

Bounds on equiangular lines and on related spherical codes

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Abstract

An L -spherical code is a set of Euclidean unit vectors whose pairwise inner products belong to the set L . We show, for a fixed $\alpha, \beta > 0$, that the size of any $[-1, -\beta] \cup \{\alpha\}$ -spherical code is at most linear in the dimension.

In particular, this bound applies to sets of lines such that every two are at a fixed angle to each another.

1 Introduction

Background A set of lines in \mathbb{R}^d is called *equiangular*, if the angle between any two of them is the same. Equivalently, if P is the set of unit direction vectors, the corresponding lines are equiangular with the angle $\arccos \alpha$ if $\langle v, v' \rangle \in \{-\alpha, \alpha\}$ for any two distinct vectors $v, v' \in P$. The second equivalent way of defining equiangular lines is via the Gram matrix. Let M be the matrix whose columns are the direction vectors. Then $M^T M$ is a positive semidefinite matrix whose diagonal entries are 1's, and each of whose off-diagonal entries is $-\alpha$ or α . Conversely, any such matrix of size m and rank d gives rise to m equiangular lines in \mathbb{R}^d .

The equiangular lines have been extensively studied following the works of van Lint and Seidel [9], and of Lemmens and Seidel [7]. Let $N(d)$ be the maximum number of equiangular lines in \mathbb{R}^d . Let N_α be the maximum number of equiangular lines with the angle $\arccos \alpha$. The values of $N(d)$ are known exactly for $d \leq 13$, for $d = 15$, for $21 \leq d \leq 41$ and for $d = 43$ [1, 5]. When d is large, the only known upper bound on $N(d)$ is due to Gerzon (see [7, Theorem 3.5]) and asserts that

$$N(d) \leq d(d+1)/2 \text{ with equality only if } d = 2, 3 \text{ or } d+2 \text{ is a square of an odd integer.}$$

A remarkable construction of de Caen [3] shows that $N(d) \geq \frac{2}{9}(d+1)^2$ for d of the form $d = 6 \cdot 4^i - 1$. A version of de Caen's construction suitable for other values of d has been given by Greaves, Koolen, Munemasa and Szöllösi [5]. See also the work of Jedwab and Wiebe [6] for an alternative construction of $\Theta(d^2)$ equiangular lines. In these constructions the inner product α tends to 0 as dimension grows.

Previously known bounds on $N_\alpha(d)$ The first bound is the so-called *relative bound* (see [9, Lemma 6.1] following [7, Theorem 3.6])

$$N_\alpha(d) \leq d \frac{1 - \alpha^2}{1 - r\alpha^2} \quad \text{if } d < 1/\alpha^2.$$

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While useful in small dimensions, it gives no information for a fixed α and large d . The second bound is

$$N_\alpha(d) \leq 2d \quad \text{unless } 1/\alpha \text{ is an odd integer [7, Theorem 3.4].}$$

This bound can be further improved to $\frac{3}{2}(d+1)$ unless $\frac{1}{2\alpha} + \frac{1}{2}$ is an algebraic integer of degree 2, see [2, Subsection 2.3].

Finally, the values of $N_{1/3}(d)$ and $N_{1/5}(d)$ for a large d have been completely determined:

$$\begin{aligned} N_{1/3}(d) &= 2d - 2 && \text{for } d \geq 15 && \text{[7, Theorem 4.5],} \\ N_{1/5}(d) &= \lfloor 3(d-1)/2 \rfloor && \text{for all sufficiently large } d && \text{[8] and [5, Corollary 6.6].} \end{aligned}$$

New bound We will show that $N_\alpha(d)$ is linear for every α . In fact, we will prove a result in greater generality. Following [4], we call a set of unit vectors P an L -spherical code if $\langle v, v' \rangle \in L$ for every pair of distinct points $v, v' \in P$. In this language, a set of equiangular lines is a $\{-\alpha, \alpha\}$ -spherical code. Let $N_L(d)$ be the maximum cardinality of an L -spherical code in \mathbb{R}^d .

