Proof of the Mumford-Shah conjecture for two shaded image segmentations

Christopher J. Larsen
Carnegie Mellon University
Proof of the Mumford-Shah Conjecture for Two Shaded Image Segmentations

Christopher J. Larsen
Carnegie Mellon University

Research Report No. 96-NA-007

May 1996

Sponsors

U.S. Army Research Office
Research Triangle Park
NC 27709

National Science Foundation
1800 G Street, N.W.
Washington, DC 20550
Proof of the Mumford-Shah Conjecture for Two Shaded Image Segmentations

CHRISTOPHER J. LARSEN
Department of Mathematics
Carnegie Mellon University
Pittsburgh, PA 15213

Abstract

In this paper, we consider minimizing the Mumford-Shah functional over two-valued functions in the plane. Existence of minimizers is straightforward and we show that the edge set of any minimizer is a finite union of $C^1$ curves.

Keywords: image segmentation, sets of finite perimeter, isoperimetric inequality

AMS Classifications: 49K10, 49N60, 49Q05, 49Q15

1 Introduction

In the variational approach to image segmentation, one seeks minimizers of the Mumford-Shah functional

$$E(u, K) = \int_{\Omega \setminus K} |u - g|^2 dx + \int_{\partial^* K} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K),$$

where $g \in L^\infty(\Omega)$ is the initial image, $u \in C^1(\Omega \setminus K)$, $\mathcal{H}^{N-1}(K)$ is the $N-1$ dimensional Hausdorff measure of the relatively closed set $K \subset \Omega$, and $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain. Minimizers of this functional are close to the initial image due to the first term, smoothed due to the second, and segmented due to all three: if the first term forces any “low energy” $u$ to have large enough gradient along some $N-1$ dimensional surface $K$, then the image is segmented across $K$, which relieves $u$ from needing to be smooth across $K$. The last term ensures that segmentations occur only when necessary.

From the point of view of image processing, the set $K$ corresponds to edges of objects in an image, placed where a smooth grey scale image is forced to change too much too quickly. Mumford and Shah [MS] conjectured that if $N = 2$ minimizers exist and the edge set $K$ of any minimizer consists of a finite number of $C^1$ curves. It was shown in [DGCL] that minimizers exist. This was done by reformulating the problem in SBV, a space introduced by De Giorgi and Ambrosio [DGA], so that weak solutions could be shown to exist using a compactness theorem due to Ambrosio [A]. They then proved that any minimizer $u$ is in $C^1(\Omega \setminus K)$, where $K$ is the closure in $\Omega$ of the jump set of $u$, i.e., the set of points that are not Lebesgue for $u$.

Attention has largely turned to the regularity of $K$ (see, e.g., [DS], [AP], [AFP]). In particular, it was shown in [AP] and [AFP] that, for $\Omega \subset \mathbb{R}^N$, optimal edge sets are $C^{1,\alpha}$ hypersurfaces outside a closed set of $\mathcal{H}^{N-1}$ measure 0. The main idea was to analyze the behavior of $|\nabla u|$ near $x \in K$ that can cause a singularity in $K$ at $x$.

In this paper, we take a step towards understanding the regularity of $K$ when there are no singularities caused by $|\nabla u|$. We consider minimizing the Mumford-Shah functional only over two-valued functions in the plane (and so also rule out singularities due to triple
junctions, i.e., singularities in $K$ that occur when three regions with different values of $u$ meet at a point), which is equivalent to minimizing over constant multiples of characteristic functions. If $S \subset \Omega$, we denote its characteristic function by $\chi_S$, and for $u = C\chi_S$, the edge set $K$ is $\partial_* S \cap \Omega$ since $\partial_* S$ is the jump set of $C\chi_S$. Our energy is then

$$E(C\chi_S) = \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(\partial_* S \cap \Omega). \quad (1.1)$$

This variational problem corresponds to finding an optimal placement of edges around clusters of overlapping objects.

We first consider the energy

$$E(C\chi_S) := \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(S) \quad (1.2)$$

and prove that for a minimizer $S$ one has $\mathcal{H}^1(\partial_* S \cap \Omega) = 0$, so that the last term in the above energy is the same as $\mathcal{H}^1(\partial_* S \cap \Omega)$. Since $\partial_* S \subset \partial_\Omega S \cap \Omega$, minimizers of $(1.2)$ coincide with minimizers of $(1.1)$. Furthermore, we show that for such an $S$, there exists an open set $A \subset \Omega$ such that $\mathcal{L}^2(S \delta A) = 0$ and $A = A^c$. Next, we prove $A = \bigcup_{i=1}^m A_i$, where $A_i$ are the connected components of $A$ and the distance between $\partial_* A_i$ and $\partial_* A_j$ is positive away from $\partial S$, if $i \neq j$. Analogous results are obtained for $S^c$, the complement of $S$ in $\Omega$, e.g., $\mathcal{L}^2(S^c \delta A_i) = 0$ where $O_i$ are connected, etc. Finally, we conclude that $\partial_* A_i \cap \partial_* O_j$ is $C^1$ for $i = 1, \ldots, m$, $j = 1, \ldots, p$, which proves the Mumford-Shah conjecture for “two shaded” image segmentations.

2 Preliminaries

We consider a bounded, simply connected, Lipschitz domain $\Omega \subset \mathbb{R}^2$, and we define the space of functions of bounded variation $BV(\Omega)$ in the usual way (see, e.g., [EG] and [Z]). For $E \subset \Omega$, $\chi_E$ stands for the characteristic function of $E$. Given two sets $A$ and $B$, the symmetric difference is given by $A \Delta B := (A \setminus B) \cup (B \setminus A)$, and if $D \subset \Omega$ then we define $\text{dist}_D(A, B) := \text{dist}(A \cap D, B \cap D)$. For $A \subset \Omega$, we denote by $A^c$ its complement, $\bar{A}$ its closure, and $A^\circ$ its interior. We write $D \subset \subset \Omega$ if $D \subset \Omega$ is open and $\bar{D} \subset \Omega$.

