Elaboration in Dependent Type Theory

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Abstract

To be usable in practice, interactive theorem provers need to provide convenient and efficient means of writing expressions, definitions, and proofs. This involves inferring information that is often left implicit in an ordinary mathematical text, and resolving ambiguities in mathematical expressions. We refer to the process of passing from a quasi-formal and partially-specified expression to a completely precise formal one as elaboration. We describe an elaboration algorithm for dependent type theory that has been implemented in the Lean theorem prover. Lean’s elaborator supports higher-order unification, type class inference, ad hoc overloading, insertion of coercions, the use of tactics, and the computational reduction of terms. The interactions between these components are subtle and complex, and the elaboration algorithm has been carefully designed to balance efficiency and usability. We describe the central design goals, and the means by which they are achieved.

1 Introduction

Just as programming languages run the spectrum from untyped languages like Lisp to strongly-typed functional programming languages like Haskell and ML, foundational systems for mathematics exhibit a range of diversity, from the untyped language of set theory to simple type theory and various versions of dependent type theory. Having a strongly typed language allows the user to convey the intent of an expression more compactly and efficiently, since a good deal of information can be inferred from type constraints. Moreover, a type discipline catches routine errors quickly and flags them in informative ways. But there is a downside: as we increasingly rely on types to serve our needs, the computational support that is needed to make sense of expressions in efficient and predictable ways becomes increasingly subtle and complex.

Our goal here is to describe the elaboration algorithm used in a new interactive theorem prover, Lean [10] [4]. Lean is based on an expressive dependent
type theory, allowing us to use a single language to define datatypes, mathematical objects, and functions, and also to express assertions and write proofs, in accordance with the propositions-as-types paradigm. Thus, filling in the details of a function definition and ensuring it is type correct is no different from filling in the details of a proof and checking that it establishes the desired conclusion.

The elaboration algorithm that we describe employs nonchronological backtracking, as well as heuristics for unfolding defined constants that are very effective in practice. Elaboration algorithms for dependent type theory are not well documented in the literature, making the practice something of a dark art. We therefore also hope to fill an expository gap, by describing the problem clearly and presenting one solution.

Lean’s elaborator is quite powerful. It solves not only first-order unification problems, but nontrivial higher-order unification problems as well. It supports the computational interpretation of terms in dependent type theory, reducing expressions as necessary in the elaboration process. It supports ad hoc overloading of constants, and it can insert coercions automatically. It supports a mechanism for type class inference in a manner that is integrated with the other components of the elaboration procedure. It also supports interaction with built-in and user-defined tactics. The interaction between the components we have just enumerated is subtle and complex, and many pragmatic design choices were made to attain efficiency and usability.

We start in Section 2 with an overview of the task of the elaborator, focusing on its outward effects. In other words, we try to convey a sense of what the elaborator is supposed to do. In Section 3 we explain the algorithm that is used to achieve the desired results. We describe related work and draw some conclusions in Section 4. Lean is an open-source project, and the source code is freely available online. Many of the features discussed below are described in greater detail in a tutorial introduction to Lean.

2 The elaboration task

What makes dependent type theory “dependent” is that types can depend on elements of other types. Within the language, types themselves are terms, and a function can return a type just as another function may return a natural number. Lean’s standard library is based on a version of the Calculus of Inductive Constructions with Universes, as are formal developments in Coq and Matita. We assume some familiarity with dependent type theory, and provide only a brief overview here.

Lean’s core syntax is based on a sequence of non-cumulative type universes and \( \Pi \)-types. There is an infinite sequence of type universes \( \text{Type}_0, \text{Type}_1, \text{Type}_2, \ldots \), and any term \( t : \text{Type}_i \) is intended to denote a type in the \( i \)th universe. Each universe is closed under the formation of \( \Pi \)-types \( \Pi x : A, B \), where \( A \) and \( B \) are type-valued expressions, and \( B \) can depend on \( x \). The idea is that \( \Pi x : A, B \) denotes the type of functions \( f \) that map any element \( a : A \)

\[ \text{http://leanprover.github.io} \]
to an element of $B[a/x]$. When $x$ does not appear in $B$, the type $\Pi x : A, B$ is written $A \to B$ and denotes the usual non-dependent function space.

Lean's kernel can be instantiated in different ways. In the standard mode, $\text{Type}_0$ is distinguished by the fact that it is impredicative, which is to say, $\Pi x : A, B$ is an element of $\text{Prop}$ for every $B : \text{Prop}$ and every $A$. Elements of $\text{Prop}$ are intended to denote propositions. When $\Pi x : A, B$ is in $\text{Prop}$, it can also be written as $\forall x : A, B$, and it is interpreted as the proposition that $B$ holds for every element $x$ of $A$. Given $P : \text{Prop}$, an element $t : P$ can be interpreted as a proof of $P$, under the propositions-as-types correspondence, or more simply as “evidence” that $P$ holds. Such a type $P$ is proof-irrelevant, which is to say, any two terms $s, t : P$ are treated as definitionally equal by the kernel.

Lean also provides a predicative mode for homotopy type theory, without any special treatment of $\text{Type}_0$. The result is a version of Martin-Löf type theory \cite{20, 30} similar to the one used in Agda \cite{7}.

In both standard and hott modes, the type universes are non-cumulative. They are treated polymorphically, which is to say, there are explicit quantifications over universes. In practice, users generally write $t : \text{Type}$, leaving it to Lean to insert an implicit universe variable and manage universe constraints accordingly.

Extensions to the core type theory inhabit a second layer of the kernel. Both modes allow one to form inductive families \cite{11}, a mechanism that can be used to define basic types like $\text{nat}$ and $\text{bool}$, and common type-forming operations, like Cartesian products, lists, $\Sigma$-types, and so on. Each inductive family declaration generates a recursor (also known as the eliminator). The standard mode includes a mechanism for forming quotient types, and the mode for homotopy type theory includes certain higher-inductive types \cite{32}. We need not be concerned with the precise details here, except to note that terms in dependent type theory come with a computational interpretation. For example, given $t : B$ possibly depending on a variable $x : A$ and $s : A$, the term $(\lambda x, t) s$ reduces to $t[s/x]$, the result of replacing $x$ by $s$ in $t$. Similarly, inductive types support definition by recursion; for example, if one defines addition on the natural numbers by structural recursion on the second argument, $t + 0$ reduces to $t$. The kernel type checker should identify terms that are equivalent under the induced equivalence relation, and, as much as possible, the elaborator should take this equivalence into account when inferring missing information. We discuss this further in Section 2.2.

2.1 Type inference and implicit arguments

The task of the elaborator, put simply, is to convert a partially specified expression into a fully specified, type-correct term. For example, in Lean, one can define a function `do_twice` as follows:

```lean
definition do_twice (f : N \to N) (x : N) : N := f (f x)
```

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One can omit any two of the three type annotations, leaving it to the elaborator to infer the missing information. Inferring types like these can be seen as a generalization of the Hindley-Milner algorithm [16, 23], which takes an unsigned term in the λ-calculus and assigns a simple type to it, if one exists.

When entering a term, a user can leave an argument implicit by inserting an underscore, leaving it to the elaborator to infer a suitable value. One can also mark function arguments as implicit by declaring them using curly brackets when defining the function, to indicate that they should be inferred rather than entered explicitly. For example, we can define the `append` function, which concatenates two lists, so that it has the following type:

```
append : Π {A : Type}, list A → list A → list A
```

Users can then write `append l₁ l₂` rather than `append A l₁ l₂`, leaving Lean to infer the first argument.

### 2.2 Higher-order unification

Many of the constraint problems that arise in type inference and the synthesis of implicit arguments are easily solved using first-order unification. For example, suppose a user writes `append l₁ l₂`, where `l₁` can be seen to have type `list T`. Temporarily naming the implicit argument to `append` as `?M`, we see that next argument to `append` should have type `list ?M`. Given that `l₁` in fact has type `list T`, we easily infer `?M = T`.

Nevertheless, it is often the case that the elaborator is required to infer an element of a Π-type, which constitutes a higher-order unification problem. For example, if `e : a = b` is a proof of the equality of two terms of some type `A`, and `H : P` is a proof of some expression involving `a`, then the term `subst e H` denotes a proof of the result of replacing some or all of the occurrences of `a` in `P` with `b`. Here, in addition to inferring the type `A`, we also need to infer an expression `T : A → Prop` denoting the context for the substitution, that is, the expression with the property that `T a` is convertible to `P`. Such an expression is inherently ambiguous; for example, if `H` has type `R (f a a) a`, then with `subst e H` the user may have in mind `R (f b b) b` or `R (f a b) a` or something else, and the elaborator has to rely on context and a backtracking search to find an interpretation that fits. Similar issues arise with proofs by induction, which require the system to infer an induction predicate.

