg}_i| \\ &\geq (8k^2)^{-1}\Delta d|V_i| \stackrel{(15)}{\geq} (16k^3)^{-1}\Delta d|V_{i-1}| = \frac{1}{16}d|V_{i-1}|, \end{aligned}$$

and so (16) holds. \square

We are ready to complete the proof of Theorem 1. If k is even, then $\lfloor k/2 \rfloor$ applications of (16) yield

$$|V_k| \geq \frac{d^k}{(400k \log k)^{k/2}}.$$

If k is odd, then $(k-1)/2$ applications of (16) yield

$$|V_k| \geq \frac{d^{k-1}}{(400k \log k)^{(k-1)/2}}|V_1| \geq \frac{d^k}{(400k \log k)^{(k-1)/2}}.$$

Either way, since $|V_k| < n$ we conclude that $d < 20\sqrt{k \log k} \cdot n^{1/k}$.

References

- [1] Clark T. Benson. Minimal regular graphs of girths eight and twelve. *Canad. J. Math.*, 18:1091–1094, 1966.
- [2] Pavle V. M. Blagojević, Boris Bukh, and Roman Karasev. Turán numbers for $K_{s,t}$ -free graphs: topological obstructions and algebraic constructions. *Israel J. Math.*, 197(1):199–214, 2013. [arXiv:1108.5254](https://arxiv.org/abs/1108.5254).
- [3] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combinatorial Theory Ser. B*, 16:97–105, 1974.
- [4] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.*, 9:281–285, 1966.
- [5] P. Erdős. On sequences of integers no one of which divides the product of two others and on some related problems. *Inst. Math. Mech. Univ. Tomsk*, 2:74–82, 1938.
- [6] P. Erdős and A. Rényi. On a problem in the theory of graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 7:623–641 (1963), 1962.
- [7] Zoltan Füredi, Assaf Naor, and Jacques Verstraëte. On the Turán number for the hexagon. *Adv. Math.*, 203(2):476–496, 2006. www.cims.nyu.edu/~naor/homepagefiles/final-hexagons.pdf.
- [8] Zoltán Füredi and Miklós Simonovits. The history of degenerate (bipartite) extremal graph problems. [arXiv:1306.5167](https://arxiv.org/abs/1306.5167), June 2013.

- [9] Peter Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011. <http://www.maths.ox.ac.uk/~keevash/papers/turan-survey.pdf>.
- [10] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.
- [11] Felix Lazebnik and Vasiliy A. Ustimenko. Explicit construction of graphs with an arbitrary large girth and of large size. *Discrete Appl. Math.*, 60(1-3):275–284, 1995. ARIDAM VI and VII (New Brunswick, NJ, 1991/1992).
- [12] Keith E. Mellinger and Dhruv Mubayi. Constructions of bipartite graphs from finite geometries. *J. Graph Theory*, 49(1):1–10, 2005. <http://homepages.math.uic.edu/~mubayi/papers/ArcConstf.pdf>.
- [13] Oleg Pikhurko. A note on the Turán function of even cycles. *Proc. Amer. Math. Soc.*, 140(11):3687–3692, 2012. <http://homepages.warwick.ac.uk/~maskat/Papers/EvenCycle.pdf>.
- [14] Alexander Sidorenko. What we know and what we do not know about Turán numbers. *Graphs Combin.*, 11(2):179–199, 1995.
- [15] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.
- [16] P. Turán. On an extremal problem in graph theory (in Hungarian). *Mat. Fiz. Lapok*, 48:436–452, 1941.
- [17] Jacques Verstraëte. On arithmetic progressions of cycle lengths in graphs. *Combin. Probab. Comput.*, 9(4):369–373, 2000. [arXiv:math/0204222](https://arxiv.org/abs/math/0204222).
- [18] R. Wenger. Extremal graphs with no C^4 's, C^6 's, or C^{10} 's. *J. Combin. Theory Ser. B*, 52(1):113–116, 1991.