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# **An Effective One-Step Hyperperpetual Reduction Strategy for Combinators**

by

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**An Effective One-Step  
Hyperperpetual Reduction Strategy  
For Combinators**  
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**Abstract:**

We construct an effective one-step reduction strategy  $\phi$  for combinators such that  $\phi$  is normalizing and if  $M$  has no normal form then the iterations  $M, \phi(M), \dots, \phi^m(M), \dots$  are all distinct.

**The Algorithm:**

We adopt the terminology and notations of [ 1 ].

Barendregt in [ 1 ] defines a perpetual reduction strategy to be a function  $\phi : CL \rightarrow CL$

such that for any combinator  $M$  we have  $M \rightarrowtail \phi(M)$  and if  $M$  is the source of non-terminating reduction then the reduction  $M \rightarrowtail \phi(M) \rightarrowtail \dots \rightarrowtail \phi^m(M) \rightarrowtail \dots$  is such a non-

terminating reduction. In addition, a function  $\phi : CL \rightarrow CL$  is said to be a normalizing reduction strategy if for any combinator  $M$  we have  $M \rightarrowtail \phi(M)$  and if  $M$  has a normal form then, for some  $m$ ,  $\phi^m(M)$  is the normal form of  $M$ . In applications, one usually wants a strategy which, in the absence of a normal form, produces, after iteration, arbitrarily long combinators. This is not done by conventional perpetual or normalizing strategies.

In the case of  $CL$  we have the following consequence of a theorem of Jan Willem Klop ([ 2 ] 1.13.(iv)); namely, if a combinator has only finitely many reducts then it has a normal form. This theorem is simply false for lambda calculus. Thus we can ask for a bit more out of a perpetual reduction strategy. We can ask that all the iterates  $M, \phi(M), \dots, \phi^m(M), \dots$  be distinct when  $M$  has no normal form. If we combine this idea with the idea of a normalizing reduction strategy we obtain the notion of a hyperperpetual reduction strategy.

**Definition:** A function  $\phi : CL \rightarrow CL$  is a hyperperpetual reduction strategy if for any  $M$

- (1)  $M \rightarrowtail \phi(M)$  and  $M \equiv \phi(M)$  iff  $M$  is normal.
- (2) if  $M$  has a normal form then, for some  $m$ ,  $\phi^m(M)$  is the normal form of  $M$ .
- (3) if  $M$  has no normal form then all the iterates  $M, \phi(M), \dots, \phi^m(M), \dots$  are distinct.

We shall prove the following theorem.

**Theorem :** There is an effective one-step hyperperpetual reduction strategy for  $CL$ .

**Proof:** We shall construct an algorithm  $\mathbb{P}$  by recursion for this purpose.

The algorithm  $\mathbb{P}$

Input: a combinator  $M$

Output: a combinator  $\mathbb{P}(M)$  such that

if  $M$  is normal then  $\mathbb{P}(M) \equiv M$   
else  $M \rightarrow \mathbb{P}(M)$

Begin:

(1) If  $M$  is head normal then

case 1;  $M$  is normal. Set  $\mathbb{P}(M) \equiv M$ .  
case 2;  $M \equiv SLN$  and  $L$  is not normal. Set  $\mathbb{P}(M) \equiv S(\mathbb{P}(L))N$ .  
case 3;  $M \equiv SLN$  and  $N$  is not normal. Set  $\mathbb{P}(M) \equiv SL(\mathbb{P}(N))$ .  
case 4;  $M \equiv KN$ . Set  $\mathbb{P}(M) \equiv K(\mathbb{P}(N))$

else

(2) If  $M \equiv SPQRM(1)...M(m)$  then  $\mathbb{P}(M) \equiv PR(QR)M(0)...M(m)$

else

(3) If  $M \equiv KPQM(1)...M(m)$  then

case 1;  $P$  is head normal. Set  $\mathbb{P}(M) \equiv PM(1)...M(m)$ .  
case 2;  $P$  is not head normal. Set  $\mathbb{P}(M) \equiv$   
 $K(\mathbb{P}(P))QM(1)...M(m)$ .

End.

We need three lemmas to prove the theorem.

Lemma : Suppose that  $M$  has no normal form. Then the reduction sequence

$M \rightarrow \mathbb{P}(M) \rightarrow ... \rightarrow \mathbb{P}^m(M) \rightarrow ...$  does not cycle.

