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Revisiting the Focal Conic Structures in Smectic A

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Revisiting the Focal Conic Structures in Smectic A

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ABSTRACT

Smectic A configurations in equilibrium display complicated focal conic textures. In the planar case, we view these configurations as extremals of a constrained Ginzburg-Landau Equation. This gives rise to a system governed by a variational principle that is subject to simple rules.

INTRODUCTION

In the hierarchy of mesophases, smectic A appears between nematic and smectics of lesser symmetry or the solid. It is characterized by the properties

- presence of a layer structure of nearly constant width and nearly incompressible
- within each layer, centers of gravity show no long range order and each layer is a two dimensional fluid
- the system is optically uniaxial with optic axis $n$ normal to the layer
- $n$ and $-n$ are equivalent

In equilibrium, thin samples often show a complicated focal conic texture, or Dupin cyclide structure [8]. Friedel [5] recognized this to be indicative of the layer properties above and also derived the rules we revisit here. Indeed, we take this opportunity to discuss our first thoughts on these issues. These configurations, in the plane, may be interpreted as singular solutions of a Ginzburg-Landau system, studied extensively by Bethuel, Brezis, and Helein [1], subject to constraints. An interesting consequence of the Ginzburg-Landau viewpoint is the tendency of the material to nucleate new smectic domains in response to defects in preference to deforming to accommodate them.

The simple conic configurations arise from a conservation condition and other properties follow from a stability condition. We do not yet have a satisfactory explanation of the assembly of many ellipses between hyperbolic arcs, but we are able to show that this configuration is consistent with our rules. Moreover, a simple symmetry property characterizes elliptical domains. All of our discussion is completely elementary.

Basic references for us have been Chandrasekhar [4]. Some additional papers of interest are Guerst [6], KJ and Sethna and Huang [12]. Jerry Ericksen is our mentor and our friend. He is convinced of the predictive authority of mathematics.

1. KINEMATICS AND LOCAL EQUILIBRIUM

A simplified continuum theory may be based on the for which we establish constraints. Following [2],

$$\int_{\Omega} n \cdot d \mathbf{x} = \text{number of layers}$$

where $d \mathbf{x}$ is a typical layer spacing,

should be independent of path. This gives rise to the

$$l_n l = 1 \quad \text{and} \quad \text{curl} \ n = 0.$$ 

Given a small region $D$ of defect free material, we interpret $f$ as the layer density of the material. From (1.2) we have that

$$\nabla n = 0 \quad \text{and} \quad \nabla n \text{ is symmetric so that}$$

$$\nabla n = -x t \otimes t,$$

where $t \cdot n = 0$.

Suppose that $(t, n)$ is a right handed system. Solve the

$$\frac{dp}{ds} = t(p), \quad p(0) = x_0.$$

Then

$$\frac{df}{ds} = \nabla f \cdot \frac{dp}{ds} = n \cdot t = 0.$$
so \( y \) is a level curve of \( f \). By using the Frenet Formulas, one sees easily that \( \kappa \) in (1.4) is the curvature of \( y \). Using a variation of this idea, Virga and Fournier [13] introduce confocal coordinates based on the fields \((t,n)\) which serves also to illustrate the equally spaced layer property of smectic \( A \).

The elementary condition for equilibrium is that

\[
\delta \int \left( \nabla \cdot n \right)^2 \, dx = 0 \quad \text{subject to} \quad \| n \| = 1,
\]

hence

\[
\nabla \cdot n \parallel n \quad \text{or} \quad t \cdot \nabla \cdot n = 0.
\]

After some manipulation, and writing \( \kappa = \kappa(s) \) for the curvature of \( y \), we see that the equation above is equivalent to

\[
\frac{d}{ds} \kappa(s) = 0.
\]

Hence in unloaded equilibrium, \( \kappa \) is constant on each level surface of \( f \). We conclude that local equilibria are characterized by circular arcs or straight segments, namely,

\[
n_x(x) = \frac{x - a}{\| x - a \|} \quad \text{with} \quad f_x(x) = \| x - a \|, \quad \text{or}
\]

\[
n(x) = n_0 \quad \text{with} \quad f(x) = n_0 x, \quad n_0 \text{ constant}.
\]

2. Energy and Ginzburg-Landau Formulation

We briefly discuss an appealing Ginzburg-Landau approximation as a means of accommodating elementary defect structures. For a mapping \( u : \Omega \rightarrow \mathbb{R}^2 \), let

\[
E_{\varepsilon}(u) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} F(u) \, dx,
\]

where \( F(\xi) \) is a smooth non-negative function which vanishes precisely if \( \| \xi \| = 1 \).

