A bound on the number of edges in graphs without an even cycle

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Abstract

We show that, for each fixed k, an n-vertex graph not containing a cycle of length $2k$ has at most $80\sqrt{k\log k}\cdot n^{1+1/k}+O(n)$ edges.

Introduction

Let $\mathrm{ex}(n,F)$ be the largest number of edges in an n-vertex graph that contains no copy of a fixed graph F. The first systematic study of $\mathrm{ex}(n,F)$ was started by Turán [16], and now it is a central problem in extremal graph theory (see surveys [14, 9]).

The function $\mathrm{ex}(n,F)$ exhibits a dichotomy: if F is not bipartite, then $\mathrm{ex}(n,F)$ grows quadratically in n, and is fairly well understood. If F is bipartite, $\mathrm{ex}(n,F)$ is subquadratic, and for very few F the order of magnitude is known. The two simplest classes of bipartite graphs are complete bipartite graphs, and cycles of even length. Most of the study of $\mathrm{ex}(n,F)$ for bipartite F has been concentrated on these two classes. In this paper, we address the even cycles. For an overview of the status of $\mathrm{ex}(n,F)$ for complete bipartite graphs see [2]. For a thorough survey on bipartite Turán problems see [8].

The first bound on the problem is due to Erdős[5] who showed that $\mathrm{ex}(n,C_4)=\Theta(n^{3/2})$. Thanks to the works of Erdős and Rényi [6], Brown [4, Section 3], and Kövari, Sós and Turán [10] it is now known that

$$\mathrm{ex}(n,C_4)=(1/2+o(1))n^{3/2}.$$

The best current bound for $\mathrm{ex}(n,C_6)$ for large values of n is

$$0.5338n^{4/3}<\mathrm{ex}(n,C_6)\le 0.6272n^{4/3}$$

due to Füredi, Naor and Verstraëte [7].

A general bound of $\mathrm{ex}(n,C_{2k})\le\gamma_k n^{1+1/k}$, for some unspecified constant γ_k, was asserted by Erdős. The first proof was by Bondy and Simonovits [3], who showed that $\mathrm{ex}(n,C_{2k})\le 20kn^{1+1/k}$ for all sufficiently large n. This was improved by Verstraëte [17] to $8(k-1)n^{1+1/k}$ and by Pikhurko [13] to $(k-1)n^{1+1/k}+O(n)$. The principal result of the present paper is an improvement of these bounds:

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Main Theorem. *Suppose G is n-vertex graph that contains no C_{2k}, and $n \geq (2k)^{8k^2}$ then*

$$\operatorname{ex}(n, C_{2k}) \leq 80\sqrt{k \log k} \cdot n^{1+1/k} + 10k^2 n.$$

It is our duty to point out that the improvement offered by the Main Theorem is of uncertain value because we still do not know if $\Theta(n^{1+1/k})$ is the correct order of magnitude for $\operatorname{ex}(n, C_{2k})$. Only for $k = 2, 3, 5$ constructions of C_{2k}-free graphs with $\Omega(n^{1+1/k})$ edges are known [1, 18, 11, 12]. The first author believes it to be likely that $\operatorname{ex}(n, C_{2k}) = o(n^{1+1/k})$ for all large k. We stress again that the situation is completely different for odd cycles, where the value of $\operatorname{ex}(n, C_{2k+1})$ is known exactly for all large n [15].

Proof method and organization of the paper Our proof is inspired by that of Pikhurko [13]. Apart from a couple of lemmas that we quote from [13], the proof is self-contained. However, we advise the reader to at least skim [13] to see the main idea in a simpler setting.

Pikhurko's proof builds a breadth-first search tree, and then argues that a pair of adjacent levels of the tree cannot contain a Θ-graph[1]. It is then deduced that each level must be at least $\delta/(k-1)$ times larger than the previous, where δ is the (minimum) degree. The bound on $\operatorname{ex}(n, C_{2k})$ then follows. The estimate of $\delta/(k-1)$ is sharp when one restricts one's attention to a pair of levels.

In our proof, we use three adjacent levels. We find a Θ-graph satisfying an extra technical condition that permits an extension of Pikhurko's argument. Annoyingly, this extension requires a bound on the *maximum degree*. To achieve such a bound we use a modification of breadth-first search that avoids the high-degree vertices.

What we really prove in this paper is the following:

Theorem 1. *Suppose $k \geq 4$, and suppose G is a biparite n-vertex graph of minimum degree at least $2d + 5k^2$, where*

$$d \geq \max(20\sqrt{k \log k} \cdot n^{1/k}, (2k)^{8k}), \tag{1}$$

then G contains C_{2k}.

