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**A MINIMIZATION PROBLEM INVOLVING
VARIATION OF THE DOMAIN**

by

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1. INTRODUCTION.

DAVINI [4] and DAVINI & PARRY [5, 6] introduced a model for slightly defective crystals where non-elastic defect-preserving deformations are called *neutral* and generally they involve some kind of rearrangement representing the slip mechanisms of the classic phenomenological plasticity theories. Neutral deformations can be factorized into components which are exclusively elastic at the macroscopic level or exclusively slip at the microscopic level. Essentially, a neutral change of state of a perfect crystal corresponds to a lattice matrix

$$L(u(x)) = Vu(x)\{Vv(x)\}^{-1},$$

where u is the elastic deformation of the reference configuration $Q \subset \mathbb{R}^3$ into $u(Q)$ and v represents the slip or plastic deformation with $\det Vv = 1$ a. e. in Q .

Taking the viewpoint that equilibria correspond to some kind of variational principle, in DAVINI & PARRY [4, 5] and in FONSECA & PARRY [10] the implications of including in the class of admissible variations the neutral changes of state were analyzed. Precisely, FONSECA & PARRY [10] considered the minimization of the total stored energy functional

$$E(u, v) := \int_{\Omega} W(Vu(x)\{Vv(x)\}^{-1}) dx \quad (1.1)$$

where W represents the strain energy density in the class of admissible pairs

$$\mathcal{K} := \{(u, v) \in W^{1,p}(\Omega; \mathbb{R}^3) \times W^{1,p}(\Omega; \mathbb{R}^3) \mid u = u_0 \text{ on } \partial\Omega, \det Vu > 0 \text{ and } \det Vv = 1 \text{ a. e. in } \Omega\}.$$

Of course, \mathcal{K} includes the elastic deformations in the case where v is the identity map. Formally, minimizing $E(\cdot, \cdot)$ in \mathcal{K} involves variations of the reference domain; indeed, setting $\tilde{u} := u \circ v^{-1}$

the integral (1.1) becomes

$$\int_{\tilde{\Omega}} W(V\tilde{u}(y)) dy.$$

As it is well known, the bulk energy W for ordered materials is not quasiconvex (see ERICKSEN [7], FONSECA [8], KINDERLEHRER [11]) and so, the functional $E(\cdot, \cdot)$ is not lower semicontinuous. Hence, we cannot use the direct methods of the calculus of variations in order to obtain existence of minimizers of the energy and in general, such minimizers exist only in the generalized sense. Using the parametrized probability measures of YOUNG and the theory of

compensated compactness of MURAT & TARTAR (see TARTAR [12]), FONSECA & PARRY [10] examined the behavior of minimizing sequences for defective crystals and their state functions.

In this paper we study the existence and regularity properties for minimizers of (1.1) where W satisfies some convexity assumption. It should be pointed out immediately that the direct methods of the calculus of variations fail to apply to this problem. Indeed, sequential weak lower semicontinuous of $E(\cdot, \cdot)$ (see Propositions 3.8 and 3.10) is not sufficient to ensure existence of minimizers. Precisely, setting

$W(X) = \|X\|^r$
 where $\|X\|^2 := \sum_{i,j=1}^N X_{ij}^2$, we shall establish that minimizers exist if and only if $r \geq N$ (See Theorem

2.2 and Proposition 4.1). This is in sharp contrast with the usual Dirichlet problem of minimizing

$$\inf \left\{ \int_{\Omega} \|\nabla u(x)\|^r dx \mid u = u_0 \text{ on } \partial\Omega, u \in W^{1,r}(\Omega) \right\}$$

which has solutions for every $r > 1$. Surprisingly the problem behaves in fact very similarly to
 (Q)
$$\inf \left\{ \int_{\Omega} |\det \nabla u(x)|^{r/N} dx \mid u = u_0 \text{ on } \partial\Omega \right\}$$

(cf. Corollary 2.5 and Proposition 4.3). This is in agreement with the continuum theory for elastic crystals where it can be shown that, due to the crystallographic material symmetries, the relaxation of the bulk energy depends only on the determinant of the deformation gradient (see ERICKSEN [7], FONSECA [8], KINDERLEHRER [11]).

Another interesting feature of this problem is that, under some convexity-type hypotheses on W satisfied by $W(X) = \|X\|^r$, $r \geq N$, there are solutions (u^*, v^*) verifying

$$\nabla u^*(x) \{ \nabla v^*(x) \}^{-1} = X_0 \text{ for every } x \in \Omega,$$

where X_0 is a constant matrix. In the case where $W(X) = \|X\|^r$, $r \geq N$, it turns out that $X_0 = \lambda R$ where R is an orthogonal transformation and $\lambda^N = \text{meas } u_0(\Omega) / \text{meas } \Omega$.

2. THE CASE $W(X) = \|X\|^r$.

Although the results obtained in this section are strictly included on the next, we present them beforehand for the sake of clarity. We start by introducing some notations.

Notations : i) $M^{N \times N}$ denotes the set of $N \times N$ matrices and if $X \in M^{N \times N}$ then $\text{adj } X$ denotes the matrix of cofactors. In particular, if A is invertible then

$$\frac{\text{adj } X}{\det X} = \frac{(\text{adj } X)T}{\det X}$$

and (2.1)

$$\langle X, \text{adj } X \rangle = N \det X$$

where

$$\langle X, Y \rangle := \sum_{i,j=1}^N X_{ij} Y_{ij}.$$

ii) Let $Q \subset \mathbb{R}^N$ be a bounded, open set with strongly Lipschitz boundary ∂Q . If $k \geq 1$ is an integer and if $0 < a < 1$ then by $\text{Diff}^{k,a}(\bar{Q})$ we mean the set of diffeomorphisms $u : \bar{Q} \rightarrow \bar{u}(Q)$ such that $u, u^{-1} \in C^{k,a}(\bar{Q}, \bar{u}(Q))$, where $C^{k,a}$ stands for the usual set of Hölder continuous functions. In the case $k = \infty$ we shall write $\text{Diff}^{\infty}(\bar{Q})$.

iii) With the above notations, if $u_0 \in \text{Diff}^a(\bar{Q})$ with $\det Vu_0 > 0$ in \bar{Q} is given we let

$$\mathcal{L}^{k,a} := \{(u, v) \in \text{Diff}^k(\bar{Q}) \times \text{Diff}^a(\bar{Q}) \mid u = u_0 \text{ on } \partial Q \text{ and } \det Vv(x) = 1 \text{ in } \Omega\}$$

and in the case $k = \infty$ we write simply \mathcal{L}^{∞} . Finally, for $r \geq 1$ consider the problem

$$(P) \quad \inf \{E(u, v) := \int_{\Omega} \|Vu(x)(Vv(x))^{-1}\|^r dx \mid (u, v) \in \mathcal{L}^{k,a}\}$$

Remark 2.1. We note that, formally, problem (P) reduces to the minimization of functional where both the domain and the deformation are varying. Indeed, if v was invertible then

$$E(u, v) = \int_{\Omega} \|Vu^*(y)\|^r dy$$

$$\bullet \int_{\Omega} \|Vu^*(y)\|^r dx,$$

where $u^*(y) := u(v^{-1}(y))$.

