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# MARGINALIA TO A THEOREM OF JACOPINI

by

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# MARGINALIA TO A THEOREM OF JACOPINI

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## INTRODUCTION

In this note we consider the problem of whether a combinator $P$ can consistently (in most cases with beta conversion) be assumed to satisfy the functional equation $Mx = Nx$. Much of the literature in this area concerns easy terms first discovered by Jacopini. These are combinators P which can consistently be assumed to be solutions to the equation $x = Q$ for any $Q$. Here we shall prove several results which might be viewed as unexpected; although given Jacopini's result the unexpected should be expected in this topic in lambda calculus.

We shall construct an identity $M = N$ which is not a beta conversion but which is consistent with any consistent set of combinator equations. By a simpler construction we shall build a functional equation $Mx = Nx$ for which there is no solution modulo beta conversion but such that for each consistent set $S$ of combinator equations there exists a combinator $P$ with $S \cup \{MP = NP\}$ consistent. Next we consider the problem of which sets of combinators are "consistency sets" i.e. sets of the form $\{P : MP = NP \text{ is consistent}\}$. Each such set is closed under beta conversion and pi-zero-one ("co-Visseral" in [5]). We produce such a co-Visseral set which is not a consistency set, in contrast to the case for first order arithmetic. Finally, we consider some questions involving compactness. We give several examples of sets of functional equations Mx=Nx such that

(*) for each finite subset there is a combinator which can be consistently assumed to be a solution

but there is no single combinator which can consistently be assumed to be a solution of the whole set.However, we show that if the condition (*) is made effective then no such examples are possible. This is in contrast to the familiar event of the effectivization of a classical theorem being false.

## PRELIMINARIES

We adopt for the most part the notation and terminology of [1]. A combinator is a closed term. The following are the usual combinators

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In this note we consider the problem of whether a combinator $P$ can consistently (in most cases with beta conversion) be assumed to satisfy the functional equation $Mx = Nx$. Much of the literature in this area concerns easy terms first discovered by Jacopini. These are combinators P which can consistently be assumed to be solutions to the equation $x = Q$ for any $Q$. Here we shall prove several results which might be viewed as unexpected; although given Jacopini's result the unexpected should be expected in this topic in lambda calculus.

We shall construct an identity $M = N$ which is not a beta conversion but which is consistent with any consistent set of combinator equations. By a simpler construction we shall build a functional equation $Mx = Nx$ for which there is no solution modulo beta conversion but such that for each consistent set $S$ of combinator equations there exists a combinator $P$ with $S \cup \{MP = NP\}$ consistent. Next we consider the problem of which sets of combinators are "consistency sets" i.e. sets of the form $\{P : MP = NP \text{ is consistent}\}$. Each such set is closed under beta conversion and pi-zero-one ("co-Visseral" in [5]). We produce such a co-Visseral set which is not a consistency set, in contrast to the case for first order arithmetic. Finally, we consider some questions involving compactness. We give several examples of sets of functional equations Mx=Nx such that

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## PRELIMINARIES

We adopt for the most part the notation and terminology of [1]. A combinator is a closed term. The following are the usual combinators

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$$
\begin{array}{lcl}
B & = & \lambda xyz.x(yz) \\
C^* & = & \lambda\ xy.yx \\
K & = & \lambda xy.x \\
K^* & = & \lambda xy.y \\
Y & = & \lambda x.(\lambda y.x(yy))(\lambda y.x(yy)) \\
O & = & \lambda xy.y(xy) \\
\Omega & = & (\lambda x.xx)(\lambda x.xx)
\end{array}
$$

but we reserve the symbol $S$ for sets of combinators. We are interested in functional equations

$$U = V$$

in a single free variable $x$, which by abstraction can be put in the form

$$(\lambda x.U)x = (\lambda x.V)x.$$

Functional equations in more than one free variable can be reduced to one by pairing. For example the equation

$$Mxy = Nxy$$

can be replaced by

$$M(zK)(zK^*) = N(zK)(zK^*)$$

with solutions $z = \langle x, y\rangle = \lambda a.axy$. Similarly, several equations can be combined into one by pairing. If $S$ is a set of combinator equations then, by the well known existence of free models ([1]), $M = N$ is inconsistent with $S$ if and only if $S \cup \{M = N\}| - K = K*$. Implicit in Jacopini's classic paper [3] is the following