The Main Theorem follows from Theorem 1 and two well-known facts: every graph contains a bipartite subgraph with half of the edges, and every graph of average degree d_{avg} contains a subgraph of minimum degree at least $d_{\text{avg}}/2$.

The rest of the paper is organized as follows. We present our modification of breadth-first search in Section 1. In Section 2, which is the heart of the paper, we explain how to find Θ-graphs in triples of consecutive levels. Finally, in Section 3 we assemble the pieces of the proof.

1 Graph exploration

Our aim is to have vertices of degree at most Δd for some $k \ll \Delta \ll d^{1/k}$. The particular choice is fairly flexible; we choose to use

$$\Delta \stackrel{\text{def}}{=} k^3.$$

[1] We recall the definition of a Θ-graph in Section 2

Let G be a graph, and let x be any vertex of G. We start our exploration with the set $V_0 = \{x\}$, and mark the vertex x as explored. Suppose $V_0, V_1, \dots, V_{i-1}$ are the sets explored in the 0th, 1st,…,$(i-1)$st steps respectively. We then define V_i as follows:

1. Let V_i' consist of those neighbors of V_{i-1} that have not yet been explored. Let Bg_i be the set of those vertices in V_i' that have more than Δd unexplored neighbors, and let $\mathrm{Sm}_i = V_i' \setminus \mathrm{Bg}_i$.

2. Define
$$V_i = \begin{cases} V_i' & \text{if } |\mathrm{Bg}_i| > \frac{1}{k+1}|V_i'|, \\ \mathrm{Sm}_i & \text{if } |\mathrm{Bg}_i| \leq \frac{1}{k+1}|V_i'|. \end{cases}$$
The vertices of V_i are then marked as explored.

We call sets $V_0, V_1, \dots$ *levels* of G. A level V_i is *big* if $|\mathrm{Bg}_i| > \frac{1}{k+1}|V_i'|$, and is *normal* otherwise.

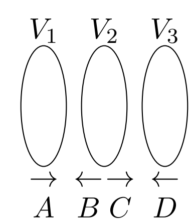
Lemma 2. *If $\delta \leq \Delta d$, and G is a bipartite graph of minimum degree δ, then each $v \in V_{i+1}$ has at least δ neighbors in $V_i \cup V_{i+2}'$.*

Proof. Fix a vertex $v \in V(G)$. We will show, by induction on i, that if $v \notin V_1 \cup \dots \cup V_i$, then v has at least δ neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$. The base case $i = 1$ is clear. Suppose $i > 1$. If $v \in \mathrm{Bg}_i$, then v has $\Delta d \geq \delta$ neighbors in the required set. Otherwise, v is not in V_i' and hence has no neighbors in V_{i-1}. Hence, v has as many neighbors in $V(G) \setminus (V_1 \cup \dots \cup V_{i-1})$ as in $V(G) \setminus (V_1 \cup \dots \cup V_{i-2})$, and our claim follows from the induction hypothesis.

If $v \in V_{i+1}$, then the neighbors of v are a subset of $V_1 \cup \dots \cup V_i \cup V_{i+2}'$. Hence, at least δ of these neighbors lie in $V_i \cup V_{i+2}'$. □

Trilayered graphs A *trilayered graph* with layers V_1, V_2, V_3 is a graph G on a vertex set V_1, V_2, V_3 such that the only edges in G are between V_1 and V_2, and between V_2 and V_3. If $V_1' \subset V_1$, $V_2' \subset V_2$ and $V_3' \subset V_3$, then we denote by $G[V_1', V_2', V_3']$ the trilayered subgraph induced by three sets V_1', V_2', V_3'. Any three sets V_{i-1}, V_i, V_{i+1}' from the exploration process naturally form a trilayered graph; these graphs and their subgraphs are the only trilayered graphs that appear in this paper.

We say that a trilayered graph has *minimum degree* at least $[A : B, C : D]$ if each vertex in V_1 has at least A neighbors in V_2, each vertex in V_2 has at least B neighbors in V_1, each vertex in V_2 has at least C neighbors in V_3, and each vertex in V_3 has at least D neighbors in V_2. A schematic drawing of such a graph is on the right.



2 Θ-graphs

A Θ-*graph* is a cycle of length at least $2k$ with a chord. We shall use several lemmas from the previous works.

