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DYNAMICAL MODELING OF PHASE TRANSITIONS BY MEANS OF VISCOELASTICITY IN MANY DIMENSIONS

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Table of Contents

1. Introduction.

In this paper we study a model equation of viscoelasticity

$$j^*_{fl} = \mathrm{div}\, G(VM) + Au_t \quad \text{in } Q \qquad (1.1)$$

in a multidimensional setting. The displacement u is vector valued, thus Vu is a matrix. We impose the 'no-traction' boundary conditions

$$a(Vu)7i+|i\frac{du_t}{\partial n}=0, \quad \text{on } 3ft$$

and initial conditions

$$u(x,0)=u_0(x), \qquad u_t(x,0)=u_1(x)$$

We assume that there exists a function W Jtf $*^{x}*$->lL such that DW(£)=a(£). For the sake of modeling phase transitions we do NOT assume that W is elliptic, Le. the condition

$$\forall \xi, \eta \in \mathbb{R}^n, \quad \sum_{\alpha,\beta,i,j} \frac{\partial^2 W(F)}{\partial F^i_\alpha \partial F^j_\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geq 0 \qquad \text{OB)}$$

may be violated. We prove a rather general existence result assuming only a growth condition on *W:* we require merely that a be globally lipschitz-continuous. We show that for arbitrary initial data, u, and div (ofVu^Vtt,) tend to zero in *appropriate* spaces as *t* goes to infinity. Finally, we prove dynamical stability for certain stationary solutions, including a class of equilibria with discontinuous gradient.

We also remark on existence of solutions to (1.1) with Dirichlet boundary conditions

$$K=0 \quad \text{on } dCl.$$

To set our analysis in the proper context we briefly review the modeling of phase transition in solids based on minimization of the energy /

$$I(u)=\int_{\Omega} W(\nabla u)dx \tag{1.2}$$

(see for instance Maddocks and Parry [27], Ball and James [5]). If the material occurs in several phases then *W* has several local minima. (If *W* happens to be frame indifferent then they must be orbits of SO *(n)* instead of being isolated points). Such *Ws* typically are not elliptic. Therefore the functional / is not sequentially weakly lower semicontinuous (swlsc). This fact forces one to study minimizing sequences in place of minimizers, since the latter may not exist. The lack of ellipticity (E) may lead to development of fine oscillation in the gradients of minimizing sequences, which prevents the minimizing sequences from converging strongly in W^{Up}.

The variational approach just described is entirely static. In order to study dynamics we could try to solve the equations of elasticity

$$u_{tt} = \operatorname{div} \tau \tag{1.3}$$

where x is the stress tensor T=O(VK)=DW(VH). We would quickly encounter an obstacle, however, which is the lack of ellipticity of *W.* (If *W* were elliptic then (13) would be hyperbolic and if $n > 1$ we would have short time existence in $W^{2*\wedge;*^{11}})$, $p >$ l+n/2; this result is due to Hughes, Kato and Marsden [22], For n=l a global existence result has been established by DiPerna [10].)

When *W* is not elliptic, a possible method of achieving well-posedness is adding to the stress tensor a higher order regularizing term corresponding to viscosity:

$$\tau' = \tau + \mu \nabla u_t.$$

Hence (1.3) becomes the equation of viscoelasticity

j*$_{lf}$=div a(Vn) + \x&u_t. (1.4)

In this paper we adopt the 'no-traction' boundary condition

du,
T71=O(VK)7H+1-^—=0,

on

and initial conditions

K(JC,O>=UO(X) I I ^ O ^ J C X) .

After scaling of time we may set |i=l. We note that more realistic viscous terms should be non-linear (see [25], [33], [34]), however we will stick to the model equation (1.4). One may also consider other regularizing terms, e.g. corresponding to *capillarity* ([7], [13]). A similar regularizing result may be achieved by introducing thermal effects, as for instance in the work of Niezg6dka, Sprekels [30]. Of course, different regularizations may lead to different dynamics.

No matter what the approach (viscoelasticity, thermoelasticity, etc.), the central questions are:

(I) Existence of solutions for all times;

(II) Stability of equilibria; and

(HI) Long time behavior

(a) do all solutions converge strongly in time?

(b) does the energy / *(u* (f)) decay to die minimum energy?

The need for global in time existence is clear without it the question of stability and long time behavior makes no sense. One particularly desires existence of dynamics in $V^{1i2}(QiE")_t$ the space of finite energy for fee functional /. Currently available results of this kind usually require ellipticity of W (cf. [17]).

As for stability, the Energy Criterion is classical. It calls an equilibrium stable if the second variation of the energy / is positive. However, the justification is difficult Only in 1982 did Potier-Ferry [33] prove that for W^{2}* equilibria of multidimensional viscoelasticity the criterion implies stability in W^CQ;!^{11}), $p > n$ (see also references in [33] for earlier results).

Apart from justifying the Energy Criterion we are interested in studying stability of local minimizers of/. The notion of local minimizer depends significantly on the underlying metric. That dependence is especially important in the case of non-elliptic W. It is not clear which type of local minimizer will be dynamically stable.

Our questions concerning behavior for large times are most interesting when / is not swlsc. In this case Ball and James [5] showed that / may not achieve its minimum, at least for some boundary conditions. On the other hand total energy decreases along trajectories. It is natural then to ask whether or not such highly oscillatory sequences are realized by dynamics.

We now briefly sketch the known results. The one-<iimensional case has been studied extensively. Existence of classical solutions and their asymptotic properties were first studied in papers by Dafermos [9], Greenberg, MacCamy, Mizel [19], and Greenberg [18]. Andrews [3], and Andrews and Ball [4] studied weak solutions. Pego [32] gave die most comprehensive answers to questions II and HI. He considered slightly different boundary conditions: his problem is

$$\%=(a(K_xH^\wedge)_x \quad «(0,r)=O, \quad (\sigma(u_x)+u_{xt})(1,t)=0.$$

His results may be summarized as follows.

- A steady state u_0 is stable if *(fiu^ytc* >0. The stable states may contain a mixture of phases; they need not be strong minimizers of energy (in the W^{1t}~-topology). The admissible perturbations for this stability result are small in the W^{1}~-topology.
- For any initial data not exceeding some value of the total energy (kinetic + elastic), the solution converges strongly to an equilibrium.

Many authors (including [8], [11], [17], [31]) have studied the problem of existence and regularity of weak solutions when the number of dimensions exceeds one. Local in time existence is shown without additional assumptions on a. But in order to prove global in time existence the authors need some extra conditions on a or *W*, typically they use ellipticity of *W* (cf. [33], [17]).

As for the issue of stability, Potier-Ferry [33] proves exponential asymptotic stability in *W^iCl'X*)* p >n* for equilibria such that the second variation of the energy functional S^2/ is positive. His stability result includes the assertion of long-time existence for initial states close to the equilibrium. In fact his existence result uses the ellipticity of *W*. He studies a quasi-linear viscoelasticity equation with Dirichlet boundary conditions.

We now describe briefly the method of Pego since we will generalize it to deal with the n-dimensional case. Pego employed a clever change of variables. (Earlier Andrews [3] also used this transformation, but his use was limited.) The new variables are

$$p(x,t)=\int_1^x u_x(y,t)\,dy \qquad q(x,t)=u_x(x,t)-p(x,t).$$

Using these variables Pego reduced system (1.5) to the following

$$p_t=p_{xx}+\sigma(p+q)$$

$$q_t=-\sigma(p+q).$$

Then he applied semigroup theory exposed in [20] to the new system (cf.[32]).

Our work generalizes Andrews-Pego's transformation to many dimensions. We set

$$\text{divP=u}, \qquad Q=Vu-P$$

where we require P and Q to be gradients, and F-/i=O at the boundary. In the new variables the equation of viscoelasticity (1.4) becomes

$$P_t=\nabla \operatorname{div} P+\pi_2\sigma(P+Q) \tag{1.6'}$$

$$Q_t=-\pi_2\sigma(P+Q) \tag{1.6''}$$

with the boundary condition $P\text{-}n=0$. Here, T^y denotes the gradient part of v in the Helmholtz decomposition. In section 2 we make these assertions rigorous.

Section 3 is devoted to proving existence of solutions of (1.6) (and therefore (1.1)). To achieve this we show that semigroup theory is applicable to system (1.6). The solution we construct is unique and defined for all times. We need only the condition that a be globally Lipschitz-continuous. Existence of a unique, global in time solution of (1.1) follows, since we can recover u by the formula

$$u(Ty=\int_0^T <ti\backslash PQ)dt+u_0.$$

If the initial data satisfy $u_oe\ W^{U2}(C1X\%\ UieL^2(Cl\backslash R^*)_9$ then our solution u is in the following spaces

$ueC([0,oo)_tw^{l}\text{-}^{2}(ClX^*))$, $\quad$ $ji_r€C([0,oo)_tL^2(Q^{*'1}))$, $\quad$ $*,€\ C^p((0,\sim),L2(ft;X''))$, $\quad$ $p>0$.

We point out that for our existence result W is permitted to be frame indifferent, but it is not required. Unfortunately, the subsequent stability analysis does not permit frame indifferent W.

We have constructed dynamics in the space $W^{l,2}(\&X^H)$ which is precisely the space of finite energy. It is natural to study stability in this space. One might expect that

proper local minimizers are dynamically stable. Indeed, this is the case for *W*s* permitting construction of potential wells, e.g. for *W* slightly better than quasiconvex (Theorem 5.1). Our proof exploits ideas of Ball and Marsden *[€*. We note that our existence result provides an essential ingredient for their considerations.

We also show *exponential* asymptotic stability of smooth equilibria for which merely the second variation of the energy $8^2/$ is positive in W^{12}. In addition, *W* evaluated at the equilibrium must be elliptic (Theorem 52). We change the underlying function spaces and we now consider perturbations in $W^{2tP}{}_t p > n$. We thus show a result corresponding to that of Potier-Ferry for Dirichlet boundary conditions.

The change of admissible perturbations is not just of technical nature, asymptotic stability is false in $W^{h2}(\&X^H)$ for equilibria merely in $W^{ia}(\&JL^H)$. To this end, for a special choice of *W* consistent with the lack of ellipticity we construct a family $R=\{q_E\}dW^{i:z}(Cl'X*)$ of equilibria with discontinuous gradient For each member of the family the second variation $6^2/$ is positive on W^{1*2}, but asymptotic stability is false. The reason is that in any W^1^-neighborhood of any member q_E of the family there is another *qg* belonging to *R* in particular, perturbations belonging to W^{1*2} may move the discontinuity of equilibria.

For that same family we nevertheless prove a stability result, under some further assumptions on *W* and admitting only $W^{2}*$ perturbations where $p > n$ (Theorem 5.7). The perturbations have continuous gradients, hence they do not move discontinuities in Vq_E. A possible physical interpretation is that at least some equilibria containing several phases are dynamically stable under perturbations not moving the interface separating the phases.

For the long time behavior for arbitrary initial data, we are only able to prove results weaker than those presently known in the one-dimensional case. We show (section^

$$u_t \to 0 \text{ in } W^{1}* \text{ and } \operatorname{div}(c\wedge VaHVu,)-\wedge \text{ in } L^2$$

The question of strong convergence of $u(t)$ in $W^{1,2}$ as $t \to \infty$ remains open. The decay of u_t in $W^{1,2}(\Omega;\mathbb{R}^n)$ (which is equivalent to decay of P in $W^{2,2}(\Omega;\mathbb{R}^{n\times n})$) supports the idea that asymptotically the dynamics is governed by the equation

$$Q_t = -\pi_2 \sigma(Q). \tag{1.7}$$

If $n=1$ then the projection π_2 is equal to identity and the problem (1.7) reduces to an ordinary differential equation (see [32]). If $n>1$ then the projection is a nonlocal operator, making the analysis of (1.7) more difficult.

This paper is concerned mainly with the existence and stability of solution to the viscoelasticity equation with Neumann boundary condition. However, it is possible to extend the existence results for other boundary conditions like homogeneous Dirichlet boundary data

$$u=0 \text{ on } \partial\Omega \tag{1.8}$$

Only minor changes in proof are required to accommodate the new boundary condition. We will make remark on those modifications at the end of sections 2, 3, 4.