**Theorem (Jacopini):** $M = N$ is inconsistent with $S$ if and only if there exist combinators $P1, \ldots, Pp$ such that $S|-K = P1M\&P1N = P2M\&\ldots\&PpN = K^*$.

Another way to state this theorem is to consider the graph whose points consist of the congruence classes of combinators modulo provable equivalence in $S$, and whose undirected edges join points of the form $PM$ to those of the form $PN$. Then $M = N$ is inconsistent $\Leftrightarrow$ $K$ and $K^*$ are connected by a path $\Leftrightarrow$ the graph is connected.

Among the congruence classes of combinators modulo equivalence in $S$ are some which do not contain any solvable terms such as the class of $K - \infty = YK$. We call the number of these classes the degree of $S$. For example Barendregt's $H^*$ has degree 1 but the empty $S$ (beta conversion) has infinite degree. Below we shall observe that $S$'s of each finite degree exist.

When it comes to functional equations $Mx = Nx$ it is possible for it to be consistent to have a solution to the equation without it being consistent for any particular combinator to be a solution. For example, in [4] we constructed a Plotkin term P such that for each combinator $M$ , $PM$ beta converts to $P$ but

$P$ does not beta convert to $KP$. It is easy to see, by using Mitschke's theorem [1] page 401, that the equation

$$Px = I$$

is consistent with beta conversion but clearly it is not consistent for any combinator value of $x$. It is also possible for a given combinator to be a consistent solution to each of several functional equations separately when the entire collection cannot have a solution. For example, Omega is a consistent solution to $x = K^*$ and to $x = Y\langle K, K^*\rangle$.

**Definition:** Suppose $S$ is a set of combinator equations. The functional equation $Mx = Nx$ is said to be consistently solvable over $S$ if there exists a combinator $P$ such that $S \cup \{MP = NP\}$ is consistent. Such a $P$ is called a consistent solution over $S$.

**Remark:** When $S$ is empty we drop the phrase "over $S$".

**Definition:** The combinator equation $M = N$ is said to be inevitably consistent if $M$ does not beta convert to $N$ but for any consistent set $S$ of combinator equations $S \cup \{M = N\}$ is consistent. The functional equation $Mx = Nx$ is said to be inevitibly consistently solvable if there is no solution in the combinators modulo beta conversion but for any consistent set $S$ of combinator equations there exists a combinator $P$ such that $S \cup \{MP = NP\}$ is consistent.

**Example:** $Y$ is a consistent solution to the equations

$$x = Ox, \ x = xO$$

since $Y$ satisfies these equations in the Bohm tree model ([1]) but there is no solution to these equations modulo beta conversion (Intrigila).

**Example (generalization):** We say that $M$ is consistently solvable if there exists $N1 \ldots Nn$ such that $MN1 \ldots Nn = I$ is consistent with beta conversion. For each e construct a combinator $P(e)$ such that

$$P(e)n = \begin{cases} \lambda x.P(e)(n+1) & \text{if the } e\text{th Turing Machine converges on input } n \\ \text{an order zero unsolvable} & \text{otherwise.} \end{cases}$$

This can be done directly or by the Visser fixed point theorem ([5]). Then P(e)0 is consistently solvable $\Leftrightarrow$ the $e$th Turing Machine is not total.

## INEVITABLY CONSISTENT AND CONSISTENTLY SOLVABLE EQUATIONS

**Theorem 1:** Suppose that $S$ is a set of combinator equations of finite degree. Then there exists a functional equation

$$Pxyz = Qxyz$$

such that for any combinator equation $M = N$.

