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MAXIMUM ENTROPY STATES AND COHERENT STRUCTURES IN MAGNETOHYDRODYNAMICS

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Abstract

We review a recently developed model of coherent structures in two-dimensional magnetohydrodynamic turbulence. This model is based upon a constrained maximum entropy principle: Most probable states are determined as maximizers of entropy subject to constraints imposed by the conservation of energy, cross-helicity, and flux under the evolution of an ideal two-dimensional magnetofluid. Predictions of the model are compared with results of high-resolution numerical simulations of magnetofluid turbulence.

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1 Introduction and Overview

In this note, we present a model of coherent structures in two-dimensional (2D) magnetohydrodynamic (MHD) turbulence. By a coherent structure, we mean a large-scale organized state that persists amidst the small-scale turbulent fluctuations of the magnetic field and the velocity field. The emergence of such macroscopic states is a dominant feature of large Reynolds number MHD flows, as has been demonstrated by direct numerical simulations. We model these structures as maximizers of an appropriate entropy functional subject to constraints dictated by the conserved quantities of the ideal (nondissipative) MHD equations. Excellent qualitative and quantitative agreement is found with recent numerical simulations of 2D MHD turbulence.

2 Ideal Magnetohydrodynamics

The equations of ideal, incompressible MHD in appropriately normalized variables are:

$$B_t = \nabla \times (V \times B), \tag{1}$$

$$V_t + V \cdot \nabla V = (\nabla \times B) \times B - \nabla p, \tag{2}$$

$$\nabla \cdot B = 0 \quad , \quad \nabla \cdot V = 0, \tag{3}$$

where $B(x,t)$ is the magnetic field, $V(x,t)$ is the velocity field, and $p(x,t)$ is the fluid pressure. Note that p is determined instantaneously in response to the incompressibility constraint on V. These equations are assumed to hold in a regular bounded spatial domain D in R^2, and $x = (x_1, x_2)$ denotes a generic point in D. The magnetic field and the velocity field take values in R^2. Boundary conditions are given by

$$B \cdot n = 0, \ V \cdot n = 0 \quad \text{on } C, \tag{4}$$

where C is the boundary of D and n is the outward normal to C. The model developed below also applies with minor modifications to the case of a fundamental period domain corresponding to periodicity of B and V in x_1 and x_2.

A 2D ideal magnetofluid conserves energy, flux, and cross-helicity. These quantities are given by, respectively,

$$E = \frac{1}{2}\int_D (B^2 + V^2)\, dx, \tag{5}$$

$$F_f = \int_D f(a)\,dx, \tag{6}$$

$$H_f = \int_D B \cdot V f'(a)\,dx. \tag{7}$$

Here a is the vector potential (or flux function), and is defined by the relation

$$B = (a_{x_2}, -a_{x_1}). \tag{8}$$

The vector potential satisfies the boundary condition

$$a = 0 \qquad \text{on } C. \tag{9}$$

The function f in (6) and (7) must satisfy certain regularity (eg. smoothness) conditions, but is otherwise arbitrary. Thus, there are infinite families of conserved flux integrals and cross-helicity integrals. These conserved functionals, which give the dynamics of the 2D magnetofluid its special characteristics, will play a fundamental role in the model sketched below.

3 Macroscopic Description of the MHD System

The high-resolution numerical simulations of Biskamp et al. [1, 2, 3] clearly display the turbulent behavior of a slightly dissipative 2D magnetofluid. As the field-flow state $Y = (B, V)$ evolves, it develops rapid fluctuations on very fine spatial scales. After a short period of time, large scale coherent structures emerge in the form of macroscopic magnetic and kinetic islands. These structures persist for a relatively long time period amidst the turbulent fluctuations before the dissipation causes them to decay. In the ideal limit of vanishing dissipation, we expect that the mixing would continue indefinitely, exciting arbitrarily small spatial scales, and that a turbulent relaxed state, consisting of a large-scale coherent structure and infinitesimal-scale local fluctuations, would be approached. Our main goal is to characterize this final turbulent relaxed state.

The field-flow state Y constitutes a *microscopic* description of the MHD system. Due to its highly intricate small-scale behavior, the microstate Y does not furnish a palpable description of the long-time behavior of the magnetofluid. For this reason, we introduce a coarse-grained, or *macroscopic* description of the system. A macrostate $(\rho(x,y))_{x \in D}$ is a family of local probability densities on the values $y \in R^4$ of the microstate Y at each point x in the domain D. That is, for each x in D, $\rho(x,y)$ represents a joint probability density on the values $y = (b, v)$ of the fluctuating field-flow pair $(B(x), V(x))$. By appealing to the methods of nonlinear analysis, it is possible to show that the macroscopic description ρ may be interpreted as a possible long-time ($t \to \infty$) weak limit of the microscopic field-flow state $Y(x,t)$ (See [4]). We say that $Y(x,t)$ converges weakly to ρ as $t \to \infty$ if for all bounded continuous functions $G(x,y)$ on $D \times R^4$ there holds

$$\int_D \int_{R^4} G(x,y)\rho(x,y)\, dy dx = \lim_{t\to\infty} \int_D G(x, Y(x,t))\, dx.$$

Technically, we may need to pass to a subsequence of times $t_n \to \infty$ in the definition.