Theorem 2.2

Let $k \geq 1$ be an integer, $0 < a < 1$, let $Q \subset \mathbb{R}^N$ be a bounded, open set with $C^{k+3,a}$ boundary and let $u_0 \in \text{Diff}^a(\bar{Q})$, $\det Vu_0 > 0$ in \bar{Q} . Then (P) attains its minimum at every $(u^*, v^*) \in S_{\pm a}$ such that

$$Vu^*(x)(Vv^*(x))^{-1} = X R \quad \text{in } Q$$

where $X^N = \text{meas } u_0(n) / \text{meas } Q$ and R is an orthogonal transformation. Thus

$$\inf \{E(u, v) \mid (u, v) \in S_{\pm a}\} = E(u^*, v^*) = N^{\frac{1}{2}} \text{meas } (Q)^{1/2} / N (\text{meas } u_0(Q))^{r/N}.$$

The proof of Theorem 2.2 is based on the following lemma.

Lemma 2.3

If $A \in M^{N \times N}$ then $\|A\|^N \geq N \text{Idet } A$. Furthermore, the equality holds if and only if $A = \lambda R$, for some $\lambda \in \mathbb{R}$ and some orthogonal transformation R .

Proof. If $\det A = 0$, then the inequality is trivially valid. Suppose that $\det A > 0$. Using the polar decomposition, we can write

$$A = RU,$$

where $U = U^T > 0$ and R is an orthogonal transformation, i. e. $R^T R = R R^T = \mathbb{1}$, $\det R = 1$. Thus, $U = Q^T \text{diag}(\lambda_1, \dots, \lambda_N) Q$, where Q is an orthogonal transformation and $\lambda_1, \dots, \lambda_N > 0$ and so

$$\|A\|^N = \|RU\|^N = \|U\|^N = \left(\prod_{i=1}^N \lambda_i \right)^N. \quad (2.2)$$

As \ln is a concave function, we have

$$\ln(\det A) = \ln \left(\prod_{i=1}^N \lambda_i \right) = \sum_{i=1}^N \ln \lambda_i \leq \sum_{i=1}^N \frac{1}{N} \ln \lambda_i^2 = \frac{1}{N} \ln \left(\prod_{i=1}^N \lambda_i^2 \right) \quad (2.3)$$

hence, by (2.2) and (2.3)

$$\det A \leq \left(\prod_{i=1}^N \lambda_i^2 / N \right)^{N/2} = \frac{\|A\|^N}{N^{N/2}}.$$

Finally, if $\det A < 0$ choose R^f to be an orthogonal transformation such that $\det R^f = -1$. Then, as $\det (R^f A) > 0$, by the first part of the proof we have

$$\|A\|^N = \|R^f A\|^N \geq N^N \wedge \text{Idet } (R^f A) = N^N \ll \text{Idet } A$$

Due to the strict concavity of the logarithmic function, it follows immediately from (2.3) that equality holds if and only if $|X| = |X^2| = \dots = |X^m|$ which case A is proportional to an orthogonal transformation.

Remark 2.4.

By abuse of language we shall call a matrix A such that $\|A\|^N = N^{N/2} \text{Idet } A$ *harmonic*. In \mathbb{R}^2 , a matrix A such that $\|A\|^2 = 2 \text{Idet } A$ is of the form

$$\text{either } \begin{bmatrix} \cdot & b \\ a & \cdot \end{bmatrix} \text{ or } \begin{bmatrix} b & \cdot \\ \cdot & J \end{bmatrix}.$$

Proof of Theorem 2.2. If $(u^*, v^*) \in S^{4^a}$ then, as $\det Vv^*(x) = 1$ in fact, as $r \geq N$ by Lemma 2.3 and by Hölder's inequality we have

$$\begin{aligned} E(u^*, v^*) &:= \int \|\nabla u^*(x)\| (\nabla v^*(x))^{r-1} dx \\ &\geq \text{meas}(\Omega)^{1-r/N} \left(\int_{\Omega} \|\nabla u^*(x)\| (\nabla v^*(x))^{r-1} dx \right)^{1/N} \\ &\geq \text{meas}(\Omega)^{1-r/N} \left(N^{N/2} \int_{\Omega} \text{Idet } Vu^*(x) dx \right)^{1/N} \\ &\geq \text{meas}(\Omega)^{1-r/N} \left(N^{N/2} \int_{\Omega} \det Vu^*(x) dx \right)^{r/N} \\ &= N^{r/2} \text{meas}(\Omega)^{1-r/N} \int_{\Omega} \det Vu^*(x) dx \end{aligned} \quad (2.4)$$

Let

$$x := \frac{\text{meas } Q}{\text{meas } u_o(Q)} \wedge v_o := x \wedge \text{Idet } u_o,$$

As $v_o \in \text{Diff}^* \cdot \bar{\wedge}$, $\text{meas } VQ(Q) = \text{meas } Q$ and since $3Q \in C^{k+3>a}$, by Theorem 1 in DACOROGNA & MOSER [3] there exists $v \in \text{Diff}^c \wedge \bar{Q}, \overline{v_o(Q)}$ such that

$$\begin{cases} \det Vv(x) = 1 & \text{in } Q. \\ v(x) = v_o(x) & \text{on } BQ \end{cases}$$

and define

$$u := \frac{v}{\lambda^{1/N}} \in C^{k,\alpha}(\bar{\Omega}).$$

Clearly

$$u = u_0 \text{ on } \partial\Omega$$

and we have

$$\begin{aligned} E(u, v) &:= \int_{\Omega} \|\nabla u(x)(\nabla v(x))^{-1}\|^r dx \\ &= \int_{\Omega} |f, III| N dx \\ &= \frac{N^{r/2} \text{meas}(f)}{\lambda^{r/N}} \\ &= N^{r/2} \text{meas}(Q)^{1-\frac{r}{N}} (\text{meas } u_0(Q))^{r/N} \end{aligned}$$

which, together with (2.4), finishes the proof.

Corollary 2.5

Under the hypotheses of Theorem 2.2, and in particular if $r \geq N$, then

$$\inf \{E(u, v) \mid (u, v) \in G \text{ } \mathbb{R}^n \text{ } \Omega\} = N^{r/2} \inf \left\{ \int_{\Omega} |f| |\det \nabla u(x)|^{r/N} dx \mid u = u_0 \text{ on } \partial Q \text{ and } u \in \text{Diff}^{k,\alpha}(\bar{\Omega}) \right\}.$$

Moreover, if (u^*, v^*) is a solution then there exist a rotation $R(\cdot)$ and a scalar $X(\cdot)$ such that

$$\nabla u^*(x) = X(x) R(x) \nabla v^*(x) \text{ for every } x \in Q.$$

Proof. As $\det \nabla u_0 > 0$, for all $u \in \text{Diff}^a(\bar{Q})$ with $u = u_0$ on ∂Q we have

$$\int_{\Omega} |\det \nabla u(x)| dx \geq \int_{\Omega} |\det \nabla u_0(x)| dx = \int_{\Omega} |\det \nabla u_0(x)| dx \geq a$$

where

$$a := \inf \left\{ \int_{\Omega} |\det \nabla u(x)| dx \mid u = u_0 \text{ on } \partial Q \text{ and } u \in \text{Diff}^a(\bar{Q}) \right\}.$$

Thus, by Theorem 2.2 we obtain

$$\inf \{E(u, v) \mid (u, v) \in G \text{ } S \text{ } a\} = N^{r/2} \text{meas}(Q)^{1-\frac{r}{N}} a^{r/N}. \tag{2.5}$$

On the other hand, as $r \geq N$ using Hölder's inequality we deduce that

$$a^{r/N} = \inf \left\{ \int_{\Omega} |\det \nabla u(x)|^{r/N} dx \mid u \in W^{1,r}(\Omega) \text{ and } u|_{\partial\Omega} = \varphi \right\} \\ \leq (\text{meas}(\Omega))^{r/N-1} \inf \left\{ \int_{\Omega} |\det \nabla u(x)|^{r/N} dx \mid u = \varphi \text{ on } \partial\Omega \text{ and } u \in \text{Diff}^{k,\alpha}(\bar{\Omega}) \right\}$$

which, together with (2.5) implies that

$$\inf \{ E(u, v) \mid (u, v) \in W_{k,\alpha} \} \leq N^{r/2} \inf \left\{ \int_{\Omega} |\det \nabla u(x)|^{r/N} dx \mid u = \varphi \text{ on } \partial\Omega \text{ and } u \in \text{Diff}^k(\bar{\Omega}) \right\}$$

and the reverse inequality follows immediately from Lemma 2.3.