Lemma 3 (Lemma 2.1 in [13], also Lemma 2 in [17])**.** *Let F be a Θ-graph and $1 \leq l \leq |V(F)| - 1$. Let $V(F) = W \cup Z$ be an arbitrary partition of its vertex set into two non-empty parts such that every path in F of length l that begins in W necessarily ends in W. Then F is bipartite with parts W and Z.*

Lemma 4 (Lemma 2.2 in [13])**.** *Let $k \geq 3$. Any bipartite graph H of minimum degree at least k contains a Θ-graph.*

Corollary 5. *Let $k \geq 3$. Any bipartite graph H of average degree at least $2k$ contains a Θ-graph.*

For a graph G and a set $Y \subset V(G)$ let $G[Y]$ denote the graph induced on Y. For disjoint $Y, Z \subset V(G)$ let $G[Y, Z]$ denote the bipartite subgraph of G that is induced by the bipartition $Y \cup Z$.

Suppose G is a trilayered graph with layers V_1, V_2, V_3. We say that a Θ-graph $F \subset G$ is *well-placed* if each vertex of $V(F) \cap V_2$ is adjacent to some vertex in $V_1 \setminus V(F)$.

Lemma 6. *Suppose G is a trilayered graph with layers V_1, V_2, V_3 such that the degree of every vertex in V_2 is between $2d + 5k^2$ and Δd. Suppose t is a nonnegative integer, and let $F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|}$. Assume that*

$$\begin{aligned} &a) && F \geq 2, \\ &b) && e(V_1, V_2) \geq 2kF|V_1|, \\ &c) && e(V_1, V_2) \geq 8k(t+1)^2(2\Delta k)^{2k-1}|V_1|, \\ &d) && e(V_1, V_2) \geq 8(et/F)^t k|V_2|, \\ &e) && e(V_1, V_2) \geq 20(t+1)^2|V_2|. \end{aligned} \tag{2}$$

Then at least one of the following holds:

I) There is a Θ-graph in $G[V_1, V_2]$.

II) There is a well-placed Θ-graph in $G[V_1, V_2, V_3]$.

The proof of Lemma 6 is in two parts: finding trilayered subgraph of large minimum degree (Lemmas 7 and 8), and finding a well-placed Θ-graph inside that trilayered graph (Lemma 9).

Finding a trilayered subgraph of large minimum degree The disjoint union of two bipartite graphs shows that a trilayered graph with many edges need not contain a trilayered subgraph of large minimum degree. We show that, in contrast, if a trilayered graph contains no Θ-graph between two of its levels, then it must contain a subgraph of large minimum degree:

Lemma 7. *Let a, A, B, C, D be positive real numbers. Suppose G is a trilayered graph with layers V_1, V_2, V_3 and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Assume also that*

$$a \cdot e(V_1, V_2) \geq (A + k + 1)|V_1| + B|V_2|. \tag{3}$$

Then one of the following holds:

I) There is a Θ-graph in $G[V_1, V_2]$.

II) There exist non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$.

III) There is a subset $\widetilde{V}_2 \subset V_2$ such that $e(V_1, \widetilde{V}_2) \geq (1-a)e(V_1, V_2)$, and $|\widetilde{V}_2| \leq D|V_3|/d$.

Proof. We suppose that alternative (I) does not hold. Then, by Corollary 5, the average degree of every subgraph of $G[V_1, V_2]$ is at most $2k$.

Consider the process that aims to construct a subgraph satisfying (II). The process starts with $V_1' = V_1$, $V_2' = V_2$ and $V_3' = V_3$, and at each step removes one of the vertices that violate the minimum degree condition on $G[V_1', V_2', V_3']$. The process stops when either no vertices are left, or the minimum degree of $G[V_1', V_2', V_3']$ is at least $[A : B, C : D]$. Since in the latter case we are done, we assume that this process eventually removes every vertex of G.

Let R be the vertices of V_2 that were removed because at the time of removal they had fewer than C neighbors in V_3'. Put

$$\begin{aligned} E' &\stackrel{\text{def}}{=} \{uv \in E(G) : u \in V_2,\ v \in V_3,\ \text{and } v \text{ was removed before } u\}, \\ S &\stackrel{\text{def}}{=} \{v \in V_2 : v \text{ has at least } 4k^2 \text{ neighbors in } V_1\}. \end{aligned}$$

Note that $|E'| \le D|V_3|$. We cannot have $|S| \ge |V_1|/k$, for otherwise the average degree of the bipartite graph $G[V_1, S]$ would be at least $\frac{4k}{1+1/k} \ge 2k$. So $|S| \le |V_1|/k$.

