A natural barrier in random greedy hypergraph matching

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A natural barrier in random greedy hypergraph matching

Patrick Bennett*       Tom Bohman†

Abstract

Let \( r \) be a fixed constant and let \( H \) be an \( r \)-uniform, \( D \)-regular hypergraph on \( N \) vertices. Assume further that \( D \to \infty \) as \( N \to \infty \) and that co-degrees of pairs of vertices in \( H \) are at most \( L \) where \( L = o(D/\log^5 N) \). We consider the random greedy algorithm for forming a matching in \( H \). We choose a matching at random by iteratively choosing edges uniformly at random to be in the matching and deleting all edges that share at least one vertex with a chosen edge before moving on to the next choice. This process terminates when there are no edges remaining in the graph. We show that with high probability the proportion of vertices of \( H \) that are not saturated by the final matching is at most \( (L/D)^{1/2(r-1)} + o(1) \). This point is a natural barrier in the analysis of the random greedy hypergraph matching process.

1 Introduction

Let \( r \) be a fixed constant and let \( H \) be an \( r \)-uniform, \( D \)-regular hypergraph on \( N \) vertices where \( r \) is a fixed constant and \( D \to \infty \) as \( N \to \infty \). We study the evolution of the random greedy matching algorithm on \( H \). This process forms a matching (i.e. a collection of pairwise disjoint edges) in \( H \) by making a series of random choices. We begin with \( M(0) = \emptyset \) and \( H(0) = H \). In iteration \( i \) an edge \( E_i \) is chosen uniformly at random from \( H(i-1) \) and added to \( M(i-1) \) to form the matching \( M(i) \). We then form \( H(i) \) by deleting from \( H(i-1) \) all edges that intersect \( E_i \). The process proceeds until the step \( M \) where \( H(M) \) is empty. We are interested in the likely value of \( M \); that is, we are interested in the number of edges in the matching produced by the random greedy process.

The random greedy packing algorithm for producing a partial Steiner system is an important special case of this process. Let \( 1 < \ell < k \) be fixed integers. Define \( H_{\ell,k} \) to be the hypergraph on vertex set \( \binom{[n]}{k} \) with edge set consisting of all sets of the form \( \binom{A}{\ell} \) where \( A \in \binom{[n]}{\ell} \). Note that a matching in \( H_{\ell,k} \) corresponds to a collection of \( k \)-element subsets of \( [n] \) with the property that the intersect of any pair

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of sets in the collection has cardinality less than \( \ell \); that is, a matching in \( \mathcal{H}_{\ell,k} \) gives a partial \((n,k,\ell)\)-Steiner system. The random greedy matching algorithm applied to \( \mathcal{H}_{\ell,k} \) is also known as random greedy packing. This process is related to the celebrated Rödl nibble \([10]\), which is a semi-random variation on random greedy packing. The Rödl nibble was introduced in the solution of the Erdős and Hanani conjecture \([6]\), which states that for every fixed \( \ell, k \) there is a matching in \( \mathcal{H}_{\ell,k} \) that saturates \((1 - o(1)) \binom{n}{\ell} \) vertices.

In this paper we study the general random greedy matching algorithm by establishing dynamic concentration of the number of edges and the vertex degrees in the remaining hypergraph \( \mathcal{H}(i) \). Let \( Q(i) \) be the number of edges in \( \mathcal{H}(i) \) and let \( d_v(i) \) be the degree of vertex \( v \) in \( \mathcal{H}(i) \). We aim to show that that \( Q(i) \) and \( d_v(i) \), appropriately scaled, are tightly concentrated around expected trajectories that we express as smooth functions on the reals. In order to describe the trajectories we introduce a continuous time \( t \) which we relate to the steps of the process by setting

\[
t = t(i) = \frac{i}{N}.
\]

Our study is guided by the following probabilistic intuition: we suspect that \( \mathcal{H}(i) \) resembles a subhypergraph of \( \mathcal{H} \) chosen uniformly at random from the collection of all subhypergraphs induced by \( N - ir \) vertices. So we anticipate that \( \mathcal{H}(i) \) resembles a subhypergraph of \( \mathcal{H} \) induced by a random subset of the vertices where each vertex is included independently with probability

\[
p = 1 - \frac{ir}{N} = 1 - rt.
\]

