Abstract. V’yugin [2, 3] has shown that there are a computable shift-invariant measure on $2^\mathbb{N}$ and a simple function $f$ such that there is no computable bound on the rate of convergence of the ergodic averages $A_n f$. Here it is shown that in fact one can construct an example with the property that there is no computable bound on the complexity of the limit; that is, there is no computable bound on how complex a simple function needs to be to approximate the limit to within a given $\varepsilon$.

Let $2^\mathbb{N}$ denote Cantor space, the space of functions from $\mathbb{N}$ to the discrete space $\{0, 1\}$ under the product topology. Viewing elements of this space as infinite sequences, for any finite sequence $\sigma$ of 0’s and 1’s let $[\sigma]$ denote the set of elements of $2^\mathbb{N}$ that extend $\sigma$. The collection $\mathcal{B}$ of Borel sets in the standard topology are generated by the set of such $[\sigma]$. For each $k$, let $\mathcal{B}_k$ denote the finite $\sigma$-algebra generated by the partition $\{[\sigma] \mid \text{length}(\sigma) = k\}$. If a function $f$ from $2^\mathbb{N}$ to $\mathbb{Q}$ is measurable with respect to $\mathcal{B}_k$, I will call it a simple function with complexity at most $k$.

Let $\mu$ be any probability measure on $(2^\mathbb{N}, \mathcal{B})$, and let $f$ be any element of $L^1(\mu)$. Say that a function $k$ from $\mathbb{Q}^+$ to $\mathbb{N}$ is a bound on the complexity of $f$ if, for every $\varepsilon > 0$, there is a simple function $g$ of complexity at most $k(\varepsilon)$ such that $\|f - g\| < \varepsilon$. If $(f_n)$ is any convergent sequence of elements of $L^1(\mu)$ with limit $f$, say that $r(\varepsilon)$ is a bound on the rate of convergence of $(f_n)$ if for every $n \geq r(\varepsilon)$, $\|f_n - f\| < \varepsilon$. (One can also consider rates of convergence in any of the $L^p$ norms for $1 < p < \infty$, or in measure. Since all the sequences considered below are uniformly bounded, this does not affect the results below.)

Now suppose that $\mu$ is a computable measure on $2^\mathbb{N}$ in the sense of computable measure theory [1, 4]. Then if $f$ is any computable element of $L^1(\mu)$, there is a computable sequence $(f_n)$ of simple functions that approaches $f$ with a computable rate of convergence $r(\varepsilon)$; this is essentially what it means to be a computable element of $L^1(\mu)$. In particular, setting $k(\varepsilon)$ equal to the complexity of $f_r(\varepsilon)$ provides a computable bound on the complexity of $f$. But the converse need not hold: if $r$ is any noncomputable real number and $f$ is the constant function with value $r$, then $f$ is not computable even though there is a trivial bound on its complexity.

It is not hard to compute a sequence of simple functions $(f_n)$ that converges to a function $f$ even in the $L^\infty$ norm with the property that there is no computable bound on the complexity of the limit, with respect to the standard coin-flipping measure on $2^\mathbb{N}$. Notice that this is stronger than saying that there is no computable bound on the rate of convergence of $(f_n)$ to $f$; it says that there is no way of
computing bounds on the complexity of any sequence of good approximations to $f$. To describe such a sequence, for each $k$, let $h_k$ be the $B_k$-measurable Rademacher function defined by

$$h_k = \sum_{\{\sigma \mid \text{length}(\sigma) = k\}} (-1)^{\sigma_k} 1_{[\sigma]},$$

where $\sigma_{k-1}$ denotes the last bit of $\sigma$ and $1_{[\sigma]}$ denotes the characteristic function of the cylinder set $[\sigma]$. Intuitively, $h_k$ is a “noisy” function of complexity $k$. Finally, let $f_n = \sum_{i \leq n} 4^{-\varphi(i)} h_i$, where $\varphi$ is an injective enumeration of any computably enumerable set, like the halting problem, that is not computable. Given any $m$, if $n$ is large enough so that $\varphi(j) > m$ whenever $j > n$, then for every $i > n$ and every $x$ we have $|f_i(x) - f_n(x)| \leq \sum_{j \geq m} 4^{-j} < 1/(3 \cdot 4^m)$. Thus the sequence $(f_n)$ converges in the $L^1$ norm. At the same time, it is not hard to verify that if $f$ is the $L^1$ limit of this sequence and $g$ is a simple function of complexity at most $n$ such that $\mu\{x \mid |g(x) - f(x)| > 4^{-(m+1)}\} < 1/2$, then $m$ is in the range of $\varphi$ if and only if $\varphi(j) = n$ for some $j < n$. Thus one can compute the range of $\varphi$ from any bound on the complexity of $f$.

