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**THE CONDITION IN THE TRICHOTOMY
THEOREM IS OPTIMAL**

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THE CONDITION IN THE TRICHOTOMY THEOREM IS OPTIMAL

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ABSTRACT. We show that the assumption $\lambda > \kappa^+$ in the Trichotomy Theorem cannot be relaxed to $\lambda > \kappa$.

1. INTRODUCTION

The Trichotomy Theorem specifies three alternatives for the structure of an increasing sequence of ordinal functions modulo an ideal on an infinite cardinal κ — provided the sequence has regular length λ and λ is at least κ^{++} .

The natural context of the Trichotomy Theorem is, of course, pcf theory, where a sequence of ordinal functions on κ usually has length which is larger than $\kappa^{+\kappa}$. However, the trichotomy theorem has already been applied in several proofs to sequences of length κ^{+n} , ($n \geq 2$) (see [4], [1] and [3]).

Therefore, a natural question to ask is, whether the Trichotomy Theorem is valid also for sequences of length κ^+ , namely, whether the condition on the minimum length of the sequence can be lowered by one cardinal.

Below we show that the assumption $\lambda \geq \kappa^{++}$ in the Trichotomy Theorem is tight. For every infinite κ , we construct an ideal I over κ and $<_I$ -increasing sequence $\bar{f} \subseteq \text{On}^\kappa$ so that all three alternatives in the Trichotomy theorem are violated by \bar{f} .

2. THE COUNTER-EXAMPLE

Let κ be an infinite cardinal. Denote by On^κ the class of all functions from κ to the ordinal numbers.

Let I be an ideal over κ . We write $f <_I g$, for $f, g \in \text{On}^\kappa$, if $\{i < \kappa : f(i) \geq g(i)\} \in I$ and we write $f \leq_I g$ if $\{i < \kappa : f(i) > g(i)\} \in I$. A sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \text{On}^\kappa$ is $<_I$ -increasing if $\alpha < \beta < \lambda$

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implies that $f_\alpha <_I f_\beta$ and is $<_I$ -decreasing if $\alpha < \beta < \lambda$ implies that $f_\beta <_I f_\alpha$.

A function $f \in \text{On}^\kappa$ is a least upper bound mod I of a sequence $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle \subseteq \text{On}^\kappa$ if $f_\alpha <_I f$ for all $\alpha < \lambda$ and whenever $f_\alpha <_I g$ for all α then $f \leq_I g$. A function $f \in \text{On}^\kappa$ is an *exact upper bound* of \bar{f} if $f_\alpha \leq f$ for all $\alpha < \lambda$, and whenever $g <_I f$, there exists $\alpha < \lambda$ such that $g <_I f_\alpha$. For subsets t, s of κ , write $t \subseteq_I s$ if $s - t \in I$.

The dual filter I^* of an ideal I over κ is the set of all complements of members of I . The relations $\leq_I, <_I$ and \subseteq_I will also be written as $\leq_{I^*}, <_{I^*}$ and \subseteq_{I^*} .

Let us quote the theorem under discussion:

Theorem 1. (*The Trichotomy Theorem*)

Suppose $\lambda \geq \kappa^{++}$ is regular, I is an ideal over κ and $\bar{f} = \langle f_\alpha : \alpha < \lambda \rangle$ is a $<_I$ -increasing sequence of ordinal functions on κ . Then \bar{f} satisfies one of the following conditions:

1. \bar{f} has an exact upper bound f with $\text{cf}f(i) > \kappa$ for all $i < \kappa$;
2. there are sets $S(i)$ for $i < \kappa$ satisfying $|S(i)| \leq \kappa$ and an ultrafilter U over κ extending the dual of I so that for all $\alpha < \lambda$ there exists $h_\alpha \in \prod_{i < \kappa} S(i)$ and $\beta < \lambda$ such that $f_\alpha <_U h_\alpha <_U f_\beta$.
3. there is a function $g : \kappa \rightarrow \text{On}$ such that the sequence $\bar{t} = \langle t_\alpha : \alpha < \lambda \rangle$ does not stabilize modulo I , where $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$.