We close this introduction with a brief list of problems that remain open, but which we hope our methods might be able to address with further work. We conjecture that for quasiconvex W (or at least under the slightly stronger assumption of Theorem 3.5) the strong limit of $u(t)$ always exists in $W^{1,2}(\Omega;\mathbb{R}^{n\times n})$. We base this conjecture on the fact that quasiconvexity dampens oscillations in minimizing sequences (cf. [12]). We hope that our stability analysis may be extended to frame indifferent W's. We think it should be possible to prove analogous existence and stability results for other boundary conditions. The key step in this direction would be a construction of the projection π appropriate for the given problem.

In order to simplify the notation we write L^p, ($W^{1,p}$, etc) instead of $L^p(\Omega;\mathbb{R}^n)$ or $L^p(\Omega;\mathbb{R}^{n\times n})$. There is little danger of confusion since always in this paper u is vector valued and P, Q have values in $\mathbb{R}^{n\times n}$.

2. The description of the problem.

We study the system of nonlinear viscoelasticity

$$u_{tt}=\text{div}\,\sigma(\nabla u)+\Delta u_t \tag{2.1'}$$

$$\sigma(\nabla u)\cdot n+\partial u_t/\partial n=0 \quad \text{on } \partial\Omega \tag{2.1''}$$

$$u(x,0)=u_0(x) \qquad u_t(x,0)=u_1(x) \tag{2.1'''}$$

with mild assumptions on σ. We assume that the reference domain Ω is a bounded, connected region in $\mathbb{R}^n$ with smooth boundary. We may assume without loss of generality that

$$\int_\Omega u_0(x)dx=\int_\Omega u_1(x)dx=0. \tag{A}$$

It is so because $u'=u+at+b$ is a solution to (2.1'-2.1'') provided u solves the problem. We thus may choose a, b to make (A) hold. Moreover, the space average of $u_t(x,t)$ is constant in time because

$$\frac{d}{dt}\int_\Omega u_t dx=\int_\Omega[\text{div}\,\sigma(\nabla u)+\Delta u_t]dx=\int_\Omega \text{div}\,(\sigma(\nabla u)+\nabla u_t)dx=\int_{\partial\Omega}(\sigma(\nabla u)+\nabla u_t)\cdot n dS.$$

The last integral vanishes because of the boundary conditions (2.1''). We thus showed

Proposition 2.1 If (A) holds then

$$\int_\Omega u_t(x,t)dx=0 \qquad \forall t\geq 0 \quad \square$$

For the purpose of solving system (2.1) we generalize the change of variables due to Andrews [3] and Pego [32]. We set

$$\text{div}\ P=u_t \quad \text{and} \quad Q=\nabla u-P \tag{2.2}$$

where P and Q are $n\times n$–matrices. These definitions will be correct only if we impose some additional restriction on P and Q. We therefore require P, Q be gradients and P satisfy the boundary condition

$$P\cdot n=0 \quad \text{at } \partial\Omega. \tag{2.3}$$

When we work with L^p spaces, the condition that P, Q are gradients means that we actually work with a closed subspace of L^p - the image of a projection n_p. The projection is closely related to the Helmholtz decomposition of vector fields: any smooth vector field may be represented as the sum of a gradient and a divergence-free field. For construction of Ti_p which is well-known (see [14]) and its properties we refer the interested reader to the Appendix.

Now we are in position to construct the new variables P_9 and Q, precisely we show

Theorem *22.* Assume that u is a weak solution to (2.1) such that

$$u_t,\ \nabla u_t \in C([0,T],L^2) \quad ueC([0.T]_9W^1*)$$

$$\pi_2\sigma(\nabla u)+\nabla u_t \in W^{1,2}$$

and the mean value of u_x is zero. Then there exists a unique pair (P_tQ) such that

$$P \in C([0,T],W^{1,2}), \quad P\text{-}n=0$$

$$Q \in C([0,T],L^2)$$

and (/\ $Q)$ is a weak solution of

$$P_t=\pi_2\sigma(P+Q)+\nabla \operatorname{div} P \qquad (2.4')$$

$$Q_t=-\pi_2\sigma(P+Q). \qquad (2.4'')$$

Thus the transformation reduces the system (2.1) a degenerate parabolic system. The advantage of the new system is we may now apply methods of semigroup theory to construct solutions. After we solve (2.4) we will recover solutions to (2.1). Before we prove the Theorem we will show a Lemma we will rely OIL

Lemma *23* The map

$$\text{div} : n2L^2nW^{U2}n\{Pe\ W^1\text{-}^2\!: P\text{-}n=0 \text{ on } dCl\} \ \text{-»}\ L^2n\{feL^2\!: \int_\Omega f/=0/$$

is an isomorphism of Banach spaces.

Proof. Obviously div is continuous. It is also one-to-one and onto, for die equation

$$\operatorname{div} P=u$$

for P =V$ is equivalent to

$$\Delta\phi=u, \qquad \frac{\partial\phi}{\partial n}=0.$$

The standard Laplace equation theory assures existence of solutions up to a constant (cf. [26]), thus F=V<() is defined uniquely. Now, the Open Mapping Theorem yields that div $\sim^{l}$ exists and it is continuous, in particular there is a positive constant y such that

TdlPlku^lldivPlt^lllPl^u Q (2.5)
γ

Proof of Theorem *22* Since the average of u_t is always zero, it follows from the Lemma that P is well defined and it is a continuous function of time into W^{ia}. Thus from (2.2) we obtain that Q is continuous into L^2. We now substitute P and Q into (2.1)

divP, = div $(\sigma(P+Q)+\nabla \operatorname{div} P)$.

We check that the normal component of $x=o(P+Q)$+Vdiv P at the boundary is zero,

$$\tau\cdot n=(\sigma(\nabla u)+\nabla u_t)\cdot n=0.$$

We see that

$$\Xi=P_t-\tau$$

is divergence-free and its normal component at the boundary of Q. is zero, it thus follows from the very definition of n_p (see the Appendix) that

$$\pi_2\Xi=0.$$

Since $\pi_2 P=P$, $\pi_2\nabla\operatorname{div} P=V\&vP$ we obtain

P,=*2CF(P-H2)+VdivP. (2.6')

The equation of evolution of Q we obtain in a simpler way: by differentiation of the second equation in (2.2)

$$QrXu\text{-}Pr\text{-}wP+Q) \qquad (2.6")$$

Remark. If we want to consider the homogeneous Dirichlet boundary conditions

$$u=0 \quad \text{on} \quad 3Q \qquad (2.7)$$

we may proceed in a similar fashion, defining new variables $P^D\backslash$ and Q^D

$$\text{divP}\wedge\text{u}, \qquad Q^D=Vu\text{-}P^D. \qquad (2.8)$$

In order to make the choice of P^D and Q^D unique, we require that they be gradients of functions vanishing at the boundary. In other words we are looking for P^D and Q^D *in* TPL^2. By TP we denote an orthogonal projection defined as follows

$$7C^{i)}v=v<t> \quad \$=0 \text{ on an}$$

and (> is such that V-TC^V is divergence-free. Thus we have another form of Helmholtz decomposition. It turns out that TU^0 is an orthogonal projection. Properties of $T\backslash P$ are summarized in the Appendix.

We may repeat the derivation of equations (2.6'), (2.6") to obtain

$$P^D{}_t=\pi^D\sigma(P^D+Q^D)+\nabla\text{div}P^D. \qquad (2.9)$$

$$Q^D{}_t=-\pi^D\sigma(P^D+Q^D)$$

$$di\backslash P^D=G \quad \text{at } dft \qquad (2.10)$$

3. An Existence Result.

In the present section we prove existence and uniqueness of strong solutions to the system

$$P_t=\pi_2\sigma(P+Q)+\nabla\text{div}\,P \qquad (3.1')$$

$$Q_t=-\pi_2\sigma(P+Q) \qquad (3.1")$$

$$P(0)=P_0 \quad G(OH2o \quad P\text{-}n=O \;\; andCl \tag{3.1'''}$$

provided *W* grows quadratically at infinity, and <r is globally Lipschitz continuous (recall that *o=DW).* In order to achieve this goal we will *apply* results of Henry [20] for abstract evolution equations with the modified definition of solution due to Miklavtit [28]. The above equation may be rewritten as

$$z_t + Az = f(z) \tag{3.2}$$

where $z=(P,Q)$

$$A = \begin{bmatrix} \text{f-Vdiv} & 6\bullet \\ 0 & (\end{bmatrix} \tag{3.3}$$

and

$$f(z) = \begin{bmatrix} \pi_2\sigma(P+Q) \\ -\pi_2\sigma(P+Q) \end{bmatrix}. \tag{3.4}$$

At the end of this section we shall show that *P* and *Q* determine solutions to the original equation (2.1).

We use several different norms in this *papa.* We always make clear which one we mean by adding an appropriate subscript We note that a subscript being number from the unit interval denotes the norm on the fractional power of the given Banach space X, we also use the convention ||ilo=ll°lbr»

We state the main result of this section

Theorem 3.1. Let us suppose a is gjobally Lipschitz continuous. We assume that QoEifeL^2, and *PoenJW*^{1}* ^{m}& P_o is such that $P_0\text{-}n=0$ at the boundary of Q. Then for any T>0 there exists a unique (strong) solution of (3.1) defined for 0£r≤T with

$$P \in C([0,T],\pi_2 W^{1,2}) \cap C^1((0,T],\pi_2 L^2) \cap C((0,T],\pi_2 W^{2,2})$$

and *P-n=0* at *XI* for $t>0$, and

$$Q \in C^1([0,T],\pi_2 L^2).$$

In order to establish existence of solutions to the abstract equation (3.2) in a Banach space X we need to know that A is sectorial on X and it is locally Iipschitz-continuous on X^a into X, for some value of a, 0<a<l. We introduce first some notation, we shall write

y,=V,', *Zp^ftsW*^{1}**: UpO}* Kp<oo,

and

$$X=Y_2xY_2.$$

For our existence result it is sufficient to establish that A given by (3.3) is sectorial on X or equivalently, B =-Vdiv is sectorial on Y_2. But in the sequel we shall need a more general result in our stability analysis of equilibria of (2.1). Thus we shall show that B is sectorial on $Y_{p>}$ \<p <» with the domain

$$D(B)=\{\nabla\phi:\ \phi\in W^{3,p},\ \partial\phi/\partial n=0\}.$$

We start with the observation that the map

$$\nabla: Z_p \to Y_p, \quad \phi \to \nabla\phi$$

is an isomorphism of Banach spaces. Obviously V is onto $Y_{p\%}$ it is also one-to-one because all the elements of *7y* have zero average. This map is clearly continuous, since II^MIL'^IMIW^1-'- On the other hand, if 4 has zero average then Poincar6's inequality yields

$$\|\phi\|_{L^p}\le c\|\nabla\phi\|_{L^p}.$$

It follows that the inverse of V is continuous.

Let us define

$$D(\Delta_N)=\{\phi\in W^{3,p}: \partial\phi/\partial n=0 \text{ at } \partial\Omega\},$$

then for *fyeD* (A#) we have

$$B\,\nabla\phi=-\nabla \mathrm{div}\,\nabla\phi=-\nabla\Delta\phi.$$

Thus B =-VA^V~^{1}, because V is an isomorphism. We also observe that

$$D(B)=\nabla D(\Delta_N).$$

In this way we reduce the question pertaining to B on Y_p to a problem concerning the Laplace operator on *Zp*. In particular the resolvent of B may be expressed in terms of the Laplace operator

$$(B-\lambda)^{-1}=\nabla(-\Delta_N-\lambda)^{-1}\nabla^{-1}.$$

It is now obvious that in order to show that B is sectorial it is enough to prove that -Atf is sectorial on Z_p. In order to accomplish this we will use the fact that the generator of a analytic semigroup is necessarily sectorial (cf [16], [20]). We are going to show that -A# generates an analytic semigroup on *Zp*. It is well known that -A_N with the homogeneous Neumann condition is sectorial on L^p (see [16]), Le. the estimate

$$\|(-\Delta_N-\lambda)^{-1}\|\leq C/|\lambda| \tag{3.5}$$

holds for *X* belonging to a sector

SA. ©=Ae C: (Q< | arg(X-A)┼<*KfcA}*

where ©e (0^c/2), and A<0; and thus -A# generates an analytic semigroup on L^p. We rather need the semigroup on L*/l where E=kerA#. It is easy to see that Z//R is invariant under the resolvent of -A#, thus the space Z//X is invariant for the semigroup generated by -A#.