The average degree condition on $G[V_1, S]$ implies that

$$e(V_1, S) \le k(|V_1| + |S|) \le (k+1)|V_1|.$$

Let u be any vertex in $R \setminus S$. Since it is connected to at least $d + C$ vertices of V_3, it must be adjacent to at least d edges of E'. Thus,

$$|R \setminus S| \le |E'|/d \le D|V_3|/d.$$

Assume that the conclusion (III) does not hold with $\widetilde{V}_2 = R \setminus S$. Then $e(V_1, R \setminus S) < (1-a)e(V_1, V_2)$. Since the total number of edges between V_1 and V_2 that were removed due to the minimal degree conditions on V_1 and V_2 is at most $A|V_1|$ and $B|V_2|$ respectively, we conclude that

$$\begin{aligned} e(V_1, V_2) &\le e(V_1, S) + e(V_1, R \setminus S) + A|V_1| + B|V_2| \\ &< (k+1)|V_1| + (1-a)e(V_1, V_2) + A|V_1| + B|V_2|, \\ a \cdot e(V_1, V_2) &< (A + k + 1)|V_1| + B|V_2|. \end{aligned}$$

The contradiction completes the proof. ☐

Remark. The preceding lemma by itself is sufficient to prove the estimate $\operatorname{ex}(n, C_{2k}) = O(k^{2/3} n^{1+1/k})$. For that, one chooses approximately $B = k^{2/3}$, $D = k^{1/3}$ and $a = 1/2$. One can then show that when applied to trilayered graphs arising from the exploration process the alternative (III) leads to a subgraph of average degree $2k$. The two remaining alternatives are dealt by Corollary 5 and Lemma 9. However, it is possible to obtain a better bound by iterating the preceding lemma.

Lemma 8. *Let C be a positive real number. Suppose G is a trilayered graph with layers V_1, V_2, V_3, and the degree of every vertex in V_2 is at least $d + 4k^2 + C$. Let $F = \frac{d \cdot e(V_1, V_2)}{8k|V_3|}$, and assume that F and $e(V_1, V_2)$ satisfy (2). Then one of the following holds:*

I) There is a Θ-graph in $G[V_1, V_2]$.

II) There exist numbers A, B, D and non-empty subsets $V_1' \subset V_1$, $V_2' \subset V_2$, $V_3' \subset V_3$ such that the induced trilayered subgraph $G[V_1', V_2', V_3']$ has minimum degree at least $[A : B, C : D]$, with the following inequalities that bind A, B, and D:

$$\begin{aligned} &B \geq 5, \qquad (B-4)D \geq 2k, \\ &A \geq 2k(\Delta D)^{D-1}. \end{aligned} \tag{4}$$

Proof. Assume, for the sake of contradiction, that neither (I) nor (II) hold. With hindsight, set $a_j = \frac{1}{t-j+1}$ for $j = 0, \dots, t-1$. We shall define a sequence of sets $V_2 = V_2^{(0)} \supseteq V_2^{(1)} \supseteq \cdots \supseteq V_2^{(t)}$ inductively. We denote by

$$d_i \stackrel{\text{def}}{=} e(V_1, V_2^{(i)})/|V_2^{(i)}|$$

the average degree from $V_2^{(i)}$ into V_1. The sequence $V_2^{(0)}, V_2^{(1)}, \dots, V_2^{(t)}$ will be constructed so as to satisfy

$$e(V_1, V_2^{(i+1)}) \geq (1 - a_i)e(V_1, V_2^{(i)}), \tag{5}$$

$$d_{i+1} \geq d_i \cdot Fa_i \prod_{j=0}^{i} (1 - a_j). \tag{6}$$

Note that (5) and the choice of $a_0, \dots, a_i$ imply that

$$e(V_1, V_2^{(i)}) \geq \tfrac{1}{t+1} e(V_1, V_2). \tag{7}$$

The sequence starts with $V_2^{(0)} = V_2$. Assume $V_2^{(i)}$ has been defined. We proceed to define $V_2^{(i+1)}$. Put

$$\begin{aligned} A &= a_i e(V_1, V_2^{(i)})/2|V_1| - k - 1, \\ B &= a_i d_i/4 + 5, \\ D &= \min(2k, 8k/a_i d_i). \end{aligned}$$

With help of (7) and (2c) it is easy to check that the inequalities (4) hold for this choice of constants.

In addition,

$$\begin{aligned} (A + k + 1)|V_1| + B|V_2^{(i)}| &= \tfrac{3}{4} a_i e(V_1, V_2^{(i)}) + 5|V_2^{(i)}| \\ &\stackrel{(2e)}{\leq} \tfrac{3}{4} a_i e(V_1, V_2^{(i)}) + \tfrac{1}{4(t+1)^2} e(V_1, V_2) \\ &\stackrel{(7)}{\leq} a_i e(V_1, V_2^{(i)}). \end{aligned}$$