Theorem 1. *For every fixed $\beta > 0$ there exists a constant c_β such that for any L of the form $L = [-1, -\beta] \cup \{\alpha\}$ we have $N_L(d) \leq c_\beta d$.*

We make no effort to optimize the constant c_β that arises from our proof, as it is huge. We speculate about the optimal bounds on $N_L(d)$ in section 3. We do not know if the constant c_β must in fact depend on β .

The rest of the paper is organized as follows. In the next section we prove Theorem 1 and in the concluding section we discuss possible generalizations and strengthenings of Theorem 1.

2 Proof of Theorem 1

Proof sketch The idea behind the proof of Theorem 1 builds upon the argument of Lemmens and Seidel for $N_{1/3}(d)$. Before going into the details, we outline the argument.

Let $L = [-1, -\beta] \cup \{\alpha\}$, and let P be an L -spherical code whose size we wish to bound. Define a graph G on the vertex set P by connecting v and v' by an edge if $\langle v, v' \rangle \in [-1, -\beta]$. In their treatment of $N_{1/3}(d)$ Lemmens and Seidel consider the largest clique in G , and carefully analyze how the rest of the graph attaches to that clique. In contrast, in our argument we consider the largest independent set I of G , and show that almost every other vertex is incident to nearly all vertices of I . Iterating this argument inside the common neighborhood of I we can build a large clique in G . As the clique size is bounded by a function of β , that establishes the theorem.

Proof details For the remainder of the section, L , P and G will be as defined as in the preceding proof sketch. The following two well-known lemmas bound the sizes of cliques and independent sets in G :

Lemma 2. *Suppose u_1, \dots, u_n are n vectors of norm at most 1 satisfying $\langle u_i, u_j \rangle \leq -\gamma$. Then $n \leq 1/\gamma + 1$.*

Proof. This follows from $0 \leq \|\sum u_i\|^2 = \sum_{i,j} \langle u_i, u_j \rangle \leq n - \gamma n(n-1)$. □

Lemma 3.

i. Every independent set in G is linearly independent. In particular, the graph G contains no independent set on more than d vertices.

ii. The graph G contains no clique on more than $1/\beta + 1$ vertices.

Proof. i) Let p_1, \dots, p_n be the points of the independent set. Suppose that $\sum c_i p_i = 0$. Taking an inner product with p_j we obtain $0 = (1 - \alpha)c_j + \alpha \sum c_i$ implying that all c 's are equal. The result follows since $(1 - \alpha) + n\alpha \neq 0$.

ii) This is a special case of the preceding lemma. \square

In the next two lemmas we analyze how the vertices of G attach to an independent set.

Lemma 4. *Suppose that M is a matrix with linearly independent column vectors p_1, \dots, p_n . Suppose that $v, v' \in \text{span}\{p_1, \dots, p_n\}$ are points satisfying $\langle p_i, v \rangle = s_i$ and $\langle p_i, v' \rangle = s'_i$ for some column vectors $s = (s_1, \dots, s_n)^T$ and $s' = (s'_1, \dots, s'_n)^T$. Then $\langle v, v' \rangle = s^T (M^T M)^{-1} s'$.*

Proof. By passing to a subspace we may assume that p_1, \dots, p_n span \mathbb{R}^n , and so M is invertible. As $s = M^T v$ and $s' = M^T v'$, we infer that $\langle v, v' \rangle = v^T v' = ((M^T)^{-1} s)^T (M^T)^{-1} s' = s^T (M^T M)^{-1} s'$. \square

The following lemma is the geometric heart of the proof. In its special case $v = v'$, the lemma bounds degrees from certain vertices into an independent set. More precisely, let I be a sufficiently large independent set. We will show later (in Lemma 6) that the vertices, the norm of whose projection on $\text{span } I$ exceeds $\alpha^{1/2}$, are few. The straightforward, but slightly messy calculations in the following lemma characterize the vertices with such projections in terms of their degree into I . The case $v \neq v'$ is not needed when P comes from the set of equiangular lines, but is required to establish Theorem 1 in its full generality.