Since B generates an analytic semigroup on L^P*IJL* it also generates an analytic semigroup cm (1*7)*^{m} ([16], [20]). To determine (Z//*)$^{1/2}$ we shall use the following fact.**

Lemma 3-2 We assume that A is sectorial on X with domain *D {A\ K=ketA*. Then

$$X^{V2}/K=(X/K)^{V2}.$$

Proof. The left-hand-side is well defined; we need to show that the right-hand-side is well defined and both sides are equal.

We observe that if *S X-*X is* linear with domain *D* (5), and *SKcK* then the map

$[S]:X/L\rightarrow X/L$

[S][xMSx]

is well-defined and the domain of *[S]* is *[D(S)]* where we denote by [x] the class of abstraction of x. If in addition *S* is bounded so is [5] and

\\[S]\\≤\\S\l

It is easy to verify that if *S* is invertible and *S~^l KczK* then [S] is also invertible and

$$[S]^{-1}=[S^{-1}].$$

Let us set *Ai=A+aI* where *a* is such that *o(A_x)>0.* We observe that *Aj^l* $K\subset K$, because if x €AT then

$$AA_1^{-1}x=A_1A_1^{-1}x-aA_1^{-1}x=A_1^{-1}A_1x-aA?x=A?\ (A\ \mathrm{j}x-ax)=A_1^{-1}Ax=0$$

We conclude that *[A]* is sectorial on *XIK.*

Similarly, we will show that *A[^{r2}KcK_%* consequently [A}^{/2}] is well defined. For xeJTwehave

$$\mathrm{AAl}^{/2}\mathrm{xKA+tf0A\}}^{/2}\mathrm{x\text{-}\wedge r\wedge}=A_1^{1/2}A_1x-aA_1^{1/2}x=A_1^{1/2}Ax=0.$$

Finally, we have to check that *[A* i]^{y2}=[Al^{/2}]. But it is enough to prove

$$[A_1^{-1/2}]=[A_1]^{-1/2}.$$

This is clear from the definition of A7^{1/2} since

$$A_1^{-1/2}=\frac{\sin(\pi/2)}{1/2}\int_0^\infty\lambda^{-1/2}(\lambda+A_1)^{-1}d\lambda.$$

Continuity of the projection x->[x] implies that we may interchange projection and Riemann integration.

By definition X^m is D (A{$^{/2}$) equipped with the graph norm, we conclude that the Lemma holds. •

Taking into account that kerA#=fc we obtain from Lemma *32* that

$$(L^p/\mathbb{R})^{1/2} = (L^p)^{1/2}/\mathbb{R}.$$

The fact that the space of Bessel potentials $L^l+(Z^H)$ is equal to $W^{ltP}(Jt^n)_t$ (see [20], [37]) and Theorem 6.7 in [15] enables us to conclude

$$(L^p(\Omega;\mathbb{R}^n))^{1/2} = W^{1,p}(\Omega;\mathbb{R}^n).$$

Summarizing, we proved

Proposition 3*3* The operator B is sectorial on Y_p for $1 < p$ <«>. •

Having established that B is sectorial on Y_p for \<*p* <«>, we turn our attention to the special case p=2. We want to show that

$$Y_2^{1/2} = \pi_2 W^{1,2},$$

but first we need to prove positivity of B on Y_2.

Proposition 3.4 B is positive definite on $n\wedge L^{2\prime}$- there is a positive constant c such that for all $P \in D(B)$

$$(BPJ>)*c\backslash\backslash P\backslash\backslash b$$

we may take c ^ , where y is as in Lemma 2.3.

Proof. Let us compute (BP,P)

$$(BP,P) = - \int_D (\text{Vdiv/>,?})=\text{J(divF, div/>>-} \int_{\&} \text{(cttvP.P-nHldivPlli*} \tag{3.6}$$

where by *the* symbol (*,*) we denote the usual inner product in K" and the inner product in $M^{RX}*$ the space of nx/i matrices defined by

$$(F,G) = \operatorname{tr} F^T G = \sum_{i,j} f_{ij} g_{ij}.$$

We apply now the result of Lemma 2.3, this yields

$$(BP,P)\geq\gamma^2\|P\|_{L^2}^2 \qquad \square$$

We will also need in the future equivalence of various norms, first we show

Lemma 3.5. There is a constant $\kappa>0$, such that for all P in $D(B)$ we have

$$\kappa^{-1}\|P\|_{W^{2,2}}\leq\|BP\|_{L^2}\leq\kappa\|P\|_{W^{2,2}}.$$

Proof. Let us set

$$V=D(B)$$

Obviously B is continuous on V into L^2. Moreover, since by Proposition 3.4

$$\|Bx\|_{L^2}\geq\gamma\|x\|_{L^2} \qquad \forall x\in V$$

the quantity

$$|x|_V=\|Bx\|_{L^2}$$

defines a norm on V. The space V equipped with the norm $|\cdot|_V$ is a Banach space, because B is closed. The identity mapping

$$Id:(V,\ \|\cdot\|_{W^{1,2}})\rightarrow(V,\ |\cdot|_V)$$

is continuous, thus it follows from the Open Mapping Theorem that the inverse of Id is continuous too. The Lemma follows. $\square$

We are now in position to determine the space $Y_2^{1/2}$. Let us take a P belonging to $D(B)$, then

$$\|P\|_{1/2}^2=(B^{1/2}P,B^{1/2}P).$$

We showed that B is positive and bounded below, hence it is self-adjoint, so is $B^{1/2}$. We obtain by (3.6)

$$\|P\|_{1/2}^2=(BP,P)=\|\operatorname{div}P\|_{L^2}^2. \tag{3.7}$$

By Lemma 3.5 we obtain that the norms $\|\cdot\|_{1/2}$ and $\|\cdot\|_{W^{1,2}}$ are equivalent. We may conclude that

$$Y_2^{1/2} = \pi_2 W^{1,2}.$$

We may also determine $X^{1/2}$, since $D(A)=D(B)xY_2$ we conclude that $X^{112}**\wedge^{1}**?_2$.

Now, we check that /given by (3.4) is globally lipschitz-continuous. If a is globally Lipschitz-continuous with the Lipschitz constant L, then

$$\|\sigma(x)-\sigma(y)\|_{L^2}^2 \le L^2 \|x-y\|_{L^2}^2 .$$

Since 712 is linear and bounded with norm 1 we conclude

$$\|f(x)-f(y)\|_0 \le L \|x-y\|_{1/2},$$

which means that / is globally Lipschitz.

We are now in position to complete the proof of Theorem 3.1

Proof of Theorem 3.1 We have already checked that the assumptions of Theorem 3.3.3 in [20] are satisfied, thus we are provided with local in time existence of a unique solution $z=(P,Q)$ which is in the following spaces

$$z(t) \in C([0,T],X^{\alpha}) \cap C^1((0,T],X).$$

Actually a closer analysis reveals that (see remarks in the proof of Thm. 3.1 in [32])

$$z(t) \in C([0,T],X^{\alpha}) \cap C^1((0,T],X) \cap C((0,T],D(A)).$$

Taking components of z we obtain the statement of our Theorem. Since we assume that a is globally Lipschitz continuous, we can obtain global existence from Corollary 3.3.5 in [20]. We have only to verify that

$$\|f(z)\|_0 \le K(1+\|z\|_{1/2}) \quad \forall z \in X^{1/2}.$$

Recall that

$$f = \begin{bmatrix} \pi_2 \sigma(P+Q) \\ -\pi_2 \sigma(P+Q) \end{bmatrix}$$

we have

$$\|f(z)\|_0 \le \sqrt{2}\, \|\pi_2 \sigma(P+Q)\|_{L^2} \le \sqrt{2}\, \|\sigma(P+Q)-\sigma(0)+\sigma(0)\|_{L^2}$$

$$\le\sqrt{2}(L\|P+Q-0\|_{L^2}+\|\sigma(0)\|_{L^2})\le\sqrt{2}(L\|P\|_{L^2}+L\|Q\|_{L^2}+\|\sigma(0)\|_{L^2})$$

$$\le C(\|P\|_{W^{1,2}}+\|Q\|_{L^2}+1)\le C(\|z\|_{1/2}+1). \qquad \square$$

We may now recover solutions of (2.1). We also determine the smoothness of the solutions constructed this way. If *P* and *Q* are the solutions to (3.1) with initial conditions P_0and2o»weset

$$u(T)=\int_0^T \mathrm{div}\, P(t)\,dt+u_0 \ . \qquad (3.8)$$

divF is continuous on [0,«) with values in I^2, thus the above integral, understood as the Riemann integral, is well-defined. We immediately obtain that

$$u\in C^1([0,\infty),L^2).$$

To establish further smoothness of *u* and to show that *u* satisfies (2.1) we note that by Theorem 3.5.2 in [20] if z is a solution to (3.2) where *A* is sectorial, / is locally Lipschitz-continuous on $X^a_{\%}$ and z_oeX^a then the time derivative *dzldt* of a solution to (3.2) is a locally H51der continuous function with values in X^T, on (f_o,f J for any Y <1» and

$$\|dz/dt\|_\gamma\le C(t-t_0)^{\alpha-\gamma-1} \qquad (3.9)$$

holds for some constant C. This fact enables us to prove:

Theorem 3.6. Let us assume that

$$u_0\in W^{1,2},\quad u_1\in L^2.$$

Then there is a unique solution *u* of the problem *(l.V-V")* such that

$$ueC([0,oo),W^1*), \qquad a,eC([0,oo)_fL^2);$$

the map *t* —>7C₂cr(V«(r)) is locally H51der continuous with values in L^2 and

$$\int_0^t t^{-1/2}\,\|\pi_2\sigma(\nabla u(s))\|_{L^2}\,ds\to 0\quad \text{as } t\to 0^+. \qquad (3.10)$$

In addition the unique solution given by (3.8) has the properties

$$\text{div } (o(Vi«\}fVtt,), *2Cr(ViO, {}^{Vu}»» *«^{e} C*((O_{t}T], I^{2}), \quad V7>0$$

for some P>0.

Remark. We note that the solution we construct is almost classical if we write the equation in the conservative form. Then all the derivatives involved are at least continuous intoL^2.

Condition (3.10) expressed in terms of variables *P* and *Q* is necessary for uniqueness of solutions to (3.1), see [28]).