Finally, by Lemma 2.3 if (u^*, v^*) is a solution then

$$\int_{\Omega} |\det \nabla u^*(x)|^{r/N} dx \geq \inf \left\{ \int_{\Omega} |\det \nabla u(x)|^{r/N} dx \mid u = \varphi \text{ on } \partial\Omega \text{ and } u \in \text{Diff}^k(\bar{\Omega}) \right\} \\ = N^{r/2} \inf \{ E(u, v) \mid (u, v) \in W_{k,\alpha} \} \\ = N^{r/2} \int_{\Omega} |\det \nabla u^*(x) \nabla v^*(x)|^{r/N} dx \\ \geq f \int_{\Omega} |\det \nabla u^*(x)|^{r/N} dx$$

and so

$$\int_{\Omega} \{ |\det \nabla u^*(x)|^{r/N} - N^{r/2} |\det \nabla u^*(x) \nabla v^*(x)|^{r/N} \} dx = 0 \quad (2.6)$$

which, together with Lemma 2.3, implies that

$$|\det \nabla u^*(x) \nabla v^*(x)|^{r/N} = N^{r/2} |\det \nabla u^*(x)|^{r/N} \quad \text{a. e. in } \Omega.$$

Thus

$$\nabla u^*(x) = X(x) R(x) \nabla v^*(x) \quad \text{a. e. in } \Omega$$

for some rotation $R(\cdot)$ and some scalar $X(\cdot)$. From (2.6), Theorem 2.2 and using Hölder's inequality we deduce that

$$(\text{meas } \Omega)^{1-N/r} \int_{\Omega} |\det \nabla u^*(x)|^{r/N} dx = \\ = N^{r/2} [(\text{meas } \Omega)^{1+r/N} \inf \{ E(u, v) \mid (u, v) \in W_{k,\alpha} \}]^{N/r} \\ = \int_{\Omega} |\det \nabla u^*(x)| dx \\ = \int_{\Omega} |\det \nabla u^*(x)| dx \leq f \int_{\Omega} |\det \nabla u^*(x)| dx \\ \leq (\text{meas } \Omega)^{1-N/r} \int_{\Omega} |\det \nabla u^*(x)|^{r/N} dx.$$

Hence

$$\int_Q [\det \nabla u^*(x) - |\det \nabla u^*(x)|] dx = 0$$

which implies that $\det \nabla u^* > 0$ in Q .

If u_0 is affine then we can obtain existence of minimizers under less restrictive hypothesis on BQ , namely

Proposition 2.6.

Let Q be a bounded, open set with Lipschitz boundary. Let $u_0(x) = Ax + b$ where $A \in M^{N \times N}$ with $\det A > 0$ and $b \in \mathbb{R}^N$. Then (P) admits a solution $(u, v) \in S \&^*$.

This result relies on the fact that any affine deformation is harmonic up to a volume preserving transformation. Precisely

Lemma 2.7.

If $\det A \neq 0$ then there exists a matrix B such that $\det B = 1$ and $\|AB\|^N = N^N \# |\det A|$.

Proof. Suppose that $\det A > 0$. As in the proof of Lemma 2.3, we can write

$$A = RQ^T \text{diag}(\lambda_1, \dots, \lambda_N)Q$$

where R and Q are orthogonal transformations and $\lambda_1, \dots, \lambda_N > 0$. Set

> 0 . Set

$$B := Q^T \text{diag}(p_1, \dots, p_N)$$

where

$$p_i := \frac{(\lambda_1 \dots \lambda_N)^{1/N}}{\lambda_i}$$

Then $\det B = 1$ and

$$\|AB\|^N = (\lambda_1 \dots \lambda_N)^{1/N} \prod_{i=1}^N \lambda_i = \lambda_1 \dots \lambda_N = \det A.$$

If $\det A < 0$, it suffices to multiply A by an orthogonal transformation R^1 with $\det R^1 = -1$ and to apply the previous case to the matrix $R^1 A$.

Proof of Proposition 2.6. Setting $u^* := u_0$ and $Vv^* := B^{-1}$, as in the proof of Theorem 2.2 it follows that (u^*, v^*) is a solution for (P).

3. EXISTENCE AND REGULARITY RESULTS.

Now we show that the results of Section 2 can be generalized in the following way.

Hypothesis (H): Let $W : M^{N \times N} \rightarrow [0, +\infty)$ be continuous and such that there exist $X_0 \in M^{N \times N}$ and X_0

$\in M^{N \times N}$ with

$$\det X_0 = \frac{\text{meas } u_0(Q)}{\text{meas } Q}$$

and

$$W(X) - X \det X \geq W(X_0) - X_0 \det X_0 \text{ for every } X \in M^{N \times N}.$$

Remark 3.1. (i) In some sense the X appearing in (H) can be seen as a Lagrange multiplier.

(ii) If $W \in C^1(M^{N \times N})$ then

$$\left\{ \begin{array}{l} \frac{\partial W}{\partial X}(X_0) = \lambda \text{ adj } X_0 \\ \lambda := \frac{\langle -\frac{\partial W}{\partial X}(X_0), X_0 \rangle \text{meas}(Q)}{N \text{meas}(Q)}. \end{array} \right. \quad (3.1)$$

Indeed, as X_0 is a minimum for $W(X) - X \det X$ we have

$$\begin{aligned} 0 &= \langle W(X) - X \det X \rangle_{X_0} \\ &= \frac{\partial W}{\partial X}(X_0) - \lambda \text{ adj } X_0 \end{aligned}$$

and so, by (2.1) and (H)

$$\begin{aligned} \langle -\frac{\partial W}{\partial X}(X_0), X_0 \rangle &= X_0 \det X_0 \\ &= \lambda N \frac{\text{meas } u_0(Q)}{\text{meas } Q} \end{aligned}$$

which proves (3.1).

Consider the problem

(P) minimize in $\mathcal{A}_{k,\alpha}$ the functional

$$E(u, v) := \int_{\Omega} W(\nabla u(x)(\nabla v(x))^{-1}) dx$$

where the class of admissible pairs is defined by

$$\mathcal{A}_{k,\alpha} := \{(u, v) \in \text{Diff}^{k,\alpha}(\bar{\Omega}) \times \text{Diff}^{k,\alpha}(\bar{\Omega}) \mid u = u_0 \text{ on } \partial\Omega \text{ and } \det \nabla v(x) = 1 \text{ a. e. in } \Omega\}$$

and, as in Section 2, $u_0 \in \text{Diff}^{k,\alpha}(\bar{\Omega})$ is such that $\det \nabla u_0 > 0$ in $\bar{\Omega}$.