Lemma 5. *Let $t = 1/\beta + 1$. There exists $n_0 = n_0(\beta)$ and $\varepsilon = \varepsilon(\beta)$ such that the following holds. Suppose that p_1, \dots, p_n is an independent set in G of size n , and suppose that a points $p, p' \in P$ are adjacent to the same m vertices among p_1, \dots, p_n . Assume $0 < m < n - t$ and $n \geq n_0$. Write $p = v + u$ and $p' = v' + u'$ where $v, v' \in \text{span}\{p_1, \dots, p_n\}$ and u, u' are both orthogonal to $\text{span}\{p_1, \dots, p_n\}$. Then $\langle v, v' \rangle \geq \alpha + \varepsilon$.*

Proof. Since points p_1, \dots, p_n are linearly independent (by Lemma 3), the condition of the preceding lemma is fulfilled. We have $M^T M = \alpha J + (1 - \alpha)I$. One can verify that its inverse is given by

$$(1 - \alpha)(M^T M)^{-1} = I - \phi J \quad \text{with} \quad \phi \stackrel{\text{def}}{=} \frac{\alpha}{1 + (n - 1)\alpha}. \quad (1)$$

Note that $\phi \leq 1/n$ since $\alpha \leq 1$.

Without loss of generality, p_1, \dots, p_m are the m vertices that p and p' are adjacent to. This means that $s \stackrel{\text{def}}{=} M^T v$ and $s' \stackrel{\text{def}}{=} M^T v'$ are of the form $s = (-\beta_1, \dots, -\beta_m, \alpha, \dots, \alpha)$ and $s' = (-\beta'_1, \dots, -\beta'_m, \alpha, \dots, \alpha)$ for some $\beta_1, \beta'_1, \dots, \beta_m, \beta'_m \in [\beta, 1]$. From Lemma 4 and (1) it follows that

$$(1 - \alpha)\langle v, v' \rangle = \alpha^2(n - m) + \sum \beta_i \beta'_i - \phi \left(\sum \beta \right) \left(\sum \beta'_i \right). \quad (2)$$

We claim that, subject to the constraint $\beta_1, \beta'_1, \dots, \beta_m, \beta'_m \in [\beta, 1]$, the right side of (2) is minimized when all the β_i 's and all the β'_i 's are equal to β . Indeed, since $[\beta, 1]^{2m}$ is compact, the minimum is actually attained. Assume that $(\beta_1, \beta'_1, \dots, \beta_m, \beta'_m)$ is the vector achieving the minimum, and let j be the index for which β'_j is the largest. Then the derivative of the right side of (2) with respect to β'_j is $\beta_j - \phi \sum \beta_i \geq \beta_j - \frac{1}{n} \sum \beta_i > \beta_j - \frac{1}{m} \sum \beta_i \geq 0$. By the optimality assumption this implies that $\beta'_j = \beta$, and so $\beta'_i = \beta$ for all i . Similarly, $\beta_i = \beta$ for all i .

We thus deduce that

$$(1 - \alpha)\langle v, v' \rangle \geq \alpha^2(n - m) + \beta^2 m - \phi((n - m)\alpha - m\beta)^2.$$

Let $R(m, n)$ denote the right side of preceding inequality. Let $t^* \stackrel{\text{def}}{=} \frac{(1-\alpha)(\alpha-\beta)}{\alpha(\alpha+\beta)}$. We have

$$t^* = \frac{(1 - \alpha)(\alpha - \beta)}{\alpha(\alpha + \beta)} < \frac{1 - \alpha}{\alpha + \beta} < \frac{1}{\beta} = t - 1.$$

Thus to prove the lemma, it is enough to show that $R(m, n) \geq (1 - \alpha)\alpha + \varepsilon$ whenever $1 \leq m \leq n - t^* - 1$ and $n \geq n_0$ for suitable n_0 and ε .