So, the condition (3) of Lemma 7 is satisfied for the graph $G[V_1, V_2^{(i)}, V_3]$. By Lemma 7 there is a subset $V_2^{(i+1)} \subset V_2^{(i)}$ satisfying (5) and

$$|V_2^{(i+1)}| \leq D|V_3|/d.$$

Next we show that the set $V_2^{(i+1)}$ satisfies inequality (6). Indeed, we have

$$\begin{aligned} d_{i+1} = \frac{e(V_1, V_2^{(i+1)})}{|V_2|} &\geq \frac{(1-a_i)e(V_1, V_2^{(i)})}{D|V_3|/d} = (1-a_i)a_i d_i \frac{d}{8k|V_3|} e(V_1, V_2^{(i)}) \\ &\geq (1-a_i)a_i d_i \frac{d \cdot e(V_1, V_2)}{8k|V_3|} \prod_{j=0}^{i-1}(1-a_j) = d_i \cdot F a_i \prod_{j=0}^{i}(1-a_j). \end{aligned}$$

Iterative application of (6) implies

$$d_t \geq d_0 F^t \prod_{j=0}^{t-1} a_j (1-a_j)^{t-j} \geq d_0 F^t \prod_{j=0}^{t-1} \frac{e^{-1}}{t-j+1} = d_0 \frac{(F/e)^t}{(t+1)!}. \tag{8}$$

If we have $|V_2^{(t)}| < |V_1|$, then the average degree of induced subgraph $G[V_1, V_2^{(t)}]$ is greater than $e(V_1, V_2^{(t)})/|V_1| \overset{(7)}{\geq} e(V_1, V_2)/(t+1)|V_1| \overset{(2c)}{\geq} 2k$, which by Corollary 5 leads to outcome (I).

If $|V_2^{(t)}| \geq |V_1|$ and $d_t \geq 4k$, then the average degree of $G[V_1, V_2^{(t)}]$ is at least $d_t/2 \geq 2k$, again leading to the outcome (I). So, we may assume that $d_t < 4k$. Since $(t+1)! \leq 2t^t$ we deduce from (8) that

$$d_0 \leq 4k(t+1)!(e/F)^t \leq 8k(et/F)^t.$$

This contradicts (2d), and so the proof is complete. □

Locating well-placed Θ-graphs in trilayered graphs We come to the central argument of the paper. It shows how to embed well-placed Θ-graphs into trilayered graphs of large minimum degree. Or rather, it shows how to embed well-placed Θ-graphs into regular trilayered graphs; the contortions of the previous two lemmas, and the factor of $\sqrt{\log k}$ in the final bound, come from authors' inability to deal with irregular graphs.

Lemma 9. *Let A, B, D be positive real numbers. Let G be a trilayered graph with layers V_1, V_2, V_3 of minimum degree at least $[A, B, d+k, D]$. Suppose that no vertex in V_2 has more than Δd neighbors in V_3. Assume also that*

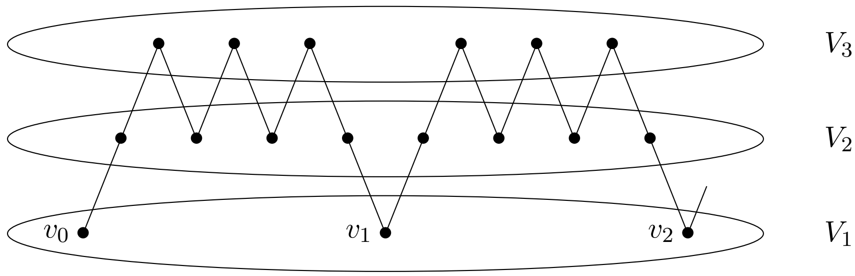
$$B \geq 5 \tag{9}$$

$$(B-4)D \geq 2k-2 \tag{10}$$

$$A \geq 2k(\Delta D)^{D-1}. \tag{11}$$

Then G contains a well-placed Θ-graph.

Proof. Assume, for the sake of contradiction, that G contains no well-placed Θ-graphs. Leaning on this assumption we shall build an arbitrary long path P of the form



where, for each i, vertices v_i and v_{i+1} are joined by a path of length $2D$ that alternates between V_2 and V_3. Since the graph is finite, this would be a contradiction.

While building the path we maintain the following property:

$$\text{Every } v \in P \cap V_2 \text{ has at least one neighbor in } V_1 \setminus P. \tag{$\star$}$$

We call a path satisfying ($\star$) *good*.

We construct the path inductively. We begin by picking v_0 arbitrarily from V_1. Suppose a good path $P = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1}$ has been constructed, and we wish to find a path extension $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow v_l$.

