1998

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REGRESSIVE RAMSEY NUMBERS ARE ACKERMANNIAN

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Research Report No. 98-208
January, 1998
REGRESSIVE RAMSEY NUMBERS ARE ACKERMANNIAN

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Abstract. We give an elementary proof of the fact that regressive Ramsey numbers are Ackermannian. This fact was first proved by Kanamori and McAloon with mathematical logic techniques.

Nous vivons encore sous le règne de la logique, voilà, bien entendu, à quoi je voulais en venir. Mais les procédés logiques, de nos jours, ne s’appliquent plus qu’à la résolution de problèmes d’intérêt secondaire. [1, 1924, p. 13] is

1. Introduction

Definition 1. 1. let A be a set of natural numbers. A coloring c : 
[A]e → N of unordered e-tuples from A is regressive if c(x) < 
min x for all x ∈ [A]e.

2. A subset B ⊆ A is min-homogeneous for a coloring c of [A]e if 
for all x ∈ [A]e the color c(x) depends only on min x.

Theorem 2 (Kanamori and McAloon). 1. For every k and e there 
exists N such that for every regressive pair coloring on {1, 2, . . . , N} 
there exists a min-homogeneous subset of size k.

2. The statement in (1) cannot be proved from the axioms of Peano Arithmetic (although it can be phrased in the language of PA).

3. Let r(k) be the least N which satisfies 1 for e = 2. The function 
ν eventually dominates every primitive recursive function.

Part (3) of Kanamori and McAloon’s result [3] was proved with mathematical logic methods. We present below an elementary proof of 3.

2. The lower bound

Define a sequence of (strictly increasing) integer functions f_i, i ≥ 1 
as follows:

The first author was partially supported by NSF grant No. DMS-9622579.
The second author was partially supported by the Binational Science Foundation.
Number 649 in list of publications.
\[ f_1(n) = n + 1 \quad (1) \]
\[ f_{i+1}(n) = f_i^{(i)}(n) \quad (2) \]

Fix an integer \( k > 2 \). Define a sequence of semi-metrics \( \{d_i : i \in \mathbb{N}\} \) on \( \{n : n \geq k^2\} \) by putting, for \( k^2 \leq m \leq n \),
\[ d_i(m, n) = |\{t \in \mathbb{N} : m \leq f_i^{(i)}(k^2) < n\}| \quad (3) \]

Let \( i(m, n) \), for \( k^2 \leq m < n \), be the greatest \( i \) for which \( d_i(m, n) \) is positive, and \( d(m, n) = d_{i(m, n)}(m, n) \).

**Claim 3.** For all \( n \geq m \geq k^2 \), \( d(m, n) \leq \sqrt{m} \).

**Proof.** Trivial. \( \square \)

Let us fix the following (standard) pairing function \( \Pr \) on \( \mathbb{N}^2 \)
\[ \Pr(m, n) = \left( \frac{m+n}{2} \right) + n \]

\( \Pr \) is a bijection between \( \mathbb{N}^2 \) and \( \mathbb{N} \) and is monotone in each variable. Observe that if \( m, n \leq 1 \) then \( \Pr(m, n) < 1^2 \) for all \( l > 3 \).

Define a pair coloring \( c \) on \( \{n : n \geq k^2\} \) as follows:
\[ c(m, n) = \Pr(i(m, n), d(m, n)) \quad (4) \]

**Claim 4.** For every \( i \in \mathbb{N} \), every sequence \( x_0 < x_1 < \cdots < x_i \) that satisfies \( d_i(x_0, x_i) = 0 \) is not min-homogeneous for \( c \).