$S \cup \{M = N\}$ is inconsistent $\Leftrightarrow PMNz = QMNz$ has a solution over $S$

**Proof:** Suppose that $S$ is given of degree $n$. Consider the graph described after the statement of Jacopini's theorem above. Now a shortest path which joins two combinator classes containing terms with distinct Bohm trees has at most $n$ intermediate points. In addition, by Bohm's theorem [1], if such a path exists then there is one which is not longer connecting the class of $K$ and the class of $K^*$. Thus for $p = n + 3$, by Jacopini's theorem, if $M = N$ is inconsistent with $S$ there exist $P1, \ldots, Pp$ such that

$$S| - K = P1M\&P1N = P2M\& \ldots \&PpN = K^*$$

in other words

$$K = x1M, x1N = x2M, \ldots\ldots, xpN = K*$$

has a solution over $S$. ■

The following corollary follows from the proof.

**Corollary:** If $S$ is a set of combinator equations of finite degree then there exists a functional equation

$$Pxyz = Qxyx$$

such that $M = N$ is inconsistent with some extension of $S \Leftrightarrow PMNz = QMNz$ is consistently solvable over $S$.

**Remark:** For the case that $S$ is empty the construction in the proof of theorem 1 does not work. This is verified in 6. However it is still the case that the theorem is true (the best proof comes from 4).

**Theorem 2:** There exist $S$ of every finite degree.

We shall present a proof of this theorem elsewhere.

**Theorem 3:** There exists an inevitibly consistent combinator equation.

**Proof:** For this we need a result from 5. A $V$-set is a set of combinators which is both $RE$ and closed under beta conversion. A $V$-partition of a $V$-set is a partition of that set whose blocks are themselves $V$-sets. A $V$-partition is said to be $RE$ if there is an $RE$ set which contains only indicies of sets which are blocks of the partition and at least one index for each block.

**Theorem [5]:** Suppose that $X$ is an $RE$ $V$-partition of a $V$-set. Then there exists a combinator $H$ such that for any combinators $M$ and $N$

$HM$ beta converts to HN $\Leftrightarrow$ $M$ beta converts to N or M and N belong to the same block of X (and thus to the V-set partitioned by X).

Indeed, in the construction of $H$, if $X$ is not a real partition in that blocks of $X$ overlap then $H$ has the same value on elements of blocks with shared members and the same values on elements once removed etc., etc., etc. Now the construction of $H$ is uniform in the $RE$ index, say $e$, of the given set of indicies for $X$; that is, there exists a combinator $G$ such that $Ge$ beta converts to $H$. To apply this theorem consider an enumeration of the finite sequences $P1, \ldots, Pp$ of combinators. Given combinators $M$ and $N$ we construct two lists of combinators

$$\begin{array}{lll} \ldots\ldots\ldots, & \langle\langle P1, \ldots, Pp\rangle, & \langle K, P1N, \ldots, PpN\rangle\rangle, \ldots\ldots\ldots \\ \ldots\ldots\ldots, & \langle\langle P1, \ldots, Pp\rangle, & \langle P1M, \ldots, PpM, K^*\rangle\rangle, \ldots\ldots\ldots \end{array}$$

where $\langle X1, \ldots, Xn\rangle$ is the usual sequencing combinator $\lambda a.aX1 \ldots Xn$. Clearly these lists share a combinator modulo beta conversion if and only if $M = N$ is inconsistent. In the case of consistent $M = N$, these lists generate a $V$-partition of the $V$-set obtained as the beta conversion closure of the two lists. As the index of this V-partition is uniform in $M = N$ there exists a combinator $F$ such that $F(\#\langle M, N\rangle)$ beta converts to the $H$ for this partition. Let $P1, \ldots, Pp$ be the first element in the enumeration of the sequences and set

$$\begin{array}{lll} L1 & = & \lambda a.\langle\langle P1, ..., Pp\rangle, \langle K, P1a, \ldots Ppa\rangle\rangle \\ L2 & = & \lambda a.\langle\langle P1, \ldots, Pp\rangle, \langle P1a, \ldots, Ppa, K^*\rangle\rangle. \end{array}$$