The need for higher-order unification even arises with common datatypes. For example, the type `Σx : A, B` denotes the type of dependent pairs `(a, b)`, where `a : A` and `b : B a`. Here `B` is in general a function `A → Type`. In the notation `(a, b)`, the arguments `A` and `B` are left implicit. The argument `A` can easily be inferred from the type of `A`, but the type of `b` will generally be an expression that involves `a` as an argument. In this case, higher-order unification is used to infer `B`.

Even second-order unification is known to be generally undecidable [13], but the elaborator merely needs to perform well on instances that come up in
practice. For example, in Lean, users can import the notation $H \triangleright H'$ for subst $H \Downarrow H'$. If we have proved

```
theorem mul_mod_mul_left {z : ℕ} (x y : ℕ) (zpos : z > 0) :
  (z * x) mod (z * y) = z * (x mod y)
we can then write
```

```
theorem mul_mod_mul_right (x z y : ℕ) :
  (x * z) mod (y * z) = (x mod y) * z :=
  !mul.comm ⊳ !mul.comm ⊳ !mul.comm ⊳ !mul_mod_mul_left
The proof applies the commutativity of multiplication three times to an appropriate instance of the theorem mul_mod_mul_left. (The symbol $!$ indicates that all arguments should be synthesized by the elaborator.) The unifier can similarly handle nested inductions and iterated recursion.
```

```text
theorem add.comm (n m : ℕ) : n + m = m + n :=
nat.induction_on m
  (nat.induction_on n rfl
    (take n, assume IH : n = 0 + n,
      show succ n = succ (0 + n), from IH ⊳ rfl))
-- induction step omitted
```

Or, in the context of homotopy type theory, where equality proofs are relevant, we can write:

```
definition concat_assoc (p : x = y) (q : y = z) (r : z = t) :
  p * (q * r) = (p * q) * r :=
  eq.rec_on r (eq.rec_on q idp)
Here rec_on denotes a form of recursion which, like induction, has to infer the relevant predicate.
```

We will see below that higher-order unification is a complex process, and places a high burden on the elaborator. It should thus be used sparingly. But it is often convenient and sometimes unavoidable, so it is important that it can be handled by the elaboration algorithm.

### 2.3 Computational behavior

The elaborator should also respect the computational interpretation of terms. It should, for instance, recognize the equivalence of the terms $(\lambda x, t) \ s$ and $t[s/x]$, as well as $(s, t).1$ (denoting the first projection of the pair) and $s$ under the relevant reduction rule for pairs. Elements of inductive types also have computational behavior; on the natural numbers, $2 + 2$ and $4$ are both definitionally equal to $\text{succ} \ (\text{succ} \ (\text{succ} \ (\text{succ} \ 0)))$, $x * 0$ is definitionally equal to $x$, and $x * 1$ is definitionally equal to $\text{succ} \ x$. The elaborator should also support unfolding definitions where necessary: for example, if $x - y$ is defined as $x + (-y)$, the elaborator should allow us to use the commutativity of addition to rewrite $x - y$ to $-y + x$. Unfolding definitions and reducing
projections is especially crucial when working with algebraic structures, where many basic expressions cannot even be seen to be type correct without carrying out such reductions. For example, given \(a : A\) and \(b : B\), the left-hand side of the expression \(\langle a, b \rangle .2 = b\) has type \(B ((a, b).1)\) and the right-hand side has type \(B a\). Both the elaborator and the type checker need to recognize these types as the same.

Determining when to unfold defined constants is a crucial part of the practice of theorem proving. It is therefore unsurprising that the naive approach of performing all such unfoldings leads to unacceptable performance, and it is an important aspect of building a practical elaboration procedure to design heuristics that limit unfolding to situations that require it. Lean allows users to annotate definitions, providing hints to the elaborator, as follows:

- An irreducible definition will never be unfolded during higher-order unification (but can still be unfolded in other situations, for example during type checking).
- A reducible definition will be always eligible for unfolding.
- A semireducible definition can be unfolded during simple decisions and won’t be unfolded during complex decisions.

For example, users can mark definitions which ought to be viewed as abbreviations as reducible. The meaning of these annotations is discussed further in Section 3.3. These annotations are used only by the elaborator; they have no bearing at all when it comes to checking the type of a fully elaborated term. As a result, the user can modify these annotations at any time, as needed, when developing a theory.

### 2.4 Type classes

Lean supports the use of Haskell-style type classes [15]. For example, we can define a class `has_mul A` of types `A` with an associated multiplication as follows:

```lean
structure has_mul [class] (A : Type) := (mul : A → A → A)
```

In other words, for every type `A`, `has_mul A` is a record with one element, `has_mul.mul`, which we should think of as a multiplication operation on `A`. We then define the generic multiplication operation,

```lean
definition mul {A : Type} [s : has_mul A] : A → A → A := has_mul.mul
```

and the notation

```lean
infix * := mul
```

The square brackets indicate that the argument `s` is implicit, and that the relevant instance of `has_mul A` should be synthesized by the class inference mechanism. We can declare a particular instance as follows:

```lean
```
definition nat_has_mul [instance] : has_mul nat :=
  has_mul.mk nat.mul

Suppose the user writes \( s \ast t \), when \( s \) is inferred to have type \( \text{nat} \). When
the elaborator is called to solve \(?M : \text{has_mul nat}\), it finds \text{nat_has_mul} on a
stored list of instances, and assigns that to \(?M\).

Instance declarations themselves can have implicit class arguments, in which
case, class inference performs a backward-chaining Prolog-like search. For ex-
ample, we can declare

structure semigroup [class] (A : Type) extends has_mul A :=
  (mul_assoc : \forall a b c, mul (mul a b) c = mul a (mul b c))

The \text{structure} declaration above automatically declares \text{semigroup} to be an
instance of \text{has_mul}, so that if, instead, \text{nat} was only declared to be an instance
of \text{semigroup}, class inference could synthesize the instance of \text{has_mul nat} in
two steps. We then get the generic theorem \text{mul.assoc} in the same way we
obtained the generic notation for multiplication.

We can then go on to define monoids, groups, rings, and commutative
versions. The \text{structure} command supports the construction of the algebraic
hierarchy by allowing the user to extend and merge multiple structures:

structure group [class] (A : Type)
  extends monoid A, has_inv A :=
  (mul_left_inv : \forall a, mul (inv a) a = one)

Users can also rename structure components on the fly. In the following example,
type class inference finds the appropriate inverse and instance of the theorem
\text{inv_inv} when processing \text{eq_inv_of_eq_inv}:

theorem inv_inv {A : Type} [s : group A] (a : A) :
  (a⁻¹)⁻¹ = a :=

...  

theorem eq_inv_of_eq_inv {A : Type} [s : group A] {a b : A}
  (H : a = b⁻¹) : b = a⁻¹ :=
  by rewrite [H, inv_inv]

Here, the \text{rewrite} tactic replaces \( a \) by \( b^{-1} \) in the goal, and then rewrites
\((b^{-1})^{-1}\) to \( b \). Since any \text{group} is an instance of a \text{monoid} and any \text{monoid}
is an instance of a \text{semigroup}, generic theorems about semigroups and monoids
can be applied to any group. The type class inference is seamless integrated in
all proof procedures implemented in Lean. We remark that the theorem above
can be proved without any user guidance using these procedures.

We can also declare so called \textit{fully bundled structures} in the style of the
Mathematical Components library \[12\]. For example:

structure Group := (carrier : Type) (struct : group carrier)

attribute Group.carrier [coercion]
attribute Group.struct [instance]
This means that whenever we have $G : \text{Group}$, we can write $g : G$, and $G$ is coerced to its carrier type. And whenever we have $g : \text{carrier } G$, type class resolution can infer that $\text{struct } G$ is the relevant group structure on $\text{carrier } G$.

In Lean, type classes can be used to infer not only notation and generic facts, but fairly complex data. For example, in the standard library, we define the class of propositions that are decidable:

\[
\text{inductive \, \text{decidable} \, [\text{class}] \, (p : \text{Prop}) : \text{Type} :=} \\
\mid \text{inl} : p \rightarrow \text{decidable} \, p \\
\mid \text{inr} : \neg p \rightarrow \text{decidable} \, p
\]

Logically speaking, since $\text{decidable} \, p$ lives in $\text{Type}$ rather than $\text{Prop}$, having an element $t : \text{decidable} \, p$ is more informative than having an element $t : p \lor \neg p$; it enables us to define values of an arbitrary type depending on the truth value of $p$. This distinction is only useful in constructive mathematics, because classically every proposition is decidable. But the $\text{decidable}$ class allows for a smooth transition between constructive and classical logic, allowing classical reasoning in suitable constructive settings as well. It is especially relevant for users interested in defining computable functions. In Lean, $(\text{if } c \text{ then } t \text{ else } e)$ is notation for $(\text{ite } c \, t \, e)$, where $\text{ite}$ is defined as:

\[
\text{definition \, \text{ite} (c : \text{Prop}) \, [H : \text{decidable } c] \, \{A : \text{Type}\} \\
(t \, e : A) : A := \\
\text{decidable.rec_on } H (\lambda Hc, t) (\lambda Hnc, e)
\]

Note that the implicit argument $H$ is automatically synthesized by type class inference. Moreover, the expression $\text{if } c \text{ then } t \text{ else } e$ computes whenever $c$ is a decidable proposition.

