Stronger Cuts from Weaker Disjunctions

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Abstract

We discuss an enhancement of the Balas-Jeroslow procedure for strengthening disjunctive cuts for mixed 0-1 programs. It is based on the paradox that sometimes weakening a disjunction helps the strengthening procedure and results in sharper cuts. When applied to a split cut derived from a source row of the simplex tableau, the enhanced procedure yields, besides the Gomory Mixed Integer cut (GMI), also inequalities that cut deeper in certain directions.

1 Introduction

Consider the q-term disjunction

$$\bigvee_{i \in Q} \left(\sum_{j \in J} a_{ij} x_j \geq a_{i0} \right), \tag{1}$$

where $x_j \geq 0, j \in J$ and $a_{i0} > 0, i \in Q$. It is well known [1] that (1) yields a disjunctive cut of the form $\beta x \geq 1$, with

$$\beta_j = \max_{i \in Q} \left\{ \frac{a_{ij}}{a_{i0}} \right\}. \tag{2}$$

If $x_j \in \mathbb{Z}, j \in J_1 \subseteq J$, and some lower bounds $b_i \geq 0$ for the quantities $\sum_{j \in J} a_{ij} x_j$ are known, i.e. we know that $\sum_{j \in J} a_{ij} x_j \geq b_i$ is satisfied for all feasible solutions and for all $i \in Q$, then monoidal cut strengthening [2] can be applied to obtain a stronger cut $\bar{\beta} x \geq 1$, with

$$\bar{\beta}_j := \begin{cases} \min_{m_j \in M} \max_{i \in Q} \left\{ \frac{a_{ij} + (a_{i0} - b_i) m_j^i}{a_{i0}} \right\} & j \in J_1 \\ \beta_j & j \in J \setminus J_1 \end{cases} \tag{3}$$

where M is the monoid $M := \{m \in \mathbb{Z}^{|Q|} : \sum_{i \in Q} m^i \geq 0\}$. The validity of $\bar{\beta} x \geq 1$ follows from the following proposition

Proposition 1.1. *Any x satisfying $x_j \geq 0, j \in J, x_j \in \mathbb{Z}, j \in J_1$ and (1) such that $\sum_{j \in J} a_{ij} x_j \geq b_i, i \in Q$, also satisfies the disjunction*

$$\bigvee_{i \in Q} \left(\sum_{j \in J_1} (a_{ij} + (a_{i0} - b_i) m_j^i) x_j + \sum_{j \in J \setminus J_1} a_{ij} x_j \geq a_{i0} \right). \tag{4}$$

(See [2] for a proof).

Applying the formula (2) to the disjunction (4) substituted for (1) yields $\bar{\beta}x \geq 1$ with coefficients (3). Now consider a Mixed Integer Program, and let

$$\begin{array}{l} x_k = a_{k0} - \sum_{j \in J} a_{kj} x_j \\ x_j \geq 0, j \in J \\ x_j \in \mathbb{Z}, j \in J_1 \subseteq J \end{array} \tag{5}$$

be a row of the simplex tableau associated with a basic solution to its linear relaxation, where $x_k \in \{0, 1\}$ and $0 < a_{k0} < 1$. The Gomory Mixed Integer (GMI) cut from (5) can be derived as a disjunctive cut from $(x_k \leq 0) \vee (x_k \geq 1)$, or

$$\left(\sum_{j \in J} a_{kj} x_j \geq a_{k0} \right) \vee \left(\sum_{j \in J} (-a_{kj}) x_j \geq 1 - a_{k0} \right) \tag{6}$$

as $\alpha x \geq 1$, with

$$\alpha_j := \max \left\{ \frac{a_{kj}}{a_{k0}}, \frac{-a_{kj}}{1 - a_{k0}} \right\}, j \in J, \tag{7}$$

which can be strengthened to $\bar{\alpha} x \geq 1$ by using the integrality of $x_j, j \in J_1$, with

$$\bar{\alpha}_j := \begin{cases} \min \left\{ \frac{a_{kj} - \lfloor a_{kj} \rfloor}{a_{k0}}, \frac{-a_{kj} + \lceil a_{kj} \rceil}{1 - a_{k0}} \right\} & j \in J_1 \\ \alpha_j & j \in J \setminus J_1. \end{cases} \tag{8}$$