Proofs of the Trichotomy Theorem are found in [5], II,1.2, in [3] or in the future version of [2].

Theorem 2. For every infinite κ there exists an ultrafilter U over κ and a $<_U$ -increasing sequence $\bar{f} = \langle f_\alpha : \alpha < \kappa^+ \rangle \subseteq \text{On}^\kappa$ such conditions 1, 2 and 3 in the Trichotomy Theorem fail for \bar{f} .

Proof. Let $\lambda = \kappa^+$.

Let us establish some notation.

We recall that every ordinal has an expansion in base λ , namely can be written as a unique finite sum $\sum_{k \leq l} \lambda^{\beta_k} \alpha_k$ so that $\beta_{k+1} < \beta_k$ and $\alpha_k < \lambda$. We limit ourselves from now on to ordinals $\zeta < \lambda^\omega$. For such ordinals, the expansion in base λ contains only finite powers of λ (that is, every β_k is a natural number).

We agree to write an ordinal $\zeta = \lambda^l \alpha_l + \lambda^{l-1} \alpha_{l-1} + \dots + \alpha_0$ simply as a finite sequence $\alpha_l \alpha_{l-1} \dots \alpha_0$. We identify an expansion with λ digits with one with $n > l$ digits by adding zeroes on the left. If $\zeta = \alpha_l \alpha_{l-1} \dots \alpha_0$, we call α_k , for $k \leq l$, the k -th digit in the expansion of ζ .

For $\alpha < \lambda$ and an integer l , define:

$$A_\alpha^l = \{\alpha_l \alpha_{l-1} \dots \alpha_0 : \alpha_k < \alpha \text{ for all } k \leq l\} \quad (1)$$

A_α^l is the set of all ordinals below λ^ω whose expansion in base λ contains $l+1$ or fewer digits from α .

Fact 3. For all $\alpha < \lambda$ and $l < \omega$,

1. $\bigcup_{\alpha < \lambda} A_\alpha^l = \lambda^{l+1}$
2. The ordinal $\sum_{k=0}^l \lambda^k = \overbrace{\alpha \alpha \dots \alpha}^{l+1}$ is the maximal element in $A_{\alpha+1}^l$
3. if $\zeta = \alpha_l \alpha_{l-1} \dots \alpha_0 \in A_\zeta^l$ is not maximal in $A_{\alpha+1}^l$, then the immediate successor of ζ in $A_{\alpha+1}^l$ is obtained from ζ as follows: let k be the first $k \leq l$ for which $\alpha_k < \alpha$. Replace α_k by $\alpha_k + 1$ and replace α_m by 0 for all $m < k$

Fix a partition $\{X_n : n < \omega\}$ of κ with $|X_n| = \kappa$ for all n . Let $n(i)$, for $i < \kappa$, be the unique n for which $i \in X_n$.

By induction on $\alpha < \kappa^+$, define $f_\alpha : \kappa \rightarrow \text{On}$ so that:

1. $f_\alpha(i) \in A_{\alpha+1}^{n(i)} - A_\alpha^{n(i)}$
2. For all $n, l < \omega$ and finite, strictly increasing, sequences $\langle \alpha_k : k \leq l \rangle \subseteq \lambda$ it holds that for every sequence $\langle \zeta_k : k \leq l \rangle$ which satisfies $\zeta_k \in A_{\alpha_k+1}^n - A_{\alpha_k}^n$, there are κ many $i \in X_n$ for which $\bigwedge_{k \leq l} f_{\alpha_k}(i) = \zeta_k$.

The first item above says that $f_\alpha(i)$ is an ordinal below λ^ω whose expansion in base λ has $\leq n(i)$ digits, at least one of which is α . The second item says that every possible finite sequence of values $\langle \zeta_k : k < l \rangle$ is realized κ many times as $\langle f_{\alpha_k}(i) : k < l \rangle$ for an arbitrary increasing sequence $\langle \alpha_k : k < l \rangle$.

The induction required to define the sequence is straightforward.