Proof of Theorem 3.6. We show first the existence. We will show that the gradient of *u* defined by (3.8) exists and VUG C ([0,O©), L^2). We claim that *u* is the limit in W^{ia} of u_5 where

$$u_\delta(T)=\int_0^T \text{div}\, P(t)\, dt+u_0.$$

Due to Theorem 3.1 the integrand is continuous with values in W^{IJ2} thus *us(T)eW*^{ia}. Since V is a continuous operation on W^{ia} we may write

$$VM_5(7>Vfdiv/\wedge r)dt+V«_0=\int^T \nabla \text{div}\, P(t)\, dt+\nabla u_0.$$

But Vdiv?=(?-H2)$_f$ hence

$$\nabla u_\delta(T)=(P+Q)(T)-(P+Q)(\delta)+P_0+Q_0$$

Because of continuity of *P* and *Q* we deduce that $V«_5$ is continuous with values in L^2. It is clear that u$ tends to *u* in L^2 as 5—>0; we will show that the convergence is actually in W^{U2}. We estimate the difference *u(T}-u$_6$(T)* in W^{U2}

$$||u-u_\delta||_{W^{1,2}}=\backslash\backslash \text{ĝVPwmdt} \quad {}_{1,2} \quad \lim \int^{6} \text{div}\, P(\,0 dt| *_{1,2} \leq \lim \int^{€} ||\text{div}\, P * || * \; dt,$$

and then

$$\leq \lim C\left(\int_\delta^6 \|\mathrm{div} P(t)\|_{L^2} dt + \int \|\nabla \mathrm{div} P(t)\|_{L^2}\right).$$

The first term is small because divP is continuous with values in L^2. The observation $\nabla \mathrm{div} P=(P+Q)_t$ helps estimating the second integral. We may apply inequality (3.9) to a solution of (3.1), we choose γ to be $0<\gamma<1/2$, $\alpha=1/2$. Since $\|z\|_0\leq c\|z\|_\gamma$ we obtain

$$\|z_t\|_0\leq c\|z_t\|_\gamma\leq Ct^{1/2-\gamma-1}.$$

Because of our choice of γ, the function $t^{-\gamma-1/2}$ is integrable over [0,1], so are the components of z_t, i.e. P_t, Q_t. Thus we obtain

$$\int_\eta^b\|\nabla \mathrm{div} P(t)\|_{L^2}dt=\int_\eta^b\|(P+Q)_t(t)\|_{L^2}dt\leq C\int_\eta^b t^{-\gamma-1/2}dt$$

$$=C(\delta^{1/2-\gamma}-\eta^{1/2-\gamma})/(1/2-\gamma)\leq C(\delta^{1/2-\gamma})/(1/2-\gamma)$$

We conclude that

$$\|\int_0^\delta \nabla \mathrm{div} P(t)\, dt\|_{L^2}\leq C\delta^{1/2-\gamma}$$

in other words $u_\delta\to u$ in $W^{1,2}$ if $\gamma<1/2$. We may now compute ∇u

$$\nabla u(T)=\lim_{\delta\to 0}\nabla u_\delta(T)=\lim_{\delta\to 0}[(P+Q)(T)-(P+Q)(\delta)+(P+Q)(0)]=(P+Q)(T).$$

Since the right-hand-side on the above equality is continuous on $[0,\infty)$ we infer that $u\in C([0,\infty),W^{1,2})$.

We note that

$$r^\wedge a^\wedge uy^\wedge Ur^\wedge h^\wedge P \quad +j2)+\nabla \mathrm{div} \quad p=p_t,$$

inequality (3.9) applied to (3.1) with $\gamma=\alpha=1/2$ yields

$$CP,.G,)eC>«p,r].lP^{1,2}\times L^2), \quad \text{for some } p>0. \qquad (3.11)$$

because $X^{1/2}\subset W^{1,2}\times L^2$. It implies that

$$\pi_2\sigma(\nabla u)+\nabla u_t\in C^\beta((0,T],W^{1,2}),\quad u_{tt}\in C^\beta((0,T],L^2),\quad \beta>0.$$

We may now see that u satisfies (2.1)

$$u_{tt}=\operatorname{div} P_t=\operatorname{div}(\pi_2\sigma(P+Q)+\nabla\operatorname{div}P)=\operatorname{div}(\sigma(\nabla u)+\nabla u_t).$$

The initial conditions are satisfied, u(x,O)=uo(x) by definition of *u,* and u,(x, 0)=div/>(x, *O)=ui(x)* by construction of *P.* The boundary condition also holds

$$(\sigma(\nabla u)+\nabla u_t)\cdot n=(\pi_2\sigma(\nabla u)+\nabla u_t)\cdot n=P_t\cdot n=(P\cdot n)_t=0.$$

The fact that

$$\int_0^t r^{1/2}\text{lto2O(Vi0llL* ->0 as r-^O*}$$

holds since Theorem 3.1 yields solutions of (3.1) satisfying

$$\int_0^t r^{-1/2}\text{H7c}_2\text{a(/}^J\text{-H2)llL}^2\text{-^0 as r-^CT}$$

The statements of smoothness of u follow from (3.11): since *Qr=-niG(Su)* and Vu_r=VdivF=/>$_f$+j2$_r$ we obtain

$$\%2<y<yu\backslash\ \text{Vu}_r\text{ec}^P((\text{o}_f\text{r}]_f\text{L}^2)\ \ \text{vr>o}\ \ (p>0).$$

Uniqueness. Suppose we have two solutions u and w of (2.1) satisfying the conditions of the Theorem. By Theorem 3.1 we may construct in a unique way P_H, Q_u, P_{wt} Q_{W9} such that

$$P_u,\ P_w\in C([0,\infty),\ W^{1,2}),$$

$$Q_u,\ Q_w\in C([0,\infty),\ L^2)$$

Since condition (3.10) is satisfied, both (P«,fij, (^w.Gw) are solutions of (3.1), thus by Theorem 3.1 they coincide, and consequently *u=w. D*

We observe that the equation does not smooth out the initial data very much: TC2<T(VK) is merely in L^2, though the stress T'=7t$_2$a(Vtt)+VK$_r$ is smoother (it is in W^{1*2}).

We close this section by showing an analogue of Theorem 3.1 for Dirichlet boundary conditions. Thus the presented above development does not depend on the type of boundary conditions, provided we can construct an appropriate projection *n.* Since our goal is limited to showing existence we may give a simpler argument. We shall establish

Theorem 3.7 Let us assume that a is globally Iipschitz continuous, $Q^D{}_0 ex?L^2$, and $P^D Q£*PW^{I:Z}$ and $P^D{}_0$ is such that divF^o^ at the boundary of ft. Then for any 7>0 there exists a unique (strong) solution erf (2.9) (2.10) defined for 0£f ≤*T* with

$$P^D \in C([0,T],\pi^D W^{1,2}) \cap C^1((0,T],\pi^D L^2) \cap C((0,T],\pi^D W^{2,2})$$

such that div $P^D=0$ at 3ft for $t>0$, and

$$G^{x>} eC^1([O,r]_f ic^D L^2).$$

Proof. The proof goes along the lines of proof of Theorem 3.1 with only minor changes. The main step is to show that the operator f^^Vdiv in UPL^2 is sectorial. The domain of B^D is

$$D(B^D)=7\wedge L^2 n W\wedge^2 n\{PeW^{2:i}:diyP=C \quad \text{at } \partial\Omega\}.$$

Since we intent to work only with p=2 we may show a simpler version of Lemma 3.3. As before, we use an array of inequalities analogous to that of Lemma 2.3 and Proposition 3.5, they are

$$\kappa^{-1}\|P\|_{W^{1,2}} \le \|\operatorname{div} P\|_{L^2} \le \kappa\|P\|_{W^{1,2}} \quad \forall P \in \pi^D L^2 \cap W^{1,2}$$

$$(BP,P)\ Z\ c\|P\|£2 \qquad VPeD\ (B^D)$$

$$\kappa^{-1}\|P\|_{W^{2,2}}\ ZWBPHLt\&UPWwv \quad VPeD(B^D)$$

We skip their proofs since the same type of argument is used.

We will prove that in fact B^D *is* self-adjoint

Proposition *3J6*B^D *is* self-adjoint

Proof. The domain $D(B^D)$ of B^D is dense in TPL^2. We note first that $TPL^2=\{V\$: 4KE W^{U2}Q\}$. Thus, if we take an element P=V<I> of iPl^2, then there exists a sequence {<I>J of functions in $C_0(Q)$ converging to 4> in W^{12}. Since ^ have compact supports then $A4>_n$ vanish at the boundary of H.

The operator B^D is closed as the inverse of a continuous operator $(£^D)$~\ this follows directly from the inequalities above. We also saw that B^D is positive

$$(B^D P,P)\geq c\|P\|_{L^2} \qquad VPeD(B^D)$$

for some positive c. Hence B^ is self adjoint (see [23]). •

Since self-adjoint operators are necessarily sectorial (see [20]) we may repeat the rest of proof of Thm 3.1 to complete Thm 3.7. D.

Existence of a unique solution to (2.1'), (1.10), (2.1") comes as a corollary to Theorem 3.7 and it is shown along the lines of the proof of Theorem 3.6. The statement of Theorem 3.6 requires only trivial modifications.

4. Long time behavior for arbitrary initial data

Having established the existence of dynamics we wish to study the long time behavior. Unfortunately, we are not able to show that the limit as t goes to infinity exists, as it was possible in the one-dimensional case. We show a partial result in this direction, namely that P and w^C^+G) converge to zero in $W^{2}*$ and in L^2, respectively. These results correspond to those of Pego [32] and Andrews and Ball [4]. We may rephrase them in terms of K, the solution to (1.1), as follows: u,->0 in $W^{1}*_9$ and div (G(VK)+VU,)-»0 in L^2. We cannot say much about the behavior of a(F-H2) because in constructing system (3.1) we lost information on the divergence-free part of a, Le. (/-7C2)a.

We assume throughout the section that W is C^4 and D^2W is globally bounded with

$$d+c|\xi|^2 \le W(\xi) \le D+C|\xi|^2 \quad c, C, D>0 \tag{B}$$

We first prove an energy estimate for (3.1). It shows that the total energy (kinetic plus elastic) is dissipated by the system.

Proposition 4.1. Let us assume that W satisfies the growth condition (B) and that (P,Q) is a solution of (3.1) as in Theorem 3.1, then

$$\int_\Omega \tfrac{1}{2}|\operatorname{div}P(y,T)|^2 + W(P+Q)(y,T)\,dy + \int_0^T\int_\Omega |\nabla \operatorname{div} P(y,t)|^2\,dy\,dt = \tag{4.1}$$

$$= \int_\Omega \tfrac{1}{2}|\operatorname{div}P_0(y)|^2 + W(P_0+Q_0)\,dy = const.$$

Proof. The proof is rather standard. We observe that since

$$P \in C([\delta,T], W^{2,2}(\Omega)) \quad P_t \in C([0,T], L^2(\Omega)) \quad \delta>0$$

(Thm 3.1), it follows that the integral

$$M(t) = \int_\Omega \tfrac{1}{2}|\operatorname{div}P|^2 + W(P+Q)\,dy + \int_\delta^t\int_\Omega |\nabla \operatorname{div} P|^2\,dy\,dt \quad \delta>0 \tag{4.2}$$

is well defined and finite. Again, from (3.9) applied with $\alpha=\gamma$ we know

$$\operatorname{div} P_t \in C^\beta([\delta,T], L^2(\Omega))$$

so that we can differentiate (4.2) with respect to time. We obtain

$$\frac{dM}{dt}(t) = \int_\Omega (\operatorname{div} P, \operatorname{div} P_t) + DW(P+Q)(P_t+Q_t) + |\nabla \operatorname{div} P|^2 (y,t)\,dy$$

We integrate by parts the first term, and use $P_t+Q_t=\nabla \operatorname{div}P$:

$$\frac{dM}{dt}(t) = \int_\Omega \nabla \operatorname{div}P\,(P_t+Q_t) - (P_t+Q_t)\nabla \operatorname{div}P\,dy + \int_{\partial\Omega} (\operatorname{div}P_t)Pn\,dS = 0.$$

The boundary integral drops out since $Pn=0$ at the boundary. The first integral is zero since (3.T) holds and (T^c^Vdiv/^cVdivP) due to properties of %. We conclude then $M(t)$ is equal to $M(\delta)$, and

$$M(\delta)=\int_\Omega [\tfrac{1}{2}|\operatorname{div}P(y,\delta)|^2+W(P+Q)(y,\delta)]\,dy.$$

divP is a continuous function cm [0,7] into L^2, and taking the norm is a continuous operation, thus r-^JldivP | *dx* is continuous. It is a well-known result (see [24]) that condition (B) implies continuity of the composition *W(P+Q)* as an operator from L^2 into L^1. We conclude that *t-*]W(P+Q)dx* is continuous since P_9 *Q* are continuous into L^2. Thus we can pass to the limit S->0. Finally

$$\int_\Omega [\tfrac{1}{2}|\operatorname{div}P(y,T)|^2+W(P+Q)(y,T)]\,dy + \int_0^T\int_\Omega |\nabla\operatorname{div}P(y,t)|^2\,dy\,dt = M(0)$$

$$= \int_\Omega [\tfrac{1}{2}|\operatorname{div}P(y,0)|^2+W(P+Q)(y,0)]\,dy \quad \square$$

Since

$$\int_0^T\int_\Omega |\nabla\operatorname{div}P(y,t)|^2\,dy\,dt$$

increases in time we see that

$$L(t)=\tfrac{1}{2}\int_\Omega [|\operatorname{div}P(y,t)|^2+W(P+Q)(y,t)]\,dy$$

is a Liapunov function for system (3.1): $dL(t)/dt\le 0$, and in particular

$$d\backslash Cl\backslash+\int_\Omega W(P+Q)(y,T)\,dy\le L(0)<\infty.$$

Before we study long time behavior of *P* and J^a we establish a preliminary, rather crude estimate.