(Note that this probability can be viewed as either a function of either \( i \) or \( t \); we pass between these interpretations without comment.) It follows from this assumption that the probability an edge \( E \in \mathcal{H} \) is in \( \mathcal{H}(i) \) should be about \( p^r \), and therefore we ought to have

\[
Q(i) \approx |\mathcal{H}|p^r = NDp^r/r.
\]

(1)

Furthermore, if a vertex \( v \) is not saturated by \( M(i) \) then we should have

\[
d_v(i) \approx Dp^{r-1}.
\]

(2)

Our main result (see Theorem 2.1 below) is that estimates (1) and (2) hold for most of the evolution of the process. This is a generalization of a recent result of Bohman, Frieze and Lubetzky \([3]\), who proved an analogous result for the special case of \( \mathcal{H}_{2,3} \).

In order to discuss our main result in more detail, we define the random variable

\[
X = X(\mathcal{H}) := 1 - Mr/N
\]

where \( M \) is the number of steps before the random greedy matching algorithm on \( \mathcal{H} \) terminates. In other words, \( X \) is the proportion of vertices left unsaturated by the matching produced by the random greedy algorithm. The following bound is a Corollary of Theorem 2.1.
Theorem 1.1. Let \( r \geq 2 \) and \( \mathcal{H} \) be an \( r \)-uniform, \( D \)-regular hypergraph on \( N \) vertices. If the maximum co-degree \( L \) of a pair of vertices in \( \mathcal{H} \) satisfies \( L = o(D/\log^5 N) \) and \( X(\mathcal{H}) \) is the proportion of vertices that are not saturated by the matching produced by the random greedy algorithm then with high probability we have

\[
X(\mathcal{H}) \leq \left( \frac{L}{D} \right)^{\frac{1}{r-1} + o(1)}.
\]

Previous analyzes of the random greedy matching algorithm due to Spencer [12] and, independently, Rödl and Thoma [10] showed that if \( L = o(D) \) then we have \( X(\mathcal{H}) = o(1) \) with high probability. Note that this result applied to the hypergraph \( \mathcal{H}_{\ell,k} \) gives an alternate proof of the Erdős–Hanani conjecture. Wormald [15] applied the differential equations method for random graph processes to show that if \( \mathcal{H} \) is an \( r \)-uniform, \( D \)-regular hypergraph on \( N \) vertices such \( D = o(N) \) but \( D \to \infty \) sufficiently quickly as \( N \to \infty \) then

\[
X(\mathcal{H}) < D^{1/r-1} + o(1) \quad \text{with high probability.}
\]

We note that Theorem 2.1 takes the analysis of random greedy matching up to a natural barrier. To describe this barrier we assume estimates (1) and (2) hold. For a fixed vertex \( v \) let \( L_v \) be the set of vertices \( u \) such that the co-degree of \( u \) and \( v \) in \( \mathcal{H} \) is \( L \). Note that \( |L_v| \) can be as large as \( D/L \). Now early in the process (when \( p = 1/2 \), say) the expected number of vertices in \( L_v \) that are not saturated by \( \mathcal{M} \) can be as large as \( pD/L \) and thus can have variation as large as \( \sqrt{D/L} \), roughly speaking. This yields variations in vertex degrees that are as large as \( \sqrt{D/L} \cdot L = \sqrt{DL} \). If these early variations in vertex degree persist then at the point when \( Dp^{r-1} = \sqrt{DL} \) these variations will be as large as the expected degree itself. So, if these variations indeed persist then when we reach this point vertex degrees could be zero even though the expected vertex degree is large. Note that this is point where Theorem 2.1 no longer holds. One would expect that in order to prove better bounds one would have to show that the variations in vertex degree decrease as the process evolves.

But where do we expect the random greedy matching algorithm to finally terminate? If we assume that estimates (1) and (2) hold all the way to termination then when \( NDp^r = Np \) the number of unsaturated vertices should be roughly the same as the number of remaining edges. At this stage a positive proportion of the unsaturated vertices should be in no remaining edges; these vertices would remain unsaturated to termination. Thus, it is natural to guess that random greedy matching terminates when the proportion of unsaturated vertices is roughly \( D^{-1/(r-1)} \). (We note in passing that this line of reasoning is suspect if \( L > D^{1-\frac{1}{r-1}} \). In this case, one suspects that we will reach a point where co-degree in \( \mathcal{H}(i) \) are larger than degrees before the supposed termination point.) In the context of random greedy packing, this line of reasoning leads to the following conjecture.