For each $i = 1, 2, \ldots, 2D-1$ we shall define a family $\mathcal{Q}_i$ of good paths that satisfy

1. Each path in $\mathcal{Q}_i$ is of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u$, where $v_{l-1} \leftrightsquigarrow u$ is a path of length i that alternates between V_2 and V_3. The vertex u is called a *terminal* of the path. The set of terminals of the paths in $\mathcal{Q}_i$ is denoted by $T(\mathcal{Q}_i)$.
2. For each i, the paths in $\mathcal{Q}_i$ have distinct terminals.
3. For odd-numbered indices, we have the inequality
$$|\mathcal{Q}_{2i+1}| \geq -3k + A\left(\frac{1}{\Delta}\right)^i \prod_{j \leq i}\left(1 - \frac{j}{D}\right).$$
4. For even-numbered indices, we have the inequality
$$e(T(\mathcal{Q}_{2i}), V_2) \geq d|\mathcal{Q}_{2i-1}|.$$

Let
$$t \stackrel{\text{def}}{=} \lceil B/2 \rceil.$$

We will repeatedly use the following straightforward fact, which we call the *small-degree argument*: whenever Q is a good path and $u \in V_2$ is adjacent to the terminal of Q, then the path Qu is adjacent to fewer than t vertices in $V_1 \cap Q$. Indeed, if vertex u were adjacent to $v_{j_1}, v_{j_2}, \ldots, v_{j_t} \in V_1 \cap Q$, then $v_{j_2} \leftrightsquigarrow u$ (along path Q) and the edge uv_{j_2} would form a cycle of total length at least $2D(t-2) + 2 \geq 2D(B/2-2) + 2 \overset{(10)}{\geq} 2k$. As uv_{j_3} is a chord of the cycle, and u is adjacent to v_{j_1} that is not on the cycle, that would contradict the assumption that G contains no well-placed Θ-graph.

The set $\mathcal{Q}_1$ consists of all paths of the form Pu for $u \in V_2 \setminus P$. Let us check that the preceding conditions hold for $\mathcal{Q}_1$. Vertex v_{l-1} cannot be adjacent to k or more vertices in $P \cap V_2$, for otherwise G would contain a well-placed Θ-graph with a chord through v_{l-1}. So, $|\mathcal{Q}_1| \geq A - k$. Next, consider any $u \in V_2 \setminus P$ that is a neighbor of v_{l-1}. By the small-degree argument vertex u cannot be adjacent to t or more vertices of $P \cap V_1$, and Pu is good.

Suppose $\mathcal{Q}_{2i-1}$ has been defined, and we wish to define $\mathcal{Q}_{2i}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i-1}$. Vertex u cannot have k or more neighbors in $Q \cap V_3$, for otherwise G would contain a well-placed Θ-graph with a chord through u. Hence, there are at least d edges of the form uw, where $w \in V_3 \setminus Q$. As we vary u we obtain a family of at least $d|\mathcal{Q}_{2i-1}|$ paths eligible for inclusion into $\mathcal{Q}_{2i}$. We let $\mathcal{Q}_{2i}$ consist of any maximal set of such paths with distinct terminals.

Suppose $\mathcal{Q}_{2i}$ has been defined, and we wish to define $\mathcal{Q}_{2i+1}$. Consider an arbitrary path $Q = v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2i}$. An edge uw is called *long* if $w \in P$, and w is at a distance exceeding $2k$ from u along path Q. If uw is a long edge, then from u to Q there is only one edge, namely the edges to the predecessor of u on Q, for otherwise there is a well-placed Θ-graph. Also, at most i neighbors of u lie on the path $v_{l-1} \leftrightsquigarrow u$. Since $\deg u \geq D$, it follows that at least $(1 - i/D) \deg u$ short edges from u that miss $v_{l-1} \leftrightsquigarrow u$. Thus there is a set $\mathcal{W}$ of at least $(1 - i/D) e(T(\mathcal{Q}_{2i}), V_2)$ walks (not necessarily paths!) of the form $v_0 \leftrightsquigarrow v_1 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw$ such that $v_{l-1} \leftrightsquigarrow uw$ is a path and w occurs only among the last $2k$ vertices of the walk.

From the maximum degree condition on V_2 it follows that walks in $\mathcal{W}$ have at least $(1 - i/D) e(T(\mathcal{Q}_{2i}), V_2)/\Delta d$ distinct terminals. A walk fails to be a path only if the terminal vertex lies on P. However, since the edge uw is short, this can happen for at most $2k$ possible terminals. Hence, there is a $\mathcal{Q}_{2i+1} \subset \mathcal{W}$ of size $|\mathcal{Q}_{2i+1}| \geq (1 - i/D) e(T(\mathcal{Q}_{2i}), V_2)/\Delta d - 2k$ that consists of paths with distinct terminals. It remains to check that every path in $\mathcal{Q}_{2i+1}$ is good. The only way that $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uw \in \mathcal{Q}_{2i+1}$ may fail to be good is if w has no neighbors in $V_1 \setminus Q$. By the small-degree argument w has fewer than t neighbors in V_1. Since w has at least B neighbors in V_1 and $B \geq t + 2$, we conclude that w has at least *two* neighbors in V_1 outside the path. Of course, the same is true for every terminal of a path in $\mathcal{Q}_{2i+1}$.