We say that a set $E \subset \Omega$ has finite perimeter in $\Omega$ if $\chi_E \in BV(\Omega)$, in which case the measure theoretic boundary in $\Omega$ is defined as

$$\partial_* E := \left\{ x \in \Omega : \lim_{\delta \to 0^+} \frac{\mathcal{L}^2(B(x, \delta) \cap E)}{\mathcal{L}^2(B(x, \delta))} > 0 \text{ and } \lim_{\delta \to 0^+} \frac{\mathcal{L}^2(B(x, \delta) \setminus E)}{\mathcal{L}^2(B(x, \delta))} > 0 \right\},$$

where $B(x, \delta)$ is the open ball in $\mathbb{R}^2$ centered at $x$ with radius $\delta$. We denote by $\nu_E(x)$ the measure theoretic normal to $E$ at $x \in \partial_* E$ (for properties of this normal, see [EG]). The reduced boundary $\partial^* E$ is the set of $x \in \partial_* E$ such that $x$ is a Lebesgue point for $\nu_E$ with respect to the Radon measure $\mathcal{H}^1|\partial_* E$.

For $u \in BV(\Omega)$, we write $Du = D_{ac}u + D_s u$, where $D_{ac}u$ and $D_s u$ stand for, respectively, the absolutely continuous and singular parts of $Du$ with respect to $\mathcal{L}^2$. We also consider the set $S(u)$ of points which are not Lebesgue points for $u$. We use the representation $D_{ac}u = \nabla u \mathcal{L}^2$. We say $u$ is a special function of bounded variation, and we write $u \in SBV(\Omega)$, if $Du = \nabla u \mathcal{L}^2 + D_s u|S(u)$. This space was introduced by De Giorgi and Ambrosio [DGA].

3 Regularity of Edge Sets

Definition 3.1 For $C > 0$ fixed, we define

$$E^g_y(S) := \int_\Omega |C\chi_S - g|^2 dx + \mathcal{H}^1(\partial_* S),$$

where $g \in L^\infty(\Omega)$ and $S \subset \Omega$ is measurable.
It follows from $BV$ compactness in $L^1$ and the lower semicontinuity of perimeter that $E^*_S$ has a minimum. Indeed, even if we let $C$ vary there is a minimum. Let $C_n \chi_{S_n}$ be a minimizing sequence, and note that we can assume $|C_n| \leq ||g||_{\infty}$, $\chi_{S_n}$ is bounded in $BV(\Omega)$, so, for a subsequence, $C_n \to C$ and $\chi_{S_n} \to \chi_S$ in $L^1(\Omega)$ for some $S \subset \Omega$. Since perimeter is lower semicontinuous, it follows that $C \chi_S$ is a minimizer.

**Lemma 3.2** Suppose that $S$ minimizes $E^*_S$. Then $\mathcal{H}^1(\partial_+ S \cap \Omega \setminus \partial_+ S) = 0$.

**Proof.** Note first that if $C \chi_S$ is a minimizer of $E$, then the conclusion holds by [DGCL]. Here, we need to show that the result is true even if $C \chi_S$ is a minimum only over characteristic functions. The basic strategy follows [AP].

**Step 1:** We claim that for $D \subset \Omega$, there exists $r_D > 0$ such that if $x \in \partial_+ S \cap D$ and $r \leq r_D$, then $\mathcal{H}^1(\partial_+ S \cap B(x, r)) \geq 2r$. Clearly, it suffices to show this for $x \in \partial_+ S \cap D$.

Let $D \subset \Omega$ and $x \in \partial_+ S \cap D$ and choose $r_D < \min\{2(C + ||g||_{\infty})^{-2}, \text{dist}(D, \partial \Omega)\}$. Suppose that $r \leq r_D$ and $\mathcal{H}^1(\partial_+ S \cap B(x, r)) < 2r$. We will show that this leads to a contradiction. Put

$$S_t := S' \cap \partial B(x, t)$$

and

$$T_t := (S^c)' \cap \partial B(x, t),$$

where $S' := \{x \in \Omega : S \text{ has density 1 at } x\}$, and similarly for $(S^c)'$.

**Step 1.1:** We claim that

$$\mathcal{H}^1(\{t \in (0, r) : \mathcal{H}^1(S_t) = 0 \text{ or } \mathcal{H}^1(T_t) = 0\}) > 0. \tag{3.1}$$

Suppose that $\mathcal{H}^1(S_t), \mathcal{H}^1(T_t) > 0$ for $\mathcal{H}^1$-a.e. $t \in (0, r)$. We can choose $\phi_n \in C^\infty(B(x, r))$ such that

$$\phi_n \overset{L^1}{\to} \chi_S \text{ on } B(x, r)$$

and

$$|D\phi_n|(B(x, r)) \to \mathcal{H}^1(\partial_+ S \cap B(x, r)). \tag{3.2}$$

It follows that for $\mathcal{H}^1$-a.e. $t \in (0, r)$ we have

$$\int_{\partial B(x, t)} |\phi_n - \chi_S| d\mathcal{H}^1 \to 0. \tag{3.3}$$

For $t \in (0, r)$ such that (3.3) holds and $\mathcal{H}^1(S_t), \mathcal{H}^1(T_t) > 0$, we then have