Before stating the main result of this section, we give examples of functions satisfying the condition (H).

Proposition 3.2

The following functions $W : M^{N \times N} \rightarrow \mathbb{R}$ verify (H).

i) Let $N \geq 2$, let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be convex, C^1 and is increasing with respect to the first variable and set

$$W(X) = g(\|X\|^N, \det X).$$

In particular, (H) holds for

$$W(X) = \|X\|^r \text{ if and only if } r \geq N.$$

ii) Let $N = 2$ and let

$$W(X) = \sum_{i,j,k,l=1,2} a_{ijkl} X_{ij} X_{kl}$$

with $a_{ijkl} = a_{klij}$ and W a strictly rank one convex function, i. e.

$$W(\lambda \otimes \mu) \geq \alpha \|\lambda \otimes \mu\|^2$$

for some $\alpha > 0$ and for all $\lambda, \mu \in \mathbb{R}^2$, where $(\lambda \otimes \mu)_{ij} := \lambda_i \mu_j$ for $i, j = 1, 2$.

Remark 3.3.

There are other examples of functions satisfying (H), namely for $N = 2$

$$W(X) = \|X\|^4 - 2(\det X)^2 \text{ and } W(X) = \frac{1}{2}(X_{11}^2 + X_{12}^2 + X_{21}^2) + \frac{1}{4}X_{22}^4.$$

Proof of Proposition 3.2. i) Set

$XQ := \mathbb{R}^N$ with $\mu^N = \text{meas } \mathbb{R}^N / \text{meas}(Q)$ and R a rotation.

As $g = g(t, s)$ is convex we have

$$W(X) - W(X_0) \geq \int (\|HX\|^N, \det X_0)(\|HX\|^N - \|X_0\|^N) + \int \|X_0\|^N, \det X_0)(\det X - \det X_0),$$

and so, as $\mu^N \geq 0$ and since by Lemma 2.3

$$\|HX\|^N \geq \mu^N \det X \quad \text{and} \quad \|X_0\|^N = \mu^N \det X_0,$$

we conclude that

$$W(X) - W(X_0) \geq \lambda(\det X - \det X_0)$$

where

$$\lambda := \mu^N \wedge (\|X_0\|^N, \det X_0) + \int \mu^N \langle \|X_0\|^N, \det X_0 \rangle.$$

If $W(X) = \|X\|^r$ then (H) is violated if $r < N$ (see also Proposition 4.1). Indeed, in this case (3.1)

reduces to

$$\|X_0\|^{r-2} X_0 = \lambda \text{adj } X_0$$

and, as X_0 must be parallel to its adjugate matrix, X_0 is a harmonic matrix and

$$X_0 = \mu R \text{ for some } \mu \in \mathbb{R}, \mu > 0 \text{ and some rotation } R.$$

If in (H) we set $X = p R$, with $p \in \mathbb{R}, p > \mu$, then we obtain

$$p^r N^{1/2} - \mu^r N^{1/2} \geq \lambda (\mu^N - \mu^N) \tag{3.2}$$

and so, either $\lambda \leq 0$ and then (3.2) fails for $p < \mu$, or $\lambda > 0$ and (3.2) is false for p large enough,

(ii) Since W is rank one convex and as $N = 2$, then W is polyconvex (see DACOROGNA [2]) and

so

$$\sup \{ \lambda \mid \det Y < 0 \} \leq \inf \{ \lambda \mid \det Y > 0 \}. \tag{3.3}$$

Choose

$$\lambda = \inf \left\{ \frac{W(Y)}{\det Y} \mid \det Y > 0 \right\} \tag{3.4}$$

and observe that the infimum is attained. Indeed, since W is quadratic there is no loss of generality

in assuming that a minimizing sequence has norm 1 and so, up to the extraction of a subsequence,

we have

$$\frac{W(Y_n)}{\det Y_n} \rightarrow \lambda, Y_n \rightarrow X, \det Y_n > 0 \text{ and } \|Y_n\| = 1. \quad (3.5)$$

Note that $\det X > 0$ otherwise $X = a \otimes b$ for some unit vectors a, b , and using the strict rank one convexity and (3.5) we would have for every $\varepsilon > 0$

$$\frac{W(Y_n)}{\varepsilon + \det Y_n} \rightarrow \frac{W(X)}{\varepsilon + \det X} \geq \frac{\alpha}{\varepsilon}.$$

Letting $\varepsilon \rightarrow 0^+$ we would obtain

$$\frac{W(Y_n)}{\det Y_n} \rightarrow +\infty$$

which contradicts (3.3), (3.4), (3.5). Hence $\det X > 0$ and setting

$$X_0 := \xi X \text{ where } \xi^2 := \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega} \frac{1}{\det X} = \frac{\lambda}{W(X)} \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega}$$

it follows from (3.3) that

$$W(Y) - \lambda \det Y \geq 0 = W(X_0) - \lambda \det X_0.$$

and

$$\det X_0 = \frac{\text{meas } u_0(\Omega)}{\text{meas } \Omega}.$$

Theorem 3.4.

Let $k \geq 1$ be an integer, $0 < \alpha < 1$, Ω a bounded, open set with $C^{k+3,\alpha}$ boundary and let $u_0 \in \text{Diff}^{k,\alpha}(\bar{\Omega})$ with $\det \nabla u_0 > 0$ in $\bar{\Omega}$. If (H) holds then (P) admits a solution $(u^*, v^*) \in \mathcal{A}_{k,\alpha}$ such that

$$\nabla u^*(x) (\nabla v^*(x))^{-1} = X_0 \text{ for every } x \in \Omega$$

and

$$\inf_{\Omega} \left\{ \int_{\Omega} W(\nabla u(x) (\nabla v(x))^{-1}) dx \mid (u, v) \in \mathcal{A}_{k,\alpha} \right\} = W(X_0) \text{ meas } \Omega.$$

Remarks 3.5.

(i) As it will become clear in Section 4, in some sense the condition (H) is optimal to guarantee existence of solution.

(ii) The set $S \&^a$ of admissible pairs of functions (u, v) was chosen so as to give immediately a regularity result as well as existence of solution.

(iii) If

$$W(X) = \|X\|^r$$

for $r \geq N$, by Proposition 3.2 i) we can take $X_0 = XR$ where R is a rotation and

$$y_N = \frac{\text{meas } \text{up}(Q)}{\text{meas } Cl}$$

Then, according to Theorem 3.4 we can find a minimizer (u^*, v^*) such that

$$\nabla u^*(x)(\nabla v^*(x))^{-1} = XR \quad \text{a. e. in } Q$$

and the minimum value of the energy functional is given by

$$\begin{aligned} W(X_0) \text{ meas } Q &= \|XR\|^T \text{ meas } Q \\ &= X^T N^r \# \text{ meas } Cl \\ &= N^r \text{ meas } Cl^{\wedge} (\text{meas } \text{uo}(ft))^{r/N} \end{aligned}$$

which is in agreement with Theorem 2.2 and Corollary 2.5.

Before giving the proof of Theorem 3.4 we state a theorem which is proved exactly as the preceding one but requires less regularity on dQ (see also Proposition 2.6).

Theorem 3.6.

