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A Change of Variables Formula for Mappings in BV

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Abstract

A change of variables formula for mappings in BV is obtained, where the usual jacobian is replaced by the determinant of the approximate differential.

Keywords: functions of bounded variation, maximal function, approximate differential

AMS Classification: 35, 49

1 Introduction

In this note, we use a result of Ambrosio [2] to extend to functions of bounded variation a well known change of variables formula for continuous mappings. In recent years, spaces of discontinuous mappings have proven to be useful in modeling deformations (displacements) of continua which exhibit defects and fracture (for example, see [4], [5], [7]). From the point of view of analysis, the natural space for the underlying deformation (or displacement) is that of functions of bounded variation, BV. Recently, De Giorgi and Ambrosio [9] proposed a smaller subclass for the study of some material instabilities, namely, the space of special functions of bounded variation, SBV.

Given $u \in BV(\Omega, \mathbb{R}^N)$, the functional gradient ∇u is the density of the absolutely continuous part (with respect to $N-$dimensional Lebesgue measure $\mathcal{L}^N$) of the distributional derivative Du. Whereas for a piecewise smooth mapping this functional gradient behaves, locally, like a true gradient, in general it

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may bare no structural resemblance to the classical curl free object. In fact, it has been shown that jump discontinuities may be dense, while ∇u may be *any* integrable tensor field (see [1]).

Suppose that $u \in SBV(\Omega, \mathbb{R}^N)$ denotes the deformation of a body Ω. What information about interpenetration of matter can be inferred from either its full derivative or its functional derivative ∇u? Suppose that we ignore the values of u at the crack site. Does the determinant of the functional derivative give any insight into this question? On the basis of the previous remarks, one is tempted to say no. However, it is well known that the functional derivative is an approximate differential almost everywhere. Using this fact, together with a Lipschitz approximation result due to Ambrosio [2], we prove that this information is still contained in the determinant of the functional gradient, as long as one is willing to disregard the behavior of the mapping on a certain null set, which, in particular, encompasses the crack sites (subset of Ω on which the mapping u experiences jump discontinuities).

2 Preliminaries

Let N be a positive integer, and let Ω be an open, bounded subset of $\mathbb{R}^N$. For μ a (vector-valued) Radon measure, we denote its total variation measure by $||\mu||$. Q is the open unit cube $(-\frac{1}{2}, \frac{1}{2})^N$, and $Q(a,r)$ is the open cube centered at a with side length r, i.e., $Q(a,r) = a + rQ$. Also, $Q_\nu(a,r) := a + rQ_\nu$, where Q_ν is a unit cube centered at the origin with two of its faces normal to ν, $\nu \in S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. As usual, H^{N-1} denotes the $N-1$-dimensional Hausdorff measure.

We state some basic definitions and properties of smooth functions, of functions of bounded variation, BV, and of functions of special bounded variation, SBV. For more details, see Ambrosio [2], Evans and Gariepy [6], Fonseca and Gangbo [8], and Ziemer [10].

Definition 2.1 *Let $\phi : \Omega \to \mathbb{R}^N$ be a continuous function. For $x \in \Omega$ we set*

$$L_\varepsilon(y) := \frac{\phi(x + \varepsilon y) - \phi(x)}{\varepsilon},$$

where $y \in B(0,1)$. Let $L : \mathbb{R}^N \to \mathbb{R}^N$ be a linear mapping.

(i) *L is is said to be the* approximate differential *of ϕ at x, and we write $L = (app)d\phi_x$, if L_ε converges to L in measure on the ball $B(0,1)$, i.e.*

$$\lim_{\varepsilon \to 0} \int_{B(0,1)} (L_\varepsilon(y) - Ly)\varphi(y)\, dy = 0$$

for all $\varphi \in C_0(B(0,1))$.

(ii) *L is called the* weak differential *of ϕ at x if $L = (app)d\phi_x$ and if there exists a sequence $\{\varepsilon_m\}$ converging to 0 when m tends to infinity such that $\{L_{\varepsilon_m}\}$ converges to L uniformly on the sphere S^{N-1}.*

(iii) *If ϕ has a weak differential at $\mathcal{L}^N$ almost every point of Ω, then we say that ϕ is* weakly differentiable *on Ω.*

Definition 2.2 *A function $\phi : \Omega \to \mathbb{R}^N$ is said to satisfy the N-*property *if*

$$\mathcal{L}^N(\phi(E)) = 0$$

for every $E \subset \Omega$ such that $\mathcal{L}^N(E) = 0$.