Conjecture 1.2 (folklore). Let \( 1 < \ell < k \) be fixed. With high probability

\[
X(\mathcal{H}_{\ell,k}) = n^{\frac{k-\ell}{\binom{k}{2} - 1} + o(1)}.
\]
The $\ell = 2, k = 3$ case of this conjecture was recently proved by Bohman, Frieze and Lubetzky [4] who establish estimates for vertex degrees in $H_{2,3}(i)$ with error bounds that decrease as the process evolves. These self-correcting estimates are proved using the critical interval method that is featured in this paper and was introduced in [3]. It should be noted that the sharp result given in [4] requires a large, carefully selected ensemble of random variables.

The related problem of proving the existence of a large matching in an $r$-uniform, $D$-regular hypergraph $H$ has been widely studied (see [9] [1] [8]). The best known results are due to Vu [14] who used a semi-random (i.e. Rödl nibble type) method to show that there exists a matching in $H$ that saturates all but at most

$$\left( \frac{L}{D} \right)^{\frac{1}{r-1} + o(1)}$$

vertices where $L$ is the maximum co-degree of pairs of vertices in $H$. Vu obtained stronger results when one adds co-degree assumptions for larger sets of vertices.

The remainder of this paper is organized as followed. In the next Section we give a precise statement of our dynamic concentration result. The proof follows in Section 3. This proof uses the critical interval method introduced by Bohman, Frieze and Lubetzky in [3], where they prove Theorem 1.1 for the special case $H_{2,3}$. In this note we show that the techniques introduced in [3] are robust enough to handle the general case (with the introduction of some delicate calculations necessitated by the large co-degrees).

## 2 Dynamic Concentration

Throughout this Section we assume that $H$ is an $r$-uniform, $D$-regular hypergraph on $N$ vertices where $r$ is a fixed constant and $D \to \infty$ as $N \to \infty$. We also assume that the maximum co-degree $L$ of a pair of vertices in $H$ satisfies $L = o(D/\log^5 N)$.

In order to make the estimates (1) and (2) precise we introduce error bounds for $Q$ and $d_v$. Define

$$e_q = 15NLp^{2-r}(\log N) \left( 1 - r\log p \right)^2$$

$$e_d = \sqrt{6rLD\log N} \left( 1 - r\log p \right)$$

Further define the stopping time $T$ to be the first step $i$ such that

$$\left| Q(i) - \frac{ND}{r}p^r \right| > e_q, \text{ or}$$

$$\left| d_v(i) - Dp^{r-1} \right| > e_d \text{ for some } v \in V(i)$$

**Theorem 2.1.** With high probability we have

$$N - Tr = O \left( N \cdot \left( \frac{L}{D} \right)^{\frac{1}{2(r-1)}} \log^{\frac{5}{2(r-1)}} N \right).$$
3 Proof

For each variable $V$ and each bound (i.e. upper and lower) we introduce a critical interval $I_V = [a_V, b_V]$. This interval varies with time and has one endpoint at the bound we are trying to establish with the other slightly closer to the expected trajectory. We only track $V$ if and when it enters a critical interval. If $V$ enters the critical interval at step $j$ we ‘start’ observing a sequence of random variables that is designed to be a sub- or supermartingale with the property that if $V$ eventually passes all the way through the interval (and thereby violates the bound in question) then this martingale has a large variation. When working with the lower bounds we consider the sequence $V - a_V$. This sequence should be a submartingale with initial value (i.e. value at step $j$) roughly $b_V - a_V$. Note that this sequence becomes negative when the bound in question is violated. Similarly, when working with the upper bound we consider the sequence $b_V - V$, which should be a supermartingale. The event that $V$ ever violates one of the stated bounds is then the union over all ‘starting’ points $j$ of the event that one of the martingales that start at this point has a large variation. We prove Theorem 2.1 by an application of the union bound, taking the union over all variables $V$ of this union over all ‘starting’ points for both the upper and lower bounds.