**Proof.** The claim is proved by induction on \( i \). If \( i = 1 \) then there are no \( x_0 < x_1 \) with \( d_1(x, y) = 0 \) at all. Suppose to the contrary that \( i > 1 \), that \( x_0 < x_1 < \cdots < x_i \) is min-homogeneous with respect to \( c \) and that \( d_i(x_0, x_i) = 0 \). Necessarily, \( i(x_0, x_j) = j < i \). By min-homogeneity, \( i(x_0, x_i) = j \) as well, and \( d_j(x_0, x_i) = d_j(x_0, x_i) \). Hence, \( (x_1, x_2, \ldots, x_i) \) is min-homogeneous with \( d_j(x_1, x_i) = 0 \) — contrary to the induction hypothesis. \( \square \)

**Claim 5.** The coloring \( c \) in (4) is regressive on the interval \( [k^2, f_k(k^2)] \).

**Proof.** Clearly, \( d_{i+1}(m, n) = 0 \) for \( k^2 \leq m < n < f_k(k^2) \) and therefore \( i(m, n) \leq k < \sqrt{m} \). From Claim 3 we know that \( d(m, n) \leq \sqrt{m} \). Thus, \( c(m, n) = \Pr(i(m, n), d(m, n)) \leq \Pr(\sqrt{m}, \sqrt{m}) < m \), since \( \sqrt{m} > 3 \). \( \square \)
We show that $f_{k}(k^2)$ grows eventually faster than every primitive recursive function by comparing the functions $f_i$ with the usual approximations of Ackermann’s function. It is well known that every primitive recursive function is dominated by some approximation of Ackermann’s function (see, e.g. [2]).

Let $A_k(n)$ be defined as follows:

\begin{align*}
A_1(n) &= n + 1 \\
A_{i+1}(n) &= A_i(A_i(n))
\end{align*}

The $A_i$-s are the usual approximations to Ackermann’s function, which is defined by $A(n) = A(n)$.

Claim 6. 1. $f_i(n) \geq 4n^2$ for $i, n \geq 4$.
2. $A_i(n) \leq f_{i+4}(4n^2) \leq f_{i+4}(n)$ for all $i$ and $n \geq 4$.
3. $A_i(n) \leq f_{i+5}(n)$ for all $i$ and $n \geq 4$.

Proof. The first item is verified directly. The second inequality in the second item is by 1. The first inequality is proved by induction on $i$. Suppose $A_i(n) \leq f_{i+4}(4n^2)$. Since $A_i(n) \leq f_{i+4}(n)$, iterating $n$ times yields $A_n(n) \leq f_{i+4}(n)$, which is $\leq f_{i+4}(4n^2) = f_{i+5}(4n^2)$. Thus

$$
A_{i+1}(n) \leq f_{i+5}(4n^2)
$$

The last item follows now trivially: $A_i(n) \leq f_{i+4}(n) \leq f_{i+5}(n)$ (as $n \geq 4$).

Corollary 7. The function $\nu(k)$ eventually dominates every primitive recursive function.

3. Discussion

3.1. Other Ramsey numbers. Paris and Harrington [8] published in 1976 the first finite Ramsey-type statement that was shown to be independent over Peano Arithmetic. Soon after the discovery of the Paris-Harrington result, Erdős and Mills studied the Ramsey-Paris-Harrington numbers in [7]. Denoting by $R^e_c(k)$ the Ramsey-Paris-Harrington number for exponent $e$ and $c$ many colors, Erdős and Mills showed that $R^e_c(k)$ is double exponential in $k$ and that $R^e_c(k)$ is Ackermannian as a function of $k$ and $c$. In the same paper, several small Ramsey-Paris-Harrington numbers were computed. Later Mills tightened the double exponential upper bound for $R^e_c(k)$ in [5].

Canonical Ramsey numbers for pair colorings were treated in [4] and were also fond to be double exponential.
The second author showed that van der Waerden numbers are primitive recursive, refuting the conjecture that they were Ackermannian, in [9] (see also [6]).

We remark that an upper bound for regressive Ramsey numbers for pairs is $R^2_k(N)$ — the Ramsey-Paris-Harrington number for triples. Let $N$ be large enough and suppose that $c$ is regressive on $\{1, 2, \ldots, N-1\}$. Color a triple $x < y < z$ red if $c(x, y) = c(x, z)$ and blue otherwise. Find a homogeneous set $A$ of size at least $k$ and so that $|A| > \min A + 1$. The homogeneous color on $A$ cannot be red for $k > 5$, and therefore $A$ is min-homogeneous for $c$.

### 3.2. Problems

The following two problems about regressive Ramsey numbers remain open:

**Problem 8.**

1. Find a concrete upper bound for regressive Ramsey numbers.
2. Compute small regressive Ramsey numbers.

### REFERENCES