The sequence $(f_n)$ just constructed is contrived, and one can ask whether similar sequences arise “in nature.” Letting $A_n f$ denote the ergodic average $\frac{1}{n} \sum_{i < n} f \circ T_n$, the mean ergodic theorem implies that for every measure $\mu$ on $2^\mathbb{N}$ and $f$ in $L^1(\mu)$, the sequence $(A_n f)$ converges in the $L^1$ norm. However, V’yugin \cite{2,3} has shown that there is a computable shift-invariant measure $\mu$ on Cantor space such that there is no computable bound on the rate of convergence of $(A_n 1_{[1]})$. In V’yugin’s construction, the limit doesn’t have the property described in the last paragraph; in fact, it is very easy to bound the complexity of the limit in question, which places a noncomputable mass on the string of 0’s and the string of 1’s, and is otherwise homogeneous. The next theorem shows, however, that there are computable measures $\mu$ such that the limit does have this stronger property.

**Theorem.** There is a computable shift-invariant measure $\mu$ on $2^\mathbb{N}$ such that if $f = \lim_n A_n 1_{[1]}$, the halting problem can be computed from any bound on the complexity of $f$.

**Proof.** If $\sigma$ is any finite binary sequence, let $\sigma^*$ denote the element $\sigma\sigma\ldots$ of Cantor space. For each $e$, define a measure $\mu_e$ as follows: if Turing machine $e$ halts in $s$ steps, let $\mu_e$ put mass uniformly on these $8s$ elements:

- all $4s$ shifts of $(1^s0^s)^*$
- all $4s$ shifts of $(1^s3^s0^s)^*$

Otherwise, let $\mu_e$ divide mass uniformly between $0^*$ and $1^*$. Each measure $\mu_e$ is shift invariant, by construction. I will show, first, that $\mu_e$ is computable uniformly in $e$, which is to say, there is a single algorithm that, given $e$, $\sigma$, and $\varepsilon > 0$, computes $\mu_e([\sigma])$ to within $\varepsilon$. I will then show that information as to the complexity needed to approximate $f$ in $(2^e, B, \mu_e)$ allows one to determine whether or not Turing machine $e$ halts. The desired conclusion is then obtained by defining $\mu = \sum_e 2^{-\varepsilon(e+1)} \mu_e$.

If Turing machine $e$ does not halt, $\mu_e([\sigma]) = 1/2$ if $\sigma$ is a string of 0’s or a string of 1’s, and $\mu_e([\sigma]) = 0$ otherwise. Suppose, on the other hand, that Turing machine $e$ halts in $s$ steps, and suppose $k < s$. Then there are $2(k-1)$ additional strings $\sigma$ with length $k$ such that $\mu_e([\sigma]) > 0$, each consisting of a string of 1’s followed by a string of 0’s or vice versa. For each of these $\sigma$, $\mu_e([\sigma]) = 1/4s$, and if $\sigma$ is a string
of 0’s or a string of 1’s of length \( k \), \( \mu_\varepsilon([\sigma]) = 1/2 - (k - 1)/4s \). Thus when \( s \) is large compared to \( k \), the non-halting case provides a good approximation to \( \mu_\varepsilon([\sigma]) \) when \( \text{length}(\sigma) \leq k \), even though \( \varepsilon \) eventually halts. Thus, to compute \( \mu_\varepsilon([\sigma]) \) to within \( \varepsilon \), it suffices to simulate the \( \varepsilon \)th Turing machine \( O(k/\varepsilon) \) steps. If it halts before then, that determines \( \mu_\varepsilon \) exactly; otherwise, the non-halting approximation is close enough.

