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Dmitry Golovaty
Carnegie Mellon University

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**Dmitry Golovaty
Department of Mathematical Sciences
Carnegie Mellon University**

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On a Γ -limit of a Family of Anisotropic Singular Perturbations

Dmitry Golovaty

Department of Mathematical Sciences,
Carnegie Mellon University,
Pittsburgh, PA 15213-3890, USA,
email: dg33@andrew.cmu.edu

We construct an example confirming that the interfacial energy density of the Γ -limit of a family of nonconvex functionals, cannot be computed, in general, by assuming that the local behavior of a sequence of vector-valued minimizers near the interface is unidirectional.

1. Introduction.

Consider the nonconvex energy functional

$$E[u] := \int_{\Omega} W(u(x)) dx, \quad (1)$$

where $\Omega \in \mathbf{R}^N$ is an open, bounded, and strongly Lipschitz domain, $u : \Omega \rightarrow \mathbf{R}^p$, and W supports two phases. Depending on the constraints placed on u , in general there are more than one solution of the minimization problem for (1). In order to identify a selection criterion for resolving this non-uniqueness, one can study the properties of the limits of minimizers for the family of perturbed and rescaled energies

$$F_{\epsilon}[u] := \frac{1}{\epsilon} \int_{\Omega} W(u) dx + \epsilon \int_{\Omega} h^2(x, \nabla u) dx. \quad (2)$$

The relevant type of convergence in this context is the Γ – convergence, as introduced by De Giorgi [6] (see also [1] or [5]).

The characterization of the Γ -limit of the sequence F_ϵ was studied under the assumption that $h = \|\cdot\|$ in the scalar-valued case by Modica ([10]), and in the vector-valued case by Baldo ([2]), Fonseca and Tartar ([8]), Kohn and Sternberg ([9]), and Sternberg ([12]). In the former case it was found that $\{F_\epsilon\}$ Γ – converges to the functional given by

$$F_0[u] := \begin{cases} \bar{K} \text{Per}_\Omega\{u = a\}, & \text{if } u \in \{a, b\} \text{ a.e.}, \\ +\infty, & \text{otherwise,} \end{cases} \quad (3)$$

where

$$\bar{K} := 2 \inf \left\{ \int_{-1}^1 \sqrt{W(g(s))} |g'(s)| ds : g \text{ is piecewise } C^1, g(-1) = a, g(1) = b \right\}. \quad (4)$$

Here $\text{Per}_\Omega\{u = a\}$ denotes the perimeter of A in Ω (see, for example, [7] for the definition). Notice that the interfacial energy density \bar{K} is constant and is defined as an infimum of the integral in (4) over the curves connecting the points a and b .

A more general choice of h was considered in the vector-valued case by Barroso and Fonseca in [3]. They found that the $\Gamma(L^1(\Omega))$ -limit of the family of functionals in (2) is given by

$$F_0[u] := \begin{cases} \int_{\Omega \cap \partial^* \{u=a\}} K(x, a, b, \nu) dH^{N-1}(x), & \text{if } u \in V_{a,b}, \\ +\infty, & \text{otherwise.} \end{cases} \quad (5)$$

Here W satisfies a certain growth condition and attains its minimum value of zero at exactly two points a and b , while h grows at most linearly in the last argument and satisfies some technical continuity conditions. The symbol H^{N-1} denotes the $N-1$ – dimensional Hausdorff measure, the set $V_{a,b}$ is defined by

$$V_{a,b} := \{f \in BV(\Omega) \mid f(x) \in \{a, b\} \text{ a.e. in } \Omega\},$$

the vector $\nu(x)$ is normal to the interface $\Omega \cap \partial^*\{u = a\}$ at the point x , where $\partial^*\{u = a\}$ is the reduced boundary of the set $\{u = a\}$ (see, for example, [7] for the definition). In addition

$$K(x, a, b, \nu(x)) := \inf_{\substack{\xi \in A(a, b, \nu(x)) \\ L > 0}} \Phi_\nu[\xi, L], \quad (6)$$

$$\Phi_\nu[\xi, L] := \int_{Q_\nu} [LW(\xi(y)) + \frac{1}{L}(h^\infty(x, \nabla \xi(y)))^2] dy, \quad (7)$$

$$A(a, b, \nu) := \left\{ \xi \in H^1(Q_\nu; \mathbf{R}^p) : \begin{aligned} &\xi(y) = a \text{ if } y \cdot \nu = -\frac{1}{2}, \\ &\xi(y) = b \text{ if } y \cdot \nu = \frac{1}{2}, \text{ and } \xi \text{ is periodic} \end{aligned} \right. \quad (8)$$

with period one in the directions of ν_1, \dots, ν_{N-1} .