$$\liminf_{n \to \infty} \int_{\partial B(x, t)} \left| \frac{\partial \phi_n}{\partial r} \right| d\mathcal{H}^1 \geq 2,$$

where $\frac{\partial \phi_n}{\partial r}$ denotes the tangential derivative of $\phi_n$ on $\partial B(x, t)$. Hence, by (3.2)

$$\mathcal{H}^1(\partial_+ S \cap B(x, r)) = \lim_{n \to \infty} |D\phi_n|(B(x, r))$$

$$\geq \liminf_{n \to \infty} \int_0^r \int_{\partial B(x, t)} \left| \frac{\partial \phi_n}{\partial r} \right| d\mathcal{H}^1 dt$$

$$\geq 2r.$$

This concludes the proof of (3.1). Since $\mathcal{H}^1(\partial_+ S)$ is a Radon measure, we can choose $t \in (0, r)$ such that, e.g., $\mathcal{H}^1(S_t) = 0$ and $\mathcal{H}^1(\partial_+ S \cap \partial B(x, t)) = 0$. Set

$$T := S' \setminus B(x, t).$$
Step 1.B: Next, we claim that \( \mathcal{H}^1(\partial_+T \setminus \partial_+S) = 0 \). Note that \( \partial_+T \setminus \partial_+S \subset \partial B(x,t) \) and if \( y \in \partial_+T \cap \partial B(x,t) \), then

\[
\lim_{r \to 0^+} \sup \frac{L^2(S \cap B(y,r))}{L^2(B(y,r))} > 0.
\]

If in addition \( S \) does not have density 1 at \( y \) (i.e., \( y \notin \mathcal{S}_t \)), then \( y \in \partial_+S \). Thus \( \partial_+T \setminus \partial_+S \subset \mathcal{S}_t \) and since \( \mathcal{H}^1(S_t) = 0 \), we have \( \mathcal{H}^1(\partial_+T \setminus \partial_+S) = 0 \).

Step 1.C: We prove that \( E_\phi^x(T) < E_\phi^x(S) \). From the isoperimetric inequality and Step 1.B we have

\[
\mathcal{H}^1(\partial_+S) - \mathcal{H}^1(\partial_+T) = \mathcal{H}^1(\partial_+[S \setminus T]) \geq 2\sqrt{\pi L^2(S \setminus T)^{\frac{1}{2}}}
\]

Since \( r \leq r_D \), we know that \( r < 2(C + \|g\|_{\infty})^{-2} \), and so

\[
L^2(S \setminus T) < \pi r^2 \\
< 4\pi(C + \|g\|_{\infty})^{-4}.
\]

Hence,

\[
L^2(S \setminus T)(C + \|g\|_{\infty})^2 < 2\sqrt{\pi L^2(S \setminus T)^{\frac{1}{2}}}
\leq \mathcal{H}^1(\partial_+S) - \mathcal{H}^1(\partial_+T).
\]

But this implies that \( E_\phi^x(T) < E_\phi^x(S) \) because

\[
\int_{\Omega} |\nabla \chi_T - g|^2 dx - \int_{\Omega} |\nabla \chi_S - g|^2 dx \leq L^2(S \setminus T) + (C + \|g\|_{\infty})^2.
\]

Since this contradicts \( S \) being a minimizer, we have proved the claim in Step 1.

Step 2: Now, following [AP], we set \( \mu := \mathcal{H}^1|\partial_+S \) and note that

\[
\lim_{r \to 0^+} \inf \frac{\mu(B(x,r))}{r} \geq 2
\]

for all \( x \in \partial_+S \). Hence,

\[
0 = \mu(\partial_+S \setminus \partial_+S') \geq \mathcal{H}^1(\partial_+S \setminus \partial_+S).
\]

Lemma 3.3 Suppose that \( S \) minimizes \( E_\phi^x \). Then there is an open set \( A \subset \Omega \) such that \( L^2(S \cap A) = 0 \) and \( A = \overline{A}^c \).

Proof. Define \( S' \) as in the previous lemma, and note that \( \chi_S \) has the same total variation measure and jump set as \( \chi_S \). We wish to show that we can take \( A = S'^c \). It is clear that

\[
\overline{S'} \supset S' \cup \partial_+S'
\]

and we claim that \( \overline{S} = S' \cup \partial_+S' \). Suppose that \( x \notin S' \cup \partial_+S' \). Then \( S \) does not have density 1 at \( x \) and we can choose an \( r > 0 \) such that \( B(x,r) \cap \partial_+S' = \emptyset \). Hence, \( |D \chi_S|(B(x,r)) = \mathcal{H}^1(\partial_+S \cap B(x,r)) = 0 \), and so \( \chi_S \) is a constant \( L^2 \)-a.e. in \( B(x,r) \). Since \( S \) does not have density 1 at \( x \), we know that \( S \) has density 0 on \( B(x,r) \), and so \( B(x,r) \cap S' = \emptyset \) and \( x \notin S' \).