The expression $R(m, n)$ is a quadratic polynomial in m . A simple calculation shows that it satisfies $R(m, n) = R(n - t^* - m, n)$, and in particular that the maximum of $R(m, n)$ for a fixed n is at the point $m_{\max} \stackrel{\text{def}}{=} (n - t^*)/2$, which is inside the interval $[1, n - t^* - 1]$. Furthermore, at the boundary points of the interval we have

$$R(1, n) = R(n - t^* - 1, n) = \alpha(1 - \alpha) + (\alpha + \beta)^2 - \frac{\alpha(1 + \beta)^2}{1 + \alpha(n - 1)}.$$

Let $n_0 = 1 + 8/\beta^2$. Since $\frac{\alpha(1+\beta)^2}{1+\alpha(n-1)} < \frac{4}{n-1}$, it follows that $R(m, n) \geq R(1, n) > \alpha(1 - \alpha) + \frac{1}{2}(\alpha + \beta)^2$ whenever $1 \leq m \leq n - t^* - 1$ and $n \geq n_0$. In particular $\langle v, v' \rangle > \alpha + \varepsilon$ holds under the same conditions, where $\varepsilon = \frac{1}{2}\beta^2$. \square

Lemma 6. *Suppose p_1, \dots, p_n is an independent set in G . Suppose $p^{(1)}, \dots, p^{(m)} \in P$ are points of the form $p^{(i)} = v^{(i)} + u^{(i)}$ with $v^{(i)} \in \text{span}\{p_1, \dots, p_n\}$ and $u^{(i)} \perp \text{span}\{p_1, \dots, p_n\}$ and $\langle v^{(i)}, v^{(j)} \rangle > \alpha + \varepsilon$ for all i, j . Then $m \leq 1/\varepsilon + 1$.*

Proof. From $\langle p^{(i)}, p^{(j)} \rangle = \langle v^{(i)}, v^{(j)} \rangle + \langle u^{(i)}, u^{(j)} \rangle$ and $\langle p^{(i)}, p^{(j)} \rangle \in [-1, -\beta] \cup \{\alpha\}$, we deduce that $\langle u^{(i)}, u^{(j)} \rangle < -\varepsilon$. The result then follows from Lemma 2. \square

The combinatorial part of the argument is contained in the next result.

Lemma 7. *Suppose $\delta > 0$ is given. Then there exists $M(\beta, \delta)$ such that the following holds. Let $U \subset P$ be arbitrary. Suppose I is a maximum-size independent subset of U . Then there is a subset $U' \subset U \setminus I$ of size $|U'| \geq |U| - M|I|$ such that every vertex of U' is adjacent to at least $(1 - \delta)|I|$ vertices of I .*

Proof. Let t, ε and n_0 be as in Lemma 5, and put $n = \max(n_0, \lceil 1/\delta \rceil)$. Denote by R the least integer such that every graph on R vertices contains either an independent set of size $n + 1$ or a clique of size at least $1/\beta + 2$ (such an R exists by Ramsey's theorem). Let

$$\begin{aligned} M &= \max(R, (1/\varepsilon + 1)2^n), \\ N &= |I|. \end{aligned}$$

If $|U| < M$, then $|U| - M|I|$ is negative, and the lemma is vacuous. So, assume $|U| \geq M$. In particular, $|U| \geq R$, and since by Lemma 3 the set U contains no clique of size greater $1/\beta + 1$, we conclude that $N \geq n + 1$.

Arrange the elements of I on a circle, and consider all N circular intervals containing n vertices of I . Let S_1, S_2, \dots, S_N be these intervals, in order.