Here, the vectors $\nu_1, \dots, \nu_{N-1}, \nu$ form an orthonormal basis of \mathbf{R}^N , the open unit cube Q_ν is centered at the origin with two of its faces normal to ν and the *recession function* h^∞ is given by

$$h^\infty(x, B) := \limsup_{t \rightarrow \infty} \frac{h(x, tB)}{t}. \quad (9)$$

Remark: Following the discussion at the beginning of this section, we note that the requirement of minimality of

$$\int_{\Omega \cap \partial^*\{u=a\}} K(x, a, b, \nu(x)) dH^{N-1}(x) \quad (10)$$

represents the selection criterion for resolving the possible non-uniqueness of the minimizers of (1).

We conclude this introduction by mentioning that the scalar versions of the problem considered in [3] were studied by Bouchitté ([4]) and Owen and Sternberg ([11]).

From the definition of the interfacial energy density K one can see that it is determined by the local structure of the sequence of minimizers of (2) near the interface $\Omega \cap \partial^*\{u = a\}$.

In particular, in the scalar-valued case the methods of convex analysis allow one to conclude that for a small ϵ the minimizers of (2) are essentially locally constant along the boundary of the set $\Omega \cap \partial^* \{u = a\}$ (see, for example, [4]). This implies that the functions ξ in (8) can be taken as depending on a single variable in the direction of the vector ν . However, the same property might not hold, in general, in a vector-valued case, as the local behavior of minimizers near the interface can be more complex.

This conclusion is confirmed in this note as we provide an example of functions W and h in (2) for which the functions ξ in (8) cannot be restricted to changing only in the direction normal to the boundary. The example is based on studying the behavior of the minimizers for the “blown-up” problem (6).

2. Main Results.

From now on, we will suppose for simplicity that $(x_1, x_2) \in \Omega \subset \mathbf{R}^2$ and that $\mathbf{u} : \Omega \rightarrow \mathbf{R}^2$. In this section we will use boldface letters to represent both vectors in \mathbf{R}^2 and \mathbf{R}^2 -valued functions. Assume that the function h in (2) is independent of \mathbf{x} , positively homogeneous of degree one, convex, and satisfies the coercivity condition

$$\frac{h(p)}{\|p\|} \geq C, \quad (11)$$

where $C > 0$ is constant. Also suppose that the function W in (2) has a superlinear growth, while its minimum is equal to 0 and is achieved at exactly two vectors, \mathbf{a} and \mathbf{b} . Observe that since h is positively homogeneous of degree one, it is equal to its own recession function, that is

$$h^\infty(p) = h(p),$$

for every $p \in M^{2 \times 2}$.

Let

$$h(\nabla \mathbf{u}) = |\operatorname{div} \mathbf{u}|, \quad (12)$$

and

$$W(\mathbf{u}) = (1 - u_2^2)^2. \quad (13)$$

It is trivial to verify that both W and h satisfy only some of the restrictions imposed in the previous paragraph. In particular, W attains its minimum value on the set $\{(u_1, u_2) \in \mathbf{R}^2 \mid u_2 = \pm 1\}$ and h , while being convex and positively homogeneous of degree one, does not satisfy (11). Later we will, however, use the small perturbations of W and h in order to extend our results to the required classes of functions.

Suppose that \mathbf{i} is the unit vector in the direction of x_1 -axis and \mathbf{j} is the unit vector in the direction of x_2 -axis. Let

$$Q_{\mathbf{j}} := \left\{ (x_1, x_2) \in \mathbf{R}^2 \mid -\frac{1}{2} \leq x_1, x_2 \leq \frac{1}{2} \right\},$$

while

$$I := \left\{ x_1 \in \mathbf{R} \mid |x_1| \leq \frac{1}{2} \right\},$$

and

$$J := \left\{ x_2 \in \mathbf{R} \mid |x_2| \leq \frac{1}{2} \right\}.$$

Define

$$\begin{aligned} \overline{A}(\mathbf{a}, \mathbf{b}, \nu) := \{ \zeta : Q_{\mathbf{j}} \rightarrow \mathbf{R}^2 \mid \exists \xi \in A(\mathbf{a}, \mathbf{b}, \nu) \text{ such that} \\ \zeta(x_1, x_2) = \xi(x_1, x_2) \text{ a.e. in } \Omega \}, \end{aligned} \quad (14)$$

to be the set of restrictions to $Q_{\mathbf{j}}$ of functions from $A(\mathbf{a}, \mathbf{b}, \nu)$ (see (8)).