Let Q be a bounded, open, Lipschitz domain, let $\text{uo}(x) = Ax + b$ where $A \in M^{N \times N}$ and $b \in \mathbb{R}^N$ and assume that W satisfies (H). If $\det A \neq 0$ then (P) admits a solution $(u^*, v^*) \in \text{Stf}^*$ with $u^* = \text{uo}$ on dQ and $\det \nabla v^*(x) = 1$ in Q . Furthermore, if $\det A = 0$ and if $W(X) \rightarrow 0$ when $\|X\| \rightarrow 0$ then (P) has no solution unless $A = 0$.

Proof of Theorem 3.4.

Let $X_0 \in M^{N \times N}$ be a matrix for which (H) holds. By Theorem 1 in DACOROGNA & MOSER [3], we find a mapping $u^* \in \text{Diff}^c(\bar{Q})$ such that

$$\begin{cases} \det Vu^*(x) = \det X_0 & \text{in } Q \\ u^*(x) = UQ(x) & \text{on } \partial Q. \end{cases}$$

Setting

$$v^* := \wedge u^*.$$

we have

$$\det V v^*(x) = 1 \text{ in } Q, Vu^*(x) (Vv^*(x))^{-1} = X_0 \text{ in } Q,$$

and, by (H), if $(u, v) \in \mathcal{E}(Q, a)$ we have

$$\int_d W(Vu(x)(Vv(x))^{-1}) dx \geq \int_d W(Vu^*(x)(Vv^*(x))^{-1}) dx + \int X J[\det Vu(x) - \det Vu^*(x)] dx. \quad (3.6)$$

As $u = u^*$ on dQ , we obtain

$$\int (\det Vu(x) - \det Vu^*(x)) dx = 0$$

which, together with (3.6) implies that

$$\int W(Vu(x)(Vv(x))^{-1}) dx \geq \int W(Vu^*(x)(Vv^*(x))^{-1}) dx = \text{meas}(Q)W(X_0).$$

The proof of Theorem 3.6 requires the following lemma.

Lemma 3.7.

If $\det A = 0$ then there exists a family of matrices B_ε such that $\det B_\varepsilon = 1$ and $\|AB_\varepsilon\| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Proof. Using the polar decomposition for A we can assume that

$$A = R \text{diag}(\lambda_1, \dots, \lambda_{n-1}, 0, \dots, \lambda_N)$$

where R is a rotation and $\lambda_i \neq 0$. Set

$$B_\varepsilon := \text{diag}(b_1(\varepsilon), \dots, b_N(\varepsilon))$$

where

$$b_j(\epsilon) := \begin{cases} 1 & \text{if } \lambda_j = 0, \\ \epsilon/\lambda_j & \text{if } \lambda_j \neq 0 \end{cases}$$

if $j \neq i$ and

$$b_i(\epsilon) := \frac{1}{b_1(\epsilon) \dots b_{i-1}(\epsilon) b_{i+1}(\epsilon) \dots b_N(\epsilon)}.$$

Clearly $\det B_\epsilon = 1$ and $\|AB_\epsilon\| \leq (N-1)^{1/2}\epsilon$.

Proof of Theorem 3.6. Suppose that $\det A \neq 0$. Setting $u^* := u_0$ and $v^*(x) := \text{sign}(\det A) |XQ|^{-1} U^*$, by (H) it follows that

$$\det X_0 = \det A$$

and so

$$\det Vv^* = 1 \text{ in } Q.$$

As in the proof of Theorem 3.4, we conclude that

$$\inf_{\{J \setminus V(Vu(x)(Vv(x))^{-1}) \text{ dx } | (u, v) \in \Lambda_{t,a}\}} W(X_0) \text{ meas } Q = E(u^*, v^*).$$

Finally, if $\det A = 0$ with $A \neq 0$, by Lemma 3.7 consider a sequence $\{B_\epsilon\}$ such that $\det B_\epsilon = 1$ and $\|AB_\epsilon\| \rightarrow 0$ as $\epsilon \rightarrow 0^+$. Setting $u_\epsilon = u_0$ and $Vv_\epsilon = B_\epsilon^{-1}$ we obtain

$$E(u_\epsilon, v_\epsilon) = \text{meas}(Q) W(AB_\epsilon) \rightarrow 0.$$

It is clear that in this case (P) admits no solution since, if

$$E(u^*, v^*) = \inf \{E(u, v) | (u, v) \in S_{ka}\} = 0,$$

then $Vu^*(x) (Vv^*(x))^{-1} = 0$ in Q , i. e. $Vu^*(x) = 0$ in \mathbb{R}^1 . Hence u^* must be constant and as $A \neq 0$, this is in contradiction with the condition $u^* = u_0$ on $\partial\Omega$.

Finally, we conclude this section with a result on the weak lower semicontinuity of $E(\cdot, \cdot)$. However, we insist that this property is not sufficient to ensure existence since, in general, no weak compactness can be obtained for the minimizing sequences regardless of the coercivity of W .

Proposition 3.8

Let $Q \subset \mathbb{R}^N$ be a bounded, open set. Let $p \geq 1$, $q \geq N$ and $1/p + (N-1)/q < 1$. If

$$(u_\epsilon, v_\epsilon) \rightharpoonup^* (u, v) \text{ weakly in } W^{1,p} \times W^{1,q}$$

and if $\det Vv_\epsilon = 1$ a. e. in Q then $\det Vv = 1$ a. e. in Q , and

$$Vu_\epsilon(x) (Vv_\epsilon(x))^{-1} \rightharpoonup Vu(x) (Vv(x))^{-1} \text{ weakly in } L^1$$

Consequently, if $W : M^{N \times N} \rightarrow [0, +\infty)$ is convex then

$$\int_{\Omega} W(\nabla u(x)(\nabla v(x))^{-1}) \, dx \leq \liminf \int_{\Omega} W(\nabla u_\epsilon(x)(\nabla v_\epsilon(x))^{-1}) \, dx.$$

Conjecture.

In Proposition 3.8 we used the convexity of W to obtain the weak lower semicontinuity of the energy functional $E(.,.)$. As, formally, $Vu (Vv)^{-1}$ is the gradient of $u \circ v^{-1}$, we conjecture that if W is quasiconvex and if (u_ϵ, v_ϵ) converges weakly to (u, v) then

$$\int_{\Omega} W(Vu(x)(Vv(x))^{-1}) \, dx \leq \liminf \int_{\Omega} W(Vu_\epsilon(x)(Vv_\epsilon(x))^{-1}) \, dx.$$

Proof of Proposition 3.8. As $\det Vv = 1$ a. e. in Ω , we have

$$(Vu(x) (Vv(x))^{-1})_{ij} = (Vu(x) (\text{adj } Vv(x))T)_{ij} = m(x) \cdot \wedge(x)$$

where r_i is the gradient of the i^* component of u and ξ_j is the j^* row of $\text{adj } Vv$. Hence

$$\text{curl } r_i = 0 \text{ and } \text{div } \xi_j = 0$$

and by the div-curl lemma (see TARTAR [12]) we conclude that if $p \geq 1$, $q \geq N$ and if $1/p + (N-1)/q \leq 1$ then

$$Vu_\epsilon(x) (Vv_\epsilon(x))^{-1} \rightharpoonup Vu(x) (Vv(x))^{-1} \text{ weakly in } L^1 \tag{3.7}$$

whenever

- $(u_\epsilon, v_\epsilon) \rightharpoonup^* (u, v)$ weakly in $W^{1,p} \times W^{1,q}$.

Finally, if W is a convex, nonnegative function then by (3.7) the functional $(u,v) \rightarrow E(u,v)$ is lower semicontinuous (see DACOROGNA [2]).