The reason that we focus our attention on these critical intervals is fact that the expected one-step changes in the variables we consider have self-correcting terms. These terms introduce a drift back toward the expected trajectory when $V$ is far from the expected trajectory. By restricting our attention to the critical intervals we make full use of these terms. See [13] and [5] for early applications of this self-correcting phenomena in applications of the differential equations method for proving dynamic concentration. As we noted above, the critical interval method we use here was introduced in [3].

We close this section with some notation conventions and a Lemma that we use below. For an arbitrary random variable $V$ we define

$$\Delta V(i) = V(i + 1) - V(i).$$

We let $\mathcal{F}_i$ be the filtration of the probability space given by the first $i$ edges chosen by the random greedy matching process.

Lemma 3.1. Suppose $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ are real numbers such that $|x_i - x| \leq \delta$ and $|y_i - y| < \epsilon$ for all $i \in I$. Then we have

$$\left| \sum_{i \in I} x_i y_i - \frac{1}{|I|} \left( \sum_{i \in I} x_i \right) \left( \sum_{i \in I} y_i \right) \right| \leq 2|I|\delta \epsilon$$

Proof. The triangle inequality gives

$$\left| \sum_{i \in I} (x_i - x)(y_i - y) \right| \leq |I|\delta \epsilon.$$
Rearranging this inequality gives
\[
\sum_{i \in I} x_i y_i = x \sum_{i \in I} y_i + y \sum_{i \in I} x_i - |I|xy \pm |I|\delta\epsilon
\]
\[
= \frac{1}{|I|} \left( \sum_{i \in I} x_i \right) \left( \sum_{i \in I} y_i \right) - |I| \left( \frac{1}{|I|} \sum_{i \in I} x_i - x \right) \left( \frac{1}{|I|} \sum_{i \in I} y_i - y \right) \pm |I|\delta\epsilon.
\]

\[
\]

3.1 Vertex degrees

Let \( v \) be a fixed vertex. As is usual in applications of the differential equations method for establishing dynamic concentration, we begin the expected one-step change in \( d_v \) (i.e. we begin with the trend hypothesis). We have
\[
E[\Delta d_v(i)|F_i] = -\frac{1}{Q} \sum_{E \in H(i):v \in E} \sum_{u \in E \setminus \{v\}} d_u(i) \pm d_v(i) \left( \frac{r}{2} \right) L^2 Q,
\]
where \( F_i \) is the filtration defined by the random greedy matching process.

We begin with the upper order \( d_v \). Our critical interval is
\[
[D p^{r-1} + e_d - f_d, D p^{r-1} + e_d].
\]
The function \( e_d \) is define above and the function \( f_d \) will be determined below. For each step \( j \) of the process we define the sequence of random variables
\[
d^+_{v,j}(i) := d_v(i) - D p^{r-1} - e(t) \quad \text{for } i \geq j
\]
with the stopping time \( T_j \) defined to be the minimum of \( T, j \), and the smallest index \( i \geq j \) such that \( d_v(i) \) is not in the critical interval. Note that if \( d_v(j) \) is not in the critical interval then we simply have \( T_j = j \). We prove dynamic concentration by considering the sequence of random variables \( d^+_{v,j}(j), \ldots, d^+_{v,j}(T_j) \). We chose \( e_d \) (with foresight) so that this sequences is a supermartingale with respect to the natural filtration \( F_i \). For \( j \leq i < T_j \) we have
\[
E[\Delta d^+_{v,j}|F_i] \leq -\frac{1}{Q} \sum_{E \in H(i):v \in E} \sum_{u \in E \setminus \{v\}} d_u(i) + \frac{Dr(r-1)}{N} p^{r-2} - \frac{1}{N} e'_d
\]
\[
+ \frac{L_d u}{Q} + \frac{D}{N^2} p^{-3} + \frac{1}{N^2} e''_d
\]
\[
\leq -\frac{(D p^{r-1} + e_d - f_d) (r-1) (D p^{r-1} - e_d)}{N D p^{r}/r + e_d} + \frac{Dr(r-1)}{N} p^{r-2}
\]
\[
- \frac{1}{N} e'_d + O \left( \frac{L_d u}{Q} + \frac{D}{N^2} p^{-3} + \frac{1}{N^2} e''_d \right)
\]
\[
\leq \frac{r(r-1)}{Np} f_d - \frac{1}{N} e'_d
\]
\[
+ O \left( \frac{(e_d - f_d) e_d}{N D p^{r}} + \frac{e_d}{N^2 p^{2}} + \frac{L_d u}{Q} + \frac{D}{N^2} p^{-3} + \frac{1}{N^2} e''_d \right)
\]
Note that we use the assumption that $d_v(i)$ lies in the critical interval. Also note that in order to get the desired supermartingale condition it is necessary to choose $f_d$ so that

$$e'_d > \frac{r(r-1)}{p} f_d.$$  
(3)