Now, suppose that \( x \in S' \setminus \partial_+S' \). Then \( S \) has density 1 at \( x \) and and we can choose \( r > 0 \) such that \( B(x,r) \cap \partial_+S' = \emptyset \), so \( S \) has density 1 on \( B(x,r) \), and \( x \in S'^c \). Clearly, \( S'^c \subset S' \setminus \partial_+S' \), thus

\[
S'^c = S' \setminus \partial_+S'.
\]

Since \( S' \) minimizes \( E_\phi^x \), we know that \( \mathcal{H}^1(\partial_+S') < \infty \) and by the previous lemma \( \mathcal{H}^1(\partial_+S' \cap \Omega) < \infty \), hence

\[
L^2(S \setminus S'^c) = 0.
\]
Clearly $S^\circ \subset (S^\circ)^\circ$. We also have $S^\circ \subset S' = S' \cup \partial_* S'$. Suppose $B \subset S' \cup \partial_* S'$ is open. If $B \cap \partial_* S' \neq \emptyset$, then $L^2(B \setminus S) > 0$. But this is a contradiction since $L^2(\partial_* S') = 0$. Therefore, $B \subset S'$ which implies $(S' \cup \partial_* S')^\circ = S^\circ$. So, $(S^\circ)^\circ \subset S^\circ$ and
\[ S^\circ = (S^\circ)^\circ. \]

**Lemma 3.4** Suppose that $S$ minimizes $E_\gamma$. Then we can write $A = \bigcup_{i=1}^m A_i$, where $A$ is the set from Lemma 3.3 and $A_i$ are disjoint, open, and connected sets. Furthermore,
\[ \text{dist}_D(A_i, A_j) > 0 \text{ if } i \neq j \text{ and } D \subset \subset \Omega. \]

**Proof.** We may write $A = \bigcup_{i=1}^m A_i$, where $A_i$ are disjoint, open, and connected sets. We first claim that $\partial^* A \cap \partial^* A_i \cap \partial^* (A \setminus A_i) = \emptyset$. We know (see, e.g., Theorem 5.6.2 of [Z], Theorem 1 in Section 5.7.2 of [EG]) that if $x \in \partial^* A \cap \partial^* A_i \cap \partial^* (A \setminus A_i)$, then
\[
\lim_{r \to 0^+} \frac{L^2(A \cap B(x, r))}{L^2(B(x, r))} = \frac{1}{2},
\]
and
\[
\lim_{r \to 0^+} \frac{L^2(A_i \cap B(x, r))}{L^2(B(x, r))} = \frac{1}{2},
\]
which is a contradiction.

We next claim that
\[ \mathcal{H}^1(\partial_* A_i \cap \partial_* [A \setminus A_i]) = 0. \quad (3.4) \]
We first show that $\partial_* A_i, \partial_* (A \setminus A_i) \subset \overline{\partial A}$. If $x \notin \overline{\partial A}$, then we can choose $r > 0$ such that $B(x, r) \cap \partial_* A = \emptyset$. This implies $|D_{\partial A}(B(x, r))| = 0$, and so $\chi_A$ is a constant $L^2$-a.e. on $B(x, r)$. Since $A = S^\circ$, it follows that $B(x, r) \subset A$ or $B(x, r) \subset A^c$, which yields $x \notin \partial_* A_i \cup \partial_* (A \setminus A_i)$. We conclude, using Lemma 3.2, that
\[
\mathcal{H}^1(\partial_* A_i \cap \partial_* [A \setminus A_i]) = \mathcal{H}^1(\overline{\partial A} \cap \partial_* A_i \cap \partial_* [A \setminus A_i]) = \mathcal{H}^1(\overline{\partial A} \cap \partial_* A_i \cap \partial_* [A \setminus A_i]) = \mathcal{H}^1(\partial^* A \cap \partial^* A_i \cap \partial^* [A \setminus A_i]) = 0.
\]
If $A_i$ is removed from $A$, then $\int_B |C x s - g|^2 dx$ is increased by at most $(C + ||g||_\infty)^2 L^2(A_i)$. It is clear from the definition of measure theoretic boundary and the proof of (3.4) that
\[ \partial_* A \subset \partial_* A_i \cup \partial_* [A \setminus A_i] \subset \overline{\partial A}. \]
So,
\[
\mathcal{H}^1(\partial_* A) = \mathcal{H}^1(\partial_* A_i \cup \partial_* [A \setminus A_i])
\]
\[ = \mathcal{H}^1(\partial_* A_i) + \mathcal{H}^1(\partial_* [A \setminus A_i]) - \mathcal{H}^1(\partial_* A_i \cap \partial_* [A \setminus A_i])
\]
\[ = \mathcal{H}^1(\partial_* A_i) + \mathcal{H}^1(\partial_* [A \setminus A_i]).
\]
Therefore, by removing $A_i$ from $A$, $\mathcal{H}^1(\partial_* A)$ is decreased by $\mathcal{H}^1(\partial_* A_i)$. Due to the minimality of $A$ it follows that
\[ \mathcal{H}^1(\partial_* A_i) \leq (C + ||g||_\infty)^2 L^2(A_i). \quad (3.5) \]
Although the relative isoperimetric inequality (Theorem 5.4.3 in [Z], Theorem 2, Section 5.6.2 in [EG]) is stated for balls, it is immediate from the proof that a relative isoperimetric inequality holds for any bounded Lipschitz domain. In particular, there exists a constant $k > 0$ such that

$$\min\{\mathcal{L}^2(E), \mathcal{L}^2(\Omega \setminus E)\}^{1/2} \leq k\mathcal{H}^1(\partial E)$$

for all $E \subset \Omega$ measurable. Let $A_i$ be a connected component of $A$, and suppose that $\mathcal{L}^2(A_i) \leq \frac{1}{2}\mathcal{L}^2(\Omega)$. It follows from our isoperimetric inequality that

$$\frac{\mathcal{L}^2(A_i)}{\mathcal{H}^1(\partial A_i)} \leq k\mathcal{L}^2(A_i)^{1/2}.$$ 

This, together with (3.5), gives

$$\mathcal{L}^2(A_i) \geq k^{-2}(C + \|g\|_{\infty})^{-4}.$$ 

Since $\Omega$ is bounded, there are finitely many $A_i$.

