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The purpose of this note is to present an extension of a theorem proved by the authors in [1]. Let ft denote the space of all continuous paths u : R^{+} -* R^n, o^O) = 0, with the topology of uniform convergence on compact subsets of R^{+}. Suppose (ft^jP) is the complete probability space with & = Borel ft and P Wiener measure on ft.

Theorem 1

Suppose that **g : R^d -+ R^d xn** *denotes a* C^m *map from* R^d *into the space* $R^{d\times n}$ **o / d a** *matrices, with bounded derivatives of all orders. Assume that there is a positive integer* **p** *and positive constants* X *and* 6 *such that*

$$\mathbf{g(v)g(v)^* \geq A\min(|v|^2P,£)I} \qquad \mathbf{(C)}$$

for all **v G R^d** *, where* | |, I, *and* * *denote respectively the Euclidean norm on* **R^d***, the* **d * d** *identity matrix, and matrix transposition. Let* **x** *denote the solution of the follwoing stochastic differential delay equation:*

$$\left.\begin{aligned} \mathbf{dx(t)} &= \mathbf{g(x(t\text{-}r))dW(t),} \quad && \mathbf{t > 0} \quad , \\ \mathbf{x(t)} &= \eta(t) \quad , && \mathbf{-r \leq t \leq 0} \end{aligned}\right\} \qquad \mathbf{(I)}$$

where **W** *is normalized n—dimensional Brownian motion on* **(ft^P)** *and i is a strictly positive time delay. Suppose the initial path rj* **6 L^{∞}([—r,0],R^d)** *and is such that*

$\int_{-r}^{0}$ $|r/(s)|^2$ds > 0. *Define* s_Q e [-r,0] *by*

$$s_Q := \sup\{s : s \text{ e } [-1,0], \int_{-r}^{s} | rfa) 1^2 du = 0\}.$$

Then for each $t > s_Q + r$, ***the random variable*** $x(t)$ ***has a distribution which is absolutely continuous and has a*** C^{00} ***density with respect to Lebesgue measure on*** R^d.

Remark:

We proved the theorem in [1] in the special case $p = 1$ using the methodology of the Malliavin calculus. The significance of the result lies in the fact that in view of the time delay r the solution of equation (I) is a non-Markov R^d—valued process and is therefore not amenable to analysis via classical PDE techniques. Indeed prior to [1] the only existing regularity result for non—Markov diffusions was a theorem of Kusuoka and Stroock [2], which requires the assumption that the diffusion coefficient g be *bounded away from* 0.

The proof of Theorem 1 relies on the following result, which we give in its most general form as we feel it may also be of interest in its own right.

Theorem 2

Suppose that y ***is a process in*** R^d ***defined by an Itb integral of the type***

$$y(t) = z + \int^{*} A(s)dW(s), \qquad t \geq 0$$

where z **e** R^d ***and*** $A : [0,\infty) \times fl \to R^{d \times n}$ ***is a bounded measurable process, adapted to the filtration of*** W. ***Let*** $h : R^d \to R$ ***be a measurable function satisfying the condition***

$$|h(v)| \geq A\min\{|v|^p, 6) \qquad v \text{ G } R^d. \tag{D}$$

Let $0 < a < b$ ***and suppose that***

$$P[\int_a^b \|y(s)\|^2 ds < \epsilon] = o(\epsilon^k) \quad \textit{as} \quad \epsilon \to 0+, \quad \textit{for all} \quad k \geq 1. \tag{E}$$

Then

$$P[\int_a^b (h(y(s))^2 \, ds < \epsilon] = o(\epsilon^k) \quad \textit{as} \quad \epsilon \to 0+, \quad \textit{for all} \quad k \geq 1 \, .$$

N.B. $\|\cdot\|$ *denotes any norm on the space* $\mathbb{R}^{d \times n}$ *of* $d \times n$ *matrices.*

Proof:

Note that we proved this theorem in [1, Lemma 3] for the case $p = 1$. Define $f(v) = |h(v)|^{1/p}$, $v \in \mathbb{R}^d$. Then f satisfies the condition

$$|f(v)| \geq \lambda^{1/p} \min(|v|, \delta^{1/p}), \qquad v \in \mathbb{R}^d.$$

Condition (E) together with Jensen's inequality and Lemma 3 of [1] now imply

$$P[\int_a^b h(y(s))^2 \, ds < \epsilon^p] \leq P[\int_a^b f(y(s))^2 \, ds < \epsilon]$$

$$= o(\epsilon^k) \quad \text{for all} \quad k \geq 1$$

from which the result clearly follows. □

Following [1] we define

$$h(v) \equiv \inf\{|g(v)^*(e)| : e \in S^{d-1}\}, \quad v \in \mathbb{R}^d.$$

Then (C) implies that h satisfies (D); thus h satisfies the conclusion of Theorem 2. We

observe that this step is the only part of the argument in [1] in which the lower bound condition on g is used. Hence the argument in [1] suffices to complete the proof of Theorem 1.

In conclusion we remark that Theorem 1 is a significant extension of the main result in [1]. For example in the one dimensional situation vanishing to some finite order at 0 occurs for any analytic function not identically zero in some neighborhood of 0, whereas in the previous version of the theorem the hypotheses imply that g has essentially linear behavior at 0.

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