Define now, for every $\alpha < \kappa^+$, a function $g_\alpha : \kappa \rightarrow \text{On}$ as follows:

$$g_\alpha(i) = \min[(A_{\alpha+1}^{n(i)} \cup \{\lambda^{n(i)+1}\}) - f_\alpha(i)] \quad (2)$$

Since $f_\alpha(i) < \lambda^{n(i)+1}$ for $i \in X_n$, the definition is good. If $f_\alpha(i)$ is not maximal in $A_{\alpha+1}^{n(i)}$, then $g_\alpha(i)$ is the immediate successor of $f_\alpha(i)$ in $A_{\alpha+1}^{n(i)}$. Let us make a note of that:

Fact 4. There are no members of $A_{\alpha+1}^{n(i)}$ between $f_\alpha(i)$ and $g_\alpha(i)$

We have defined two sequences:

$$\begin{aligned}\bar{f} &= \langle f_\alpha : \alpha < \lambda \rangle \\ \bar{g} &= \langle g_\alpha : \alpha < \lambda \rangle\end{aligned}$$

Next we wish to find an ideal modulo which \bar{f} is $<_I$ -increasing and \bar{g} is a $<_I$ -decreasing sequence of upper bounds of \bar{f} .

Claim 5. *For every finite increasing sequence $\alpha_0 < \alpha_1 < \dots < \alpha_l < \lambda$ there exists $i < \kappa$ such that for all $k < l$*

$$f_{\alpha_k}(i) < f_{\alpha_{k+1}}(i) < g_{\alpha_{k+1}}(i) < g_{\alpha_k}(i) \quad (3)$$

Proof. Suppose $\alpha_0 < \alpha_1 < \dots < \alpha_l < \lambda$ is given and choose $n > l$. Let $\zeta_0 = \overbrace{\alpha_0 \alpha_0 \dots \alpha_0}^{l+1} \in A_{\alpha_0}^n$. Let ζ_{k+1} be obtained from ζ_k by replacing the first $l+1-k$ digits of ζ_k by α_{k+1} :

$$\begin{aligned}\alpha_0 \alpha_1 \dots \alpha_k \alpha_{k+1} \dots \alpha_l &= \zeta_l \\ &\vdots \\ \alpha_0 \alpha_1 \dots \alpha_{k-1} \alpha_k \dots \alpha_k &= \zeta_k \\ &\vdots \\ \alpha_0 \alpha_1 \dots \alpha_1 \dots \alpha_1 &= \zeta_1 \\ \alpha_0 \alpha_0 \dots \alpha_0 \dots \alpha_0 &= \zeta_0\end{aligned}$$

Thus $\zeta_0 < \zeta_1 < \dots < \zeta_l$ and $\zeta_k \in A_{\alpha_k}^l \subseteq A_{\alpha_k}^n$ is not maximal in $A_{\alpha_k}^n$ (because $n > l$). Let ξ_k be the immediate successor of ζ_k in $A_{\alpha_k}^n$.

By Fact 3 above, we have

$$\begin{aligned}\overbrace{1 \ 0 \ \dots \ \dots \ 0 \ \dots \ \dots \ 0}^{l+1} &= \xi_0 \\ (\alpha_0 + 1) 0 \ \dots \ \dots \ 0 &= \xi_1 \\ &\vdots \\ \alpha_0 \alpha_1 \dots (\alpha_{k-1} + 1) 0 \dots 0 &= \xi_k \\ &\vdots \\ \alpha_0 \alpha_1 \dots \dots (\alpha_{l-1} + 1) 0 &= \xi_l\end{aligned}$$

Therefore $\zeta_0 < \zeta_1 < \dots < \zeta_l < \xi_l < \xi_{l-1} < \dots < \xi_0$. To complete the proof it remains to find some $i \in X_n$ for which $f_{\alpha_k}(i) = \zeta_k$ for

$k \leq l$, and, consequently, by the definition (2) above, $g_{\alpha_k}(i) = \xi_k$. The existence of such $i \in X_n$ is guaranteed by the second condition in the definition of \bar{f} . \square