Proposition 4.2. If the growth condition (B) is fulfilled then P, Q and consequently %a(P+2) are bounded in L^2. the bound being independent of time.

Proof. Lemma 3.5 states that

IT[1] llPHwu^lldivFlt^KllPl^u.

From Proposition 4.1 we know that HdivPll^a is bounded independent of time.

Combining these two facts we obtain that the L^2-norm of P is bounded independent of time.

The boundedness of Q follows from Proposition 4.1 and boundedness of P

$$M(0)+|d|\,|\Omega|\geq\int_\Omega[W(P+Q)-d]\geq\int_\Omega c\,|P+Q|^2\geq c\,|\,\|P\|_{L^2}-\|Q\|_{L^2}|^2$$

thus

$$\|Q\|_{L^2}\leq const+\|P\|_{L^2}.$$

Finally the boundedness of $\pi_2\sigma(P+Q)$ is a result of σ being globally Lipschitz-continuous:

$$\|\pi_2\sigma(P+Q)\|_{L^2}\leq\|\pi_2\sigma(P+Q)-\pi_2\sigma(0)+\pi_2\sigma(0)\|_{L^2}\leq L\|P+Q\|_{L^2}+\|\sigma(0)\|_{L^2}\leq C \quad \Box$$

The following lemma is a very useful source of estimates. This is refined version of Theorem 3.5.2 in [20] and it is due to Pego [32].

Lemma 4.3. (Lemma A.3 in [32]). We assume that A is sectorial on a Banach space X, $f:U\to X$ is locally Lipschitz continuous on an open set $U\subset\mathbb{R}\times X^\alpha$ for some $0\leq\alpha<1$, and $z(t)$ is a solution on $(t_0,T+t_0]$ of

$$z_t+Az=f(t,z), \quad z(t_0)=z_0$$

with $(t_0,z_0)\in U$. We assume

$$\|f(t,z(t))-f(s,z(s))\|\leq K\,|t-s|+L\|z(t)-z(s)\|_\alpha.$$

Then for any $0<\gamma<1$, there exists $C_*=C_*(\alpha,\gamma,T,L)$ so that for $0\leq t_0<t<T+t_0$

$$\|z_t(t)\|_\gamma\leq C_*((t-t_0)^{\alpha-\gamma-1}\|z(t_0)\|_\alpha+(t-t_0)^{-\gamma}(\sup_{t_0\leq\tau\leq T+t_0}\|f(\tau,z(\tau))\|+K)) \quad \Box$$

One of the consequences of the Lemma is the following bound which we will need later

Lemma 4.4.

$$\|\nabla\operatorname{div}P\|_{L^2}\leq const<\infty, \quad \text{for } t\geq1.$$

Proof. The proof is an application of Lemma 43 to system (3.1) where A is given by (33) and / i s defined by (3.4). We set oe=¥=V4, and 7=1, in our context £=0. In Proposition *42* we showed that IfaoCP+gHk* is bounded, hence so is ll/"(z)llo. A bound on IMI1/2 is also known: from (3.7) we obtain

$$\|z\|_{1/2}\leq c(\|\mathrm{div}\,P\|_{L^2}+\|Q\|_{L^2});$$

next, Proposition *A2* and (4.1) give a bound on the right-hand-side. If we add up the components of *z{={P,,Q,)* we obtain a bound on (? -fO^VdivP. •

The plan of the proof that P goes to zero in W^{2}* follows the idea of Pego [32]. We start with the observation that in order to show that a continuous function v(r) with values in a normed space X goes to 0 as r->» it suffices to prove veL^2(*$^+$;X) and $\frac{d}{dt}$HKOIli^ <°°- We will apply this fact to prove ||P|byu ->0 and IIJ^C+OIIL^2 ->0- Then we apply Lemma 43 to show decay in W^{2}*^{1}. Our proof that *n2<y(P+Q)eL^{2}QL*; L^2)* generalizes an argument of Andrews and Ball [4].

Proposition 4.5. If in addition to the assumptions of the existence Theorem (Thm. 3.1) we impose the growth condition (B), then the following is true

$$\lim_{t\to\infty}\|\pi_2\sigma(P+Q)\|_{L^2}=0.$$

Proof. From (3.9) we know that P, is sufficiently smooth to take the divergence for $t>0$:

$$\mathrm{div}\,P_t=\mathrm{div}\,(\pi_2\sigma(P+Q)+\nabla\mathrm{div}\,P).$$

We can take the inner product with a function *<beC^{l}dS$_9$T],W^1*)* and integrate over Qx[8,r]. After integrating by parts we obtain

$$-\int_{\delta}^{T}\!\!\int_{\Omega}(*_{t}\,\mathrm{div}\,P)\,dx\,dt+\int_{\Omega}(*\wedge,\mathrm{div}\,P(T))dx-\int_{\Omega}(\Phi(\delta),\mathrm{div}\,P(\delta))dx= \tag{4.3}$$

$$-\int\!\!\int[(V<fr,7i2a(P-H2)HV\wedge,V\mathrm{div}P)]dxdt+\int\!\!\int(<\&,(*_2a(P+fiKV\mathrm{div}P)-n)dSdt$$

We take Φ such that VCfc=n$_2$^(^H2) and to make flie choice unique we impose

$$\int_\Omega \Phi = 0. \tag{4.4}$$

Theorem 3.1 guarantees that Φ has the desired properties. Then the first term on the RHS of (4.3) is just what we want to study

$$-\int_0^T\int_\Omega |\pi_2\sigma(P+Q)|^2 dx dt = -\int_0^T \|\pi_2\sigma(P+Q)\|_{L^2}^2 \, dt$$

The third term on the RHS drops out due to the boundary conditions. The second is

r r
Jf7t2CT(/^5^)VdivPdxdf=JJ%a(P^X^r^r)<i«i=J^(i'-H2X7>ix-fW(P-H2)(5)ic
in ta a a

and it is bounded since the elastic energy is bounded. We used here the fact that (7t2v,V4>Hv,V<t>).

The third term on the LHS is a constant The second is bounded since

$$\left|\int_\Omega (\Phi(T), \operatorname{div} P(T)) dx\right| \leq \|\Phi(T)\|_{L^2} \|\operatorname{div} P(T)\|_{L^2} \leq$$

$$C\|\nabla\Phi(T)\|_{L^2} \|\operatorname{div} P(T)\|_{L^2} \leq C\|\pi_2\sigma(P+Q)(T)\|_{L^2} \|\operatorname{div} P(T)\|_{L^2} \leq C$$

We used here (4.4), Poincar6*s inequality and boundedness of lldivPl^a (Proposition 4.1) and \\K2C(P -H2)IL2 (Proposition *A2).*

We estimate the first term on the LHS in the following way. We observe

$$\int_\Omega \Phi_t = 0$$

which is a consequence of (4.4). Then by Poincar6's inequality we have

$$\|\Phi_t\|_{L^2} \leq C\|\nabla\Phi_t\|_{L^2}.$$

Since $\pi_2\sigma(P+Q)$ is differentiate in time and

$$\frac{d}{dt}\pi_2\sigma(P+Q) = \pi_2[D\sigma(P+Q)(P_t+Q_t)].$$

Hence by Schwarz's inequality

$$\left|\int_0^T\int_\Omega \Phi_t \operatorname{div} P \, dx dt\right| \leq C\int_0^T \| \quad [D\sigma(P+Q)(\nabla \operatorname{div} P)]\|_{L^2} \cdot \|\operatorname{div} P\|_{L^2}.$$

Since we assumed that the derivative of a is bounded, we obtain

T T
I ff<D,div/> I^LCfllVdivPlta-lldivFti.

The mean value of div P over Cl is zero, so by Poincarfe's inequality

$$\text{lldivPI^^CHVdiv?^!.} \qquad (4.5)$$

Then

$$|\int_0^T\int_\Omega \langle b_t, \operatorname{div} P\rangle dx\,dt \;|\le LC\int_0^T \|\nabla \operatorname{div} P\|_{L^2}^2 dt \le constant < \infty.$$

We have proved that

$$\int_0^T \|\pi_2\sigma(P+Q)\|_{L^2}^2 dt \le C < \infty,$$

with a bound that is independent of T.

The time derivative of Ite̲aC̣P+G̲)!!!̲^{2} is also bounded:

$$\frac{d}{dt}\|\pi_2\sigma(P+Q)\|_{L^2}^2 = 2(\pi_2\sigma(P+Q), \pi_2[D\sigma(P+Q)(P_t+Q_t)]).$$

By Schwarz's inequality and Lemma 4.4 we get

$$|\frac{d}{dt}\|\pi_2\sigma(P+Q)\|_{L^2}^2| =$$

$$\le 2\|\pi_2\sigma(P+Q)\|_{L^2}\cdot\|\pi_2[D\sigma(P+Q)(\nabla\operatorname{div} P)\|_{L^2} \le 2L\|\pi_2\sigma(P+Q)\|_{L^2}\cdot\|\nabla\operatorname{div} P\|_{L^2} \le const. < \infty.$$

Knowing that

$$\int_0^\infty \|\pi_2\sigma(P+Q)\|_{L^2}^2 < \infty \quad \text{and} \quad |\frac{d}{dt}\|\pi_2\sigma(P+Q)\|_{L^2}^2| < \infty$$

we deduce that

$$\lim_{t\to\infty} \text{lt\%a(P+2)11^2=0.} \quad \text{Q}$$

Unfortunately, we are not able to determine the behavior of cr(P-H2). The problem is

that we do not have any information on *(J-ih)o(P+Q).*

We proceed to study *P.* The following Theorem is another application of Lemma 4.3.

Theorem 4.6. Assume that *W* satisfies the growth condition (B) stated above. Then for any initial data *(Po*Qo)eihW*^{l}xji2L*^{2} the solution *(P,Q)* of (3.1) has the properties**

it$_2$a(/$^{>}$+j2)+VdivP->0 in *W*^{l}* , P-»0 *inW*^{2}*** asr-*0**

Proof. We will show first that *P* decays in *W*^{l}^-topology. We know by (4.5) that

T ***T***
\\\6i\P\\$^{2}$dxdt<£\\1VdivP\\$^{2}$dxdt<£{M+\d\\Cl\).
oa **OQ**

which combined with Lemma 3.5 yields PeL^{2}(E^{+}; *W*^{ia}*(Q)).* It remains to prove the boundedness of the time derivative of ll^lto^ •

We will now invoke Lemma 4.3: we set

$$f(t,z)=\begin{bmatrix}\pi_2\sigma(P+Q)\\ -\pi_2\sigma(P+Q)\end{bmatrix}$$

α=γ=½, and *K=O,* we take 7=1. We have already established in Lemma 4.4 and in Proposition 4.2 that

supllz|li/2<C iupljr(z(r))||<C
t≥ t≥

the constants are independent of time. Thus Lemma 4.3 gives us that

n/MhvwSC Vr>8.