Next, we introduce the space of *functions of bounded variation.*

Definition 2.3 *A function $u \in L^1(\Omega; \mathbb{R}^N)$ is said to be of* bounded variation, *$u \in BV(\Omega; \mathbb{R}^N)$, if for all $i, j \in \{1, \ldots, N\}$ there exists a finite Radon measure μ_{ij} such that*

$$\int_\Omega u_i(x) \frac{\partial \varphi}{\partial x_j}(x)\, dx = -\int_\Omega \varphi(x)\, d\mu_{ij}$$

for every $\varphi \in C_0^1(\Omega)$. The distributional derivative Du is the matrix-valued measure with components μ_{ij}. We denote by $||Du||$ the total variation of the gradient measure, i.e., $||Du||(\Omega) := \sum_{i=1}^N ||Du_i||(\Omega)$, where

$$||Du_i||(\Omega) := \sup_\varphi \left\{ \int_\Omega u_i \operatorname{div} \varphi\, dx \ : \ \varphi \in C_0^1(\Omega, \mathbb{R}^N), |\varphi|_\infty \le 1 \right\}.$$

We write

$$Du = \nabla u\, \mathcal{L}^N + D_s u,$$

where ∇u is the Radon-Nikodym derivative of Du with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N$, and $D_s u$ and $\mathcal{L}^N$ are mutually singular. Whenever there is no possibility of confusion, we abbreviate $\mathcal{L}^N(A)$ as $|A|$.

We denote by $S(u)$ the complement of the Lebesgue set of u; precisely, $x_0 \notin S(u)$ if and only if there exists $\tilde{u}(x_0) \in \mathbb{R}^N$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{|B(x_0, \varepsilon)|} \int_{B(x_0,\varepsilon)} |u(y) - \tilde{u}(x_0)|\, dy = 0.$$

Clearly $\tilde{u}$ is uniquely determined on $\Omega \setminus S(u)$.

Theorem 2.4 *If $u \in BV(\Omega; \mathbb{R}^N)$ then*

(i) for $\mathcal{L}^N$ a.e. $x \in \Omega$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N+1}} \left\{ \int_{Q(x,\varepsilon)} |u(y) - u(x) - \nabla u(x) \cdot (y-x)|^{\frac{N}{N-1}} dy \right\}^{\frac{N-1}{N}} = 0;$$

(ii) for H^{N-1} a.e. $x \in S(u)$, there exists a unit vector $\nu(x) \in S^{N-1}$, normal to $S(u)$ at x, and there exist vectors $u_-(x), u_+(x) \in \mathbb{R}^N$, such that

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon) : (y-x)\cdot\nu(x) > 0\}} |u(y) - u_+(x)|^{\frac{N}{N-1}} dy = 0,$$

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^N} \int_{\{y \in Q_{\nu(x)}(x,\varepsilon) : (y-x)\cdot\nu(x) < 0\}} |u(y) - u_-(x)|^{\frac{N}{N-1}} dy = 0;$$

(iii) for H^{N-1} a.e. $x_0 \in S(u)$

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{N-1}} \int_{Q_\varepsilon \cap S(u)} |u_+(x) - u_-(x)| dH^{N-1}(x) &= |u_+(x_0) - u_-(x_0)| \\ &> 0, \end{aligned}$$

where $Q_\varepsilon := Q_{\nu(x_0)}(x_0, \varepsilon)$.

In view of the latter theorem, $S(u)$ is called the *singular set*, or *jump set* of u. It is well known that $S(u)$ is $N-1$ rectifiable, i.e.,

$$S(u) = \bigcup_{n=1}^{\infty} K_n \cup \mathcal{N},$$

where K_n is a compact subset of a C^1 hypersurface for each n, and $H^{N-1}(\mathcal{N}) = 0$.

If $u \in BV(\Omega; \mathbb{R}^N)$, then the measure Du may be represented as

$$Du = \nabla u \, \mathcal{L}^N + (u_+ - u_-) \otimes \nu \, H^{N-1} \lfloor S(u) + C(u), \tag{2.1}$$

where $C(u)$ is the so-called *Cantor part*. The three measures in (2.1) are mutually singular: if $H^{N-1}(B) < +\infty$ then $||C(u)||(B) = 0$, and there exists a Borel set E such that $\mathcal{L}^N(E) = 0$ and $||C(u)||(X) = ||C(u)||(X \cap E)$ for all Borel sets $X \subset \Omega$. A function $u \in BV(\Omega, \mathbb{R}^N)$ is said to be of *special bounded variation* if $C(u) = 0$. We write $u \in SBV(\Omega; \mathbb{R}^N)$. This subspace of BV was introduced by De Giorgi and Ambrosio in [9].