For example, we can prove, constructively, that equality on the natural numbers is decidable:

\[
\text{nat.decidable_eq \, [\text{instance}] : } \forall x \, y : \text{nat}, \text{decidable } (x = y)
\]

We can do the same for inequality relations on $\text{nat}$, and moreover show that decidability is preserved under boolean operations and bounded quantification. We moreover make the following definitions:

\[
\text{definition \, \text{is_true} (c : \text{Prop}) \, [H : \text{decidable } c] : \text{Prop} :=} \\
\text{if } c \text{ then true else false}
\]

\[
\text{definition \, \text{of_is_true} \{c : \text{Prop}\} \, [H_1 : \text{decidable } c] \\
(H_2 : \text{is_true } c) : c := \\
\text{decidable.rec_on } H_1 (\lambda Hc, Hc) (\lambda Hnc, \text{!false.rec } (\text{if_neg } Hnc \, H_2))
\]

\[
\text{notation \, } \text{`dec_trivial`} := \text{of_is_true } \text{trivial}
\]

What is going on here is subtle. The expression $\text{is_true } c$ infers a decision procedure for $c$, and returns either $\text{true}$ or $\text{false}$. Assuming $H_2 : \text{is_true } c$,
of_is_true H₂ is a proof of c. But if is_true c evaluates to true, it has the canonical proof trivial. Thus, given a proposition c, the notation dec_trivial does the following:

- infers a decision procedure for c, and
- tries to use trivial to prove is_true c.

If it succeeds — that is, if the resulting term type checks — the result is a proof of c.

With these definitions, we can write the following proof:

```lean
example : ∀ x : nat, x < 10 → x ≠ 10 ∧ x < 12 := dec_trivial
```

Type class resolution infers the decision procedure for the proposition in question, and computational reduction evaluates it. Problems like this can appear anywhere in a proof or an expression, and type class inference will solve them at appropriate times within the elaboration process.

### 2.5 Overloading

We have seen that the standard library relies on type class inference to support the use arithmetic operations like + and * for different number classes. This is sometimes known as *parametric polymorphism*. Lean also supports *ad hoc polymorphism* by allowing us to overload identifiers and notation. For example, the notation ++ is used for concatenation of both lists and tuples:

```lean
import data.list data.tuple
open list tuple

variables (A : Type) (m n : ℕ)
variables (v : tuple A m) (w : tuple A n) (s t : list A)

check s ++ t
check v ++ w
```

Where it is necessary to disambiguate, Lean allows us to precede an expression with the notation #<namespace>, to specify the namespace in which notation is to be interpreted.

```lean
check λ x y, (#list x + y)
check λ x y, (#tuples x + y)
```

We can also overload identifiers. Every identifier in Lean has a full name that is unique, but identifiers can be grouped into namespaces, and opening the namespace produces a shorter alias. For example, if we define foo in namespaces a and b, we obtain identifiers named a.foo and b.foo respectively. If we open both namespaces, however, the alias foo is an overloaded reference to both, leaving the elaborator to resolve the ambiguity.
Ad hoc overloading is more flexible than type class overloading, in that the overloaded constants can denote entirely different kinds of objects. It adds ambiguity and choice points to the elaboration process, and should therefore be used sparingly. But it is often useful, especially when we want to reuse notation for expressions that do not have the same shape. For example, in the Lean standard library, we use \(-1\) above to denote the inverse function for algebraic structures that support it, as well as for the symmetry operation for equalities. It can also be used to invert bijections, used group isomorphisms, ring isomorphisms, isomorphisms in a category, or equivalences between categories.

2.6 Coercions

The treatment of coercions in Lean is as one would expect. One can, for example, coerce a `bool` to a `nat` and a `nat` to an `int`, and Lean will insert coercions in list expressions `[n, i, m, j]` and `[i, n, j, m]` when `n` and `m` have type `nat` and `i` and `j` have type `int`. One can also coerce axiomatic structures, so that the user can provide a group as input anywhere a semigroup is expected. One can also coerce from a suitable family of types to `Type` or to a \(\Pi\)-type.

In fact, just as in Coq, Lean allows us to declare three kinds of coercions:

- from a family of types to another family of types
- from a family of types to the class of sorts
- from a family of types to the class of function types

The first kind of coercion allows us to view any element of a member of the source family as an element of a corresponding member of the target family. The second kind of coercion allows us to view any element of a member of the source family as a type. The third kind of coercion allows us to view any element of the source family as a function. For details, see [4].

2.7 Tactics and structuring mechanisms

Finally, definitions and proofs can invoke tactics, that is, user-defined or built-in proof-finding procedures that construct various subterms. The constraint solver described in this paper invokes user provided tactics to construct terms that cannot be synthesized by solving unification constraints and type class resolution. Lean’s tactic language is similar to those found in other LCF-style theorem provers. Describing the tactic language here would take us too far afield; we only wish to point out that our implementation makes the use of tactics continuous with the act of writing terms. Anywhere a term is expected, a user can used the keywords `begin` and `end` to enter a tactic block:

```lean
theorem test (p q : Prop) (Hp : p) (Hq : q) : p ∧ q ∧ p :=
begin
  apply and.intro,
  exact Hp,
```

10
apply and.intro,
exact Hq,
exact Hp
end

One-line tactic proofs can be specified with the keyword by:

```lean
theorem test (p q : Prop) (Hp : p) (Hq : q) : p \land q \land p :=
  by apply (and.intro Hp); exact (and.intro Hq Hp)
```

Conversely, when in tactic mode, one can use the exact tactic to specify an explicit term, as in the example above. The keywords have and show make it possible to do that in an elegant and structured way.

```lean
theorem card_image_eq_of_inj_on {f : A → B} {s : finset A}
  (H1 : inj_on f (ts s)) :
  card (image f s) = card s :=
begin
  induction s with a t H IH,
  {rewrite [card_empty]},
  {have H2 : ts t ⊆ ts (insert a t),
    by rewrite [-subset_eq_to_set_subset];
    apply subset_insert,
    have H3 : card (image f t) = card t,
      from IH (inj_on_of_inj_on_of_subset H1 H2),
    have H4 : f a \notin image f t,
      from ..., -- proof suppressed
    show card (image f (insert a t)) = card (insert a t),
      from ... -- proof suppressed
  }
end
```

Thus one can pass freely between the two modes. This yields a tradeoff between two different strategies for elaboration: tactics build an expression using local information in a surgical way, whereas the elaborator solves constraints involving global information, spread out across the entire term.

The availability of tactic mode also provides a convenient way of sectioning long proof terms: the construct `proof t qed` is syntactic sugar for `by+ exact t`. Including this in a long proof terms forces the elaborator to process the surrounding expression independent of `t`, and then process `t`, separately, using information from the surrounding term. Thus we can treat the processing of a long proof as one large elaboration problem or the composition of smaller ones, balancing the advantages of the local and global approaches in a convenient and flexible way.

### 2.8 Combining the various components

Any given definition or theorem in Lean can draw on many of the features just described. Consider the following, which defines the composition of two natural
transformations between functors (and establishes that it is, indeed, a natural transformation):

\[
\text{variables } \{C, D : \text{Category} \} \{F, G, H : C \Rightarrow D\}
\]

\[
\text{definition nt_compose } (\eta : G \Rightarrow H) (\theta : F \Rightarrow G) : F \Rightarrow H := \\
\\text{natural_transformation.mk}
\text{(take a, } \eta a \circ \theta a) \\
\text{(take a b f, calc}
\begin{align*}
H f \circ (\eta a \circ \theta a) &= (H f \circ \eta a) \circ \theta a : \text{assoc} \\
\ldots &= (\eta b \circ G f) \circ \theta a : \text{naturality} \\
\ldots &= \eta b \circ (G f \circ \theta a) : \text{assoc} \\
\ldots &= \eta b \circ (\theta b \circ F f) : \text{naturality} \\
\ldots &= (\eta b \circ \theta b) \circ F f : \text{assoc}
\end{align*}
\]

Here the functors \(F, G\), and \(H\) are coerced to their action on morphisms, and the natural transformations \(\eta\) and \(\theta\) are coerced to their first component. The composition symbol \(\circ\) for functions is overloaded to denote composition of morphisms as well, and type class inference infers the category in which the composition takes place. The appropriate substitution contexts in the calculation are inferred, as are the arguments to the theorems that are invoked.