Proof: Suppose that the lemma is false. Of all the counter-examples to the lemma let  $M$  be the shortest in length. Let  $M \equiv AN(1)...N(n)$  for  $A$  an atom i.e.  $M \in \{S, K\}$ . In the cycle  $M \rightarrow \mathbb{P}(M) \rightarrow ... \rightarrow \mathbb{P}^m(M) \equiv M$  some trace of  $N(n)$  must be the last argument of a redex. For otherwise we can shorten  $M$  to  $AN(1)...N(n-1)$ . Moreover, after zero or more  $S$  reductions there must be at least one  $K$  reduction whose last argument is the last component of the term, for otherwise the last components have the form  $N(n)$ ,  $L(1)N(n)$ ,  $L(2)(L(1)N(n))$ , ...,  $L(k)(...L(1)N(n))$  which can never repeat  $N(n)$ . Thus there is some  $\mathbb{P}^k(M) \equiv KPQ$  and  $\mathbb{P}^{k+1}(M) \equiv P$ . Hence  $P$  is head normal and this contradicts the choice of  $M$ . End of proof.

Definition; If  $M$  has a head normal form then the principal head normal form of  $M$  is the one obtained by performing all possible head reductions of  $M$ .

Indeed if  $P$  is the principal head normal form of  $M$  and  $H$  is any other head normal form then  $P \rightarrow\!\!\!>(internal) H$ .

Lemma : If  $M$  has a head normal form then, for some  $m$ ,  $\mathbb{P}^m(M)$  is the principal head normal form of  $M$ .

Proof ; by induction on the number of head reductions from  $M$  to its principal head normal form. The basis case is when  $M$  is already head normal and this case is trivial.

Next suppose that  $M$  has a head redex. When this head redex is an  $S$  redex then the result follows directly from the induction hypothesis applied to  $\mathbb{P}(M)$ . Now suppose that  $M$  begins with a  $K$  redex  $M \equiv KPQM(1)...M(m)$ . Then  $P$  has a head normal form and if  $P \rightarrow P(1) \rightarrow ... \rightarrow P(p)$  the head reduction of  $P$  to principal head normal form then the head reduction of  $M$  to principal head normal form begins  $M \rightarrow PM(1)...M(m) \rightarrow P(1)M(1)...M(m) \rightarrow ... \rightarrow P(p)M(1)...M(m)$ . By induction hypothesis there is an  $n$  such that  $\mathbb{P}^n(P) \equiv P(p)$  and there is an  $m$  such that  $\mathbb{P}^m(P(p)M(1)...M(m))$  is the principal head normal form of  $M$ . But then  $\mathbb{P}^n(M) \equiv KP(p)QM(1)...M(m)$ ,  $\mathbb{P}^{(n+1)}(M) \equiv P(p)M(1)...M(m)$ ,

and  $\mathbb{P}^{(n+m+1)}(M)$  is the principal head normal form of  $M$  as desired. End of proof.

**Lemma :** If  $M$  has a normal form then, for some  $m$ ,  $\mathbb{P}^m(M)$  is the normal form of  $M$ .

**Proof :** By the standardization theorem if  $M$  has a normal form then there is a standard reduction to the normal form of  $M$ . Let  $\#M$  be the length of such a standard reduction. The proof is by induction on  $\#M$ . The basis case is when  $M$  is already normal and this is trivial. For the induction step we distinguish several cases.

Case 1;  $M$  is head normal.

When  $\#M$  is fixed we prove this by induction on the length of  $M$ . We have either  $M \equiv SPQ$  or  $M \equiv KP$  with one or the other induction hypothesis applying to  $P$  and  $Q$ .

Here the result follows from (1) in the definition of  $\mathbb{P}$ .

Case 2;  $M$  begins with a head redex.

By the previous lemma there exists an  $n$  such that  $\mathbb{P}^n(M)$  is the principal head normal form of  $M$  and moreover  $\#\mathbb{P}^n(M) < \#M$ . Thus the induction hypothesis applies to  $\mathbb{P}^n(M)$  and there is an  $m$  such that  $\mathbb{P}^m(\mathbb{P}^n(M)) \equiv \mathbb{P}^{(n+m)}(M)$  is the normal form  $\mathbb{P}^n(M)$  i.e. the normal form of  $M$ . This completes the proof of the lemma and the theorem.

#### References:

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|----------------|---|
| [1] Barendregt | The Lambda Calculus<br>North Holland 1980   |
| [2] Klop       | Reduction cycles in combinatory logic<br>in To H.B. Curry : Essays on Combinatory Logic, Lambda Calculus,<br>and Formalism<br>Hindley and Seldin eds.<br>Academic Press 1980<br>pages 193-214 |