Remark 2.5 In view of Definition 2.1 and Theorem 2.4 (i), if $u \in BV(\Omega, \mathbb{R}N)$ then ∇u is an approximate gradient, and so (see [6], 6.13, Theorem 3) given $v \in BV(\Omega, \mathbb{R}^N)$

$$\nabla u = \nabla v \quad \mathcal{L}^N \, a.e. \, x \in \{u = v\}.$$

3 A Change of Variables Formula and Applications

Let μ be a finite, positive, scalar-valued Radon measure on Ω. We define the *maximal function of* μ to be

$$M(\mu)(x) := \sup\left\{\frac{\mu(B_\rho(x))}{|B_\rho(x)|} : 0 < \rho < \operatorname{dist}(x, \partial\Omega)\right\}.$$

This notion was introduced explicitly by Ambrosio [2] (see also [6] Section 6.62 for similar ideas). It can be shown that (see [2])

$$|\{x \in \Omega : M(\mu)(x) > \lambda\}| \leq \frac{C(N)\mu(\Omega)}{\lambda}, \tag{3.1}$$

and so

$$|\{x \in \Omega : M(\mu)(x) = +\infty\}| = 0. \tag{3.2}$$

The proof of (3.1) is a simple consequence of the corresponding result obtained by Ambrosio ([2], Proposition 2.2) in the case where Ω is a ball, together with Besicovitch's Covering Theorem (see [2], Theorem 2.1).

Let $u \in BV(\Omega, \mathbb{R}^N)$ and let

$$H(x) := M(||Du||)(x).$$

Define

$$E_\lambda := \{x \in \Omega : H(x) < \lambda\} \qquad \text{and} \qquad E := \{x \in \Omega : H(x) < \infty\}.$$

Note that H is measurable and, by (3.2), $|\Omega \backslash E| = 0$.

The following result was obtained by Ambrosio in [2] in the case where Ω is a ball, and the proof for an arbitrary open, bounded set Ω may be carried out exactly in the same way.

Theorem 3.1 *Let $u \in BV(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$, $\lambda > 0$. Given $\rho > 0$, define $\Omega_\rho := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \rho\}$. Then there exists a Lipschitz function $u_{\lambda,\rho} : \Omega_\rho \to \mathbb{R}^N$ such that*

$$\tilde{u}(x) = u_{\lambda,\rho}(x) \qquad \text{for every } x \in (\Omega_\rho \cap E_\lambda) \backslash S(u).$$

The Lipschitz constant is bounded above by $C(N)\lambda + \frac{2|u|_\infty}{\rho}$.

We note that the maximal function H is infinite at H^{N-1} a.e. $x \in S(u)$. Hence in the latter theorem $(\Omega_\rho \cap E_\lambda) \backslash S(u)$ reduces to removing from $\Omega_\rho \cap E_\lambda$ a set of H^{N-1} measure zero. See Remark 3.6 for more details.

Next we state a change of variables formula (see [8], Theorem 5.23) for a class of continuous functions, where, for a given set $A \subset \mathbb{R}^N$, $\sharp\{A\}$ stands for the number of elements of A if A is a finite set, and is $+\infty$ otherwise.

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set and assume that $\phi \in C(\Omega;\mathbb{R}^N)$ has a weak differential almost everywhere, it has the N-property, and* $\det\nabla\phi \in L^1_{loc}(\Omega)$. *If $v \in L^\infty(\mathbb{R}^N)$, then for every measurable set $A \subset \Omega$,*

$$\int_A v \circ \phi(x)|\det\nabla\phi(x)|\,dx = \int_{\mathbb{R}^N} v(y)N(\phi, A, y)\,dy, \tag{3.3}$$

where $N(\phi, A, y) = \sharp\{x \in A : \phi(x) = y\}$.

We note that Lipschitz functions are weakly differentiable and satisfy the $N-$ property.

Finally, we state and prove a change of variables formula for BV. Roughly speaking, this is a change of variables formula for a mapping in BV away from the crack site, in the unfractured zone.

Theorem 3.3 *Let $u \in BV(\Omega, \mathbb{R}^N)$, $v \in L^\infty(\mathbb{R}^N)$. Then for any measurable subset $A \subset \Omega$, the function $N(\tilde{u}, (E \cap A)\backslash S(u), \cdot)$ is measurable in $\mathbb{R}^N$, and we have*

$$\int_A v \circ u(x)\,|\det\nabla u|\,dx = \int_{\mathbb{R}^N} v(y)\,N(\tilde{u}, (E \cap A)\backslash S(u), y)\,dy, \tag{3.4}$$

whenever one of the two integrals in meaningful.