Summarizing, we know

$$\int_0^{\infty}\|P\|^2_{W^{1,2}}\,dt\le C$$

and the time derivative of IIPIIW^{2,1,2} is bounded,

$$\left|\frac{d}{dt}\|P\|^2\right|=2\,|(P,P_t)|\le\|P\|\cdot\|P_t\|\le C.$$

We conclude that

$$\lim_{t\to\infty} \|P\|_{W^{1,2}}=0.$$

In order to show the decay of $\|P\|_{W^{2,2}}$ we claim that Lemma 4.3 is applicable to the system

$$P_t+BP=g(t,P)$$

where $B=-\nabla\mathrm{div}$, and $g(t,P)=\pi_2\sigma(Q(t)+P)$. We already know that the operator B is sectorial on Y_2 (Proposition 3.3), also we have established

$$Y_2^{1/2}=\pi_2 W^{1,2}.$$

The non-linear term is Lipschitz-continuous,

$$\|g(t,P(t))-g(s,P(s))\|_{L^2}=\|\pi_2\sigma(P(t)+Q(t))-\pi_2\sigma(P(s)+Q(s))\|_{L^2}$$

$$\leq L\|P(t)-P(s)+Q(t)-Q(s)\|_{L^2}\leq L\|P(t)-P(s)\|_{W^{1,2}}+L\sup_{t>s}\|Q_t\|_{L^2}|t-s|$$

$$\leq L\|P(t)-P(s)\|_{W^{1,2}}+L\sup_{t>s}\|\pi_2\sigma(P(t)+Q(t))\|_{L^2}|t-s|=L\|P(t)-P(s)\|_{W^{1,2}}+LK(s)|t-s|.$$

where we set

$$K(s)=\sup_{t>s}\|Q_t\|(t)=\sup_{t>s}\|\pi_2\sigma(Q+P)\|(t).$$

We may then apply Lemma 4.3 with $\gamma=\alpha=\frac{1}{2}$ and $T=1$. Then we have for $t_0\leq t\leq t_0+1$

$$\|P_t\|_{W^{1,2}}\leq C_*((t-t_0)^{-1}\|P(t_0)\|_{W^{1,2}}+(t-t_0)^{-1/2}(\sup_{0\leq\tau-t_0\leq 1}\|g(\tau,P(\tau))\|_{L^2}+K(t_0))).$$

We let t_0 go to infinity. Then the right-hand-side of the above inequality goes to zero, since $\|P\|_{W^{1,2}}\to 0$, and $\|\pi_2\sigma(P+Q)\|_{L^2}\to 0$ (Proposition 4.5). As a result we obtain

$$\|\pi_2\sigma(P+Q)+\nabla\mathrm{div}\,P\|_{W^{1,2}}\to 0 \quad\text{and}\quad \|\nabla\mathrm{div}\,P\|_{L^2}\to 0.$$

In virtue of Lemma 3.5 the $W^{2,2}$-norm of P is bounded by $\kappa\|\nabla\mathrm{div}\,P\|_{L^2}$ thus the Theorem is proved. □

We may rephrase the result in terms of *u:*

Corollary 4.7.

$$*,\to 0 \quad \text{in} \quad W^{1}*^{2} \quad \text{and} \quad \text{div}\,(o(Vj<)+Vj<f)\to 0 \quad \text{in} \quad L^2 \quad Q$$

Remark. We note that the results of this section, in particular Corollary 4.7, are valid also for Dirichlet boundary conditions, the proofs presented here do not need changes. We have used extensively Poincait's inequality, we may do so again due to the boundary conditions which elements of TPL^1 satisfy and the fact that for solution P^D to (2.9) and (2.10) the condition

$$\text{div}\, P^D = 0$$

holds at the boundary (Theorem 3.7). Only in Proposition 4.5 we replace the normalizing condition (4.4) by the following one

$$0 = 0 \quad \text{at} \quad 3tt.$$

5. Stability of equilibria

In the present section we study stability of certain equilibria of (2.1). In particular we are interested in showing stability of local strong minimizers of the energy /

$$I(u \quad \int_Q W(\nabla u)\, dx.$$

We have constructed dynamics in W^{12} the space of finite eoergy, provided that the initial data (u (x, 0),u,(x, 0)) are in $W^{ltZ} x L^2$. Thus it is natural to consider stability in this space. Furthermore, our dynamics provides an essential ingredient for a potential-well argument for proving stability. Actually, we show that proper local strong minimizers of / are stable. For the argument to work we need an assumption on W consistent with existence of energy minimizers. Roughly, we require that W be "strong quasiconvex". Strong quasiconvexity of W dampens oscillations in gradient of minimizing sequences of /, thus it forces weak and strong convergence to be equivalent (cf [12]). In our proof we use a potential-well argument presented by Ball and Marsden [6].

We also show exponential asymptotic stability of smooth equilibria for which merely the second variation of the energy $6^2/$ is positive. Moreover we need W evaluated at the equilibrium state to be elliptic. This result corresponds to that of Potier-Ferry [33] with the exception he worked with Dirichlet boundary conditions.

We prove our result by using the linearized Stability Principle (LSP), one of the tools available within the framework of semigroup theory. That is, to show stability of an equilibrium it suffices to establish that the spectrum of the linearized operator is in the right-hand half-plane separated away firom the imaginary axis. For the LSP to work we have to change the underlying function spaces. Now, the admissible perturbations must be in $W^{2**}p > n$, Le. gradients of perturbations must be continuous. We are thus forced to show a new existence result, at least for initial values close to the equilibrium.

The method of proving asymptotic stability works not only for smooth solutions but also for a family R={qE) of states with discontinuous gradients. The family R is in W^{lt2}, gradients of all elements q_E have at most two values F, GeM***. Working with equilibria firom the family R imposes some restrictions on behavior of W near the minima F and G. For the linearization argument to be correct W must be of the same shape in some neighborhoods of F and G.

The physical interpretation of our result is that at least for some equilibria of (2.1) which contain two or more phases are asymptotically stable under perturbation, provided the perturbation does not move the interface separating the phases.

It turns out that our asymptotic stability result for R is false if we admit perturbations merely in W^{1*1}. The reason for that is, if n>l then W^{1*2} is not contained in the space of continuous functions. In fact for any q_E^R we can find in any W^{1*2}-neighborhood of q_E another dement of the family R .

We show first stability of proper local minimizers of /. We do not touch upon the issue of existence of such minimizers which is beyond the scope of this paper. Our precise result is this.

Theorem 5.1 We assume that W satisfies the conditions:

$W(\&=G£\}+\|k\|$ SeAf**", X>0 **(a)**

G is quasiconvex, $0\leq G(£)\leq A(1+1£|^2)$ **(P)**

for some constant A. We also assume that the equilibrium point u_0 of (2.1) is a proper local minimum of the functional /. Then for a given e there is a 8 such that if the initial data (u (0),u,(0))e $W^{ia}xL^2$ for equation (3.1) satisfy

$$\|u_0-u(0)\|_{W^{1,2}}<\delta \quad \text{and} \quad \int_\Omega[Vi|u,(0)|^2+W(\nabla u(0))] < \int_\Omega W(\nabla u_0)+\delta$$

then

$$\|u_t\|_{W^{1,2}}+\|u-u_0\|_{W^{1,2}}<\varepsilon \qquad V/\geq 0.$$

Our result is in the spirit of BaD and Marsden, who prove a similar result for polyconvex W.

We will first recall the notion of potential well. According to Ball and Marsden [6], we call ueW^{lt2} a proper local minimum of / if there exists e>0 such that /(v)>/(w) whenever 0<||v-tt||wu££. An element $ue\ W^{ia}$ lies in a potential well if for all e>0 sufficiently small there exists Y(e)>0 such that

/(v)-/(K)>Y(e) whenever Hv-K|tau=€

The key observation in the proof of the Theorem is Proposition 4.3 in *[6]* rephrased as follows.

Proposition 5*2* (Proposition 4.3 in [6]). Let *ue* fae W^{1}-^{2}: j^=O} lie in a potential well. Given e>0, there exists 8>0 such that if (u (0),u_r(0))€ W ^ x L^2 with

$$\|u(0)-u_0\|_{W^{1,2}}<\delta \quad \text{and} \quad \int_\Omega[\wedge|u_t(0)|^2+\wedge(V_w(0))] < \int_\Omega \wedge(Vi/_0)+5 \qquad (5.1)$$

then $\|u(t)-u\|_{L^2} < \varepsilon$ for all $t \geq 0$. □

Before we give the proof of our Theorem we recall the definition of quasiconvexity. We say that $W(\xi)$ is quasiconvex if for some fixed p, $1<p<\infty$ (we take $p=2$) W satisfies

$$0 \leq W(\xi) \leq \Gamma(1+|\xi|^p) \tag{5.2}$$

for some constant F and all £e M***, and

$$\int_{\Omega} W(A) \leq \int_{\Omega} W(A+\nabla\phi)$$

for all open 0 c E \ AeM^{KX}", $\wedge W \wedge iO,$**). It is a well-known result that if W is continuous, and it satisfies *(52)* then the functional / is weakly sequentially lower semicontinuous on $W^{ltP}(QLjL^*)$ if and only if W is quasiconvex (cf. Morrey [29], Acerbi-Fusco [1]).

We will see that undo* the hypothesis of Theorem 5.1, weak convergence is improved to strong. The following Proposition may be found in [12].

Proposition *S3* Let us suppose IP satisfies assumptions (a) and (p) of Theorem 5.1 Then

$$u_k \rightharpoonup u \text{ weakly in } W^{1,p} \text{ and } I(u_k) \to I(u)$$

implies

$$u_k \to u \text{ strongly in } W^{1,p} \qquad \square$$

Proof of Theorem 5.1 We have to show that conditions *(fi)* and (a) imply that u lies in a potential well. Let us suppose that it is false, we can find then a sequence $u_k \in W^{1,2}$ such that

$$I(u_k) \to I(u) \text{ and } \lim_{k\to\infty} \|u_k - u\|_{W^{1,2}} = \varepsilon > 0.$$

Because of the growth condition (p) we info- that u_k *art* bounded in W^{ia}, thus we can subtract a weakly convergent sequence, again denoted by u_{kj} with limit v. The norm in W^{U2} is weakly sequentially lower semicontinuous, thus $\|w-v\|_w u \wedge e$. We assumed that u is a proper minimum of / and / is weakly sequentially lower semicontinuous hence «=v.

We may apply now Proposition 53 to conclude that $u_k \to u$ strongly in W^{ia}, which is a contradiction. Thus u lies in a potential well. We observe that Theorem 4.6 guarantees that $u_t \to 0$ in W^{ia}. Finally, we apply Proposition 5.2 to complete the proof of Theorem 5.1 •

Having proved stability of proper local minimizers of / we turn our attention to those smooth equilibria u for which the second variation of the energy $8^2/(u)$ is positive. We also relax our assumption on W. We no longer need W to be strongly quasiconvex (i.e. we no longer assume (a) and (p)). Instead, we require that W evaluated at u be elliptic. Since we relaxed our assumption on W we restrict admissible perturbations of u, they must be in W^{2*}, p>n, i.e. gradient of pertuibations are now continuous. We first observe that equilibria of (2.1) satisfy

divcr(Vu)=0 in ft, a(Vu)7i=0 at 3ft.

It follows from the construction of n_p that the above equation is equivalent to

$$\pi_p \sigma(\nabla u)=0.$$

Hence equilibria of (3.1) must satisfy

$$\pi_p \sigma(Q)=0, \qquad P=0.$$

We may now formulate our stability result for equilibria of (3.1). We set

$$X_p = Y_p \times \pi_p W^{1,p}.$$

Theorem 5.4 We assume that W: M*ˣ"-»l is smooth, and it satisfies the growth condition of Theorem 4.6

rf+c|^|²<wa)^D-K:|^|² c,C,£>>0. (B)

A smooth equilibrium state $(0,Q_0)$ of (3.1) is asymptotically stable in $X\%_{\%}$ for any p >n, l>ool/2if

$$\int_\Omega D^2 W(Q_0)(\nabla h, \nabla h) \geq \beta \|\nabla h\|^2 \quad \forall h \in W^{1,2}; \qquad (y)$$

$$\forall \xi, \eta \in \mathbb{R}^n, \quad \sum_{\alpha,\beta,i,j} \frac{\partial^2 W(Q_0)}{\partial F^i_\alpha \partial F^j_\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geq \beta |\xi|^2 |\eta|^2. \tag{ε}$$

hold.