Fix a function $u_2 : Q_{\mathbf{j}} \rightarrow \mathbf{R}$ such that $u_2(x_1, \cdot) \in C^2(J)$ for every $x_1 \in I$, while $\langle 0, u_2 \rangle \in \overline{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$, and

$$\frac{\partial u_2}{\partial x_2} \left(x_1, \pm \frac{1}{2} \right) \equiv 0. \quad (15)$$

Now consider any $u_1 : Q_j \rightarrow \mathbf{R}$ that satisfies

$$\frac{\partial u_1}{\partial x_1}(x_1, x_2) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_2}{\partial x_2}(s, x_2) ds - \frac{\partial u_2}{\partial x_2}(x_1, x_2), \quad (16)$$

$$u_1 \left(x_1, \pm \frac{1}{2} \right) \equiv 0. \quad (17)$$

It is easy to verify that u_1 is a local minimizer of the functional

$$\Pi_{u_2}[w] := \int_{Q_j} \left(\frac{\partial w}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right)^2 dx,$$

where $\langle w, u_2(0, \cdot) \rangle \in C^1(Q_j; \mathbf{R}^2) \cap \bar{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$. By evaluating Π_{u_2} at u_1 , we obtain that

$$\Pi_{u_2}[u_1] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial u_2}{\partial x_2} dx_1 \right)^2 dx_2.$$

Choose $\phi \in C^2(\mathbf{R})$ to be monotone increasing and such that $\phi(x) \cdot \text{sgn}(x) = 1$, whenever $|x| \geq \frac{1}{2}$. For every $x \in Q_j$ and every small $\epsilon > 0$ set

$$\bar{u}_2(x_1, x_2) = \phi \left(\frac{2x_2 - |x_1|}{\epsilon} \right), \quad (18)$$

$$\bar{u}_1(x_1, x_2) = x_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\partial \bar{u}_2}{\partial x_2}(s, x_2) ds - \int_0^{x_1} \frac{\partial \bar{u}_2}{\partial x_2}(s, x_2) ds. \quad (19)$$

Then the function $\bar{\mathbf{u}} = \langle \bar{u}_1, \bar{u}_2 \rangle$ satisfies the conditions (15-17) and $\bar{\mathbf{u}} \in \bar{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j})$. Hence we can extend $\bar{\mathbf{u}}$ periodically with period one in the direction of the x_1 - axis to obtain $\bar{\mathbf{U}} \in A(-\mathbf{j}, \mathbf{j}, \mathbf{j})$.

One can calculate both \bar{u}_1 and $\text{div } \bar{\mathbf{u}}$ explicitly in terms of ϕ . In particular, for every $x \in Q_j$ we have

$$\begin{aligned} \bar{u}_1(x_1, x_2) &= 4x_1 \left(\phi \left(\frac{2x_2}{\epsilon} \right) - \phi \left(\frac{2x_2}{\epsilon} - \frac{1}{2\epsilon} \right) \right) \\ &\quad - 2 \text{sgn}(x_1) \left(\phi \left(\frac{2x_2}{\epsilon} \right) - \phi \left(\frac{2x_2 - |x_1|}{\epsilon} \right) \right), \end{aligned}$$

$$\operatorname{div} \bar{\mathbf{u}} = 4 \left(\phi \left(\frac{2x_2}{\epsilon} \right) - \phi \left(\frac{2x_2}{\epsilon} - \frac{1}{2\epsilon} \right) \right).$$

Then, by definition of ϕ ,

$$|\operatorname{div} \bar{\mathbf{u}}| \leq 8. \quad (20)$$

Next, define

$$\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) = \inf_{\substack{\xi \in \mathcal{A}(-\mathbf{j}, \mathbf{j}, \mathbf{j}) \\ L > 0}} \Phi_{\mathbf{j}}[\xi, L], \quad (21)$$

where $\Phi_{\mathbf{j}}$ is as defined in (7) and

$$\mathcal{A}(\mathbf{a}, \mathbf{b}, \nu) := \{ \xi \in A(\mathbf{a}, \mathbf{b}, \nu) \mid \nabla \xi \times \nu = \mathbf{0} \text{ a.e. in } Q_{\mathbf{j}} \},$$

consists of those functions in $A(\mathbf{a}, \mathbf{b}, \nu)$ that remain constant in any direction, perpendicular to the vector ν . Then, following [8] one can find that

$$\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) = 2 \int_{-1}^1 \sqrt{W(s)} ds,$$

which implies that $\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) = \frac{8}{3}$ for our choice of W .