Proof. It suffices to prove the result for $v \in L^\infty(\mathbb{R}^N; [0, +\infty))$. First assume that $u \in L^\infty$. Fix $\lambda > 0$ and let u_λ be the Lipschitz function given by Theorem 3.1 with $\rho = \frac{1}{\lambda}$. Let A be a measurable subset of Ω. By Theorem 3.2 and Remark 2.5,

$$\begin{aligned}
\int_{(\Omega_\rho \cap E_\lambda)\cap A} v \circ u(x)\,|\det\nabla u|\,dx &= \int_{((\Omega_\rho \cap E_\lambda)\cap A)\backslash S(u)} v \circ u_\lambda(x)\,|\det\nabla u_\lambda|\,dx \\
&= \int_{\mathbb{R}^N} v(y)N(u_\lambda, ((\Omega_\rho \cap E_\lambda) \cap A)\backslash S(u), y)\,dy \\
&= \int_{\mathbb{R}^N} v(y)\,N(\tilde{u}, ((\Omega_\rho \cap E_\lambda) \cap A)\backslash S(u), y)\,dy.
\end{aligned}$$

By Theorem 5.5 in [8],

$$N(u_\lambda, ((\Omega_\rho \cap E_\lambda) \cap A)\backslash S(u), \cdot) = N(\tilde{u}, ((\Omega_\rho \cap E_\lambda) \cap A)\backslash S(u), \cdot)$$

is measurable. Thus, letting $\lambda \to \infty$ we conclude that $N(\tilde{u}, (E \cap A)\backslash S(u), \cdot)$ is measurable, and by virtue of the Monotone Convergence Theorem we have

$$\int_A v \circ u(x)\,|\det\nabla u|\,dx = \int_{\mathbb{R}^N} v(y)\,N(\tilde{u}, (E \cap A)\backslash S(u), y)\,dy. \tag{3.5}$$

It remains to remove the L^∞ constraint on u. To this end let $u \in BV(\Omega, \mathbb{R}^N)$ be arbitrary and for each positive integer n we define the Lipschitz function

$$\phi_n(x) := \begin{cases} x & \text{if} \quad |x| \leq n \\ n\frac{x}{|x|} & \text{if} \quad |x| > n. \end{cases}$$

Let $A \subset B$ be measurable. For each n, we have

$$\begin{aligned} \int_A v(\phi_n \circ \tilde{u}(x))|\det \nabla(\phi_n \circ u)|dx &= \int_{\mathbb{R}^N} v(y)N((\phi_n \circ \tilde{u}), (E \cap A)\backslash S(\phi_n \circ u), y)dy \\ &= \int_{B(0,n)} v(y)N(\tilde{u}, (E \cap A)\backslash S(u), y)\, dy, \end{aligned}$$

because $\phi_n(\mathbb{R}^N) = \overline{B}(0, n)$. By the chain rule formula, valid for the composition of Lipschitz functions with BV functions (see [3]), and since $\det\nabla\phi_n(y) = 0$ if $|y| > n$, we obtain

$$\int_{A\cap\{|u(x)|\leq n\}} v(u(x))|\det \nabla u(x)|dx = \int_{B(0,n)} v(y)N(\tilde{u}, (E \cap A)\backslash S(u), y)dy.$$

Letting $n \to \infty$ on both sides of the above equation, we conclude (3.4) for arbitrary $u \in BV(\Omega, \mathbb{R}^N)$. ∎

As an immediate consequence of (3.4), we have

Corollary 3.4 *Let $u \in BV(\Omega, \mathbb{R}^N)$. Then $\det \nabla u \in L^1(\Omega)$ if and only if $N(\tilde{u}, E\backslash S(u), y) \in L^1(\mathbb{R}^N)$.*

In particular, if $\det \nabla u$ is not integrable then there is *no* representative of u, $\tilde{u}$, for which $N(\tilde{u}, \Omega, \cdot)$ is bounded and $\mathcal{L}^N(\tilde{u}(\Omega)) < +\infty$. It is well known that there are functions $u \in W^{1,1}(\Omega, \mathbb{R}^N) \subset BV(\Omega, \mathbb{R}^N)$ for which $\det \nabla u$ is not integrable. Corollary 3.4 characterizes this failure in terms of the integrability of the multiplicity function calculated on the "continuous" representative $\tilde{u}$, with domain set $E \setminus S(u)$.