We declare a vertex $p \in U \setminus I$ to be *i-bad* if it is adjacent to between 1 and $n - t^*$ vertices of S_i . For a set $T \subset S_i$, we call an *i-bad* vertex p to be *of type T* if T is precisely the set of neighbors of p in the set S_i . Let $B_{i,T}$ be the set of all *i-bad* vertices of type T , and let $B_i = \bigcup_T B_{i,T}$ be the set of all *i-bad* vertices. By Lemmas 5 and 6 we have $|B_{i,T}| \leq 1/\varepsilon + 1$ for every T , and so

$$|B_i| \leq (1/\varepsilon + 1)(2^n - 1).$$

Let $B = \bigcup B_i$ be the set of bad vertices. Hence, $|B| \leq N(1/\varepsilon + 1)(2^n - 1)$, and $|B \cup I| \leq MN$.

Consider a vertex $p \in U \setminus I$ that is good, i.e., $p \notin B$. Since I is a maximal independent set, p is adjacent to at least one vertex of I . Say p is adjacent to a vertex of S_i for some i . Since p is good, p must in fact be adjacent to at least $n - t$ vertices of S_i . As S_i shares $n - 1$ vertices with both S_{i-1} and S_{i+1} , we are impelled to conclude that p must be adjacent to some of the vertices of S_{i-1} and of S_{i+1} . Repeating this argument we conclude that p is non-adjacent to at most t elements from among any interval of length n . In particular, p is adjacent to at least $N(1 - t/n)$ vertices of I . As p is an arbitrary good vertex and $t/n \leq \delta$, the lemma follows. \square

We are now ready to complete the proof of Theorem 1. Indeed, with foresight we set

$$\begin{aligned} B &= \lceil 1/\beta + 1 \rceil, \\ \delta &= 1/(B + 1)^2. \end{aligned}$$

and let M be as in the preceding lemma. Put $U_0 = P$ and let I_0 be a maximal independent set in U_0 . By the preceding lemma, there exists $U_1 \subset U_0 \setminus I_0$ such that every vertex of U_1 is adjacent to $(1 - \delta)|I_0|$ vertices of I_0 and $|U_1| \geq |U_0| - M|I_0|$. In view of Lemma 3, $|U_1| \geq |U_0| - Md$. Let I_1 be a maximal independent set in U_1 . Repeating this argument, we obtain a nested sequence of sets $U_0 \supset U_1 \supset \dots$ and a corresponding sequence of independent sets I_0, I_1, \dots such that

- i. $|U_i| \geq |U_{i-1}| - Md$ for each $i = 1, 2, \dots$,
- ii. For $r < s$, each vertex in I_s is adjacent to at least $(1 - \delta)|I_r|$ vertices of I_r .

We claim that $|P| \leq BMd$, which would be enough to complete the proof of Theorem 1. Indeed, suppose for the sake of contradiction that $|P| > BMd$. Then I_0, \dots, I_B are non-empty. Pick vertices v_0, \dots, v_B uniformly at random from I_0, \dots, I_B respectively. Since, for every $i \neq j$, the pair $v_i v_j$ is an edge with probability at least $1 - \delta$, it follows that v_0, \dots, v_B is a clique with probability at least $1 - \binom{B+1}{2}\delta > 0$. In particular, G then contains a clique of size $B + 1 > \beta/2 + 1$, contrary to Lemma 3. The contradiction shows that $|P| \leq BMd$, completing the proof of Theorem 1.