Proof. We first linearize system (3.1). If we subtract from the equation (3.1)

Pr=Vdiv P+$K2O(P+Q)$

$$Q_t = -\pi_2 \sigma(P+Q)$$

the steady state equations then we obtain the system for (5P.5Q) where 5P=P-P$_0$, &Q=Q H2 o, we know that P_0=0,

$$\delta P_t = \pi_2 [D\sigma(P_0+Q_0)(\delta P+\delta Q)] + \nabla\text{div } 5P + \wedge (5\wedge+52) \tag{5.3}$$

$$\delta Q_t = -\pi_2 [D\sigma(P_0+Q_0)(\delta P+\delta Q)] - g(\delta P+\delta Q)$$

the term ^ is defined below

$$g(h) = \pi_2 [\sigma(Q_0+h) - \sigma(Q_0) - D\sigma(Q_0)(h)].$$

We may rewrite this system as

$$z_t + (A-S)z = \tilde{g}(z)$$

where $\tilde{g}(z)=(g(z),-g(z))$, and z=(67>,6j2). We also define S as follows

$$S = \begin{bmatrix} & \\ -V_p & -V_p \end{bmatrix}$$

where

$$V_p = \pi_p D\sigma(Q_0). \tag{5.4}$$

We proved in Proposition 2.1 that B is sectorial on Y_p and consequently A is sectorial on X_{pt} where X_p=Y_px$n_p$$W^{ljf}$. Since 5 is a bounded operator on X_p we conclude that A -5 is sectorial as welL We may also apply Theorem 1.4.6 in [20] to conclude that the domains of fractional powers of A and A -S are identical. Thus in particular

$$X_p^\alpha = Y_p^\alpha \times \pi_p W^{1,p}.$$

We want to show that $\tilde{g}$ is locally Lipschitz-continuous on X_p^α we first establish the following Lemma

Lemma 5.5 The function $g: W^{1,p} \to W^{1,p}$ as defined by

$$g(h) = \pi_p \sigma(Q_0 + h) - \pi_p \sigma(Q_0) - \pi_p [D\sigma(Q_0)(h)]$$

where the composition $D\sigma(Q_0)$ is smooth, has the properties

1^0 g is locally Lipschitz-continuous on $W^{1,p}$ with values in $W^{1,p}$;

2^0 If $h \in Y_p^\alpha$ then $\|g(h)\|_{W^{1,p}} = o(\|h\|_{Y_p^\alpha})$ $\alpha > 1/2$

Proof. We use in the proof the standard results on differentiability of composition operators in Sobolev spaces (see [36]).

We set

$$\Xi(h) = \sigma(Q_0 + h) - \sigma(h) - D\sigma(Q_0)(h)$$

for $h \in W^{1,p}$ in an neighborhood of zero. Since the map $h \to \sigma$ is differentiable we obtain that

$$\|\Xi(h)\|_{W^{1,p}} = o(\|h\|_{W^{1,p}}).$$

Moreover, Ξ itself is continuously differentiable and $D\Xi(0) = 0$ so Ξ is Lipschitz continuous in a neighborhood of 0.

We note that from the construction of the projection π_p and from the elliptic regularity theory follows that π_p is continuous not only on L^p but also on $W^{1,p}$. Thus the first statement of the Lemma follows. In order to complete the proof of 2^0 we observe that embedding theorem 1.6.1 in [20] implies $\|x\|_{W^{1,p}} \le c\|x\|_{Y_p^\alpha}$ if $\alpha > 1/2$. Thus

$$\|g\|_{W^{1,p}} = \|\pi_p \Xi(h)\|_{W^{1,p}} = o(\|h\|_{Y_p^\alpha}) \quad \square$$

Since $X_p^\alpha \subset (\pi_p W^{1,p})^2$ for $\alpha > 1/2$, it follows from Lemma 5.5 that $\tilde{g}$ is locally Lipschitz-continuous on X_p^α and

$$\|\tilde{g}(z)\|_0 = o(\|z\|_\alpha). \tag{5.5}$$

We may now invoke Theorem 3.3.3 in [20] to conclude existence of kxal in time solutions to (5.3). Proceeding as in the proof of Theorem 3.1 we conclude that

$$\delta P\in C([0,T],Y_p^{\alpha})\cap C^1((0,T],\pi_p L^p)\cap C((0,T],\pi_p W^{2,p})$$

$$\delta Q\in C^1([0,T],\pi_p W^{1,p}).$$

It follows from (53) that we may linearize (5.3). Thus we turn our attention to study of the spectrum of *AS*. Our ultimate goal is to show that the real part of spectrum of *(AS)* is positive and separated away from 0. But first we have to investigate sp(A). Certainly, *sp(A)=sp(B)v{0}*. It is natural to expect that *B* being equivalent to the *Laplace* operator will have pure point spectrum. As we saw the problem

$$B\nabla\phi-\lambda\nabla\phi=\nabla f,\quad \partial\phi/\partial n=0$$

is equivalent to

-A<H4=/, 3<t>/an=0, **(5.6)**

where J/=0. We can take Ao belonging to the sector *Sj^* $_W$ then the estimate (3.5)

$$\|\phi\|_{W^{2,p}}\le C\|f\|_{L^p}/|\lambda_0|$$

holds. The a priori estimates [2] give us

$$\|\phi\|_{W^{3,p}}\le C(\|f\|_{W^{1,p}}+\|\phi\|_{L^p}).$$

Combining these inequalities we obtain

$$\|\phi\|_{W^{3,p}}\le C\|f\|_{W^{1,p}}. \qquad (5.7)$$

In other words the map /—>4> is continuous. It means that the resolvent operator (-A-Xo)""^{1} exists and it is continuous from *W^l to *W*** and consequently it is compact, because the embedding *W^3*c:W^{ltP}* is compact by the Rellich-Kondrachov Theorem. It follows that (B-Ao)"^{1} exists and it is a compact operator from Y_p into itself. The spectrum of a compact operator consists entirely of eigenvalues of finite multiplicity and at most one limit point at 0. Therefore the spectrum of *B* is pure point, and the limit point is**

at infinity. We have to rule out possibility of negative eigenvalues of *B*. We assumed that $p > n \geq 2$, then $L^p c L^2$. If there is an eigenvalue *X*, with non-positive real part, and

$$Bv-\lambda v=O$$

holds for some vector v*0, then we can compute the inner produa of the above equation with v. The result is

CBvfv)-X||v|lL»=O.

After applying Proposition 3.4 we obtain

$$(\gamma^2-\lambda)\|v\|_{L^2}\leq 0$$

which forces *X* to be real and $\gamma\leq\lambda$.

We also need to know that

/teC:ReX<p;cp(V,) (5.8)

for *V* defined by (5.4) acting on $n_pW^{lj>}$. We introduce the notation

$$M=DG(Q_0)$$

What we already know is that

(7C2Mx^c)=(Mx>x)^Pllr|ll2 , xeY_2

which is due to assumption (y), so (5.8) holds on Y_2 .

It follows directly from the definition on T^ that the problem

*,(M-X)x=y (5.9)

where x, yen_pW^{ltP} is equivalent to the following one, where we set x=V$, y=V/

. div((M-X)V<)>)=A/ (5.10)

$$(M-\lambda)\nabla\phi\cdot n=\frac{\partial f}{\partial n}.$$

Because of assumption (e) the above problem is elliptic. Since (5.8) holds for p=2, so if ReX<p, yen_pW^{ltP} then we have a unique solution x to problem (5.10) belonging to Y_2.

We have to show that in fact x is in $n_p W^{1}*$. We can take the inner product of (5.9) with any element t of Y_2

$$(\pi_p(M-\lambda)x,t)=(y,t).$$

Since fev,Vy)=(v, Vy) we obtain for r=V\p

$$\int_{\Omega}((M-\lambda)\nabla\phi,\nabla\psi)=\int_{\Omega}(\nabla f,\nabla\psi).$$

Because ye $W^{1}*$ we can integrate by parts the second integral

$$\int_{\Omega}((M-\lambda)\nabla\phi,\nabla\psi)+\int_{\Omega}(\Delta f,\nabla\psi)-\int_{\partial\Omega}(\frac{\partial f}{\partial n},\psi)=0.$$

We thus obtain that ǂe $W^{1}*$ is a weak solution to equation (5.10). The standard elliptic regularity theory (see [21]) implies thai 4> *is* in $W^{2}*$. The a priori estimates give continuity of the map V/-»V<|>. Hence, the inclusion

$$\{\lambda\in\mathbb{C}:\mathrm{Re}\lambda<\beta\}\subset\rho(V_p)$$

is valid for V *acting* on $n_p W^{1,p}$.

Our proof that *sp(A S)≥c* >0 takes advantage of some ideas of Pego (cf.proof of 4.1 in [32]) in a simplified form. First of all we establish that the essential spectrum of *AS* is bounded away from the imaginary axis. Thai we shall show the same thing for eigenvalues of *AS.* Following [32] we decompose S as *SPSQ* where

$$S_P=\begin{bmatrix}V_p & 0\\ -V_p & 0\end{bmatrix}\qquad S_Q=\begin{bmatrix}0 & V_p\\ 0 & -V_p\end{bmatrix}.$$

Since 0 does not belong to spectrum of B nor V_p it is possible to write down explicitly the inverse operator for

$$A-S_Q=\begin{bmatrix}B & -V_p\\ 0 & V_p\end{bmatrix}$$

which is

$$(A-S_Q)^{-1}=\begin{bmatrix}B^{-1} & B^{-1}\\ 0 & V_p^{-1}\end{bmatrix}$$

We have remarked earlier that due to a priori estimates (see (5.7)) the operator B^{-1} is compact. In virtue of the form of the operator $(A-S_Q)^{-1}$ the composition $S_P(A-S_Q)^{-1}$ is also compact. Thus, S_P is a compact perturbation of $A-S_Q$ and hence by Theorem A.1 in ch.5 in [20], $\mathrm{sp}_{ess}(A-S)\subset\mathrm{sp}_{ess}(A-S_Q)$. Due to the block structure of $A-S_Q$ we conclude that $\mathrm{sp}_{ess}(A-S_Q)\subset\mathrm{sp}(V)$. In addition, we know by (5.8) that the spectrum of V_p is separated away from 0.

At last we have to check that there is no eigenvalues of $A-S$ in the left half plane. Let us write the equation for eigenvalues for $A-S$

$$(A-S)\begin{bmatrix}\phi\\ \psi\end{bmatrix}=\lambda\begin{bmatrix}\phi\\ \psi\end{bmatrix}$$

or

$$(B-V_p)\phi-V_p\psi=\lambda\phi \tag{5.11'}$$

$$V_p\phi+V_p\psi=\lambda\psi. \tag{5.11''}$$

We may assume $\lambda\neq 0$ otherwise we conclude $V_p(\phi+\psi)=0$ and $B\phi=0$, it follows $\phi=\psi=0$.

We can add together equations (5.11)

$$B\phi=\lambda(\phi+\psi)$$

and solve the above equation for ψ

$$\psi=(B-\lambda)\phi/\lambda.$$

We insert the result into (5.11')

$$B\phi-V_pB\phi/\lambda=\lambda\phi.$$

Since L^p is a subspace of L^2 we can take the inner product with $B\phi$

$$\|B\phi\|^2_{L^2}-(V_pB\phi,B\phi)/\lambda=\lambda(B\phi,\phi). \tag{5.12}$$

If the real part of λ is non-positive, then the real part of the RHS of (5.12) is non-positive while the real part of the LHS of (5.12) is strictly positive, because $\|B\phi\|_{L^2}\geq\gamma^2\|\phi\|_{L^2}$ (Proposition 3.4) and V_p is positive. This contradiction proves claim that there is no eigenvalues in the left half plane.

We may finally invoke Theorem 5.1.1 in [20] to complete the proof of our Theorem. For initial data sufficiently close in *X%* to the equilibrium (0, Go) solutions to (3.1) exist for all times and we have the estimates

$$\|z(t)-z_0\|_\alpha \leq Ce^{-\gamma\beta}\|z(0)-z_0\|_\alpha$$

where z_o=(O, Go)- In other words,

W (Oliry^C^die *(O}-Qo\\wi*+||*P* (0)|| ^)

and

112*(tyQ*$_o$\\$_w$^<*Ce^(\\Q(Oy-Qo\\w^\\P*(0%),

provided ^(O^oll^+I^O)!^ is small. •

Now, stability of equilibria of (2.1) comes as a corollary to Theorem 5.4.

Corollary 5.6 If u_0 is a smooth equilibrium of (2.1) such that the conditions (y) and (e) hold, then u_0 is exponentially asymptotically stable. Precisely, if the perturbation *(u* (x, *0y-*$u_{0i}u_t$*(x,* 0)) is small in *W^xW*^{1}*** then

$$\|u-u_0\|_{W^{2,p}} \leq Ce^{-\theta t}$$

and

$$\|u_t\|_{L^p} \leq Ce^{-\theta t}$$

for some positive C and 9.