Remark 3.9.

Let $p \geq 1$, $q \geq N$ and $1/p + (N-1)/q \leq 1$ and consider the class of admissible pairs to be given by

$$\mathcal{E}_{p,q} := \{(u, v) \in W^1 \times Q; [\mathbb{R}^N] \times W^{1, \infty}; \mathbb{R}^N \mid u = u_0 \text{ on } \partial Q, \det \nabla v(x) = 1 \text{ a. e. in } Q$$

$$\text{and } \int_{\Omega} Jv(x) \, dx = 0 = \int_{\Omega} Jx \, dx\}.$$

Suppose that W is convex, $W(x) \geq C_i H_i |x|^r - C_0$ with $C_i > 0$. By Proposition 3.8 (P) has a solution in $\mathcal{E}_{p,q}$ if there is a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ bounded in $W^{1,p} \times W^{1,\infty}$. Suppose that $r \geq p \frac{q-1}{q}$ and that $\{v_\varepsilon\}$ is bounded in $W^{1,\infty}$. Let s be such that

$$\frac{q}{q-1} \leq s \leq \frac{r}{p}.$$

By Hölder's inequality

$$\begin{aligned} \int_{\Omega} \| \nabla u_\varepsilon(x) \|^p dx &= \int_{\Omega} H v_\varepsilon(x) C(x) X V v_\varepsilon(x) W v_\varepsilon(x) \|^p dx \\ &\leq \left(\int_{\Omega} H v_\varepsilon(x) (V v_\varepsilon(x))^{1/s} dx \right)^{1/s} \left(\int_{\Omega} \| \nabla v_\varepsilon(x) \|^s dx \right)^{1/s} \\ &\leq \text{Const} \left(\int_{\Omega} H v_\varepsilon(x) X V v_\varepsilon(x) W v_\varepsilon(x) dx \right)^{1/r} \left(\int_{\Omega} \| \nabla v_\varepsilon(x) \|^q dx \right)^{1/q} \end{aligned}$$

and so $\{u_\varepsilon\}$ is bounded in $W^{1,p}$. We conclude that if there exists a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ where $\{v_\varepsilon\}$ is bounded in $W^{1,\infty}$ then (P) admits a solution in $\mathcal{E}_{p,q}$.

We next show that the set of solutions is weakly closed.

Proposition 3.10.

Let Q be an open, bounded, Lipschitz domain, let W be a convex function, let $u_0 \in W^{1,p}(Q; \mathbb{R}^N)$ and let $p \geq 1$, $q \geq N$, $1/p + (N-1)/q \leq 1$ and $r \geq 1$. If $\{(u_n, v_n)\}$ is a sequence of solutions of (P) in S_{VA} and if (u_n, v_n) converges weakly to (u, v) in $W^{1,p} \times W^{1,\infty}; (\mathbb{R}^N) \times W^1(Q; \mathbb{R}^N)$ then (u, v) is also a solution of (P).

Proof. As $q \geq N$ standard results imply that $\text{adj } Vv_n$ converges weakly in L^{-1} to $\text{adj } Vv$.

Moreover, as

$\det Vv_n$ converges in the sense of distributions to $\det Vv$,

we must have

$$\det Vv = 1 \text{ a. e. in } Q.$$

and so

$$(u, v) \in \text{Stf}_{p,q}.$$

Finally, using the div-curl lemma we deduce that

$$Vu_n(x) (Vv_n(x))^{-1} \rightharpoonup Vu(x) (Vv(x))^{-1} \text{ weakly in } L^1$$

and as W is convex we conclude that

$$\begin{aligned} \int_{\Omega} W(Vu(x)(Vv(x))^{-1}) \, dx &\leq \liminf_{d} \int_{\Omega} W(Vu_n(x)(Vv_n(x))^{-1}) \, dx \\ &= \inf \{ E(u, v) \mid (u, v) \in \wedge_{p,q} \}. \end{aligned}$$

4. NON EXISTENCE RESULTS.

In this section we present two types of non existence results showing that, despite the resemblance of our problem to the classic Dirichlet problem of minimizing $\int_{\Omega} W \nabla u$, problem (P) is

in fact very different in nature. It turns out to be much closer to

$$(Q) \quad \inf_{\Omega} \left\{ \int_{\Omega} \det \nabla u \, dx \mid u = u_0 \text{ on } \partial Q, u \in \text{Diff}^{k,\alpha}(\bar{\Omega}) \right\}$$

as already seen in Corollary 2.5 and as it will be illustrated below. Indeed, restricting our attention to

$$W(X) = |X|^r, r > 1,$$

Theorem 3.6 provides a first type of non existence result. Namely, if $u_0(x) = Ax$ for some $A \in M^{N \times N}$, $A \neq 0$ with $\det A = 0$, then (P) does not admit a solution. This is in sharp contrast with the minimization of $\int_{\Omega} W \nabla u$.

We have seen in Theorem 2.2 and Corollary 2.5 that if $r \geq N$ then (P), as well as (Q) (with $p = r/N \geq 1$), admit solutions. Now we show the second type of non existence result, proving that if $r < N$ then neither (P) (see Proposition 4.1) nor (Q) (see Proposition 4.3) have solutions.

Proposition 4.1.

Let $Q = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, let $u_0(x) = x$ and let $0 < r < 2 = N$. Then $\inf\{ \int_Q |Vu(x)(Vv(x))^{-1}|^r dx \mid (u, v) \in W^{1,\infty}(Q) \times W^{1,\infty}(Q), u = u_0 \text{ on } \partial Q, \det Vv = 1 \text{ a. e. in } Q\} = 0$ and hence the infimum is not attained.

Remarks 4.2.

- i) In order to avoid some technicalities, in the previous proposition we considered u and v in $W^{1,\infty}(Q)$. However, the result remains valid if instead we assume that the admissible pairs $(u, v) \in \text{Diff}^{k,\alpha}(\bar{Q}) \times \text{Diff}^{k,\alpha}(\bar{Q})$.
- ii) Similarly, we take the boundary condition $u_0(x) = x$ just for the sake of illustration, since it could be replaced by any $UQ \in \text{Diff}^a(\bar{Q})$.

Proof of Proposition 4.1. Using polar coordinates we define

$$u_n(x, y) := \begin{cases} \frac{1}{e}(x, y) & \text{if } re \in (0, e) \\ \frac{1}{r}(x, y) & \text{if } re \in (e, 1), \end{cases}$$

where $e := (2n)^{k/(*-2)} \rightarrow 0$, and

$$v_n(x, y) = \frac{r}{\sqrt{2n}} \begin{pmatrix} \cos 2n\theta \\ \sin 2n\theta \end{pmatrix}$$

where $r = \sqrt{x^2 + y^2}$. A direct computation gives

$$\det Vv_n(x) = 1,$$

$$\|\nabla u_n(x)(\nabla v_n(x))^{-1}\|^k = \begin{cases} \mathbf{1}(\mathbf{1} & , \mathbf{f} \cdot , \mathbf{n} < \mathbf{r} < \mathbf{e} \\ \varepsilon^k & \\ \frac{1}{r^k} \left(\frac{1}{2n}\right)^{k/2} & \text{if } e < r < 1, \end{cases}$$

and

$$E(u_n, v_n) = n(2n + \dots)^{k/2} (2n)^{-r} + \pi \frac{2(2n)^{-k/2}}{2-r} (1 - (2n)^{-k}) \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Finally, we conclude this section with a similar result on problem (Q).