The interactions between the components of the elaboration task are subtle, and the challenge is to deal with them all at the same time. A definition or proof may give rise to hundreds of constraints requiring a mixture of higher-order unification, disambiguation of overloaded symbols, insertion of coercions, type class inference, and computational reduction. The net effect is then a difficult constraint-solving problem with a combinatorial explosion of options. Lean’s elaborator manages to solve such problems, and it is quite fast. (See, for example, the data presented at the end of Section 3.3.) In the next section, we explain how the elaborator processes the constraints and navigates the search space in an effort to balance completeness and efficiency.

## 3 The elaboration procedure

### 3.1 Overview

Section 2 describes what we want the elaboration algorithm to do. It needs to infer types and implicit arguments in expressions, including sometimes higher-order functions and predicates. It needs to support type class inference that is robust enough to work with structures in an algebraic hierarchy in a uniform and convenient way. It needs to disambiguate overloaded notation and identifiers, and it needs to insert coercions where appropriate. Moreover, it needs to respect the computational behavior of expressions while performing all these tasks, since, in general, the constraints can only be solved up to equivalence of terms. The goal of this section is to describe an algorithm that does this.

The task of Lean’s parser, which we do not describe here, is to convert a user’s input to a preterm, a formal but incomplete reflection of that input. The
The process for getting from a preterm to a fully elaborated term has two main steps: preprocessing and constraint resolution. The preprocessing phase takes a preterm and creates a partially specified term, with “holes,” or metavariables, representing the information that needs to be inferred. At the same time, the preprocessor generates a list of constraints that these metavariables need to satisfy. Some of the constraints are unification constraints, for example, the constraint that the type of an argument to a function matches the function’s argument type. Others are choice constraints, for example, that an inferred value is among a finite set of possible overloads. Finally, after all constraints have been solved, Lean invokes the tactic blocks associated with the remaining holes to produce the terms necessary to fill them. This is a recursive procedure because some tactics may contain nested preterms that must be also elaborated, and these preterms may additional tactic blocks, and so on.

The constraint resolution phase aims to find a consistent solution to all the unification and choice constraints. Heuristically, “straightforward” constraints should be solved first, providing useful information to guide the rest of the search. Choice points result in backtracking, which needs to be handled carefully to avoid duplication of work. Failures need to be carefully tracked in order to provide informative error messages to the user.

The simple division into the preprocessing phase and constraint resolution phase is slightly too simplistic: even the preprocessing phase has to process and simplify constraints, in order to detect the possibility of inserting a coercion. Thus both phases make use of a constraint simplification procedure that performs preliminary reductions.

We spell out the details below. Sections 3.2 and 3.3 describe the main data structures and some of the support functions used by the algorithm, and Section 3.4 describes the constraint simplification procedure. The preprocessing step is described in Section 3.5 and the constraint solving procedure, which is both the heart of the elaboration algorithm and the most complex component, is described in Sections 3.6 and 3.7.

3.2 Main data structures

In this section, we describe the term representation and the main data structures used in our elaboration procedure. We assume the term language is a dependent λ-calculus in which terms are described by the following grammar:

\[
t, s = \ell \mid x \mid f \mid ?m \mid \text{Type } u \mid t \mid \lambda x : s, t \mid \Pi x : s, t
\]

where

- \(\ell\) a free variable (also called a local constant)
- \(x\) is a bound variable
- \(f\) is a constant (parametrized by a list of universe terms)
- \(?m\) is a metavariable
• $u$ is a universe term

We adopt a locally nameless variable binding style: free variables have a unique identifier and a type, while bound variables are simply represented by a number, a de Bruijn index. We store the type with each free variable, thereby removing the need to carry around contexts in the type checker and normalizer. As described in [21], this representation style simplifies the implementation considerably, as it minimizes the number of places where explicit calculations with de Bruijn indices must be performed. We use the notation $t[x := s]$ to represent the substitution of $x$ for $s$ in $t$, where $x$ is a bound variable, free variable, or metavariable. When $x$ is a bound variable, the operation also lowers all bound variables with index greater than $x$.

While the locally nameless approach simplifies many aspects of the code, the operations of abstracting and instantiating variables can be costly. Fortunately, we found a simple optimization that completely eliminates any performance concerns. The problem, and our solution, are described at the end of Section 3.3.

An environment stores a sequence of declarations. The Lean kernel supports three different kinds of declarations: axioms, definitions and inductive families. Each has a unique identifier, and can be parametrized by a sequence of universe parameters. Every axiom has a type, and every definition has a type and a value. A constant is just a reference to a declaration.

A user's input to the elaborator can generally be viewed as partial constructions, i.e., constructions containing holes that must be filled by the system. Internally, each hole is represented by a metavariable. Each metavariable has a unique identifier and a type. The main operation on metavariables is instantiation. In our implementation, only closed terms can be assigned to metavariables. This design decision guarantees that operations such as $\beta$-reduction and metavariable instantiation commute. Since only closed terms can be assigned to metavariables, on creation a metavariable is applied to the variables in the context where it appears. For example, we encode a hole in the context $(x : A) (y : B)$ as $?m x y$, where $?m$ is a fresh metavariable. The type of $?m$ is $\Pi (x : A) (y : B), C$, where $C$ is the expected type for the hole at that position. If the expected type is also unknown at preprocessing time, we create another fresh metavariable $?m_t : \Pi (x : A) (y : B), C$. This gives us $?m : \Pi (x : A) (y : B), ?m_t x y$. We say a term is fully elaborated if it does not contain metavariables.

We say a term is $\beta$-reducible if it is of the form $(\lambda x : A, s)t$, and $\iota$-reducible if it is of the form C.rec $s$ (C.mk $i$ r) t, where C.rec is the recursor/eliminator for an inductive datatype C. Here, the sequence $s$ represents the parameters, minor premises and indices, and (C.mk $i$ r) is the main premise (where C.mk is the $i$-th constructor of C). The function reduce$_{\beta, \iota}$ $s$ applies head $\beta$ and $\iota$ reduction to $s$. We say a term $t$ is stuck if computation cannot occur without instantiating a metavariable $?m$, where $?m s$ is a sub-term of $t$. In that case, we say $?m s$ is the reason for $t$ being stuck. More formally, a term is stuck
when the head symbol is a metavariable (i.e., it is of the form \(?m \, s\)), or it is a
recursor application where the main premise is stuck. We say the first case is a
\textit{stuck application}, and the second a \textit{stuck recursor}.

During the preprocessing step, \textit{unification} and \textit{choice} constraints are gen-
erated. Unification constraints are used to enforce typing constraints, and \textit{choice}
constraints are for overloading, coercion resolution, and triggering the type class
mechanism.

A unification constraint \(t \approx s\) is annotated with a \textit{justification}, which rep-
resent the facts and assumptions that gave rise to the constraint. Justifications
are used to assist the generation of error messages when a term fails to be elabo-
rated, and to implement \textit{non-chronological backtracking} \cite{25}. Non-chronological
backtracking allows exploring the (possibly infinite) tree of potential solutions
more efficiently, by eliminating branches which we know cannot possibly contain
an actual solution.

There are three kinds of justifications: \textit{asserted}, \textit{assumption} and \textit{join}. An
asserted justification is used to annotate constraints generated during the pre-
processing phase. Whenever the solver has to perform a choice (also known as a
\textit{case split}), it annotates each choice with a fresh assumption. A join justifi-
cation \(j_1 \bowtie j_2\) represents the “union” of the justifications \(j_1\) and \(j_2\). We use \(\langle t \approx s, j \rangle\) to denote the unification constraint justified by \(j\). A \textit{substitution}
is a finite collection of \textit{assignments} from metavariables to pairs \(\langle t, j \rangle\), written
\(?m \mapsto \langle t, j \rangle\), where \(t\) is a closed term and \(j\) is a justification for the assignment.

Assignments are generated when solving unification constraints. For example,
the constraint \(\langle ?m \approx t, j \rangle\) is solved by adding the assignment \(?m \mapsto \langle t, j \rangle\).
Whenever we apply a substitution we use a join justification to track its effect.
For example, the result of applying the assignment \(?m \mapsto \langle t, j_m \rangle\) over the con-
straint \(\langle r \approx s, j \rangle\) is the new constraint \(\langle r[?m := t] \approx s[?m := t], j \bowtie j_m \rangle\). We also use \(\langle s \approx t, j_1 \bowtie j_2 \rangle\) to denote the constraint \(\langle s \approx t, j_1 \bowtie j_2 \rangle\). Moreover, if \(a\) is a list of constraints \([c_1, \ldots, c_n]\), \(a \bowtie j\) is \([c_1 \bowtie j, \ldots, c_n \bowtie j]\).