Finally, we prove that $\partial A_i \cap \partial A_j = \emptyset$ if $i \neq j$ and so $\text{dist}_D(A_i, A_j) > 0$ for $D \subset \subset \Omega$. Suppose that $x \in \partial A_i \cap \partial A_j \cap D$, where $D \subset \subset \Omega$ and $i \neq j$. Then

$$\mathcal{H}^1(\partial A_i \cup \partial A_j \cap B(x, r)) \geq 4r \quad (3.6)$$

for $r < r_D$, where we have applied Step 1 in the proof of Lemma 3.2 to $\partial A_i$ and $\partial A_j$, and we used the fact that, by an argument just like that proving (3.4), we know these sets intersect on a set of $\mathcal{H}^1$ measure 0.

However, note that if for $r > 0$ we take $T := A \setminus B(x, r)$, then

$$\int_{\Omega} |C\chi_T - g|^2 \, dx - \int_{\Omega} |C\chi_A - g|^2 \, dx$$

is at most $(C + \|g\|_{\infty})^2\mathcal{L}^2(B(x, r))$, while

$$\mathcal{H}^1(\partial A \cap B(x, r)) = \mathcal{H}^1(\partial B(x, r) \cap A) - \mathcal{H}^1(\partial A \cap B(x, r)).$$

Since $E^\chi_T(T) \geq E^\chi_A(A)$, we have

$$\mathcal{H}^1(\partial A \cap B(x, r)) \leq (C + \|g\|_{\infty})^2\mathcal{L}^2(B(x, r)) + \mathcal{H}^1(\partial B(x, r) \cap A).$$

A similar argument can be made for $T := A \cup B(x, r)$, and as

$$\min\{\mathcal{H}^1(\partial B(x, r) \cap A), \mathcal{H}^1(\partial B(x, r) \setminus A)\} \leq \pi r,$$

it follows that

$$\mathcal{H}^1(\partial A \cap B(x, r)) \leq (C + \|g\|_{\infty})^2\mathcal{L}^2(B(x, r)) + \pi r, \quad (3.7)$$

contradicting (3.6) for sufficiently small $r$.

Now, note that

$$E^\chi_{C - g}(S^c) = \int_{\Omega} |C\chi_{S^c} - (C - g)|^2 \, dx + \mathcal{H}^1(\partial S^c)$$

$$= \int_{\Omega} |C(1 - \chi_S) - (C - g)|^2 \, dx + \mathcal{H}^1(\partial S)$$

$$= E^\chi_S(S).$$

Hence, $S^c$ minimizes $E^\chi_{C - g}$ if $S$ minimizes $E^\chi_T$, and so we may write

$$\mathcal{L}^2(S^c \Delta \bigcup_{i=1}^p O_i) = 0,$$

dist$_D(O_i, O_j) > 0$ if $i \neq j$ and $D \subset \subset \Omega$, and all properties obtained for $S$ and $A_i$ hold also for $S^c$ and $O_i$.

We will need the following lemma in order to prove the regularity theorem, Theorem 3.6.
Lemma 3.5 Let $A \subset \subset \Omega$ be a simply connected domain with Lipschitz boundary. Suppose that $E \subset \subset \Omega$ has finite perimeter. Suppose further that there are $a \neq b \in \partial A$ so that the connected components $C, D$ of $\partial A \setminus \{a, b\}$ are such that $E \cap A$ has density $0$ $\mathcal{H}^1$-a.e. on $C$ and $A \setminus E$ has density $0$ $\mathcal{H}^1$-a.e. on $D$. Then

$$\left| \int_{\partial_s E \cap A} \nu_E \cdot e_1 d\mathcal{H}^1 \right| = |b_2 - a_2|,$$

and similarly for $e_2$ and $|b_1 - a_1|$.

Proof. The proof is a natural generalization of the proofs of equations (6.7) and (6.8) in [L].

Theorem 3.6 Suppose that $S$ minimizes $E_\gamma^2$. Then $\partial_s S \cap \Omega$ is a finite union of $C^1$ curves.

Proof. Set $C_{i,j} := \partial_s A_i \cap \partial_s O_j \cap \Omega$ and note that $\partial_s S \cap \Omega = \bigcup_{i=1}^n \bigcup_{j=1}^p C_{i,j}$. We claim that $C_{i,j}$ is a $C^1$ curve.

In Step 1, for $D \subset \subset \Omega$ we find a constant $\bar{c} \in (0, 1)$ depending on $D$ such that for sufficiently small $r > 0$, given any $x \in C_{i,j} \cap D$ we can choose $t \in (\bar{c}r, r)$ with the following property: we can find $a, b \in \partial B(x, t)$ so that one connected component of $\partial B(x, t) \setminus \{a, b\}$ does not intersect $A_i$, and the other connected component does not intersect $O_j$. In Step 2, we get an estimate for the maximum distance between the line $L$ connecting these points and $C_{i,j} \cap B(x, t)$. In particular, we find a constant $c' > 0$ such that this maximum is bounded above by $c't^2$. In Step 3, we show that $\nu_{A_i}$ is locally uniformly continuous on $\partial^* A_i$, and so it can be extended continuously to $\partial_s A_i \cap \Omega$. Step 4 consists of proving that $C_{i,j}$ is locally the graph of a $C^1$ function, and finally we prove in Step 5 that $C_{i,j}$ are the connected components of $\partial_s S \cap \Omega$.