Proposition 1: *The inequality*

$$\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) > K(-\mathbf{j}, \mathbf{j}, \mathbf{j}) \quad (22)$$

holds for the functions h and W defined in (12) and (13).

Proof: In order to prove (22) we will use the following lower bound in L on $\Phi_{\mathbf{j}}[\bar{\mathbf{U}}, L]$:

$$\begin{aligned} K(-\mathbf{j}, \mathbf{j}, \mathbf{j}) &\leq \inf_{L > 0} \Phi_{\mathbf{j}}[\bar{\mathbf{U}}, L] \\ &= \Lambda[\bar{\mathbf{u}}] := 2 \left(\int_{Q_{\mathbf{j}}} W(\bar{\mathbf{u}}) dx \right)^{1/2} \left(\int_{Q_{\mathbf{j}}} h^2(\nabla \bar{\mathbf{u}}) dx \right)^{1/2}, \end{aligned} \quad (23)$$

and verify that $\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) > \Lambda[\bar{\mathbf{u}}]$ for a particular value of the parameter ϵ in (18). Indeed, by (20)

$$\begin{aligned} \Lambda[\bar{\mathbf{u}}] &= 2 \left(\int_{Q_j} W(\bar{u}_2) dx \right)^{1/2} \left(\int_{Q_j} (\operatorname{div} \bar{\mathbf{u}})^2 dx \right)^{1/2} \\ &\leq 16 \left(\int_{Q_j} W(\bar{u}_2) dx \right)^{1/2}. \end{aligned} \quad (24)$$

Then, by the definition of W and ϕ , the right-hand side of (24) can be made arbitrarily small by choosing $\epsilon > 0$ small enough in the definition (18) of \bar{u}_2 . Hence there exists a $\bar{\mathbf{u}}$ such that

$$K(-\mathbf{j}, \mathbf{j}, \mathbf{j}) \leq \Lambda[\bar{\mathbf{u}}] < \frac{8}{3} = \kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}).$$

■

This example can be extended to a more general choice of the functions W and h . In particular, by considering

$$\begin{aligned} W(\mathbf{u}) &:= \alpha u_1^4 + (1 - u_2^2)^2, \\ h(\nabla \mathbf{u}) &:= |\operatorname{div} \mathbf{u}| + \beta |\nabla \mathbf{u}|, \end{aligned}$$

for small $\alpha > 0$ and $\beta > 0$, we obtain W and h that satisfy the requirements imposed at the beginning of this section. That is, W attains its minimum at exactly two vectors \mathbf{a} and \mathbf{b} and has a superlinear growth, while h is convex, positively homogeneous of degree one, and coercive. The inequality (22) can be proven for this new set of functions W and h by using Proposition 1, the continuity of $\Lambda[\bar{\mathbf{u}}]$ with respect to α and β , the boundedness of the function $\bar{\mathbf{u}}$, and the easily verifiable fact that $\kappa(-\mathbf{j}, \mathbf{j}, \mathbf{j}) = \frac{8}{3}(1 + \beta)$. Here Λ , $\bar{\mathbf{u}}$, and κ are as defined in (23), (18-19), and (21).

Therefore, when the family of functionals in (2) is defined over a set of the vector-valued functions, the interfacial energy density K cannot be computed, in general, by restricting the set A in (8) to its proper subset \mathcal{A} of the functions that are independent of x_1 -variable. In other words, unlike in the scalar-valued

case, it is not, in general, optimal for an element of a sequence of the vector-valued minimizers of (2) to be locally constant along the interface $\Omega \cap \partial^* \{\mathbf{u} = \mathbf{a}\}$ (The isotropic case (3-4) can be considered here as one of the exceptions). Hence, we conclude that the local behavior of such vector-valued minimizers near the interface can be significantly more complex when compared to the behavior of the minimizers that are scalar-valued.

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