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Research Report No. 96-191
July, 1996
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Introduction

In this note we shall construct an effective one-step Church-Rosser conversion strategy F. We wish to emphasize that our strategy is not a reduction strategy since it on occasion expands rather than contracts; however, F is indeed a Church-Rosser strategy since X = Y => there exist n and m such that X \rightarrow F \uparrow n(X) = F \uparrow m(Y) \leftarrow Y. Our strategy only works for combinators, since it makes use of our effective one-step cofinal reduction strategy [3] which only works for combinators; however, it does yield an effective one-step conversion strategy for lambda terms which the reader will easily see.

In short F has the following properties;
1. F is effective
2. either K \rightarrow F(K) or F(K) \rightarrow K
3. if X beta converts to Y then for some n and m
   X \rightarrow F \uparrow n(X) which is identical to F \uparrow m(Y) \leftarrow Y.

Preliminaries

Below '=' denotes beta conversion and '=:' denotes syntactic identity.

A combinator is an applicative combination of S and K. D is a the digraph whose points consist of the combinators and whose lines are defined by the one-step reduction relation X \rightarrow Y. The depth d(X) of a combinator X is defined by
d(S) = d(K) = 1 and d(XY) = \max \{d(X), d(Y)\} + 1.
Church-Rosser

\( D(m) \) is the subgraph of \( D \) induced by \{ \( X : d(X) \leq m \) \} and \( D(X) \) is the weakly connected (i.e. connected in the undirected sense) component of \( D(d(X)) \) containing \( X \). We assume that the combinators have been ordered by \( < \) so that 
\[ d(X) < d(Y) \Rightarrow X < Y. \]
Let \( t(X) \) be the least element of \( D(X) \). In \( [5] \) we defined an effective one-step cofinal reduction strategy \( C \). The \( n \)th iterate of \( C \) on \( X \) is denoted \( C^ \langle n \rangle (X) \). Here we recall that either there are infinitely many \( n \) such that 
\[ X, C(X), C^ \langle 1 \rangle (X), \ldots, C^ \langle n \rangle (X) \]
belong to \( D(C^ \langle m \rangle (X)) \) (these \( C^ \langle n \rangle (X) \) are called the principal reducts of \( X \) ) or there is some \( n \) such that ,for all \( m > n \), \( C^ \langle m \rangle (X) \) belongs to \( D(C^ \langle n \rangle (X)) \) (such a \( C^ \langle n \rangle (X) \) is called a sink for \( X \)). If there is a sink in \( D(X) \) we let \( s(X) \) be the least such sink. Given a reduction sequence \( R = X_1 \rightarrow X_2 \rightarrow \ldots \rightarrow X_n \) we define 
\[ \text{lh}(R) = n, \quad \text{df}(R) = \sum_{i=1}^{n} \max \{ d(X_{i+1}) - d(X_i), 0 \}, \quad \text{wk}(R) = | \{ X_i : t(X_i) > t(X_1) \} |. \]
Now we order the triples 
\[ \text{trip}(R) = (\text{df}(R), \text{wk}(R), \text{lh}(R)) \]
lexicographically and observe that among all the reduction sequences from \( X_1 \) to \( X_n \) there are only finitely many paths \( R \) with \( \text{df}(R) < m \) for any fixed \( m \). This is because any term in such a path has depth at most \( d(X_1) + m \). We shall assume that all of these paths have been ordered by \( < \) so that \( \text{trip}(R_1) < \text{trip}(R_2) \Rightarrow R_1 < R_2 \). Now given \( X \) find \( p(X) \) the least reduction path from \( X \) to a principal reduct of \( t(X) \) or a sink of \( t(X) \) which ever exists. Let \( q(X) \) be the least reduction path from \( X \) to \( s(X) \) if this exists. \( p(X) \) and \( q(X) \) can be effectively constructed from \( X \). Finally we set 
\[ \text{ord}(X) = (t(X), \text{trip}(p(X))) \quad \text{and} \quad \text{ord}'(X) = (s(X), \text{trip}(q(X))) \]
if the latter exists. These quadruples are ordered lexicographically.
The algorithm

We now give the definition of $F(H)$. We assume that we have a Gödel numbering of combinators $X$ such that $d(X) <$ the Gödel number of $X$.

First we determine whether $D(X)$ contains a sink and if one exists we compute $s(H)$ and $q(H)$. In addition, we find a principal reduct or a sink of $t(X)$, $Y$, such that $X \rightarrow Y$. This can be done by iterating $C$ on $t(X)$ while simultaneously enumerating the reduction paths beginning with $X$. By the definition of $C$ ([3]), a sink for $t(X)$ can be effectively recognized. Thus either a sink will be found or a reduction to a principal reduct. Next we find $p(H)$; this can be found by the above remark from any reduction to $Y$. Let

$A_1 = S(KK)(S(SKK)(SKK))(S(KK)(S(SKK)(SKK)))$

(this is just a combinatoric fixed point of $K$)

$A_2 = KK(S(KK)(S(SKK)(SKK)))A_1$

We distinguish several cases.

Case 1; $s(H)$ exists.

In case either $X = Ks(H)(K \text{nfl} 1)$ or $X = K(s(H)) (K \text{nfl} 2)$ we put resp. $F(X) = Ks(H)(K \text{nfl} 1)$ and $F(X) = Ks(H)(K \text{nfl} 2)$. Similarly if $X = s(H)$ we put $F(X) = KA_1$. Otherwise let $H^+$ be the next point on $q(X)$. If $s(H^+) < s(H)$ then put $F(X) = KXN$ for $N$ a combinatoric integer representing the Gödel number of $H$.