Corollary 3.5 *Let $u \in BV(\Omega, \mathbb{R}^N)$. Assume that $|\det \nabla u(x)| = 1$ for a.e. $x \in \Omega$ and $|\tilde{u}(E\backslash S(u))| = |E\backslash S(u)| = |\Omega|$. Then for any measurable set $A \subset E\backslash S(u)$ we have*

$$|\tilde{u}(A)| = |A|.$$

Proof. By Theorem 3.3, given a measurable set $A \subset E \backslash S(u)$, the function $N(\tilde{u}, A, y)$ is measurable, and so $\tilde{u}(A) = (N(\tilde{u}, A, \cdot))^{-1}(0, +\infty)$ is measurable. Using Theorem 3.3 with $v = 1$ we have

$$\begin{aligned} |A| &= \int_A |\det \nabla u| \, dx \\ &= \int_{\mathbf{R}^N} N(\tilde{u}, A, y) \, dy \\ &\geq |\tilde{u}(A)|. \end{aligned} \tag{3.6}$$

Similarly,

$$|A^c \cap (E \setminus S(u))| \geq |\tilde{u}(A^c \cap (E \setminus S(u)))|. \tag{3.7}$$

If either (3.6) or (3.7) were strict inequalities then

$$\begin{aligned} |\Omega| &\geq |A| + |A^c \cap (E \setminus S(u))| \\ &> |\tilde{u}(A)| + |\tilde{u}(A^c \cap (E \setminus S(u)))| \\ &= |\tilde{u}(E \setminus S(u))|, \end{aligned}$$

contradicting the hypothesis. ∎

Corollary 3.5 implies that the mapping $\tilde{u}$ cannot destroy volume but may create it, depending on whether the N-property holds on the complement of $E \backslash S(u)$.

Remark 3.6 It follows from Theorem 2.4 that, setting

$$\mathcal{R}(u) := \left\{ \begin{array}{l} x \in S(u) : u_+(x), u_-(x) \quad \text{exist}, \\ \qquad \text{and} \lim_{\varepsilon \to 0} \dfrac{(|u_+ - u_-| H^{N-1} \lfloor S(u))(B(x_0, \varepsilon))}{\varepsilon^N} = +\infty \end{array} \right\},$$

then

$$\mathcal{R}(u) \cap E = \emptyset, \quad \text{and} \, H^{N-1}(S(u) \setminus \mathcal{R}(u)) = 0.$$

Therefore, formula (3.4) may be reformulated as

$$\int_A v \circ u(x) \, |\det \nabla u| \, dx = \int_{\mathbf{R}^N} v(y) \, N(\tilde{u}, (E \cap A) \backslash (S(u) \setminus \mathcal{R}(u)), y) \, dy.$$

Given that there is no natural way of extending BV functions to sets of H^{N-1} measure zero, we may rewrite this formula as

$$\int_A v \circ u(x) \, |\det \nabla u| \, dx = \int_{\mathbf{R}^N} v(y) \, N(u^*, E \cap A, y) \, dy,$$

where

$$u^*(x) := \begin{cases} \tilde{u}(x) & \text{if } x \in \Omega \setminus (S(u) \setminus \mathcal{R}(u)) \\ 0 & \text{if } x \in S(u) \setminus \mathcal{R}(u). \end{cases}$$

In order to simplify even further the change of variables formula in such a way (3.4) may be written as (3.3), i.e. $E \cap A$ may be replaced by A, one has to redefine u on the complement of the set E. Clearly, $\mathcal{L}^N(E^c \cup S(u)) = 0$; setting u to be 0 on $E^c \cup S(u)$, and denoting this representative of u by $\hat{u}$, we would obtain

$$\int_A v \circ \hat{u} \, |\det \nabla \hat{u}| \, dx = \int_{\mathbb{R}^N} v(y) \, N(\hat{u}, A, y) \, dy,$$

for any $A \subset \Omega$ measurable, $v \in L^\infty(\mathbb{R}^N)$. However, there may be too much information, present in the original representation $\tilde{u}$, which is lost after performing this redefinition. For example, let u be the Cantor-Vitali function. Here, $\nabla u = 0$, $S(u) = \emptyset$, and the range of u is the entire unit interval. After redefinition, $N(\hat{u}, A, y) = 0$ a.e. for every $A \subset [0,1]$ and $\mathcal{L}^1(\hat{u}([0,1])) = 0$. This is an example where the N-property fails on the complement of E. Indeed, E^c is the Cantor middle third set, and by (3.4) we have

$$0 = \int_{(0,1)} N(\tilde{u}, E, y) \, dy$$

implying that $\mathcal{L}^1(\tilde{u}(E)) = 0$, and so $\mathcal{L}^1(\tilde{u}(E^c)) = 1$.

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