By the fixed point theorem [1] there exists a pair $M, N$ such that $\langle M, N\rangle$ beta converts to $\langle F(\#\langle M, N\rangle)(L1N), F(\#\langle M, N\rangle)(L2M) >$. Now if $M = N$ is inconsistent then $F(\#\langle M, N\rangle)(L1N)$ beta converts to $F(\#\langle M, N\rangle)(L2M)$ and thus $M$ beta converts to $N$. Conversely if $M$ beta converts to $N$ then by the construction of $H$ we have $M = N$ inconsistent. Thus $M$ does not beta convert to $N$. Similarly if $M = N$ is inconsistent with $S$ then $S| - F(\# < M, N > )(L1N) = F(\# < M, N >)(L2M)$ i.e. $S| - M = N$. Thus $M = N$ is inevitibly consistent. ■

**Remark:** It is easy to see from Mitschke's theorem [1] that any inevitibly consistent equation must contain a universal generator. This is indeed the case for our example. The following theorem follows from theorem 4; however, it has a simpler proof.

**Theorem 4:** There exist inevitibly consistently solvable functional equations.

**Proof:** We can restate Jacopini's theorem for the empty $S$ as follows. $M = N$ is inconsistent with beta conversion $\Leftrightarrow$ there exists a combinator $P$ of the form $\lambda a.\ aP1 \ldots Pp$ such that $B(C^*K^*)(PM)$ beta converts to $B(PN)(C*K)$. Now

by [4] there exists a combinator $R$ such that $RP$ beta converts to $R$ if and only if $P$ beta converts to the form $\lambda a.aP1...Pp$ for combinators $P1, \ldots, Pp$. Thus the equations

$$(*) \qquad Rx = R \ , \ B(C^*K^*)(xM) = B(xN)(C^*K)$$

have a solution modulo beta conversion if and only if $M = N$ is inconsistent with beta conversion. Moreover, if $S$ is a consistent set of combinator equations then (*) has a solution over $S$ if $M = N$ is inconsistent with $S$. Hence the equations

$$(**) \qquad Rx = R \ , \ B(C^*K^*)(x(Omega)) = B(xy)(C^*K)$$

once the two variables are replaced by one variable through pairing, are inevitible. For given an $S$ either Omega is inconsistent with every solvable term or Omega is consistently inconsistent with at least one unsolvable term, and Omega is easy. This completes the proof. ■

## CONSISTENCY SETS

Clearly if $S$ is $RE$ then the set of consistent solutions to $Mx = Nx$ is a co-Visseral ([5]) set. It is natural to ask if every co-Visseral set is representable in this manner as a "consistency set". By [5] it suffices to consider only co-Visseral sets of the form $\{P : P$ does not beta convert to $Q\}$.

**Theorem 5:** Let $Mx = Nx$ be given. Then there exists a combinator $P$ not beta convertible to Omega such that either $M(\text{Omega}) = N(\text{Omega})$ is consistent or $MP = NP \Rightarrow M(\text{Omega}) = N(\text{Omega})$.

**Proof:** Suppose that $Mx = Nx$ is given and $M(\text{Omega}) = N(\text{Omega})$ is inconsistent. Then $M(\text{Omega})$ and $N(\text{Omega})$ have beta eta distinct Bohm trees ([1] page 504 and page 244). Without loss of generality we may assume that $M(\text{Omega})$ and $N(\text{Omega})$ are not separable. Thus $M(\text{Omega})$ and $N(\text{Omega})$ have reducts with equivalent subterms one of which is unsolvable and the other of which has a head normal form. Symmetrically assume that the unsolvable one is in a reduct of $M$. By the Bohm-out technique there exists a possibly open term $X$ such that

$$\begin{array}{lll} X(Mx) & \beta \text{ converts to} & \left\{ \begin{array}{l} \text{no head normal form} \\ \lambda y1 \ldots yr.\ xY1 \ldots Ys \end{array} \right. \\ X(Nx) & \beta \text{ converts to} & \qquad y \end{array}$$