Proposition 4.3.

If an $e \in C^{3,a}$ for some $0 < a < 1$, if $u_0 \in \text{Diff}^1(\bar{Q})$ with $\det \nabla u_0 > 0$ in \bar{Q} and if $0 < \beta <$

1 then for all $p \geq 1$

$$\inf_{u \in \text{Diff}^{1,a}(\bar{Q})} \int_{\bar{Q}} |\det \nabla u(x)|^p dx \quad |u = UQ \text{ on } dQ \text{ and } u \in \text{Diff}^{1,a}(\bar{Q})| = 0$$

and thus the infimum is not attained¹.

Proof. Let $x_0 \in Q$ and let $\bar{B}(x_0, 2\varepsilon) \subset Q$. Let φ_n be a family of smooth functions such that $0 \leq \varphi_n \leq 1$ and

$$\varphi_n(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t \geq e^{1/n} \end{cases}$$

and define

$$f_n(x) := \varphi_n\left(\left|\frac{x-x_0}{\varepsilon}\right|^2\right) \left(1 + \left|\frac{x-x_0}{\varepsilon}\right|^{2n}\right) + 1 - \varphi_n\left(\left|\frac{x-x_0}{\varepsilon}\right|^2\right).$$

Clearly, $f_n \geq 0$, f_n are smooth and $f_n \geq 1$. In addition,

$$\int_a^b f_n(x) dx > \int_{|x-x_0|<\varepsilon} f_n(x) dx > \text{Const.} \left(\int_{y_0}^1 R^{N-1} R^{2n} dx \right)^{\beta}$$

¹The same result holds for $u \in W^{1,p}$, with $p \geq PN$.

$$= \text{Const.} \frac{1}{(N + 2n)^\beta} \quad (4.1)$$

and

$$\begin{aligned} \int_{\Omega} f_n(x)^\beta dx &\leq \text{Const.} + \int_{|x-x_0|<\varepsilon} \left| \frac{x-x_0}{\varepsilon} \right|^{2n\beta} dx \\ &= \text{Const.} + \text{Const.} \frac{1}{N + 2n\beta}, \end{aligned}$$

and so, from (4.1) we conclude that

$$\lim_{n \rightarrow \infty} \frac{\int_{\Omega} f_n(x)^\beta dx}{\left(\int_{\Omega} f_n(x) dx \right)^\beta} \leq \lim_{n \rightarrow \infty} \text{Const.} \left(1 + \frac{1}{N + 2n\beta} \right) (N + 2n)^\beta = 0. \quad (4.2)$$

Using Theorem 1 in DACOROGNA & MOSER [3], we find a sequence $u_n \in \text{Diff}^{1,\alpha}(\bar{\Omega})$ such that

$$\begin{cases} \det \nabla u_n(x) = \frac{\text{meas } u_0(\Omega)}{\int_{\Omega} f_n(x) dx} f_n(x) & \text{in } \Omega \\ u_n(x) = u_0(x) & \text{if } x \in \partial\Omega. \end{cases}$$

From (4.2) it follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} |\det \nabla u_n(x)|^\beta dx = \text{meas } u_0(\Omega)^\beta \lim_{n \rightarrow \infty} \frac{\int_{\Omega} f_n(x)^\beta dx}{\left(\int_{\Omega} f_n(x) dx \right)^\beta} = 0.$$

5. QUALITATIVE PROPERTIES.

We remark that if (P) has one solution then, if $\partial\Omega$ is sufficiently smooth², there are uncountably many solutions. In fact, if

$$\min E(u, v) = E(u_1, v_1)$$

²If the class of admissible functions is $\mathcal{A}_{k,\alpha}$ then $\partial\Omega$ must be $C^{k+3,\alpha}$. If we are considering the set $\mathcal{A}_{p,q}$ then we assume that $\partial\Omega$ is Lipschitz.

and if f is such that³

$$\begin{cases} \det \nabla f(x) = 1 & \text{in } \Omega \\ f(x) = x & \text{on } \partial\Omega, \end{cases}$$

then $(u_1 \circ f, v_1 \circ f)$ is admissible and, as $f(\Omega) = \Omega$ we obtain

$$\begin{aligned} E(u_1 \circ f, v_1 \circ f) &= \int_{\Omega} W(\nabla u_1(f(x)) \nabla f(x) (\nabla f(x))^{-1} (\nabla v_1(f(x))))^{-1} dx \\ &= \int_{\Omega} W(\nabla u_1(f(x)) (\nabla v_1(f(x))))^{-1} dx \\ &= \int_{f(\Omega)} W(\nabla u_1(y) (\nabla v_1(y)))^{-1} dy \\ &= E(u_1, v_1). \end{aligned}$$

In Remark 3.9 we noted that if $W(X) = \|X\|^r$, $r \geq N$, and if there exists a minimizing sequence $\{(u_\varepsilon, v_\varepsilon)\}$ where $\{v_\varepsilon\}$ is bounded in $W^{1,q}$ then (P) admits a solution in $\mathcal{A}_{p,q}$. By the preceding remark, it would suffice to show that given a sequence $\{v_\varepsilon\}$ in $W^{1,q}$ then there exists a sequence $f_\varepsilon \in W^{1,\infty}(\Omega, \Omega)$ such that

$$\begin{cases} \det \nabla f_\varepsilon(x) = 1 & \text{in } \Omega \\ f_\varepsilon(x) = x & \text{on } \partial\Omega, \end{cases}$$

and $\{v_\varepsilon \circ f_\varepsilon\}$ is bounded in $W^{1,q}$. However, such sequence $\{f_\varepsilon\}$ may fail to exist since (P) has no solution if $u_0(x) = Ax + b$, $b \in \mathbb{R}^N$, $A \in M^{N \times N}$, $\det A = 0$ and $A \neq 0$ (see Theorem 3.6).

As we mentioned before, the minimization of (P) corresponds, formally, to a minimization of a functional where the domain is varying. Theorem 3.4 provides a sufficient condition under which there is existence of solution. Here, $v(\Omega)$ becomes the domain of the solution. It is natural to ask what type of domains may correspond to solutions of (P). The following proposition partially answers that question.

³Here $f \in C^{k,\alpha}$ in the case where the class of admissible functions is $\mathcal{A}_{k,\alpha}$ and f is Lipschitz if we are considering the class $\mathcal{A}_{p,q}$.

Proposition 5.1.

Let $k \geq 1$ be an integer, $0 < c < 1$, let $Q \subset \mathbb{R}^2$ be a bounded, open set with C^{k+3} boundary and let $u_0 \in \text{Diff}^k(\overline{Q})$ with $\det Vu_0 > 0$ in \overline{Q} . Let $W(X) = |X|^2$ and assume that $3u_0(f_2)$ is an analytic Jordan curve. If $Y \subset \mathbb{R}^2$ is such that $\text{meas } Y = \text{meas } Q$ and if ∂Y is an analytic Jordan curve then there exists a minimizer (u, v) of $E(\cdot, \cdot)$ on \mathcal{K}_c such that $v(Q)$ is a translation of Y .