A \textit{choice constraint} is of the form \(\langle ?m \, \ell : t \in f, j \rangle\), where:

\begin{itemize}
  \item \(?m\) is a metavariable,
  \item \(\ell\) are free variables representing the context where \(?m\) was created,
  \item \(t\) is the type of \(?m \, \ell\), and
  \item \(f\) is a procedure that, given the term \(?m \, \ell\), its type \(t\), and a substitu-
  tion, produces a (possibly unbounded) stream of constraints representing
  possible ways of synthesizing \(?m\), and a justification \(j\).
\end{itemize}

Note that each alternative is itself a list of constraints, and is not necessarily
just a \textit{single} unification constraint.

Whereas some constraints should be solved eagerly, other constraints should
be solved only when there is sufficient information to process them in a reliable
way. To that end, a choice constraint \(\langle ?m \, \ell : t \in f, j \rangle\) may be marked as \textit{on-
demand}. When the flag \textit{ondemand} is set, the constraint solver will try to invoke
function \(f\) only after all metavariables in \(t\) have been instantiated. We say a
ondemand choice constraint is ready when \( t \) does not contain metavariables, and postponed otherwise. We will later describe how this feature is used to implement the type class mechanism and coercions. If a choice constraint is not marked as ondemand, we say it is a regular choice constraint. We use regular choice constraints to specify overloaded symbols. The result of applying the assignment \(?m \mapsto \langle s,j_m \rangle\) over the choice constraint \(?n \ell : t \in f,j\) is the new constraint \(?n \ell : t[?m := s] \in f,j \bowtie j_m\). We also use the notation \( c \bowtie j \) when \( c \) is a choice constraint.

3.3 Support functions

In this section, we describe some auxiliary functions that are used throughout the elaboration algorithm.

The function \( \text{typeof} \ r \) returns the inferred type of a term \( r \), where \( r \) may contain metavariables. Specifically, it returns a pair \( \langle t, S \rangle \) where \( t \) is the type of \( r \) and \( S \) is a set of constraints on the metavariables. If \( r \) does not contain metavariables, then \( S \) is empty.

Given \((\ell_1 : A_1) \ldots (\ell_n : A_n)\), the operation \( \text{abstract}_\lambda [\ell_1 \ldots \ell_n].t \) returns

\[
\lambda(x_1 : A_1) \ldots (x_n : A_n[\ell_1 := x_1, \ldots, \ell_{n-1} := x_{n-1}]), \ t[\ell_1 := x_1, \ldots, \ell_n := x_n]
\]

We also have \( \text{abstract}_\Pi \), the equivalent operation for \( \Pi \)-abstraction.

The function \( \text{unfold} (f \ t_1 \ldots t_n) \) applies a \( \delta \)-reduction, i.e. it unfolds the definition of constant \( f \). In practice, however, it is not feasible to apply \( \delta \)-reduction to all constants in a constraint solving problem. To cope with this performance issue, we allow the user to annotate definitions with the hints irreducible, semireducible or reducible, as described in Section 2.3. Recall that an irreducible definition is never unfolded by the constraint solver, while a semireducible or reducible definition may be unfolded or not depending on the constraint being solved. Roughly, a semireducible definition is unfolded only if the decision to unfold is “simple,” which is to say, if the unfolding does not require the procedure to consider an extra case split. When a decision is not simple, the unfolding produces at least one extra case, and consequently increases the search space. When no annotation is provided, the system assumes the definition is semireducible. Note that when the kernel type checks fully elaborated definitions, these annotations are ignored; they are only relevant during the elaboration process.

The function \( \text{whnf} \ r \) returns a pair \( \langle w, S \rangle \) where \( w \) is a term convertible to \( r \) that is in weak head normal form (whnf) or is stuck, and \( S \) is a set of unification constraints. (In this paper, we can assume the set \( S \) returned by \( \text{whnf} \) is always empty. In our implementation, constraints in \( S \) arise because the elimination rule for equality has extra computational reductions when proof irrelevance is enabled, corresponding to Streicher’s axiom K [29]. Specifically, given \( H_1 : a = a \), the term \( \text{eq.rec} A \ a \ C \ a \ H_2 \ H_1 \) reduces to \( H_2 \) even when \( H_1 \) is not the constructor \( \text{eq.refl} A \ a \), where eq.rec is the recursor for the Leibniz equality. To that end, the elaborator has to infer the type of \( H_1 \), which
can give rise to unification constraints.) The function \texttt{whnf} does not unfold irreducible definitions, and during type class resolution, it also does not unfold semireducible ones.

The procedure \texttt{error} $j$ throws an exception tagged with a justification $j$.

Finally, the function \texttt{ensurefun} $s$ ensures that $s$ has a function type. Specifically, it infers the type $t$ of $s$ (using \texttt{typeof}) and then reduces $t$ to $t'$ in weak head normal form. If $t'$ is a $\Pi$-term, then it returns $t'$ and any new unification constraints. If $t'$ is not a $\Pi$-term and is not stuck, then it generates an error with justification $j$. Otherwise, if $?m s$ is the reason that $t'$ is stuck, where $?m : (\Pi x : A, B)$, we create two fresh metavariables: $?m_1 : (\Pi x : A, \text{Type $?u_1$})$ and $?m_2 : (\Pi (x : A) (y : ?m_1 x), \text{Type $?u_2$})$, and the new constraint

$$\langle t \approx (\Pi x : ?m_1 s, ?m_2 s x), j \rangle.$$  

This ensures that $s$ has a function type, and defers the problem of figuring out what that type is.

Recall that we use the \textit{local nameless} approach for representing terms, in which free variables have a unique identifier and a type, while bound variables are represented by a de Bruijn index. We say a term $t$ has a \textit{dangling} bound variable if there is a bound variable in $t$ that is not in the scope of any $\lambda/\Pi$-expression binding it. For example, the $\lambda$-expression $\lambda x : \text{Type}, f x (g x)$ has no dangling bound variables, but $x$ is a dangling bound variable in $f x (g x)$. In the locally nameless approach, all major operations (such as type inference, normalization, and unification) assume there are no dangling bound variables, that is, no bound variables that "point out of the scope." This invariant is enforced by replacing bound variables with fresh free variables whenever we visit the body of a $\lambda/\Pi$-expression. In the lambda expression above, we would replace $x$ in $f x (g x)$ with the free variable $\ell : \text{Type}$. This operation is called \texttt{instantiate} in \cite{21}. The operation \texttt{abstract} $\ell t$ is the inverse; it replaces the free variable $\ell$ in $t$ with the bound variable with de Bruijn index 0. These two operations are essentially the only ones that have to deal with de Bruijn indices.

Although the locally nameless approach greatly simplifies the implementation effort, there is a performance penalty. Given a term $t$ of size $n$ with $m$ binders, it takes $O(nm)$ time to visit $t$ while making sure there are no dangling bound variables. In \cite{21}, the authors suggest that this cost can be minimized by generalizing \texttt{abstract} and \texttt{instantiate} to process sequences of free and bound variables. This optimization is particularly effective when visiting terms containing several consecutive binders, such as $\lambda x_1 : A_1, \lambda x_2 : A_2, \ldots, \lambda x_n : A_n, t$. Nonetheless, we observed that these two operations were still a performance bottleneck for several files in the Lean standard library. We have addressed this problem using a very simple complementary optimization. For each term $t$, we store a \textit{bound} $B$ such that all de Bruijn indices occurring in $t$ are in the range $[0, B)$. This bound can easily be computed when we create new terms: the bound for the de Bruijn variable with index $n$ is $n + 1$, and given terms $t$ and $s$ with bounds $B_t$ and $B_s$ respectively, the bound for the application $(t \ s)$ is $\max(B_t, B_s)$, and the bound for $(\lambda x : t, s)$ is $\max(B_t, B_s - 1)$. We use the bound
B to optimize the instantiate operation. The idea is simple: B enables us to decide quickly whether any subterm contains a bound variable being instantiated or not. If it does not, then our instantiate procedure does not even visit the subterm. Similarly, for each term \( t \), we store a bit that is set to “true” if and only if \( t \) contains a free variable. We use this bit to optimize the abstract operation, since it enables us to decide quickly whether a subterm contains a free variable.

These optimizations are crucial to our implementation. The Lean standard library currently contains 172 files, and 41,700 lines of Lean code. With the optimizations, the whole library can be compiled in 71.06 seconds using an Intel Core i7 3.6Ghz processor with 32Gb of memory. Without the optimizations, it takes 2,189.97 seconds to compile the same set of files.