Step 1: Let $D \subset \subset \Omega$ and $x \in C_{i,j} \cap D$, and set

$$m(x, t) := \mathcal{H}^0(C_{i,j} \cap \partial B(x, t))$$

and

$$\alpha(x, r) := \mathcal{H}^1(\{t \in (0, r) : m(x, t) \geq 4\}).$$

Choose $D' \subset \subset \Omega$ such that $D \subset \subset D'$ and set

$$\bar{r}_D := \min\{r_D, \text{dist}(D, \partial D'), \text{dist}_{D'}(C_{i,j}, \partial_s S \setminus C_{i,j})\} > 0.$$  

For $r < \bar{r}_D$ we know from the fact that (3.1) led to a contradiction that

$$\mathcal{H}^1(\{t \in (0, r) : \mathcal{H}^1(A_i \cap \partial B(x, t)), \mathcal{H}^1(O_j \cap \partial B(x, t)) > 0\}) = r. \quad (3.8)$$

If $y \in A_i \cap \partial B(x, t) \cap O_j \cap \partial B(x, t)$, then for all $\delta > 0$, $\chi_S$ is not a constant $L^2$-a.e. on $B(y, \delta)$, and so $y \in C_{i,j}$. For $t \in (0, r)$ such that $\mathcal{H}^1(A_i \cap \partial B(x, t)), \mathcal{H}^1(O_j \cap \partial B(x, t)) > 0$, it is immediate that either $m(x, t) = \infty$ or

$$\mathcal{H}^0(A_i \cap \partial B(x, t) \cap O_j \cap \partial B(x, t)) \geq 2,$$

and so we have $m(x, t) \geq 2$. By the definition of $\bar{r}_D$, we know that if $r < \bar{r}_D$, then

$$\mathcal{H}^1((C_{i,j} \Delta \partial_s A_i) \cap B(x, r)) = 0$$

and

$$\mathcal{H}^1((\partial_s A_i \Delta \partial_s A_i) \cap B(x, r)) = 0.$$
By (3.7) we have, for \( c := (C + ||\theta||_\infty)^2 \),
\[
c\pi r^2 + \pi r \geq H^1(C_{i,j} \cap B(x, r)) \geq \int_0^r m(x, t) \, dt \geq 4\alpha(x, r) + 2(r - \alpha(x, r)),
\]
which implies that \( \alpha(x, r) \leq \frac{1}{4} c\pi r^2 + (\frac{3}{2} - 1) r \). If necessary, we can redefine \( \tau_D > 0 \) to guarantee that we can find \( \bar{c} \in (0, 1) \) such that \( r - \alpha(x, r) > \bar{c} r \) for all \( r \leq \tau_D \). Choose \( t \in (\bar{c}r, r) \) such that \( m(x, t) \in \{2, 3\} \) and, by (3.8), \( H^1(A_i \cap \partial B(x, t), \partial B(x, t)) > 0 \). For either value of \( m \), we can choose \( a, b \in A_i \cap \partial B(x, t) \cap \partial B(x, t) \), so that one connected component of \( \partial B(x, t) \setminus \{a, b\} \) does not intersect \( A_i \), and the other does not intersect \( O_j \).

**Step 2:** Let \( L \) be the straight line segment connecting \( a \) and \( b \), and let \( l \) be its length. Assume, without loss of generality, that \( e_2 \) is normal to \( L \) in the \( O_j \) direction. We can consider adding the \( A_i \) "side" of \( L \) to \( A_i \), and similarly for \( O_j \), which must not reduce \( E^\chi_g \). That is, we set

\[
T := (A_i \cup \text{the } A_i \text{ "side" of } L \text{ in } B(x, t)) \setminus \text{the } O_j \text{ "side" of } L \text{ in } B(x, t)
\]
and note that
\[
E^\chi_g(T) \leq E^\chi_g(S) - H^1(C_{i,j} \cap B(x, t)) + l + c\pi t^2.
\]
Since \( E^\chi_g(T) \geq E^\chi_g(S) \), it follows from Step 1 in Lemma 3.2 that
\[
c\pi t^2 + l \geq H^1(C_{i,j} \cap B(x, t)) \geq 2t.
\]
Set
\[
d(x, t) := \sup \{\text{dist}(y, L) : y \in C_{i,j} \cap B(x, t)\}.
\]
We claim that we can find \( c' > 0 \) depending only on \( c \) and \( \tau_D \) such that
\[
d(x, t) \leq c' t^2. \quad (3.9)
\]
We know, for \( T \) as above, that
\[
E^\chi_g(T) \leq E^\chi_g(S) - H^1(C_{i,j} \cap B(x, t)) + l + 4cd(x, t)t,
\]
so
\[
4cd(x, t) + l \geq H^1(C_{i,j} \cap B(x, t)) \geq 2t. \quad (3.10)
\]
We claim also that
\[
4d(x, t)^2 \leq H^1(C_{i,j} \cap B(x, t))^2 - l^2. \quad (3.11)
\]
Since \( |\nu_{A_i}| = 1 \) \( H^1 \)-a.e. on \( \partial_A A_i \), it follows that
\[
H^1(C_{i,j} \cap B(x, t)) = \int_{C_{i,j} \cap B(x, t)} |\nu_{A_i}|^2 dH^1 = \int_{C_{i,j} \cap B(x, t)} (\nu_{A_i} \cdot e_1)^2 dH^1 + \int_{C_{i,j} \cap B(x, t)} (\nu_{A_i} \cdot e_2)^2 dH^1.
\]
By Jensen's inequality, we know
\[
\int_{C_{i,j} \cap B(x, t)} (\nu_{A_i} \cdot e_1)^2 dH^1 \geq \left( \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_1 dH^1 \right)^2,
\]
and similarly for $e_2$. Hence, we have

$$\mathcal{H}^1(C_{i,j} \cap B(x, t)) =$$

$$\mathcal{H}^1(C_{i,j} \cap B(x, t)) \left[ \int_{C_{i,j} \cap B(x, t)} (\nu_{A_i} \cdot e_1)^2 d\mathcal{H}^1 + \int_{C_{i,j} \cap B(x, t)} (\nu_{A_i} \cdot e_2)^2 d\mathcal{H}^1 \right]$$

$$\geq \mathcal{H}^1(C_{i,j} \cap B(x, t)) \left[ \left( \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_1 d\mathcal{H}^1 \right)^2 + \left( \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_2 d\mathcal{H}^1 \right)^2 \right].$$