Proof. By the Riemann Mapping Theorem there exists a conformal equivalence $f \in \text{Diff}^k(\overline{Y}, \overline{u_0(\mathbb{D})})$. Thus we have $f = (f_1, f_2)$ where

$$\begin{cases} \frac{\partial f_1}{\partial y_1} = \frac{\partial f_2}{\partial y_2} \\ \frac{\partial f_1}{\partial y_2} = -\frac{\partial f_2}{\partial y_1} \end{cases} \quad (5.1)$$

Set

$$v_0 = f^{-1} \circ u_0 \circ i_Q \in \text{BCO}(\mathbb{R}^2).$$

As $v_0 \in \text{Diff}^k(\overline{\mathbb{D}})$, we have

$$\text{meas } v_0(Q) = \text{meas } Y = \text{meas } Q$$

and since $dQ \in C^{k+3-\alpha}$, by Theorem 1 in DACOROGNA & MOSER [3] there exists $v_1 \in \text{Diff}^{k,\alpha}(S; \overline{Y})$ such that

$$\begin{cases} \det Vv_1(x) = 1 & \text{in } Q \\ v_2(x) = v_0(x) & \text{on } \partial\Omega. \end{cases}$$

Finally, set

$$v(x) := v_1(x) + C, \text{ where the constant } C \text{ is such that } \int_{\Omega} v(x) dx = 0,$$

and define

$$u := f \circ (v - C) \in C^{k,\alpha}(\Omega).$$

Clearly

$$u = u_0 \text{ on } \partial\Omega$$

and as v is invertible (see BALL [1], Theorems 1 and 2), we have

$$\begin{aligned} E(u, v) &:= \int_{\Omega} H \nabla u(x) (\nabla v(x))^{-1} dx \\ &= \int_{\Omega} H \nabla u(v^{-1}(y)) \nabla v^{-1}(y) dy \\ &= \int_{Jv(Q)} H \nabla f(y) dy. \end{aligned}$$

Therefore, by (5.1) we deduce that

$$\begin{aligned} E(u, v) &= 2 \int_{\Omega} \left[\left(\frac{\partial f_1}{\partial y_1}(y) \right)^2 + \left(\frac{\partial f_1}{\partial y_2}(y) \right)^2 \right] dy \\ &= 2 \int_{\Omega} \det \nabla f(y) dy \\ &= 2 \int_{\Omega} \det \nabla u(v^{-1}(y)) \det \nabla v^{-1}(y) dy \\ &= 2 \int_{\Omega} \det \nabla u(x) dx. \end{aligned} \tag{5.2}$$

From (2.4) and (5.2) we deduce that (u, v) is a solution of (P) and

$$v(Q) = v_0(Q) + C = Y + C.$$

Next, and pursuing the discussing of the nature of the set of solutions of (P), we give some uniqueness results.

Proposition 5.2.

Let Q be an open, bounded, Lipschitz domain in \mathbb{R}^N , let $r \geq N$ and let $p \geq r \geq N$, $q \geq N$.

$W(X) = W(X)^r$ and if (u_1, v) and (u_2, v) are solutions of (P) in $P_{p,q}$ then $u_1 = u_2$ a. e. in Q .

Proof. Clearly, if $\delta \in (0, 1)$ then $(\delta u_1 + (1-\delta)u_2, v)$ is admissible and

$$\int_{\Omega} |V(\delta u_1 + (1-\delta)u_2)(x) - Vv(x)|^p dx < \delta \int_{\Omega} |Vu_1(x) - Vv(x)|^p dx \\ + (1-\delta) \int_{\Omega} |Vu_2(x) - Vv(x)|^p dx$$

unless $Vu_1(x) = Vu_2(x)$ a. e. in Q , and so, as $u_1 = u_2$ on ∂Q , we conclude that $u_1 = u_2$ a. e. in Q .

Proposition 5.3.

Let Q be an open, bounded, Lipschitz domain in \mathbb{R}^N , let $r \geq N$ and let $p \geq r \geq N$, $q > N$. If $W(x) = |X(x)|^r$ and if (u, v_1) is a solution of (P) in L^p such that v_1 is invertible and $v_1(Q)$ is a Lipschitz domain⁴, then (u, v_2) is another solution of (P) if and only if there exist a constant rotation R and a constant $C \in \mathbb{R}^N$ such that $v_2(x) = Rv_1(x) + C$ a. e. in Q .

Proof. Suppose that $v_2(x) = Rv_1(x) + C$ a. e. in Q . By Corollary 2.5

$$Vu(x) = \lambda(x) Q(x) Vv_1(x) \quad \text{a.e. in } Q$$

for some rotation $Q(\cdot)$ and some scalar $\lambda(\cdot)$. Hence,

$$Vu(x) = \lambda(x) Q(x) R^T Vv_2(x) \quad \text{a. e. in } Q$$

and so

$$\int_{\Omega} |Vu(x) - Vv_2(x)|^p dx = \int_{\Omega} |\lambda(x)|^p dx \\ = \int_{\Omega} |Vu(x) - Vv_1(x)|^p dx$$

and so, (u, v_2) is also a minimizer. Conversely, if (u, v_1) and (u, v_2) are solutions of (P) then by Corollary 2.5 we must have $\det Vu > 0$, $Vu(x) = \lambda_1(x) Q_1(x) Vv_1(x)$ and $Vu(x) = \lambda_2(x) Q_2(x) Vv_2(x)$ a. e. in Ω , where $\lambda_1, \lambda_2 \in \mathbb{R}$ and Q_1, Q_2 are rotations. Thus $\lambda_1 \lambda_2 > 0$,

⁴Here we will use the fact that if $v \in W^{1,p}$, $q > N$, v is invertible, $v(\Omega)$ is a strongly Lipschitz domain and if $\det Vv = 1$ a. e. then

- (i) $v^{-1} \in W^{1,q}$, $Vv^{-1}(y) = (Vv(x))^{-1}$ a. e., where $y = v(x)$;
- (ii) $Vw \in W^{1,p}$ and $V(w \circ v)(x) = Vw(v(x)) Vv(x)$ a. e. in Q , whenever $w \in W^{1,p}$, $p \geq q/(q-1)$.

$$\mathbf{V}v_i(\mathbf{x}) = \mathbf{X}_2(\mathbf{x})\mathbf{Q}_y(\mathbf{x})\mathbf{Q}_2(\mathbf{x})\mathbf{V}v_2(\mathbf{x}) \quad \text{a. e. in } Q$$

and as $\det \mathbf{V}v_i(\mathbf{x}) = 1$ we have

$$\mathbf{X}_2(\mathbf{x}) = \mathbf{I} \quad \text{a. e. in } Q.$$

We conclude that

$$\mathbf{V}v_2(\mathbf{x})(\mathbf{V}v_1(\mathbf{x}))^{-1} = \mathbf{R}_0(\mathbf{x}) \tag{5.3}$$

for some rotation $\mathbf{R}(\cdot)$. Setting

$$\mathbf{v}_2(y) := \mathbf{v}_2(\mathbf{v}_1^{-1}(y)) \quad \text{and} \quad \tilde{\mathbf{R}}_0(y) := \mathbf{R}_0(\mathbf{v}_1^{-1}(y)),$$

(5.3) reduces to

$$\mathbf{v}_2(y) = \tilde{\mathbf{R}}_0(y) \quad \text{a. e. } y \in v_1(Q)$$

and we conclude that (see FONSECA [9], Proposition A.I)

$$\tilde{\mathbf{R}}_0(\cdot), \text{ and therefore } \mathbf{R}_0, \text{ must be constantly equal to a fixed rotation } \mathbf{R}$$

which, together with (5.3) implies that

$$v_2(\mathbf{x}) = \mathbf{R}v_1(\mathbf{x}) + C \quad \text{a. e. in } Q.$$

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