3.4 The constraint simplification procedure

The preprocessing step and the constraint-solving procedure rely on a constraint-simplification procedure, which we now describe. The idea of the simp procedure is to decompose all the constraints that can “straightforwardly” be decomposed to simpler ones, and to detect quickly any constraints that simply cannot be solved. Thus, given a unification constraint, the simp procedure produces a set of (potentially) simpler unification constraints or throws an error. Moreover, if the input constraint does not contain metavariables, then the result is the empty set \{\} or an error.

In the pseudocode below, \( s \) and \( t \) denote arbitrary terms, \( \ell \) is a free variable, and \( f \) and \( g \) are constants. The procedure mklocal \( A \) creates a fresh free variable with type \( A \). To simplify the presentation, we assume there is a global unique name generator. The function depth \( f \) returns the definition depth of the constant \( f \), which is equal to 0 if \( f \) is not a definition, and 1 + max\{depth \( g \) | \( g \) appears in the definition of \( f \)} otherwise. To save space, we do not list symmetric cases; for example, we present a case for \( \langle \ell s_1 \ldots s_n \approx \ell t_1 \ldots t_n, j \rangle \) but not \( \langle \ell t_1 \ldots t_n \approx \ell s_1 \ldots s_n, j \rangle \).

\[
\begin{align*}
\text{simp} \ (r \approx r, j) &= \{\} \\
\text{simp} \ (s \approx t, j) \quad \text{when} \ s \ \text{is } \beta/\iota \text{-reducible} &= \text{simp} \ (\text{reduce}_{\beta/\iota} \ s \approx t, j) \\
\text{simp} \ (\ell \ s_1 \ldots s_n \approx \ell \ t_1 \ldots t_n, j) &= \bigcup_{i=1}^n \text{simp} \ (s_i \approx t_i, j) \\
\text{simp} \ (f \ s_1 \ldots s_n \approx f \ t_1 \ldots t_n, j) &= 
\begin{cases} 
\text{simp} \ (\text{reduce}_{\beta/\iota} \ s \approx t, j) 
& \text{if } s_1 \ldots s_n \text{ and } t_1 \ldots t_n \text{ do not contain metavariables} \\
\text{simp} \ (\text{reduce}_{\beta/\iota} \ s \approx t, j) 
& \text{if } f \text{ is not reducible} \\
\text{simp} \ (\langle f \ s_1 \ldots s_n \approx f \ t_1 \ldots t_n, j \rangle) 
& \text{else} \\
\text{simp} \ (\langle f \ s_1 \ldots s_n \approx f \ t_1 \ldots t_n, j \rangle) 
& \text{else} \\
\text{simp} \ (\langle f \ s_1 \ldots s_n \approx f \ t_1 \ldots t_n, j \rangle) 
& \text{else} \\
\end{cases}
\end{align*}
\]
\[\text{simp } (\text{unfold } (f \ s) \approx \text{unfold } (g \ t), \ j)\]
\[\text{let } \ell = \text{mklocal } A \text{ in simp } (A \approx B, \ j) \cup \text{simp } (s[x := \ell] \approx t[y := \ell], \ j)\]
\[\text{let } \ell = \text{mklocal } A \text{ in simp } (A \approx B, \ j) \cup \text{simp } (s[x := \ell] \approx t[y := \ell], \ j)\]
\[\text{simp } (\lambda x : A, s \approx (\lambda y : B, t), \ j) = \text{let } \ell = \text{mklocal } A \text{ in simp } (A \approx B, \ j) \cup \text{simp } (s[x := \ell] \approx t[y := \ell], \ j)\]
\[\text{simp } (\lambda x : A, s \approx (\lambda y : B, t), \ j) = \text{let } \ell = \text{mklocal } A \text{ in simp } (A \approx B, \ j) \cup \text{simp } (s[x := \ell] \approx t[y := \ell], \ j)\]
\[\text{simp } (\text{fold } (\lambda x : C, s) \approx t, \ j) = \text{if } s \text{ or } t \text{ is stuck then } \{ (s \approx t, \ j) \} \text{ else error } j\]

In the actual implementation, we also use a heuristic optimization for the case \(\text{simp } (f \ s_1 \ldots s_n \approx f \ t_1 \ldots t_n, \ j)\), where \(s_1 \ldots s_n\) and \(t_1 \ldots t_n\) do not contain metavariables, and \(f\) is not a projection. In this case, we first try \(\text{simp } (s_1 \approx t_1, \ j) \ldots \text{simp } (s_n \approx t_n, \ j)\), and if no error is thrown, we return \(\{\}\).

Each unification constraint returned by \(\text{simp}\) is in one of the following categories:

- **delta**: \( (f \ s \approx f \ t, \ j)\). Note that, based on the definition of \(\text{simp}\), \(f\) must be a reducible definition.

- **pattern**: \( (?m \ell_1 \ldots \ell_n \approx t, \ j)\), where \(\ell_1, \ldots, \ell_n\) are pairwise distinct free variables, \(t\) only contains free variables in \(\{\ell_1, \ldots, \ell_n\}\), and \(?m\) does not occur in \(t\).

- **quasi-pattern**: \( (?m \ell_1 \ldots \ell_n \approx t, \ j)\), where all \(\ell_1, \ldots, \ell_n\) are free variables, but are not pairwise distinct.

- **flex-rigid**: \( (?m s_1 \ldots s_n \approx t, \ j)\), where at least one of \(s_1, \ldots, s_n\) is not a free variable.

- **flex-flex**: \( (?m_1 s \approx ?m_2 t, \ j)\).

- **recursor**: \( (t \approx s, \ j)\), where \(t\) or \(s\) is a stuck recursor.

In the literature, **pattern**, **quasi-pattern**, and **flex-rigid** are simply called flex-rigid constraints, and the category **pattern** corresponds to Miller patterns [22]. Note that flex-flex constraints are badly underconstrained, and we typically expect that other constraints will do more to limit the interpretation of the metavariables.

### 3.5 Preprocessing

The preprocessor is a recursive procedure that, given a preterm and a context, returns a term \(t\) (potentially containing metavariables) and a set of unification and choice constraints. The basic idea is that if the constraints are solved, their solution should contain an assignment for all metavariables in \(t\). The preprocessor must carry a context, a list of free variables, to be able to create fresh metavariables. This is the only procedure in our implementation that
“carries contexts around.” The preprocessor only creates asserted justification objects.

Applications \((rs)\) are the main source of unification constraints. After a preterm \(p\) in a context \(\ell\) is converted into the application \((rs)\), the preprocessor uses \texttt{ensurefun} to make sure that the type of \(r\) is of the form \(\Pi x : A, B\), and \texttt{simp} to enforce that the type \(C\) of \(s\) is convertible to \(A\). If \(C\) is not convertible to \(A\), the preprocessor checks the database of available coercions. If there is a coercion \(c\) from \(C\) to \(A\), it replaces the application \((rs)\) with \((r(c)s))\). If \(A\) is stuck, but there are coercions \(\{c_1, \ldots, c_n\}\) from \(C\) to \(A\), the preprocessor creates a fresh metavariable ?\(m\) : \texttt{abstract}\(\Pi \ell A\), replaces the application with \((r(?m\ell))\), and creates a \texttt{ondemand} choice constraint \((?m\ell : A\texttt{in}f,j)\), where the choice function \(f\) produces one of the following alternatives \(s, c_1 s, \ldots, c_n s\). If possible, the solver will only invoke \(f\) after all metavariables in \(A\) have been instantiated. In this ideal situation, \(f\) returns at most one solution, and no case-analysis is needed. The same process is performed when \(C\) is stuck and there are coercions to \(A\). We currently do \texttt{not} try to inject coercions when both \(A\) and \(C\) are stuck at preprocessing time.

As noted in Section 2.6 Lean supports parametric coercions, and coercions to sorts and function classes. Ad hoc overloading is also realized using choice constraints. The idea is the same, but we create a \texttt{regular} choice constraint, where the choice function \(f\) produces the different interpretations for the overloaded symbol.

In a context \(\ell\), a placeholder “ _ “ is simply replaced by ?\(m\ell\), where ?\(m\) is a fresh metavariable.

Finally, to handle implicit arguments, when we infer the type \(t\) of a term \(r\), if \(t\) is of the form \(\Pi\{x : A\}, B\), then we create a fresh metavariable ?\(m\) : \texttt{abstract}\(\Pi \ell A\) and replace \(r\) with the application \((r(?m\ell))\). If the implicit argument is marked with square brackets to indicate it should be synthesized by the type class mechanism, we also create an \texttt{ondemand} choice constraint \((?m\ell : A\texttt{in}f,j)\), where the choice function \(f\) invokes the type class resolution procedure. This procedure is essentially a simple \(\lambda\)-Prolog interpreter [22], where the Horn clauses are the user-declared instances.