So,

$$\mathcal{H}^1(C_{i,j} \cap B(x, t))^2 \geq \left( \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_1 d\mathcal{H}^1 \right)^2 + \left( \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_2 d\mathcal{H}^1 \right)^2. \quad (3.12)$$

Note that the same holds if $\nu_{A_i} \cdot e_k$ is replaced by $|\nu_{A_i} \cdot e_k|$. From Lemma 3.5, with $E = A_i$ and $A = B(x, t)$, we know that

$$\int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_1 d\mathcal{H}^1 = 0 \quad \text{and} \quad \left| \int_{C_{i,j} \cap B(x, t)} \nu_{A_i} \cdot e_2 d\mathcal{H}^1 \right| = l. \quad (3.13)$$

Hence $\int_{C_{i,j} \cap B(x, t)} |\nu_{A_i} \cdot e_2| d\mathcal{H}^1 \geq l$, and to prove the claim (3.11), it is sufficient by (3.12) and (3.13) to prove $2d \leq \int_{C_{i,j} \cap B(x, t)} |\nu_{A_i} \cdot e_1| d\mathcal{H}^1$. Let $\varepsilon > 0$ be given and choose $v \in \partial^* A_i \cap B(x, t)$ such that $\text{dist}(v, L) > d(x, t) - \varepsilon$. Since $A_i$ and $O_j$ are connected, we can find $z \in A_i \cap \partial B(x, t)$, such that $d(z, v) > d(x, t) - \varepsilon$. Since $K$ and $M$ are connected, we can then apply Lemma 3.5 to both “sides” of $K \cup M$ in $B(x, t)$, yielding, together with the arbitrariness of $\varepsilon$, $2d \leq \int_{C_{i,j} \cap B(x, t)} |\nu_{A_i} \cdot e_1| d\mathcal{H}^1$.

Now, we have

$$4d(x, t)^2 \leq \mathcal{H}^1(C_{i,j} \cap B(x, t))^2 - l^2 \quad (by \ (3.11))$$

$$= (\mathcal{H}^1(C_{i,j} \cap B(x, t)) - l)(\mathcal{H}^1(C_{i,j} \cap B(x, t)) + l)$$

$$\leq 4\mathcal{H}^1(C_{i,j} \cap B(x, t))^2$$

$$= (4\varepsilon d(x, t))^2 \leq 16\varepsilon d(x, t)$$

$$= 16\varepsilon^2 2d(x, t)^2 + 16\varepsilon d(x, t), \quad (since \ l \leq 2t)$$

which gives

$$d(x, t) \leq 4\varepsilon^2 (1 + \varepsilon).$$

We label this last constant multiplying $t^2$ by $c'$. 

**Step 3:** We claim that $\nu_{A_i}$ is locally uniformly continuous on $\partial^* A_i$. Let $y \in C_{i,j} \cap B(x, \frac{1}{4}t)$. Let $n \in \mathbb{N}$ and choose $t(y) \in (\frac{1}{2}n, \frac{1}{2}n + \frac{1}{4})$ as for $x$. Choose $a$ and $b$ for $y$, and denote the normal to $L(a, b)$ by $\nu(y)$. We may then find $t(a) \in (\varepsilon t(y), t(y))$ such that $\partial B(a, t(a))$ intersects $C_{i,j}$ two or three times, with $A_i$ and $O_j$ separated in $\partial B(a, t(a))$ by $a'$ and $b'$, and $b' \in B(y, t(y))$. Since $\text{dist}(b', L(a, b)) \leq c' t(y)^2$, similarly for $\text{dist}(a, L(a', b'))$, and $\text{dist}(a, b')$, $\text{dist}(a', a) > \varepsilon t(y)$, we see that

$$|\nu(a) - \nu(y)| \leq c\varepsilon t(y)$$

$$|\nu(a') - \nu(a)| \leq c\varepsilon t(y),$$
for some \( c > 0 \), where \( \nu(a) \) is normal to \( L(a,y) \) and \( \nu(a') \) is normal to \( L(a',a) \).

We may proceed similarly \( n \) times, each time picking \( a^k \in \partial B(a^{k-1}, t(a^{k-1})) \), \( t(a^k) \in (\delta t(a), t(a)) \), with \( \nu(a^k) \) normal to \( L(a^k, a^{k-1}) \). Since \( nt(a) < \frac{1}{2} t \), we know that we stay in \( B(x,t) \). It follows that we have

\[
|\nu(a^k) - \nu(a^{k-1})| \leq \frac{ct}{2n}.
\]

Setting

\[
\beta := \nu(y) \cdot e_1,
\]

we see that

\[
\nu(a^k) \cdot e_1 \geq \beta - \frac{ct}{2}
\]

for all \( k \in \{1, \ldots, n\} \). We have

\[
a_2^n - y_2 = \sum_{k=1}^{n} (a_2^k - a_2^{k-1}),
\]

where \( a^0 := y \). Assuming \( \beta > 0 \) without loss of generality, and further assuming \( \beta - \frac{ct}{2} > 0 \), we also have

\[
a_2^k - a_2^{k-1} = (\nu(a^k) \cdot e_1)(a^k, a^{k-1}) \geq (\beta - \frac{ct}{2}) \eps^2 t(y),
\]

so

\[
a_2^n - y_2 \geq n(\beta - \frac{ct}{2}) \eps^2 t(y) \geq \eps^2 (\beta - \frac{ct}{2}) \frac{1}{2} t.
\]