4 Constraints on Macrostates

The conservation of energy, flux and cross-helicity under the ideal dynamics translates into corresponding constraints on admissible macrostates. These constraints are formulated in a manner consistent with the above-mentioned weak convergence of $Y(x,t)$ to ρ. They take the forms (see [4, 5] for mathematical details):

$$E(\rho) \equiv \frac{1}{2}\int_D \int_{R^4} (b^2 + v^2)\rho(x,y)\, dy dx = E^0, \tag{10}$$

$$F_f(\rho) \equiv \int_D f(\overline{a}(x))\, dx = F_f^0, \tag{11}$$

$$H_f(\rho) \equiv \int_D \int_{R^4} b \cdot v f'(\overline{a}(x))\rho(x,y)\, dy dx = H_f^0, \tag{12}$$

where E^0, F_f^0, and H_f^0 are the values of energy, flux, and cross-helicity fixed by the initial state of the MHD system; the local mean magnetic field $\overline{B}(x)$ is defined by the relation

$$\overline{B}(x) = \int_{R^4} b\rho(x,y)\, dy, \tag{13}$$

and $\overline{a}(x)$ is the vector potential corresponding to $\overline{B}(x)$. For future reference, we also define the local mean velocity field

$$\overline{V}(x) = \int_{R^4} v\rho(x,y)\, dy \tag{14}$$

We note that both the mean field-flow and the fluctuations contribute to the energy and cross-helicity integrals, whereas only the mean field contributes to the flux integrals. The latter is a consequence of the smoothing property of the operator $B \to a$ [4]. We might say that energy and cross-helicity are cascaded to infinitesimal scales, while flux is cascaded to large scales.

5 Most Probable States: The Maximum Entropy Principle

There have been previous applications of maximum entropy principles to determine most probable states in MHD turbulence. An interesting model along those lines was proposed by Montgomery *et al.* [6]. Our own research into this subject began as an attempt to build upon the ideas presented in [6]. The classical statistical mechanical theory of MHD turbulence, as set forth by Fyfe and Montgomery [7] utilizes a truncated Fourier series representation of the field-flow state, together with a canonical ensemble on the Fourier amplitudes.

Our approach is inspired in part by the recent maximum entropy model of Robert *et al.* [8, 9] for coherent structures in 2D hydrodynamics. A novelty of our model is that it incorporates the complete list of conserved integrals of the ideal MHD dynamics, unlike the above mentioned theories. Another new feature of our model is that it provides a scheme for determining analytical expressions for both the large-scale mean field-flow and the infinitesimal-scale fluctuations inherent in the long-evolved state.

The entropy functional that we use is essentially the classical Gibbs-Boltzmann-Shannon entropy:

$$S(\rho) = -\int_D \int_{R^4} \rho(x,y) \log \rho(x,y)\, dy dx. \tag{15}$$

As such, S is a measure of (the logarithm) of the number of microstates corresponding to the macrostate ρ. Implicit in its definition as an integral over D is the assumption that fluctuations at two separated points in D are statistically independent. A detailed discussion of the rationale behind this assumption is provided in [10].

In accordance with the principles expounded by Jaynes [11], we now determine the most probable macrostate ρ as a maximizer of the entropy (15) subject to the constraints (10)-(12) on energy, flux, and cross-helicity. That is, we solve the constrained entropy maximization problem

$$\textbf{(MEP)} \qquad S(\rho) \to \max, \text{ subject to } E(\rho) = E^0,\ F_f(\rho) = F_f^0,\ H_f(\rho) = H_f^0,$$

where f varies over all (sufficiently smooth) functions on the invariant range of the flux function a.

In [4], a slightly different entropy functional (a Kullback relative entropy functional with a Gaussian reference measure) was employed. However, identical results are obtained with either the entropy used in [4] or the entropy (15) used in the present note. Our maximum entropy formulation may be partially justified by appealing to the theory of large deviations, as was done in [4], or by the methods of [10], in which a discrete system that satisfies a Liouville property was used to approximate the continuous MHD system.

6 Calculation of Equilibrium States

For sake of economy, we consider here the simplified problem (**SMEP**):

$$S(\rho) \to \max,$$

subject to the constraints

$$E(\rho) = E^0,$$

$$F_i(\rho) = \int_D f_i(\overline{a})\, dx = F_i^0, \; i = 1, \ldots, M,$$

$$H(\rho) = \int_D \int_{R^4} b \cdot v \rho(x,y)\, dy dx = H^0.$$

Here, $f_i, i = 1, \ldots, M$, may be chosen from some convenient family of basis functions. Such a discretization of the flux constraints approximates quite accurately the infinite family of constraints [12].

In taking into account only the quadratic cross-helicity constraint, we are simplifying considerably the full statistical equilibrium problem (**MEP**). However, this simplified problem does capture the essence of the correlation effects between the field and the flow that result from the conservation of cross-helicity. For an analysis of the consequences of the complete family of cross-helicity integrals, the reader is referred to [10].