The usual choice is

\[
F(\xi) = \frac{1}{4} (1 - \| \xi \|^2)^2.
\]

For any such \( u \),

\[
1 |\nabla u|^2 = (\nabla u)^2 + (\text{curl} \, u)^2 + 2
\]

If \( u \) satisfies the constraints (1.2), then

\[
E_{\varepsilon}(u) = \frac{\varepsilon}{2} \int_{\Omega} (\nabla u)^2 \, dx,
\]

which is the basic energy mentioned in (1.5) for a d the Ginzburg-Landau formulation is that we know \( m \) and how it can be used to systematically account Bethuel, Brezis, and Helein [1]. A sequence of \( \rho \) Dirichlet boundary condition \( u_\rho \) of degree \( M_0 \) relabelled, such that

\[
u_\rho \rightarrow u^* \quad \text{in} \quad \Omega \setminus \{a_1, ..., a_M\}
\]

where \( a_1, ..., a_M \) are special points of \( \Omega \), cf. [1], and

\[
u^*(z) = \frac{z - a_1}{\| z - a_1 \|} ..., \frac{z - a_M}{\| z - a_M \|} \text{ and}
\]

\[\Delta h = 0 \quad \text{in} \quad \Omega \quad \text{and} \quad h \text{ is a real valued}
\]

Given \( b_1, ..., b_M \in \Omega \) and \( \rho > 0 \), let \( B_\rho(b) \) denote \( \rho, \Omega = \Omega \cup B_\rho(b) \), and

\[
A_{\varepsilon}(b) = \{ v \in H'(\Omega, \mathbb{S}^1) : \deg(v, \partial B_\rho(b)) = \}
\]

where \( \deg \) denotes the topological degree or winding

\[
\min_{A_{\varepsilon}(b)} \frac{\varepsilon}{2} \int_{\Omega} |\nabla v|^2 \, dx = \kappa M \varepsilon \log \varepsilon
\]

where the first term on the right reflects the presence refer to as an excess energy. Also,

\[
\inf_{A_{\varepsilon}(b)} \frac{\varepsilon}{2} \int_{\Omega} |\nabla v|^2 \, dx = \kappa M \varepsilon \log \varepsilon + KU(a), \quad A = \{ v \in U(a) = \lim U_j(a) \quad \text{and} \quad a \text{ is the set of defects equilibrium positions in (2.2).}
\]
Now if $u^*$ is also, locally, an equilibrium smectic A configuration, then in (2.2) $M = 1$, $h = 0$, and $u^* = n_*$ as in (1.7).

3. THE ELLIPSE AND THE HYPERBOLA

In view of the conclusion above, we may anticipate difficulties in seeking configurations with more than one defect or even configurations with a single defect not satisfying the special condition detailed in [1]. Assume initially a configuration in equilibrium with a single defect at $a \in \Omega$,

$$n(z) = n_a(z), \quad z \in \Omega,$$

and that this defect is displaced nearby to $b \in \Omega$ without alteration of the loading environment. Thus

$$\min_{A_{a,b}} \frac{K}{h} \int_{A_{a,b}} |\nabla u|^2 \, dx < \min_{A_{a,b}} \frac{K}{h} \int_{A_{a,b}} |\nabla v|^2 \, dx$$

and a convergent subsequence of minimizers $(u_k)$ of $E$ converges to $n_a$.

We may envision two scenarios. First the system may fail to be in equilibrium with the displaced defect $b$ because of $|Vv|^2$. Second, we may nucleate a region $D$, with $a,b \in D$, so that, for example, the resulting configuration is given by

$$n(z) = \begin{cases} n_a(z) & z \in D \\ n_b(z) & z \in \Omega \setminus D \end{cases}$$

What sort of region can $D$ be? Assume that $\Gamma = \partial D$ is a simple closed curve. Let us simply impose the condition (1.1) on $\Gamma$, that the number of traversed layers is the same on any subarc on approach from $D$ and from $\Omega \setminus D$. Accounting for orientation, and reverting to real notation,

$$\int_{\Gamma_{1,z_1}} n_{a}(z) \, dx' - \int_{\Gamma_{1,z_2}} n_{b}(z) \, dx' = f_{1}(z_1) - f_{1}(z_2) = -(f_{1}(z_1) - f_{1}(z_2)).$$

Hence

$$f_{1}(z) + f_{1}(z) = f_{1}(z_1) + f_{1}(z_2) = C \quad \text{for all} \quad z \in \Gamma,$$

and $\Gamma$ is an ellipse with foci at $a$ and $b$. Moreover the symmetry of the ellipse that

$$\frac{K}{h} \int_{\Omega \setminus D} |\nabla n_{a}|^2 \, dx = \frac{K}{h} \int_{\Omega \setminus D} |\nabla n_{b}|^2 \, dx$$

where $U(a)$ is the minimum possible value of the exoenergy.