Case 2 $s(H)$ does not exist.

In case $X = KC^n(Y)N$ where $N$ is the combinatoric integer representing the Gödel number of $Y$ a principal reduct of $t(X)$ and none of the $C^j(Y)$ for $j = 1, ..., n$ are principal reducts of $t(X)$ then we put $F(X) = KC^n(Y)N$. Otherwise, we distinguish several subcases.

Subcase 1. $\text{lh}(p(H)) = 1$.

If $C(X)$ is a principal reduct of $t(X)$ then we set $F(X) = C(X)$. Otherwise, we set $F(X) = KN$ for
N a combinatory integer representing the Gödel number of X

Subcase 2. \( \ln(p(X)) > 1 \).

Let \( X^+ \) be the next point on \( p(X) \). If \( t(X^+) \leq t(X) \)
then we put \( F(X) = X^+ \) unless \( X = KN \) for \( N \) a
combinatory integer representing the Gödel number of \( X^+ \). In the latter case we put \( F(X) = KXN \).
Otherwise, we set \( F(X) = XN \) where \( N \) is a combinatory
integer representing the Gödel number of \( X \).

A correctness proof

First consider the sequence of iterations of \( F \)
\( X, F(X), F(F(X)), ..., F^n(X), ... \)

We claim that this sequence is unbounded in depth.
Indeed if \( s(Y) \) is defined for any \( Y = F^n(X) \) then for a \( < \)
smallest such \( s(Y) \) we observe that there are two cases.
If \( Y = s(Y) \), \( KS(Y)(K \cdot m1) \), or \( KS(Y)(K \cdot m2) \) then \( F(Y) = KS(Y)(K \cdot m1) \), \( KS(Y)(K \cdot m2) \), or \( KS(Y)(K \cdot (m+1) A1) \) and \( s(F(Y)) = s(Y) \).
Otherwise \( s(F(Y)) = q(F(Y)) < q(Y) \). Thus the
first case eventually comes up and, once it is established
it persists. Otherwise \( s(Y) \) is not defined for any \( Y = F^n(X) \).
Let \( Y1 = F^m(X) \) be such that \( t(Y1) \) is \( < \) smallest
and among those such that \( p(Y1) \) is \( \leq \) least. We claim that
there is some principal reduct of \( t(Y1) \) in the original
iterative sequence. Write \( p(Y1) = Y1 \rightarrow Y2 \rightarrow ... \rightarrow Yk \).
If for some \( i > 1 \) \( t(Y(i+1)) > t(Y1) \), for smallest such \( i \) we have
\( F(Y1) = Y2, F(Y2) = Y3, ..., F(Y(i-1)) = Yi, F(Yi) = KyN \), and \( KyN \)
\( \rightarrow Ky(i+1)N \rightarrow ... \rightarrow KykN \rightarrow Yk \). Thus \( p(F(Yi)) < p(Yi) \)
contradicting the choice of \( Y1 \). Thus for \( j = 1, ..., K-2 \) \( F(Yj) = Yj \) and the principal reduct \( Yk \) of \( t(Y1) \)
is in the original iterative sequence. It is possible that
\( F(Y(k-1)) = Yk \). In this case, \( F(Y(k-1)) = KY(k-1)N \) for \( N \)
the Gödel number of \( Y(k-1) \) and the next principal reduct
of \( t(Y1) \) is a member of the original iterative sequence.
For, if the next principal reduct of \( t(Y1) \) is \( C \cdot r(Yk) \) we have
\[ F(Y(k-1)) = KY(k-1)N, \quad F^2(Y(k-1)) = KYkN, \]
\[ F^3(Y(k-1)) = KC(Yk)N, \ldots, F^{r+2}(Y(k-1)) = C^r(Yk). \]

Once a principal reduct of \( t(V1) \) is found in the original iterative sequence, the sequence of iterates alternates between the first part of case 2 and subcase 1 of case 2 forever. Since \( s(V) \) never exists the sequence must grow unbounded in depth.

Since the sequence of iterates is unbounded in depth there exists an infinite sequence of points \( F^n(t(Y)) \) such that \( K, F(K), \ldots, F^\lceil(n-1)\rceil(K) \) belong to \( D(F^n(t(Y))) \). Let all of these points in order be \( Y1, Y2, \ldots, Yn, \ldots \). Now if there exists a sink \( Z = K \) then there exists some \( Yn \) such that \( Z \) belongs to \( D(Yn) \). Thus \( s(Yn) \) exists and for all but finitely many \( n \)
\( s(F^n(t(Y))) \) is the smallest sink beta convertible to \( K \). Similarly, for all but finitely many \( n \), \( t(F^n(t(Y))) \) is the smallest combinator beta convertible to \( K \).

Finally, if there is some sink \( Z = K \) then, for all but finitely many \( n \), \( F^n(t(Y)) \) alternates between \( KZ(K^{m1}) \) and \( KZ(K^{m2}) \) where \( Z \) is the smallest sink \( = K \). And, if there is no such sink then, for all but finitely many \( n \), \( F^n(t(Y)) \) alternates between the principal reducts \( Y \) of the smallest combinator \( = K \) and the terms \( K(C^m(t(Y)))N \) for \( N \) the combinatory integer representing the Gödel number of \( Y \) (and all but finitely many such \( Y \) are included). It follows that \( F \) is a Church–Rosser strategy.

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