But,

\[
|a_2^n - y_2| \leq 2d(x, t) \leq 2c't^2,
\]

which implies that \( \beta \leq t(4\frac{\eps^2 t^2}{c^2} + \frac{1}{2}) \). If \( \beta - \frac{ct}{2} \leq 0 \), we still have \( \beta \leq t\frac{\eps^2}{2} \). A similar argument can be made for \( \nu(y) \cdot e_2 \), so that

\[
|\nu(y) - \nu(z)| \leq \epsilon t,
\]

for some \( \epsilon > 0 \). Since, for \( x, y \in \partial^* A \), we can choose \( r \) small enough so that \( \nu(x) \) is arbitrarily close to \( \nu_A(x) \) and \( n \) large enough so that \( \nu(y) \) is arbitrarily close to \( \nu_A(y) \), (3.14) implies local uniform continuity of \( \nu_A \), and so \( \nu_A \) can be extended continuously from \( \partial^* A \) to \( \partial \bar{A} \cap \Omega \). In particular, this shows that \( \partial B(x, r) \) intersects \( C_{i,j} \) exactly twice for sufficiently small \( r > 0 \), and furthermore that \( \partial \bar{A} \cap \Omega = \partial^* A \).

**Step 4:** We show that \( C_{i,j} \) is locally the graph of a \( C^1 \) function. Let \( x \in C_{i,j} \) be given and by Step 3, choose \( r > 0 \) such that \( C_{i,j} \) intersects \( \partial B(x, r) \) twice, at \( a \) and \( b \), and \( \nu_A(y) \cdot e_2 > 0 \) for all \( y \in C_{i,j} \cap B(x, r) \), where \( e_2 = \nu(z) \). Let \( L \) be the line segment connecting \( a \) and \( b \), and let \( l \) be its length, and assume that \( a \) and \( b \) are oriented so that \( b - a = l \cdot e_1 \). For \( \lambda \in (0, l) \), consider the line \( L_\lambda \) through \( a + \lambda e_1 \) in the direction \( e_2 \). Since \( \nu_A(y) \cdot e_2 > 0 \) for all \( y \in B(x, r) \cap C_{i,j} \), we know that \( L_\lambda \) intersects \( C_{i,j} \cap B(x, r) \) just once. We label the intersection \( \gamma(\lambda) \). We therefore can define \( f : (0, l) \to \mathbb{R} \) by

\[
\lambda \mapsto [\gamma(\lambda) - (a + \lambda e_1)] \cdot e_2
\]

and \( C_{i,j} \cap B(x, r) \) is the graph of \( f \). Let \( \lambda_1 > \lambda_2 \in (0, l) \) and take \( |\lambda_1, \lambda_2| \) to be the region in \( B(x, r) \) between \( L_{\lambda_1} \) and \( L_{\lambda_2} \). We will denote \( \nu_A(\gamma(\lambda_1)) \) by \( \nu_A(\lambda_1) \). We have
where the first equality follows just as Lemma 3.5, the second follows from (3.14), and the third from an argument similar to the proof of (3.9). Hence, \( f \in C^1 \).

**Step 5:** Finally, we show that \( C_{i,j} \) is a connected component of \( \partial \Omega \cap \Omega \). Let \( x, y \in C_{i,j} \). By the regularity of \( C_{i,j} \) and the connectedness of \( A_i \) and \( O_j \), we may choose smooth curves, one in \( A_i \) and one in \( O_j \), that connect \( x \) and \( y \) and are normal to \( C_{i,j} \) at \( x \) and \( y \). The union of these curves is Jordan, and so we may consider the interior region, \( R \). Since \( \Omega \) is simply connected, we have \( R \subset \subset \Omega \). We can choose \( r \in (0, \bar{r}_R) \) so that, for \( x \) and \( y \), and for \( z \in R \), we have, e.g., \( B(z, r) \cap C_{i,j} \) is the graph of a continuous function on a line segment, and so it is connected. The curve \( B(z, r) \cap C_{i,j} \) can be shown to continue, as before, by choosing balls with radius \( r \) centered at points in \( C_{i,j} \cap \partial B(z, r) \). It can be continued in \( R \) as long as these balls stay in \( R \) and the curve does not self intersect. But by the choice of these balls, and since \( \partial R \) is normal to \( C_{i,j} \) at \( x \), the connected curve begun at \( x \) cannot self intersect in \( R \cup \{x\} \). Since \( \mathcal{H}^1(C_{i,j}) \) is finite and each ball adds \( r \) to \( \mathcal{H}^1(C_{i,j}) \), the connected curve begun at \( x \) must leave \( R \). Because \( \partial R \cap C_{i,j} = \{x, y\} \), the connected curve must cross \( \partial R \) at \( y \), and so \( x \) and \( y \) are connected in \( C_{i,j} \). Hence, \( C_{i,j} \) is connected. Since the \( C_{i,j} \) are closed in \( \Omega \) and mutually disjoint, they are the connected components of \( \partial \Omega \), and hence each connected component of \( \partial \Omega \), of which there are finitely many, is \( C^1 \).

**Acknowledgements**

This research was partially supported by the Army Research Office and the National Science Foundation, through the Center for Nonlinear Analysis at Carnegie Mellon University.

**References**