### 3.6 The constraint solving procedure

Given a set of constraints, our solver returns a failure, or a substitution \(S\) and set of \texttt{flex-flex} constraints of the form \((?m_1 s \approx ?m_2 t, j)\) such that neither \(?m_1\) nor \(?m_2\) are assigned in \(S\). In other words, it is required to solve all the constraints that are presented to it, but it does not assign metavariables whose solutions are underconstrained.

The solver uses the following data structures:

- a priority queue \(Q\) of constraints,
- a mapping \(U\) of metavariables to constraints,
- a substitution \(S\), and
To simplify the presentation, we assume $Q$, $U$, $S$ and $C$ are global variables.

The priorities for the $Q$ are computed using the following total order, $\prec$, on constraint categories:

\[
\text{pattern} \prec \text{ready} \prec \text{regular} \prec \text{delta} \prec \text{quasi-pattern} \prec \text{flex-rigid} \prec \text{recursor} \prec \text{postponed} \prec \text{flex-flex}
\]

Recall that $\text{ready}$, $\text{regular}$, and $\text{postponed}$ are all choice constraints. If two constraints are in the same category, we use the first-in-first-out method.

The mapping $U$ works as follows: for each metavariable $?m$, $U[?m]$ is the finite subset of the constraints in $Q$ such that for each $c$ in $U[?m]$, $c$ is a unification constraint stuck because of $?m$, or $c$ is an ondemand choice constraint $\langle ?n \ell : t \text{ in } f, j \rangle$ and $?m$ occurs in $t$. The set $U[?m]$ contains the set of constraints that need to be (re-)visited whenever $?m$ is assigned. We remark that a unification constraint in $U[?m]$ may become simpler after we replace $?m$ with its assignment. Similarly, an ondemand choice constraint $\langle ?m \ell : t \text{ in } f, j \rangle$ in $U[?m]$ is ready to be processed when all metavariables in $t$ have been assigned.

Given a set of constraints $s$, for each constraint $c$ in $s$, the procedure $\text{visit } s$ simply invokes $\text{visiteq } c$ if $c$ is a unification constraint, and $\text{visitchoice } c$ otherwise. The procedure $\text{visiteq } \langle r \approx s, j \rangle$ is defined as follows:

\[
\begin{align*}
\text{if } r \text{ or } s \text{ is stuck by some } ?m \text{ and } ?m \mapsto \langle t, j_m \rangle \text{ in } S \\
\quad \text{then visit (simp } \langle r[?m := t] \approx s[?m := t], j \mapsto j_m \rangle) \\
\text{else if the constraint is a pattern } \langle ?m \ell \approx t, j \rangle \text{ then} \\
\quad \text{add the assignment } ?m \mapsto \langle (\text{abstract } \ell t), j \rangle \text{ to } S \\
\quad \text{for each } c \text{ in } U[?m], \text{ visit } (c) \\
\text{else update } U, \text{ and insert constraint into } Q
\end{align*}
\]

The procedure $\text{visitchoice } \langle ?n \ell : t \text{ in } f, j \rangle$ just substitutes any assigned metavariable $?m$ occurring in $t$, updates $U$, and inserts the constraint into $Q$.

To implement a backtracking search, we need a mechanism for restoring the state of the solver during a backtrack operation. We use a very simple approach where $Q$, $U$, and $S$ are implemented using pure data structures (red-black trees) that provide a constant time copy operation. Whenever we need to create a case split, we simply create copies of $Q$, $U$ and $S$. An alternative approach is to use a trail stack [25] which stores operations that “undo” the destructive updates performed during the search. We have determined that our simpler approach for implementing backtracking is not a bottleneck in our implementation.

When solving a non-pattern constraint $c$, the solver creates a case split, and stores it on the stack $C$. Each case split is a tuple of the form $(Q_c, U_c, S_c, j_a, j_c, z)$, where

- $Q_c$, $U_c$ and $S_c$ store the state of the solver when the case split was created,
- $j_a$ is a fresh assumption justification used to track the case split,
• $j_c$ is the justification for $c$, and
• $z$ is a lazy list containing the remaining alternatives, where each alternative is a list of constraints.

We use pull $z$ to denote the operation that destructively extracts the head of the lazy list $z$ and returns it, or returns none when $z$ is empty. The solver catches any error $j$ thrown by the simp procedure, and uses the error resolution procedure resolve $j$ defined as follows:

$$\begin{align*}
\text{while } C \text{ is not empty} \\
\text{let } (Q_c, U_c, S_c, j_a, j_c, z) = \text{top } C \text{ in} \\
\text{if } j \text{ depends on } j_a \text{ then} \\
\text{restore state } Q := Q_c, U := U_c, S := S_c \\
\text{if pull } z = \text{some } a \text{ then visit } (a \bowtie j_a \bowtie j_c) \text{ and return pop } C \\
\text{failed to solve constraints since } C \text{ is empty}
\end{align*}$$

In the procedure above, visit $(a \bowtie j_a \bowtie j_c)$ may throw another error $j'$. If this happens it recursively invokes resolve $j'$.

### 3.7 Processing constraints

At the very core of the algorithm is the procedure for processing the constraints in the queue $Q$, which we now describe. We use an auxiliary procedure process $z j$, where $z$ is a lazy list of alternatives, and $j$ is a justification. If $z$ is empty, it just invokes resolve $j$. Otherwise, it pulls the head $a$ of $z$, creates a fresh assumption justification $j_a$, pushes the new case split $(Q, U, S, j_a, j_c, z)$ on the stack $C$, and invokes visit $(a \bowtie j_a \bowtie j_c)$.

For choice constraints $(?m \ell : t \in f, j)$, whether they are ready, regular or postponed, we just invoke process $(f \langle ?m \ell \rangle t S) j$.

For delta constraints $(f s_1 \ldots s_n \approx t_1 \ldots t_n, j)$, we try two alternatives. In the first one, we assume $f$ is opaque, and try to avoid the potentially expensive δ-reduction step by using $a_1 = \bigcup_{i=1}^n \text{simp} (s_i \approx t_i, j)$. If it fails, as our next alternative, we unfold $f$ and try $a_2 = \text{simp}(\text{unfold} (f s_1 \ldots s_n) \approx \text{unfold} (f t_1 \ldots t_n), j)$. We use the operation tolazy to convert the list $[a_1, a_2]$ into a lazy list, and process the delta constraint using process (tolazy $[a_1, a_2]$) $j$. This case split is a heuristic optimization and is not necessary for completeness.

The two constraint categories quasi-pattern and flex-rigid are handled in the same way; we use different categories only to ensure that easier constraints occur first in the priority queue. We undertake an incomplete search for solutions to these constraints using a variation of the flex-rigid case of Huet's unification algorithm [17]. Given a flex-rigid constraint $(?m s_1 \ldots s_p \approx t, j)$, the main idea behind Huet’s algorithm is the observation that $t$ must be a term of the form $f r_1 \ldots r_n$, where $f$ is a free variable or a constant. The next idea is the observation that any solution for $?m$ is convertible to one in eta-long normal form, which allows us to consider only solutions for $?m$ that are of the form

$$\lambda x_1 \ldots x_n, h (?m_1 x_1 \ldots x_n) \ldots (?m_p x_1 \ldots x_n) \quad (*)$$
where \(?m_i\) are fresh metavariables, and \(h\) is a constant or one of the bound variables \(x_1 \ldots x_n\).