Note that $\mathcal{Q}_{2D-1}$ is non-empty. Let $Q = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow u \in \mathcal{Q}_{2D-1}$ be an arbitrary path. Note that since $2D - 1$ is odd, $u \in V_2$. By the property of terminals of V_i (odd i) that we noted in the previous paragraph, there are two vertices in $V_1 \setminus Q$ that are neighbors of u. Let v_l be any of them, and let the new path be $Qv_l = v_0 \leftrightsquigarrow \cdots \leftrightsquigarrow v_{l-1} \leftrightsquigarrow uv_l$. This path can fail to be good if there is a vertex w on the path Q that is good in Q, but is bad in Qv_l. By the small-degree argument, w is adjacent to fewer than t vertices in $Q \cap V_1$ that precede w in Q. The same argument applied to the reversal of the path Qv_l shows that w is adjacent to fewer than t vertices in $Q \cap V_1$ that succeeds w in Q. Since $2t - 2 < B$, the path Qv_l is good.

Hence, it is possible to build an arbitrarily long path in G. This contradicts the finiteness of G. $\square$

Lemma 6 follows from Lemmas 8 and 9 by setting $C = d + k$, in view of inequality $4k^2 + k \leq 5k^2$.

3 Proof of Theorem 1

Suppose G has minimum degree of at least $2d + 4k^2 + k$ and contains no C_{2k}. Pick a root vertex x arbitrarily, and let $V_0, V_1, \dots, V_{k-1}$ be the levels obtained from the exploration process in Section 1.

Lemma 10. *For $1 \le i \le k-1$, the graph $G[V_{i-1}, V_i, V_{i+1}]$ contains no well-placed Θ-graph.*

Proof. The following proof is almost an exact repetition of the proof of Claim 3.1 from [13] (which is also reproduced as Lemma 11 below).

Suppose, for the sake of contradiction, that a well-placed Θ-graph $F \subset G[V_{i-1}, V_i, V_{i+1}]$ exists. Let $Y = V_i \cap V(F)$. Since F is well-placed, for every vertex of Y there is a path of length i to the vertex x. The union of these paths forms a tree T with x as a root. Let y be the vertex farthest from x such that every vertex of Y is a T-descendant of y. Paths that connect y to Y branch at y. Pick one such branch, and let $W \subset Y$ be the set of all the T-descendants of that branch. Let $Z = V(F) \setminus W$. From $Y \neq V_i \cap V(F)$ it follows that Z is not an independent set of F, and so $W \cup Z$ is not a bipartition of F.

Let ℓ be the distance between x and y. We have $\ell < i$ and $2k - 2i + 2\ell < 2k \le |V(F)|$. By Lemma 3 in F there is a path P of length $2k - 2i + 2\ell$ that starts at some $w \in W$ and ends in $z \in Z$. Since the length of P is even, $z \in Y$. Let P_w and P_z be unique paths in T that connect y to respectively w and z. They intersect only at y. Each of P_w and P_z has length $i - \ell$. The union of paths P, P_w, P_z forms a $2k$-cycle in G. $\square$

The same argument (with a different Y) also proves the next lemma.

Lemma 11 (Claim 3.1 in [13])**.** *For $1 \le i \le k-1$, neither of $G[V_i]$ and $G[V_i, V_{i+1}]$ contains a bipartite Θ-graph.*

The next step is to show that the levels $V_0, V_1, V_2, \dots$ increase in size. We shall show by induction on i that

$$e(V_i, V_{i+1}) \ge d|V_i|, \tag{12}$$

$$e(V_i, V_{i+1}) \le 2k|V_{i+1}|, \tag{13}$$

$$e(V_i, V'_{i+1}) \le 2k|V'_{i+1}|, \tag{14}$$

$$|V_{i+1}| \ge (2k)^{-1} d|V_i|, \tag{15}$$

$$|V_{i+1}| \ge \tfrac{d^2}{400k \log k} |V_{i-1}|. \tag{16}$$

Clearly, these hold for $i = 0$ since each vertex of V_1 sends only one edge to V_0.