The solution ρ of (**SMEP**) follows from the Lagrange multiplier rule:

$$S'(\rho) = \beta E'(\rho) + \sum \alpha_i F_i'(\rho) + \gamma H'(\rho), \tag{16}$$

where β, α_i, and γ are Lagrange multipliers corresponding to the constraints on energy, flux, and cross-helicity, respectively. The derivatives in (16) are functional derivatives. From (16) it follows that

$$\rho = Z^{-1} \exp(-\beta E'(\rho) - \sum \alpha_i F_i'(\rho) - \gamma H'(\rho)),$$

where $Z(x)$ is the partition function which enforces the normalization constraint

$$\int_{R^4} \rho(x,y)\, dy = 1, \text{ for all } x \text{ in } D.$$

After algebraic manipulations, we arrive at the expression

$$\rho = \frac{\beta^2(1-\mu^2)}{4\pi^2} \exp\left(-\frac{\beta}{2}(1-\mu^2)(b-\overline{B}(x))^2 - \frac{\beta}{2}(v-\mu b)^2\right), \tag{17}$$

where $\mu = -\gamma/\beta$. We note that $-1 < \mu < 1$ (see [4]).

7 Analysis of Equilibrium States

A glance at equation (17) reveals that the most probable macrostate ρ is for each x in D a Gaussian distribution on the field-flow pair $(B(x), V(x))$. On closer inspection we find that Var $B_i(x)$=Var $V_i(x) = 1/(\beta(1-\mu^2))$, corr $(B_i(x), V_i(x)) = \mu$, for $i = 1, 2$, and for each x in D. The other components are uncorrelated. The mean field-flow can be shown to satisfy the equations (see [4])

$$\overline{V}(x) = \mu \overline{B}(x), \tag{18}$$

$$\overline{J}(x) = \sum \lambda_i f_i'(\overline{a}(x)), \tag{19}$$

where

$$\overline{J}(x) = \nabla \times \overline{B}(x) = -\nabla^2 \overline{a}(x),$$

is the current density corresponding to $\overline{B}(x)$, and $\lambda_i = -\alpha_i/(\beta(1-\mu^2))$.

In particular, it follows from (18)-(19) that the mean field-flow is a stationary solution of the ideal MHD equations (1)-(3). The theory predicts, therefore, that the ideal magnetofluid will evolve to a turbulent relaxed state consisting of a stationary mean field-flow (the coherent structure) and Gaussian fluctuations. We also see from (19) that the mean field $\overline{B}$ is a critical point of the (deterministic) magnetic energy, $\frac{1}{2}\int_D B^2\, dx$, subject to the flux constraints, $\int_D f_i(a)\, dx = F_i^0$.

8 Comparison with Numerical Simulations

In general, the predictions of our maximum entropy model are in good agreement with the numerical simulations of Biskamp *et al.* [1, 2, 3]. They observe local Gaussian distributions on the magnetic field and velocity field, and a cascade of flux to large scales, which is indicative of the formation of macroscopic magnetic structures. They also report a cascade of energy to small-scales.

A particularly remarkable prediction of our model is that the ratio of kinetic to magnetic energy in statistical equilibrium is less than 1, regardless of the initial ratio. This follows from straightforward calculations and the fact that the correlation μ satisfies $-1 < \mu < 1$. Indeed, we have for the magnetic energy E_m and the kinetic energy E_k the following expressions

$$E_m = \frac{1}{2}\int_D \int_{R^4} b^2 \rho(x,y)\, dydx = \frac{1}{2}\int_D \overline{B}^2\, dx + \text{volume}\,(D)/(\beta(1-\mu^2)),$$

$$E_k = \frac{1}{2}\int_D \int_{R^4} v^2 \rho(x,y)\, dydx = \frac{\mu^2}{2}\int_D \overline{B}^2\, dx + \text{volume}\,(D)/(\beta(1-\mu^2)).$$

This prediction is also in accord with the numerical studies of Biskamp *et al.* [1, 2, 3], in which they observed the rapid relaxation of E_k/E_m to an almost constant value less than 1, even for initial ratios as large as 25.

For more detailed discussions of the predictions of our model and for further comparisons with the numerical simulations of Biskamp *et al.* [1, 2, 3], the reader is referred to [4, 5, 10].

9 Related Results

The maximum entropy model for 2D MHD turbulence proposed above has been derived by Jordan and Turkington [10] as a continuum limit of a discrete model that utilizes a spatial discretization of the field-flow state $Y(x,t)$. This discrete model is based on a discrete Fourier transform together with an *implicit* canonical ensemble on the discretized variables. We also wish to bring to the attention of the reader the very interesting work of Isichenko and Gruzinov [13], who have obtained results similar to those reported here. Their approach utilizes a canonical ensemble for a truncated spectral representation of the MHD system. A clever rescaling of the inverse temperatures enables them to formally pass to a continuum limit, thereby obtaining statistics that respect the complete set of ideal MHD invariants. Their model also predicts that the ideal magnetofluid will evolve to a state consisting of a stationary coherent structure and Gaussian fluctuations.

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