Consider again a configuration with a defect in $\Omega \setminus D$, the boundary condition remains of topological degree inconsistent with the presence of two defects, but we may impose the condition (1.1) on $\Gamma$, the conclusion that $\Gamma$ is an arc of a hyperbola with foci at $a$ and $b$. Interestingly, from the viewpoint of free boundary problems, martensitic like materials, the coherence condition (1.1) has a particular form of boundary. The solution to the nucleation problem...
of confocal elliptical domains, one focus governing the nucleated region and the other focus the exterior, or (b) a family of confocal hyperbolic arcs separating a and b. Finally, the point of view given here is not completely novel but may be viewed as a somewhat more systematic formulation of considerations already present in the literature, cf. de Gennes [7].

4. LOCAL STABILITY

Applying the coherence condition in the form (3.4) to two equilibrium domains $\Omega_a$ and $\Omega_b$ in contact on an arc $\alpha$ leads to $(n_a + n_b) \cdot v = 0$ on $\alpha$, where $n_a$ and $n_b$ denote the respective directors and $v$ is the external normal referred to one of the domains. If the region of contact $\alpha$ now shrinks to a point $z$, we obtain the stability condition

$$(n_a + n_b) \cdot v = 0 \text{ at } z. \quad (4.1)$$

This leads to one of Friedel's rules, cf. [7] p. 468. If $\Omega_a$ and $\Omega_b$ are ellipses tangent at $z$, then $z$ is the intersection of the straight lines joining their foci. Just recall that the normal to the ellipse at $z$ bisects the angle $\angle aza'$. $= \theta_a$ so that

$$\theta_a = \theta_a = \varphi.$$

Since the normal is common to both domains, $czc^*$

This places stringent restrictions on the placement example, unless the point of tangency lies on the axial disc.

More generally, if two domains of arbitrary shape $\Omega_a$ and $\Omega_b$ are in contact, the visible focus of one domain, $(4.1)$ determines the location of $b$ or $b'$ nor which of the two is the visible focus $\alpha$.

5. MANY DEFECTS AND THE PLAGES A ÉVENTAIL

In an equilibrium configuration with two defects, $\Omega_a$ and $\Omega_c$, the stability condition (4.1), so in general, once two defects are present, there are infinitely many. This leads to the problem of finding all $\dot{\Omega}_a$ and of establishing a suitable variational criterion. The best way to do this, that is how to pack the domains with discs with centers $\{ a_i \}$, so that

$$\Omega = \bigcup D_i$$

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$$\Omega = \bigcup D_i$$

One solution may always be found by choosing for $D_i$ a covering of $\Omega$ by discs with centers $\{ a_i \}$. In such a covering of $\Omega$ by discs with centers $\{ a_i \}$, it corresponds to an ensemble of tori, which are degenerate tori. This argues that these are energetically unfavorable and, rarely seen.

Ellipses may be characterized by a simple symmetry condition:

$$\sum n_{i} \cdot v = 0, \quad v \text{ normal to } \partial D,$$

and $D$ is symmetric about the perpendicular bisector of $a b$.
A frequently observed configuration is the lenticular region between two hyperbolas. This arrangement has the property that the fine structure, represented by infinitely many defects, is limited to the lens while outside the possible defects are the foci of the hyperbola. To construct this solution requires verifying that (4.1) can be satisfied by ellipses tangent to the hyperbolas and each other. We elucidate the constraints and show how this is possible in a subsequent and more complete paper.

An interesting feature is that the general form depicted in Figure 9.1, is a consequence of the construction: the ellipses above the segment joining the foci of the hyperbola point downwards. The lines joining the tangential points of the ellipses intersect at a focus of the hyperbola, no surprise in view of the discussion in §4, but these segments do not extend to the visible foci of the ellipses. We have drawn Figure 3 to illustrate the situation; its geometric configuration is very special just to give the idea.

6. REMARKS

In Figure 4, we depict the appearance of two ellipses question, not completely resolved in our minds, of segments joining the foci of the ellipses. We are investigating fluctuations in the texture and the stability condition weak solutions, but at this writing have not been able...

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REFERENCES