Now consider \( f = \lim_n A_n1_{[1]} \) in \( (2^\omega, \mathcal{B}, \mu_\varepsilon) \). Note that \( (A_n1_{[1]})(\omega) \) counts the density of 1’s among the first \( n \) bits of \( \omega \). If Turing machine \( \varepsilon \) does not halt, \( f(\omega) = 1 \) if \( \omega \) is the sequence of 1’s, and \( f(\omega) = 0 \) if \( \omega \) is the sequence of 0’s. Up to a.e. equivalence, these are all that matters, since the mass concentrates on these two elements of Cantor space. If Turing machine \( \varepsilon \) halts in \( s \) steps, then \( f(\omega) = 1/4 \) on the shifts of \((1^s0^3)^*\), and \( f(\omega) = 3/4 \) on the shifts of \((1^s0^*\varepsilon)^*\).

Suppose \( g \) is \( \mathcal{B}_k \)-measurable. If Turing machine \( \varepsilon \) halts in \( s \) steps and \( k \) is much less than \( s \), then roughly \( 3/4 \) of the shifts of \((1^s0^3)^* \) lie in \([0^k]\) and roughly \( 1/4 \) lie in \([1^k]\); and roughly \( 3/4 \) of the shifts of \( (1^k3^0)^* \) lie in \([1^k]\) and roughly \( 1/4 \) lie in \([0^k]\). But \( f(\omega) \) only takes on the values \( 1/4 \) and \( 3/4 \), and \( g \) is constant on \([0^k]\) and \([1^k]\). So if \( k \) is much less than \( s \), \( \mu_\varepsilon(\{\omega \mid |f(\omega) - g(\omega)| > 1/8\}) > 1/4 \). Turning this around, given the information that \( \mu_\varepsilon(\{\omega \mid |f(\omega) - g(\omega)| > 1/8\}) \leq 1/4 \) for some \( g \) of complexity at most \( k \) enables one to determine whether or not Turing machine \( \varepsilon \) halts; namely, one simulates the Turing machine for \( O(k) \) steps, and if it hasn’t halted by then, it never will.

Set \( \mu = \sum_{e} 2^{-(\varepsilon + 1)} \mu_\varepsilon \). Since, for any \( g \),

\[
\mu_e(\{\omega \mid |f(\omega) - g(\omega)| > 1/8\}) \leq \mu(\{\omega \mid |f(\omega) - g(\omega)| > 1/(8 \cdot 2^{\varepsilon + 1})\})
\]

knowing a \( k_e \) for each \( \varepsilon \) with the property that \( \mu(\{\omega \mid |f(\omega) - g(\omega)| > 1/(8 \cdot 2^{\varepsilon + 1})\}) < 1/4 \) for some \( g \) of complexity at most \( k_e \) enables one to solve the halting problem. But such a \( k_e \) can be obtained from a bound on the complexity of \( f \). Thus \( \mu \) satisfies the statement of the theorem. \( \square \)

The proof above relativizes, so for any set \( X \) there is a measure \( \mu \) on \( 2^N \), computable from \( X \), such that no bound on the rate of complexity of \( f \) can be computed from \( X \). As the following corollary shows, this implies that \( \lim_n A_n1_{[1]} \) can have arbitrarily high complexity.

**Corollary.** For any \( v : \mathbb{Q}^+ \to \mathbb{N} \) there is a measure \( \mu \) on \( 2^N \) such that if \( f = \lim_n A_n1_{[1]} \) and \( k(\varepsilon) \) is a bound on the complexity of \( f \), then \( \limsup_{\varepsilon \to 0} k(\varepsilon)/v(\varepsilon) = \infty \).

**Proof.** Let \( \mu \) be such that no bound on the complexity of \( f \) can be computed from \( v \). If the conclusion failed for some \( k \), then there would be a rational \( \varepsilon' > 0 \) and \( N \) such that for every \( \varepsilon < \varepsilon' \), \( k(\varepsilon) < N \cdot v(\varepsilon) \). But then \( k'(\varepsilon) = N \cdot v(\min(\varepsilon, \varepsilon')) \) would be a bound on the complexity of \( f \) that is computable from \( v \), contrary to our choice of \( \mu \). \( \square \)

**References**


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