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**REGULARITY OF STOCHASTIC DELAY EQUATIONS
UNDER Ptk ORDER DEGENERACY**

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The purpose of this note is to present an extension of a theorem proved by the authors in [1]. Let C^0 denote the space of all continuous paths $u : \mathbb{R}^+ \rightarrow \mathbb{R}^n$, $u(0) = 0$, with the topology of uniform convergence on compact subsets of \mathbb{R}^+ . Suppose (Ω, \mathcal{F}, P) is the complete probability space with $\mathcal{F} = \text{Borel } C^0$ and P Wiener measure on C^0 .

Theorem 1

Suppose that $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times n}$ denotes a C^m map from \mathbb{R}^d into the space $\mathbb{R}^{d \times n}$ of $d \times n$ matrices, with bounded derivatives of all orders. Assume that there is a positive integer p and positive constants X and δ such that

$$g(v)g(v)^* \geq \delta \min(|v|^2, X) I \quad (C)$$

for all $v \in \mathbb{R}^d$, where $|\cdot|$, I , and * denote respectively the Euclidean norm on \mathbb{R}^d , the $d \times d$ identity matrix, and matrix transposition. Let x denote the solution of the following stochastic differential delay equation:

$$\left. \begin{aligned} dx(t) &= g(x(t-r))dW(t), & t > 0, \\ x(t) &= \eta(t), & -r \leq t \leq 0 \end{aligned} \right\} \quad (I)$$

where W is normalized n -dimensional Brownian motion on (Ω, \mathcal{F}, P) and r is a strictly positive time delay. Suppose the initial path $\eta \in L^2([-r, 0], \mathbb{R}^d)$ and is such that

$\int_{-r}^0 |\eta(s)|^2 ds > 0$. Define $s_Q \in [-r, 0]$ by

$$s_Q := \sup\{s : s \in [-1, 0], \int_{-1}^s |rfa| \, 1^2 du = 0\}.$$

Then for each $t > s_Q + r$, the random variable $x(t)$ has a distribution which is absolutely continuous and has a C^{00} density with respect to Lebesgue measure on \mathbb{R}^d .

Remark:

We proved the theorem in [1] in the special case $p = 1$ using the methodology of the Malliavin calculus. The significance of the result lies in the fact that in view of the time delay r the solution of equation (I) is a non-Markov \mathbb{R}^d -valued process and is therefore not amenable to analysis via classical PDE techniques. Indeed prior to [1] the only existing regularity result for non-Markov diffusions was a theorem of Kusuoka and Stroock [2], which requires the assumption that the diffusion coefficient g be *bounded away from 0*.

The proof of Theorem 1 relies on the following result, which we give in its most general form as we feel it may also be of interest in its own right.

Theorem 2

Suppose that y is a process in \mathbb{R}^d defined by an Itô integral of the type

$$y(t) = z + \int_0^* A(s) dW(s), \quad t \geq 0$$

where $z \in \mathbb{R}^d$ and $A : [0, \infty) \times \Omega \rightarrow \mathbb{R}^{d \times n}$ is a bounded measurable process, adapted to the filtration of W . Let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function satisfying the condition

$$|h(v)| \geq A \min\{|v|^p, \delta\} \quad \forall v \in \mathbb{R}^d. \quad (D)$$

Let $0 < a < b$ and suppose that

$$P\left[\int_a^b \|y(s)\|^2 ds < \epsilon\right] = o(\epsilon^k) \text{ as } \epsilon \rightarrow 0+, \text{ for all } k \geq 1. \quad (\text{E})$$

Then

$$P\left[\int_a^b (h(y(s)))^2 ds < \epsilon\right] = o(\epsilon^k) \text{ as } \epsilon \rightarrow 0+, \text{ for all } k \geq 1.$$

N.B. $\|\cdot\|$ denotes any norm on the space $\mathbb{R}^{d \times n}$ of $d \times n$ matrices.

Proof:

Note that we proved this theorem in [1, Lemma 3] for the case $p = 1$. Define $f(v) = |h(v)|^{1/p}$, $v \in \mathbb{R}^d$. Then f satisfies the condition

$$|f(v)| \geq \lambda^{1/p} \min(|v|, \delta^{1/p}), \quad v \in \mathbb{R}^d.$$

Condition (E) together with Jensen's inequality and Lemma 3 of [1] now imply

$$\begin{aligned} P\left[\int_a^b h(y(s))^2 ds < \epsilon^p\right] &\leq P\left[\int_a^b f(y(s))^2 ds < \epsilon\right] \\ &= o(\epsilon^k) \quad \text{for all } k \geq 1 \end{aligned}$$

from which the result clearly follows. □

Following [1] we define

$$h(v) \equiv \inf\{|g(v)^*(e)| : e \in S^{d-1}\}, \quad v \in \mathbb{R}^d.$$

Then (C) implies that h satisfies (D); thus h satisfies the conclusion of Theorem 2. We

observe that this step is the only part of the argument in [1] in which the lower bound condition on g is used. Hence the argument in [1] suffices to complete the proof of Theorem 1.

In conclusion we remark that Theorem 1 is a significant extension of the main result in [1]. For example in the one dimensional situation vanishing to some finite order at 0 occurs for any analytic function not identically zero in some neighborhood of 0, whereas in the previous version of the theorem the hypotheses imply that g has essentially linear behavior at 0.

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