For $\alpha < \beta < \lambda$, let

$$C_{\alpha,\beta} = \{i < \kappa : f_\alpha(i) < f_\beta(i) < g_\beta(i) < g_\alpha(i)\} \quad (4)$$

Claim 6. $\{C_{\alpha,\beta} : \alpha < \beta < \kappa^+\}$ has the finite intersection property

Proof. Suppose that $\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_l, \beta_l$ are given and $\alpha_k < \beta_k < \lambda$ for $k \leq l$. Let $\langle \gamma_m : m < m(*) \rangle$ be the increasing enumeration of $\bigcup_{k \leq l} \{\alpha_k, \beta_k\}$. To show that $\bigcap_{k \leq l} C_{\alpha_k, \beta_k}$ is not empty, it suffices to find some $i < \kappa$ for which the sequence $g_{\gamma_m}(i)$ is decreasing in m and $f_{\gamma_m}(i)$ is increasing in m . The existence of such an $i < \kappa$ follows from Claim 5. \square

Let U be any ultrafilter extending $\{C_{\alpha,\beta} : \alpha < \beta < \lambda\}$. Since for every $\alpha < \beta$ it holds that $f_\alpha <_U f_\beta <_U g_\beta <_U g_\alpha$, we conclude that \bar{f} is $<_U$ -increasing, that \bar{g} is $<_U$ -decreasing and that every g_α is an upper bound of $\bar{f} \bmod U$.

Claim 7. There is no exact upper bound of $\bar{f} \bmod U$.

Proof. It suffices to check that there is no $h \in \text{On}^\kappa$ that satisfies $f_\alpha <_U h <_U g_\alpha$ for all $\alpha < \kappa^+$. Suppose, then, that $h \in \text{On}^\kappa$ satisfies this. Since $h <_U g_0$, we may assume that $g(i) < \lambda^{n(i)+1}$ for all $i < \kappa$ (by changing h on a set outside of U).

Let $i < \kappa$ be arbitrary. Since $\bigcup_{\alpha < \lambda} A_\alpha^{n(i)} = \lambda^{n(i)+1}$, there is some $\alpha(i)$ so that $h(i) \in A_{\alpha(i)}^{n(i)}$. By regularity of λ it follows that there is some $\alpha(*) < \lambda$ such that $h(i) \in A_{\alpha(*)}^{n(i)}$ for all $i < \kappa$. By our assumption about h , $f_{\alpha(*)} <_U h <_U g_{\alpha(*)}$. Thus, there is some $i < \kappa$ for which $f_{\beta(*)}(i) < h(i) < g_{\beta(*)}(i)$. However, all three values belong to $A_{\alpha(*)+1}^{n(i)}$, while by Fact 4 there are no members of $A_{\alpha(*)+1}^{n(i)}$ between $f_{\beta(*)}(i)$ and $g_{\alpha(*)}(i)$ — a contradiction. \square

Claim 8. there are no sets $S(i) \subseteq \text{On}$ for $i < \kappa$ which satisfy condition 2 in the trichotomy for \bar{f} and U .

Proof. Suppose that $S(i)$, for $i < \kappa$, and $h_\alpha \in \prod_{i < \kappa} S(i)$ satisfy 2. in the Trichotomy Theorem. Find $\alpha < \lambda$ such that $S(i) \subseteq A_\alpha^{n(i)}$ for all i . Thus $f_\alpha <_U h_\alpha <_U g_\alpha$ — contradiction to 4. \square

Claim 9. there is no $g : \kappa \rightarrow \text{On}$ such that g, \bar{f} and the dual of U satisfy condition 3. in the Trichotomy Theorem.



Proof. Let $g : \kappa \rightarrow On$ be arbitrary, and let $t_\alpha = \{i < \kappa : f_\alpha(i) > g(i)\}$. As f is $<_U$ -increasing, for every $\alpha < \beta < \lambda$ necessarily $t_\alpha \subseteq_U t_\beta$. Since U is an ultrafilter, every \subseteq_U -increasing sequence of sets stabilizes. \square

\square

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