Proof. We assumed that $u_t(x_t$ 0) is in $W^{1}*_9$ it implies that *P(x,* 0) is in W^{Xp} and thus in *Y%* for col/2. We may use the results of Theorem 5.4.

The solution *u* of (2.1) is given by the formula

$$u(T)=\int_0^T ldi\backslash P(t)dt+u_0.$$

We take gradient of u,

$$\nabla u=\int_0^T \nabla \mathrm{div}\, P(t)\, dt+\nabla u_0.$$

We showed in Theorem 3.6 that it is possible to interchange integration and differentiation. Since Vdiv*P=(P+Q)*$_t$ we obtain

$$\nabla u=P(T)+Q(T)-P(0)-Q_0+Q_0=P(T)+Q(T),$$

where we set Q_0=*V*u_0. Theorem 5.4 yields

IIVU *oy-vu*$_0$*w**$=\|P(T)+Q(T)-Q_0\|_{W^{1,p}}\le C\|P(T)\|_{Y^-_p}+\|Q(T)-Q_0\|_{W^{1,p}}$

$$\le C'e^{-\theta T}(\|P(0)\|_{Y^-_p}+\|Q(T)-Q_0\|_{W^{1,p}})\le C''e^{-\theta T}(\|u_1\|_{W^{1,p}}+\|u(0)-u_0\|_{W^{2,p}})$$

By Proposition 2.1 it follows that the space average of *u* must be equal to the space average of u_0. We also set the average of u_0 to be zero (assumption (A)) thus by PoincarS's inequality we estimate the the difference $u-u_0$

$$\|u-u_0\|_{L^p}\le C\|\nabla(u-u_0)\|_{L^p}$$

finally

I" *Cn-u*$_o$*bp*ZCe-*$^{-\theta t}(\|u_1\|_{W^{1,p}}+\|u(0)-u_0\|_{W^{2,p}}),$

and

$$\|u_t\|_{L^p}=\|\mathrm{div}\, P\|_{L^p}\le C\|P\|_{W^{1,p}}\le C'\|P\|_{Y^-_p}\le C''e^{-\theta t}.\square$$

The method of the proof of Theorem 5.4 yields another result, namely stability of a family *R* to be constructed, of equilibria with discontinuous gradients. The idea is to make the composition *Dc(Q$_0$)* smooth. We will achieve our goal but at the expense of an additional assumption on *W*. But first we construct the family *R* .

Let us suppose that *W* has two local minima at *F* and *GeM****, where *F* and *G* are rank-one related, i.e. the condition

rank(F-G)=l

holds. Since DW(F)=CT(F)=0=<T(G)=D^(G), then (0,F) and (0,G) are steady states of

(3.1). Because F and G are rank-one related then there exist vectors a and *ne*^{n} such that

$$F\text{-}G\text{=}a®n.$$

Let us choose E an open bounded subset of E, E is then at most countable union of open intervals. We set

$$Q_E(x) = F\chi_E(x\cdot n)+(1-\chi_E(x\cdot n))G \quad x\in\Omega$$

We claim that Q_E is a gradient of an absolutely continuous function. If so, $(0,Q_E)$ form a family of equilibria of (3.1) since aG2£>=0. In particular, if $Vq_E=Q_E$ then q_E are steady states for (2.1) because the boundary condition (2.1") is satisfied.

If |E-E'| is small, (|-| is the Lebesgue measure), then *WQETQETWL*^{1} is also small because of the continuity of the integral with respect to the set of integration. This means that in any *L* ^neighborhood of a fixed Q_E we can find a steady state *Q?* . It follows that in $W^{1}*^{2}$ asymptotic stability fails for q_E. We note that the example is valid for any bounded domain £1

We prove now our claim. We define

$$\psi_E(t) = \int_{-\infty}^{t}\chi_E(s)ds$$

The definition is valid since E is bounded. Then $\backslash y_E{}'$ = Xs a.e. We also set

$$\tilde{q}_E = Gx + a\psi_E(n\cdot x);$$

$$q_E = \tilde{q}_E - \frac{1}{|\Omega|}\int_\Omega \tilde{q}_E.$$

We see that q_E has zero mean, and

$$Dq_E = G + a\otimes n\psi'_E(x\cdot n) = G + (F-G)\chi_E(x\cdot n) = Q_E \quad a.e.$$

The L^2-perturbations of Q_E (the W^{I}^-perturbations of q_E) allow the discontinuities of the equilibrium point to move, and this is responsible for the lack of asymptotic

stability. We observe that if *W(F)±W(G)* then the q_E need not be strong local minimizers, despite positivity of 5^2/(^£). Nonetheless, we show a kind of stability for q_E, but only for continuous perturbations, under which discontinuities do not move. We first prove stability of *QE*.

Theorem 5.7 We assume that *W: M *->&* is smooth, it has two local minima at *F* and G, such that rank(F-G)=l. The growth conditions as in Theorem 5.4 are satisfied. In addition we assume**

$$\exists\delta>0\ \forall\xi\in M^{n\times n},\quad |£|<8 \qquad W(F+\backslash y=W\{G \wedge constant \backslash \tag{5}$$

$$\backslash D^2W(Q_E)(yhyh)>V\backslash m\backslash b \qquad \text{VfceW}^{1}\text{-}^{2}; \tag{y}$$

$$\forall\xi,\eta\in\mathbb{R}^n,\qquad \sum_{\alpha,\beta,i,j}\frac{\partial^2W(Q_E)}{\partial F^i_\alpha\partial F^j_\beta}\xi_i\xi_j\eta_\alpha\eta_\beta\geq\beta|\xi|^2|\eta|^2. \tag{ε}$$

Then all the equilibria *(0,QE)* are asymptotically stable under perturbations in *X%, p* >n, ool/2.

Remark. Since ool/2 the space *X%* is embedded in $(W^1*)^2$. Because *p >n* the allowed perturbations are continuous, and they do not move discontinuities of Q_E. The condition (5) means that in a neighborhood of the minima the stored energy function has the same shape. Since Q_E has only two values the assumptions (y) and (e) pertain only to the properties of *W* at the local minima *F* and G.

Proof. As in the proof of Theorem 5.4 we may subtract from (3.1) the equations of steady states to obtain

$$\delta P_t=\pi_2[D\sigma(P_0+Q_0)(\delta P+\delta Q)]+\nabla\text{div}\,\delta P+g\,(\delta P+\delta Q) \tag{5.13}$$

$$\delta Q_t=-\pi_2[D\sigma(P_0+Q_0)(\delta P+\delta Q)]-g\,(\delta P+\delta Q)$$

where

$$g\,(h)=\pi_2[\sigma(Q_0+h)-\sigma(Q_0)-D\sigma(Q_0)(h)].$$

We observe first that due to assumption (6) and the fact that Q_o has only two values, the

composition *DG(Q₀)* is smooth. Moreover, if ||A||w^l* is so small that PIIL-<6 then the function *g (h)* is in W^{ltP} and Lemma 5.5 is applicable. Thus the above system considered for z=<5/\5j2) with the initial data in *X%* has local in time solutions by Theorem 3.3.3 in [20], and

$$\delta P\in C([0,T],Y_p^{\alpha})\cap C^1((0,T],\pi_p L^p)\cap C((0,T],\pi_p W^{2,p})$$

$$\delta Q\in C^1([0,T],\pi_p W^{1,p}).$$

We want to study stability of zero solution to (5.13). By Lemma 5.5 the linearized system is

$$z_t+(A-S)z=0.$$

We may proceed as in the proof of Theorem 5.4 to establish that

sp(A *-S)<z{ReXtc >0}.*

We may do so since in the proof of this fact we only used smoothness of $Do(Q_0)$, (8) and (e). Hence we conclude existence of solutions to (5.13) for all times and existence of positive constants *M* and p such that

IK8P (0,52 *(tMaZMe* $^{-t\beta}||(\delta P(0),\delta Q(0)||_{\alpha}$

provided ||(SP (0),52 (0))H_a is sufficiently small.

Let us define

$$F=5P, \qquad Q=\delta Q+Q_0$$

If we add to (5.13) the equation of steady state

$$0=\pi_2\sigma(Q_0)$$

$$(Q_0)_t=-\pi_2\sigma(Q_0),$$

then taking into account the definition of *g* we obtain

$$\delta P_t=\pi_2\sigma(\delta P+\delta Q+Q_0)+\nabla\operatorname{div}\delta P$$

$$(\delta Q+Q_0)_t=-\pi_2\sigma(\delta P+\delta Q+Q_0),$$

in other words, P, Q as defined above, satisfy (3.1).

Let us assume that at the initial time the data for (3.1) are (SP,62+Qo)(O) where (6P, 62)(0)eX£. Theorem 3.1 guarantees existence of solution *(P9Q)* to (3.1) far all times. Due to uniqueness of solutions of (3.1) we conclude that (SP.SgM^Q-Go) is in *X%.* Thus we have showed that if initially a small perturbation belongs to *X%* then it stays in this space for positive times.

Since we have already proved that (5P.52) decays exponentially to zero in X£ we have completed the proof of die Theorem. •

The result of Theorem 5.7 may be readily used to prove stability of q_E belonging to /?.

Corollary 5.8 Under the assumptions of Theorem 5.4 die equilibria q_E of (2.1) are exponentially asymptotically stable. Precisely, if the perturbed state (K(X,0),U,(X,0)) is such that *(u* (x, *0}*-q_E,u_t(x_t 0)) is small in *W^xW¹* p >n* then

$$\|u-q_E\|_{W^{2,p}}\le Ce^{-\theta t}$$

and

$$\|u_t\|_{L^p}\le Ce^{-\theta t}$$

for some positive C and G.

Proof. Proof is entirely analogous to the proof of Corollary 5.6. •

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Appendix

We briefly recall the results of Fujiwara and Morimoto on Helmholtz deccxnposition [14]. They showed existence of the decomposition for vector fields in L^p. In fact, their ultimate goal is to construct a continuous map P_p±^{p}-+$L^p_\$$ \<p <*©, the projection onto the closure of the set of smooth divergence free vector fields vanishing on the boundary of a region. However, it follows from section 3 in [14] that if we set

$$\pi_p = I - P_p$$

we obtain the desired projection. n_p is a continuous projection with the following properties, if v is in L^p then $n_p v = Vfa$ for some *$e* W^{ltP}. The 4> is the sum *fa+fa.* Here *fa* is the solution to

A4>i=divv, in a, $=0 on 3Q, (A.1)

moreover, the estimate

IWhllw^^CIML/ (A.2)

holds. And *fa* the unique solution to

Afa=4) in Q,(7UH_L/∂n=v*/i^*Y1/∂n at *XI*

satisfying the estimate

$$\|\Phi_2\|_{W^{1,p}(\Omega)} \le C \|v \cdot \frac{\partial}{\partial n}\|_{W^{-1/p,p}(\partial\Omega)}.$$

It follows from the construction (see [14]) that

(v-JCpV)7i=0 on *dCl* and div *(y-$n_p$vy=O* in *Cl.*

We also have

$$L^p = \pi_p L^p \oplus (I - \pi_p) L^p,$$

and

$$\pi_p{}' = \pi_q$$

where $1/p+1/q=1$. If $p=2$ then π_2 is an orthogonal projection and $\pi_2 L^2$ is orthogonal to $(I-\pi_2)L^2$.

We define π^D in the following way, for $v \in L^2$ we set

$$\pi^D v = \nabla \phi_1$$

where ϕ_1 is a unique solution of (A.1). The estimate (A.2) establishes continuity of π^D. It is a matter of easy integration by parts to show that π^D is an orthogonal projection.

We may also define the projection for tensor fields. Suppose we are given a tensor field $V \in L^p(\Omega; M^{n\times n})$

$$V = \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

where v_j are the rows of V, then we set

$$\pi_p V = \nabla(\phi_1, \cdots, \phi_n)$$

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We study the equations of viscoelasticity in a multidimensional setting for the "no-traction" boundary" data. For the sake of modeling phase transitions we do not assume ellipticity of the stored energy function W. We construct dynamics in $W^{\wedge Z}(\wedge, IR^{n})$ globally in time. Next, we study the question of stability for a class of equilibria. Moreover, we show certain kind of decay in time of solutions for arbitrary initial conditions.

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