In Huet’s algorithm, only opaque constants are considered, so if \(h\) is a constant different from \(f\) of the rigid term \(t\), the solution would lead to an unsolvable constraint. Therefore, we say that Huet’s procedure has two kinds of case splits: *imitation* (when \(h\) is the constant \(f\) of the rigid term), and *projection* (when \(h\) is one of the bound variables \(x_1 \ldots x_n\)). However, there are two complications in our setting. First, we do not eagerly unfold \(fr_1 \ldots fr_n\) when \(f\) is a constant. For example, assume that \(\text{sub}\ a\ b\) (subtraction for integers) is defined as \(\text{add}\ a\ (\text{uminus}\ b)\). Then \(\langle ?m\ (\text{uminus}\ a) \approx \text{sub}\ b\ a, j \rangle\) has a solution \(?m = \lambda x, \text{add}\ b\ x\), but we would miss it if we did not unfold \(\text{sub}\) before trying to imitate. Second, we have recursors in our language, and even if \(f\) is an opaque constant, it is not the only constant that can be used for \(h\). For example, given the constraints \(\langle ?m\ \text{zero} \approx \text{true}, j \rangle, \langle ?m\ (\text{succ}\ \text{zero}) \approx \text{false}, j \rangle\), a possible solution is \(?m = \lambda x, \text{nat.rec}\ (\lambda n, \text{bool}\ \text{true}\ (\lambda n\ r, \text{false})\ x)\), where \(\text{nat.rec}\) is the recursor for the type \(\text{nat}\) (of the natural numbers). We cope with the first problem using an approach similar to the one used for delta-constraints when \(f\) is a reducible constant. The idea is to have two imitation steps, one where \(f\) is not unfolded, and another one where the term \(fr_1 \ldots fr_n\) is put into weak head normal form before performing the imitation. In our implementation, it is currently infeasible to consider the extra imitation step (after \(\text{whnf}\)) for all constants. Even using non-chronological backtracking, the search space becomes too big. The main problem is that the system may spend a huge amount of time traversing the whole search space when the user provides an incorrect partial construction. As to the second issue, we currently simply ignore this possibility, since the search space would become too big if we considered recursors for \(h\). Moreover, if \(h\) is a recursor, the constraint obtained after replacing \(?m\) would be a stuck recursor.

As in most higher-order unification procedures, we try first the projection case splits because they generate more general solutions. We remark that the number of case splits can usually be greatly reduced for quasi-patterns, which is the case the arises most commonly in practice. In this case, if \(f\) is a constant (not marked as reducible), then we do not need to consider any projections. Any projection would fail immediately: if we take \(h\) to be \(\ell_i\) and substitute (*) for \(?m\) in the original constraint, we obtain an unsolvable constraint \(\langle \ell_i\ (?m_1\ \ell) \ldots (?m_p\ \ell) \approx fr_1 \ldots fr_n, j' \rangle\). Finally, if \(f\) is a free variable \(\ell\), then we only need to consider the projection where \(h\) is \(x_i\) if \(\ell_i = \ell\).

For flex-rigid constraints \(\langle ?m\ s_1 \ldots s_n \approx t, j \rangle\), we only consider the case \(h\) is \(x_i\) when \(s_i\) is a free variable \(\ell_i\), or \(s_i\) is convertible to \(t\). In the second case, where \(s_i\) is convertible to \(t\), we simply assign \(\lambda x_1 \ldots x_n, x_i\) to \(?m\). This is a heuristic for reducing the size of the state, and minimizing the number of instances where the procedure exhibits nonterminating behavior. We note that in the second-order case, the solver does not miss solutions by using this heuristic. Finally, our solver has a threshold on the number of steps that can be performed.

We also use an approximate solution for recursor constraints \(t \approx s, j\).
If the head of \( t \) and \( s \) is the same recursor \( C.\text{rec} \), then we try to solve the constraint by treating \( C.\text{rec} \) as a regular opaque constant which has no computational behavior associated with it. If \( t \) or \( s \) is of the form \( ?m \ r \), then we treat it as a flex-rigid constraint. In a previous implementation of our algorithm, when the recursor \( C.\text{rec} \) was stuck because of a term \( ?m \ r \), we tried to perform a case split for each constructor \( C.\text{mk}_i \) of \( C \), assigning \( ?m \) to terms of the form \( \lambda x. C.\text{mk}_i (?m_1 x) \ldots (?m_n x) \). However, this provides only a minor improvement on the usability of the system: only three theorems in our library broke after we removed this feature, and all of them could be easily fixed by providing implicit arguments explicitly.

4 Related work and conclusions

The elaboration algorithm we have described above has been developed and tuned in conjunction with the development of Lean’s standard and homotopy type theory libraries. Although these libraries are still under development, they provide ample evidence that the approach we describe here is effective in practice. At the time of writing, the standard library consists of about 42k lines of code, with core datatypes including products, lists, sets, multisets (bags), tuples, subtypes, and vectors; core number systems, namely, the natural numbers, integers, rationals, reals, and complex numbers; algebraic structures, including orders, (ordered) groups, (ordered) rings, (ordered) fields, and so on; elementary finite group theory, through Sylow’s theorem; elementary number theory, such as the unique factorization theorem; the beginnings of analysis, including the completeness of the reals and elementary properties of limits. The homotopy type theory library consists of more than 25k lines of code, including most of the first seven chapters of the Homotopy Type Theory book [30], and a substantial development of category theory. Specifically, it includes core datatypes and constructions, such as paths, fibrations, equivalences, and pathovers; higher inductive types, such as the circle, sphere, torus, quotients, pushouts, suspensions; the calculation of the homotopy group of the circle; and category theory through the Yoneda lemma. Lean also supported a substantial development in nonabelian algebraic topology [33], carried out by Jakob von Raumer in the homotopy type theory framework.

We attempt to put our work in the context of recent work on elaboration in dependent type theories. Abel and Pientka present an extension of Miller-style pattern unification [1] which can handle a larger class of problems (in addition to \( \Sigma \)-types) by a method they call pruning, which, intuitively, removes arguments to metavariables which fall outside of the Miller pattern fragment, allowing for more solutions to be found. They also give a bi-directional inference system for a dependently typed \( \lambda \)-calculus, which together with the unification algorithm yields an outline for a practical implementation. They show the soundness of the unification algorithm with respect to this type system. They do not, however, treat the case of defined constants, with or without recursion.

Building upon this is recent work by Ziliani and Sozeau [35] that describes
a unification algorithm for the Coq theorem prover which features defined constants and recursively defined functions. They attempt to describe the practicalities of such an algorithm for a realistic dependently typed language, outlining the heuristics and efficiency compromises inherent in this task. In that respect, their motivations are very similar to ours.

In addition to Abel and Pientka’s pruning, Ziliani and Sozeau add a more aggressive form of dependency erasure for metavariables, in an attempt to solve more unification problems at the cost of uniqueness of solutions. One example is the problem \{ ?t \text{ true} \approx \text{nat}, ?t \text{ false} \approx \text{nat} \}. This problem is solved in their framework by dropping the dependency of ?t on its argument, and returning the constraint ?t’ \approx \text{nat} which gives the solution ?t \mapsto \lambda x, \text{nat}. They also add a resolution rule called \textit{first order approximation}, in which for example the constraint ?f ?y \approx S 0 is solved with the assignment ?f \mapsto S, ?y \mapsto 0

Since we have no qualms about allowing multiple solutions and backtracking search our algorithms can handle both of these problems easily, in the first case by a special case of \textit{projection}, and in the second by an \textit{imitation} step. Our approach to free variables in metavariables is simple: there are none. In contrast, Ziliani and Sozeau carry around a suspended substitution with every metavariable, that needs to be managed in each resolution step. The heuristics outlined in their paper for unfolding constants are similar to ours: constants are unfolded only after an attempt has been made to apply type-class resolution, and constants are unfolded to a pattern match or fixpoint only in last resort. More study is needed to examine the trade-offs of these various choices. Finally, their system does not allow postponement of constraints, relying on pruning and dependency erasure to treat most cases up-front. They argue that great efficiency gains are obtained in this manner. Again, more study is required to assess the trade-offs of this approach.

Various algebraic developments in Coq make use of type classes \cite{27, 28, 14} and \textit{canonical structures} \cite{26, 12, 19}; see also \cite{2} for the use of \textit{unification hints} in Matita. Many of the features we have described are also implemented in systems based on simple type theory. For example, Isabelle uses axiomatic typeclasses \cite{34} and parameterized contexts (locales) \cite{5} to deal with algebraic structures. It also has mechanisms to insert coercions \cite{31}. The reliance on simple type theory, however, makes the elaboration problem quite different from ours. For example, an algebraic structure that depends on a parameter, such as the integers modulo \( m \), cannot be represented as a type, and so cannot be an instance of an axiomatic type class. In contrast to Lean, Isabelle uses different languages to construct expressions and assertions, build proofs, and express relationships between structures.

In a different vein, recent work by Brady on the dependently typed language Idris \cite{8} describes the elaboration process by analogy with theorem proving (and in the context of pure functional programming). Our work is in stark contrast with his, as our tactic language is completely disjoint from the methods with which we specify the constraint resolution for the unification problems. In Lean, the problems are quite different: in unification, metavariables can be very non-local, appearing in disparate contexts and the solutions can be an infinite stream.
rather than a simple finite case split.

To summarize: we have described the elaboration procedure used in the new open source interactive theorem prover Lean \cite{lean}. Our procedure uses methods found in state-of-the-art constraints solvers, such as nonchronological backtracking, indexing, and justification tracking. We have also described how coercions, type classes and ad-hoc polymorphism can be smoothly integrated in our framework using choice constraints. Our procedure has been tested with the development of more than 65k lines of Lean’s formal library, and the experience has shown that it provides powerful and effective support for the formalization process.

References