Proof of (12): By Lemma 2 the degree of every vertex in V_i is at least $d + 3k + 1$, and so

$$e(V_i, V'_{i+1}) \ge |V_i|(d + 3k + 1) - e(V_{i-1}, V_i) \overset{\text{induc.}}{\ge} (d + k + 1)|V_i|.$$

We next distinguish two cases depending on whether V_{i+1} is big (in the sense of the definition from Section 1). If V_{i+1} is big, then $e(V_i, V_{i+1}) = e(V_i, V'_{i+1})$, and (12) follows. If V_{i+1} is normal, then Corollary 5 implies that

$$e(V_i, \mathrm{Bg}_{i+1}) \le k(|V_i| + |\mathrm{Bg}_{i+1}|) \le (k+1)|V_i|$$

and so

$$e(V_i, V_{i+1}) = e(V_i, V'_{i+1}) - e(V_i, \mathrm{Bg}_{i+1}) \geq d|V_i|$$

implying (12). $\square$

Proof of (13): Consider the graph $G[V_i, V_{i+1}]$. Inequality (12) asserts that the average degree of V_i is at least $d \geq 2k$. If (13) does not hold, then the average degree of V_{i+1} is at least $2k$ as well, contradicting Corollary 5. $\square$

Proof for (14): The argument is the same as for (13) with $G[V_i, V'_{i+1}]$ in place of $G[V_i, V_{i+1}]$. $\square$

Proof for (15): This follows from (13) and (12). $\square$

Proof of (16) in the case V_i is a normal level: We assume that (16) does not hold and will derive a contradiction. Consider the trilayered graph $G[V_{i-1}, V_i, V'_{i+1}]$. Let $t = 2\log k$. Suppose momentarily that the inequalities (2) in Lemma 6 hold. Then since V_i is normal, the degrees of vertices in V_i are bounded from above by Δd, and so Lemma 6 applies. However, the lemma's conclusion contradicts Lemmas 10 and 11. Hence, to prove (16) it suffices to verify inequalities (2a–d) with $F = d \cdot e(V_{i-1}, V_i)/8k|V'_{i+1}|$.

We may assume that

$$F \geq 2e^2 \log k, \tag{17}$$

and in particular that (2a) holds. Indeed, if (17) were not true, then inequality (12) would imply $|V'_{i+1}| \geq (d^2/16e^2 k \log k)|V_{i-1}|$, and thus

$$|V_{i+1}| \geq (1 - \tfrac{1}{k})|V'_{i+1}| \geq (d^2/32e^2 k \log k)|V_{i-1}|,$$

and so (16) would follow in view of $32e^2 \leq 400$.

Inequality (2b) is implied by (15). Indeed,

$$e(V_{i-1}, V_i) = 8k|V_{i+1}|F/d \overset{(15)}{\geq} 4F|V_i| \overset{(15)}{\geq} 2k^{-1}dF|V_{i-1}|,$$

and $d \geq k^2$ by the definition of d from (1).

Inequality (2c) is implied by (1) and (12).

Next, suppose (2d) were not true. Since $F/t \geq e^2$ by (17), we would then conclude

$$\begin{aligned} |V_{i+1}| &\overset{(15)}{\geq} (2k)^{-1}d|V_i| \geq (16k^2)^{-1}(F/et)^t e(V_{i-1}, V_i) \\ &\geq (16k^2)^{-1}e^{2\log k} e(V_{i-1}, V_i) \overset{(12)}{\geq} \tfrac{1}{16}d|V_{i-1}|, \end{aligned}$$

and so (16) would follow.

Finally, (2e) is a consequence of (12). $\square$

Proof of (16) in the case V_i is a big level: We have

$$\begin{aligned}|V_{i+1}| &\geq \tfrac{1}{2}|V'_{i+1}| \overset{(14)}{\geq} (4k)^{-1}e(V_i, V'_{i+1}) \geq (4k)^{-1}e(\mathrm{Bg}_i, V'_{i+1}) \geq (4k)^{-1}\Delta d|\mathrm{Bg}_i| \\ &\geq (8k^2)^{-1}\Delta d|V_i| \overset{(15)}{\geq} (16k^3)^{-1}\Delta d|V_{i-1}| = \tfrac{1}{16}d|V_{i-1}|,\end{aligned}$$

and so (16) holds. □

We are ready to complete the proof of Theorem 1. If k is even, then $\lfloor k/2 \rfloor$ applications of (16) yield

$$|V_k| \geq \frac{d^k}{(400k\log k)^{k/2}}.$$

If k is odd, then $(k-1)/2$ applications of (16) yield

$$|V_k| \geq \frac{d^{k-1}}{(400k\log k)^{(k-1)/2}}|V_1| \geq \frac{d^k}{(400k\log k)^{(k-1)/2}}.$$

Either way, since $|V_k| < n$ we conclude that $d < 20\sqrt{k\log k} \cdot n^{1/k}$.

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