(Of course, this equation also guided the choice of $e_d$.) A function $f_d$ that satisfies this equation is also sufficient to give the supermartingale condition as, assuming the given error functions $e_d, e_q$, we have

$$\left( e_d - f_d \right) e_d + \frac{e_q}{N^2 p^2} + \frac{Dd_v}{Q} + \frac{D}{N^2} p^{r-3} + \frac{1}{N^2} e''_d,$$

$$\leq \frac{e_d}{Np} \cdot O \left( \frac{e_d p^{1-r}}{D} + \frac{e_q}{e_d Np} + \frac{L}{e_d} + \frac{D}{N e_d} p^{r-2} + \frac{1}{N p \log N} \right)$$

$$\leq \frac{e_d}{Np} \cdot O \left( \frac{\sqrt{L} (\log N)^{3/2} p^{1-r}}{\sqrt{D}} \right) + \frac{e_d}{Np} \cdot o \left( \frac{\sqrt{L}}{\sqrt{D}} + \frac{\sqrt{D}}{N \sqrt{L}} + \frac{1}{\sqrt{N}} \right).$$

(We note that these estimates make repeated use of the simple inequality $D < NL$.)

By assuming that $p$ is a sufficiently large constant times $(L/D)^{1/(r-1)} \log^2 (r-1) N^{-1}$, we see that the expression in (4) can be made smaller than any constant times $e_d/(Np \log N)$. For the time being we assume that a function $f_d$ satisfying (3) can be chosen. Thus, the supermartingale condition is satisfied.

We use a supermartingale inequality to bound the probability that the random variable $d^+_{v,j}(T_j)$ is positive. The lemma we use is as follows:

**Lemma 3.2.** Let $X(i)$ be a supermartingale, such that $-\Theta \leq \Delta X(i) \leq \theta$ for all $i$, where $\theta < \frac{\Theta}{10}$. Then for any $a < \theta m$ we have

$$Pr(X(m) - X(0) > a) \leq \exp \left( -\frac{a^2}{3 \theta^2 m} \right).$$

Since $d_v$ is non-increasing, $Dp^{r-1}$ is decreasing and $e_d$ is increasing, the one step change in $d^+_{v,j}$ is bounded above by the one step change in $Dp^{r-1}$, which is at most

$$\theta = \frac{D(r-1)}{N} (1 + o(1)).$$

For a lower bound on $\Delta d^+_{v,j}$, note that the one step change in $e_d$ is negligible compared to the maximum possible one step change in $d_v$, which occurs when we pick an edge containing a vertex that has codegree $L$ with $v$. So we can set $\Theta = rL(1+o(1))$.

Now, if $d_v$ crosses the upper boundary of its critical interval at the stopping time $T$, then there is some step $j$ (with $T = T_j$) such that

$$d^+_{v,j}(j) \leq -f_d(t(j)) + \frac{D(r-1)}{N} (1 + o(1))$$
and $d^+_{v,j}(T_j) > 0$. Applying the lemma (and assuming $D/N = o(f_d)$) we see that the probability of the supermartingale $d^+_{v,j}$ having such a large upward deviation has probability at most

$$\exp \left\{ -\frac{f_d^2}{3D(r-1)/N}(rL)(Np/r)(1 + o(1)) \right\}. $$

As there are $O(N^2)$ such supermartingales, we would like the above expression to be $o(N^{-2})$. Thus, it suffices to take

$$ f_d = \sqrt{6rLD \log N}. $$

Furthermore this choice also satisfies (3). (Note that, in fact, this condition together with (3) essentially determines the error functions $e_d$.)