Clearly we may assume that the second alternative for $X(Mx)$ does not occur. In addition we can arrange it so that $X(Mx)$ has the property $X(Mx)$ either has infinite order or order zero. By the fixed point theorem there exists a

combinator $P$ such that $P$ beta converts to $(\lambda z.\ z((\lambda y.\ X(MP))z))\ (\lambda z.\ zz)$. By the standardization theorem $P$ does not beta convert to Omega. However whenever $MP = NP$ we have $P =$ Omega. This completes the proof. ■

**Corollary:** The set $\{\ P : P$ does not beta convert to Omega $\}$ is not a consistency set.

## FINITELY CONSISTENTLY SOLVABLE SETS OF EQUATIONS

**Definition:** If $S$ is a set of functional equations then $S$ is said to be (effectively) finitely consistently solvable if there is a partial (recursive) function $f$ defined on exactly the finite subsets of $S$ such that if $F$ is a finite subset of $S$ then

$$\{M(f(F)) = N(f(F)) : Mx = Nx \text{ in } F\}$$

is consistent with beta conversion.

**Remark:** The effectiveness condition in the definition really has two parts

(a) $S$ is $RE$

(b) consistent solutions can be computed for finite subsets.

Next we show that neither of these restrictions can be relaxed.

**Theorem 6:** There exists a finitely consistently solvable set which is not consistently solvable.

**Proof:** We shall actually build two variations on the same example only one of which is $RE$. The $RE$ example goes as follows. For each combinator $M$ we shall use two "local" variables $y$ and $z$ which actually depend on $M$. For each such combinator we take the equations

$$zx = K, zM = yM, yx = K^*.$$

Our example consists of all these equations with all the local variables replaces by one global variable through pairing.Clearly this set is not consistently solvable. However for any finite subset correcponding to the combinators $M1, \ldots, Mm$ we can find a consistent solution as follows.

Let $M$ have a head normal form distinct from the head normal forms of the solvable members of $\{M1, \ldots, Mm\}$. Let $N1, \ldots, Nn$ be such that

$$MN1 \ldots Nn \twoheadrightarrow I,$$

then for each of the sets

$$zx = K, zMi = yMi, yx = K^*$$

we have the solution of $M$ for $x$ and

| | |
|---|---|
| if $Mi$ is solvable | then there exists a Bohm-out term $P$ such that $PM$ beta converts to $K^*$ and $PMi$ converts to $K$ and p $P$ for $y$ and $KK$ for $z$ |
| if $Mi$ is unsolvable | then there exists a fixed point $P$ without head normal form such that $P$ beta converts to $PN1 \ldots NnK^*$. Putm $\lambda x.\ xN1 \ldots Nn$ for $y$ and $I$ for $z$. This works in the Bohm tree model where all the unsolvable are equal; in particular $Mi = P$. |

Clearly computiong the finite consistent solution requires determining the solvablitiy of $Mi$. Computing a finite consistent solution can be simplified by passing to a non-$RE$ example. We keep the above equations for those terms $M$ which are unsolvable and add the following for terms $N$ in head normal form

$$yN = K, yx = K^*.$$

It should be clear how to solve for the variables in any finite subset of these equations. This completes the construction.

**Theorem 7:** If $S$ is effectively finitely consistently solvable then $S$ is consistently solvable.

**Proof:** Suppose that $S$ is effectively finitely consistently solvable and the function $f$ is as above. For each finite subset $F$ of $S$ define $T(F) = \{Mf(F) = Nf(F) : Mx = Nx$ belongs to $F\}$. By Visser's theorem 3.8 ([7]) there exists a combinator $P$ such that for each finite subset $F$ of $S$ $T(F)U\{P = f(F)\}$ is consistent. Thus by the compactness theorem the set $\{\ MP = NP : Mx = Nx$ belongs to $S\ \}$ is consistent. This completes the proof. ■

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