3 Open problems

- I know of only one asymptotic lower bound on N_L . It is a version of [5, Proposition 5.12] that is also implicit in the bound for $N_{1/3}(d)$ in [7]. Denote by I_n the identity matrix of size n , and by J_n the all-one matrix of size n . Then the matrix $M = (r-1)I_{rt} - (J_r - I_r) \otimes I_t$ is a positive semidefinite matrix of nullity t , it has $(r-1)$'s on the diagonal, and its off-diagonal entries are 0 and -1 . Hence, $\frac{1}{r-1+\tau}(M + \tau J_{rt})$ is a Gram matrix of a $\{-\frac{1-\tau}{r-1+\tau}, \frac{\tau}{r-1+\tau}\}$ -code in $\mathbb{R}^{(r-1)t+1}$ of size rt . So, $N_L(d) \geq \frac{r}{r-1}d + O(1)$ for $L = \{-\frac{1-\tau}{r-1+\tau}, \frac{\tau}{r-1+\tau}\}$. For $\tau = 1/2$, this yields a family of equiangular lines. The results in [7, 8, 5] suggest that this bound is sharp.

Conjecture 8. *The number of equiangular lines with a given angle is $N_{1/(2r-1)}(d) = \frac{r}{r-1}d + O(1)$ as d tends to infinity.*

In contrast, one can show that the bound implicit in the proof of Theorem 1 is $2^{O(1/\beta^2)}d$.

- Informally, it is natural to think of Theorem 1 as a juxtaposition of two trivial results from Lemma 3: $N_{[-1, -\beta]}(d) = O(1)$ and $N_{\{\alpha\}}(d) = O(d)$. Since $N_{\{\alpha_1, \dots, \alpha_k\}}(d) = O(d^k)$ for any real numbers $\alpha_1, \dots, \alpha_k$ (see [2, Proposition 1]) this motivates the following conjecture.

Conjecture 9. *Suppose $\alpha_1, \dots, \alpha_k$ are any k real numbers, and $L = [-1, -\beta] \cup \{\alpha_1, \dots, \alpha_k\}$. Then $N_L(d) \leq c_{\beta, k}d^k$.*

It is conceivable that in this case even $N_L(d) \leq c_{\beta}N_{\{\alpha_1, \dots, \alpha_k\}}(d)$ might be true.

- I cannot rule out the possibility that for a fixed α the size of any $[-1, 0] \cup \{\alpha\}$ -code is at most linear in the dimension.

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References

- [1] Alexander Barg and Wei-Hsuan Yu. New bounds for equiangular lines. In *Discrete geometry and algebraic combinatorics*, volume 625 of *Contemp. Math.*, pages 111–121. Amer. Math. Soc., Providence, RI, 2014. [arXiv:1311.3219](https://arxiv.org/abs/1311.3219).
- [2] Boris Bukh. Ranks of matrices with few distinct entries. [arXiv:1508.00145](https://arxiv.org/abs/1508.00145), 2015.
- [3] D. de Caen. Large equiangular sets of lines in Euclidean space. *Electron. J. Combin.*, 7:Research Paper 55, 3 pp. (electronic), 2000. http://www.combinatorics.org/Volume_7/Abstracts/v7i1r55.html.
- [4] P. Delsarte, J. M. Goethals, and J. J. Seidel. Spherical codes and designs. *Geometriae Dedicata*, 6(3):363–388, 1977.

- [5] Gary Greaves, Jacobus H. Koolen, Akihiro Munemasa, and Ferenc Szöllösi. Equiangular lines in Euclidean spaces. [arXiv:1403.2155](#), January 2015.
- [6] Jonathan Jedwab and Amy Wiebe. Large sets of complex and real equiangular lines. *J. Combin. Theory Ser. A*, 134:98–102, 2015. [arXiv:1501.05395](#).
- [7] P. W. H. Lemmens and J. J. Seidel. Equiangular lines. *J. Algebra*, 24:494–512, 1973.
- [8] A. Neumaier. Graph representations, two-distance sets, and equiangular lines. *Linear Algebra Appl.*, 114/115:141–156, 1989.
- [9] J. H. van Lint and J. J. Seidel. Equilateral point sets in elliptic geometry. *Nederl. Akad. Wetensch. Proc. Ser. A 69=Indag. Math.*, 28:335–348, 1966.