Thus, the probability that $T$ is less than bound stated in Theorem 2.1 due to a violation of the upper bound on $d_v$ is goes to zero as $N$ tends to infinity.

The lower bound for $d_v$ is similar.

### 3.2 Number of edges

We again begin with the trend hypothesis. We have

$$ E[\Delta Q(i)|F_i] = -\frac{1}{Q} \sum_{A \in H(i)} \sum_{v \in A} d_v(A) + O(L) = -\frac{1}{Q} \sum_{v \in V(i)} d_v^2(i) + O(L) $$

For $i < T$ we have

$$ \sum_{v \in V(i)} d_v^2 = \frac{(rQ)^2}{Np} \pm 2Npe_d^2, $$

by an application of Lemma 3.1 and therefore

$$ E[\Delta Q(i)|F_i] = -\frac{r^2Q}{Np} \pm \frac{2Npe_d^2}{Q} + O(L). $$

We work with the upper bound on $Q(i)$. Our critical interval is

$$ \left[ \frac{ND}{r} p^r + e_q - f_q, \frac{ND}{r} p^r + e_q \right]. $$

The function $e_q$ is define above and the function $f_q$ will be determined below. For each step $j$ of the process we define the sequence of random variables

$$ Q^+_j(i) := Q(i) - \frac{ND}{r} p^r - e_q $$
with the stopping time $T_j$ defined to be the minimum of $T$, $j$, and the smallest index $i \geq j$ such that $Q(i)$ is not in the critical interval. We begin by showing that $Q_j^+(j), \ldots, Q_j^+(T_j)$ is a supermartingale. For $j \leq i < T_j$ we have

$$E \left[ \Delta Q_j^+(i) | F_i \right] \leq -\frac{r^2 Q}{Np} + rDp^{r-1} - \frac{1}{N} e'_q + \frac{2NPe_d}{Q} + O\left( L + \frac{D}{N} p^{r-2} + \frac{1}{N^2} e''_q \right)$$

$$\leq -\frac{r^2(e_q - f_q)}{Np} - \frac{1}{N} e'_q + \frac{(2r + o(1))p^{1-r}e_d^2}{D}$$

$$+ O\left( L + \frac{D}{N} p^{r-2} + \frac{1}{N^2} e''_q \right)$$

In order to get the supermartingale condition this requires, up to constant facts, $e_q > e'_q Np^{2-r}/D$. Note that this determines the main terms in the choice of $e_q$ above. We set

$$f_q = NL \log Np^{2-r}.$$ 

Then we have

$$-\frac{r^2(e_q - f_q)}{Np} + \frac{(2r + o(1))p^{1-r}e_d^2}{D} \leq -Lp^{1-r}(\log N)(1 - r \log p)^2.$$ 

This clearly dominates the remaining error terms (note that $e'_q > 0$) and therefore the sequence $Q_j^+(j) \ldots Q_j^+(T_j)$ a supermartingale.

Now we apply the Hoeffding-Azuma inequality to bound the probability that the random variable $Q_j^+(T_j)$ is positive. Since $i < T$ implies bounds on degrees, we have

$$|\Delta Q|^+ = O(e_d) = O(\sqrt{LD\log N}(1 - \log p)).$$ 

Thus, if $Q$ crosses its upper boundary at the stopping time $T$, then there is some step $j \ (\text{with} \ T = T_j)$ such that

$$Q_j^+(j) \leq f_q(t(j)) + O(\sqrt{LD\log^3/2 N})$$ 

and $Q_j^+(T_j) > 0$. Applying the Hoeffding-Azuma we see that the probability of the supermartingale $Q_j^+$ having such a large upward deviation has probability at most

$$\exp\left\{ -\Omega\left( \frac{(NL \log Np^{2-r})^2}{(Np)(LD \log N(1 - \log p)^2)} \right) \right\}$$

$$\leq \exp\left\{ -\Omega\left( \frac{NL D \log Np^{3-2r}}{(1 - \log p)^2} \right) \right\} = o(N^{-1})$$

where $p = p(j)$. As there are at most $O(N)$ such supermartingales, the probability that $T$ is less than the bound stated in Theorem 2.1 due to $Q(i)$ breaching the upper bound tends to zero as $N$ tends to infinity.

The lower bound for $Q$ is similar.
References