As (6) and (7) are a special case of (1) and (2), the coefficients $\bar{\alpha}_j$ of (8) should be obtainable by monoidal cut strengthening. Indeed, as in this case $b_1 = a_{k0} - 1, b_2 = -a_{k0}$, we have $a_{k0} - b_1 = 1, 1 - a_{k0} - b_2 = 1$, and (8) becomes (9)

$$\bar{\beta}_j := \begin{cases} \min_{(m_j^1, m_j^2) \in M} \max \left\{ \frac{a_{kj} + m_j^1}{a_{k0}}, \frac{-a_{kj} + m_j^2}{1 - a_{k0}} \right\} & j \in J_1 \\ \alpha_j & j \in J \setminus J_1 \end{cases} \tag{9}$$

which is a special case of (3). It is not hard to see that the minimum in the expression for $\bar{\beta}_j, j \in J_1$, is attained for the smaller of $\frac{a_{kj} + \lfloor a_{kj} \rfloor}{a_{k0}}$ and $\frac{-a_{kj} + \lceil a_{kj} \rceil}{1 - a_{k0}}$. A glance at expression (9) suggests that the role of the integers $m_j^i, i = 1, 2$, in strengthening $\bar{\beta}_j$ consists in reducing the value of the larger of the two terms in the brackets while increasing the value of the smaller term. The reduction obtainable in the value of the larger term comes "at the cost" of increasing the value of the smaller term, this limit being enforced by the condition $m_j^1 + m_j^2 \geq 0$. Similarly, in (3) the integers $m_j^i, i \in Q$, can be used to reduce the value of the largest term in brackets "at the cost" of increasing the values of several smaller terms, and the more such terms there are, the lesser the amount by which the value of each term has to be increased in order to offset a given decrease in the value of the largest term. This suggests that from the point of view of monoidal strengthening, there may be an advantage in weakening a disjunction by adding extra terms to it. While a weaker disjunction can only yield a weaker (unstrengthened) cut, applying to such a cut the monoidal strengthening procedure may result in a stronger cut than the one obtained by applying the same strengthening procedure to the cut from the original disjunction. This is what we mean by "stronger cuts from weaker disjunctions".

The next section applies this idea to the simplest type of disjunctive cut, namely the one derived from a split disjunction of the form $x_k \leq \lfloor a_{k0} \rfloor \vee x_k \geq \lfloor a_{k0} \rfloor + 1$, also known as the GMI cut. For this case we formulate a specific cut generating rule which takes maximum advantage of the possibilities outlined above.

2 Lopsided cuts

Consider again a mixed integer program, and let (5) be a row of the simplex tableau associated with a basic solution to its LP relaxation.

Theorem 2.1. *$\alpha^+ x \geq 1$ and $\alpha^- x \geq 1$ are valid cuts, with*

$$\alpha_j^+ := \begin{cases} \frac{-a_{kj}+1}{1-a_{k0}} & j \in J_1^+ := \{j \in J_1 : a_{kj} > 1\} \\ \min\left\{\frac{a_{kj}-\lfloor a_{kj} \rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj} \rceil}{1-a_{k0}}\right\} & j \in J_1^> := \{j \in J_1 : a_{k0} - 1 \leq a_{kj} \leq 1\} \\ \max\left\{\frac{a_{kj}}{a_{k0}}, \frac{-a_{kj}}{1-a_{k0}}\right\} & j \in (J \setminus J_1) \cup \{j \in J_1 : a_{kj} < a_{k0} - 1\} \end{cases} \tag{10}$$

and

$$\alpha_j^- := \begin{cases} \frac{a_{kj}+1}{a_{k0}} & j \in J_1^- := \{j \in J_1 : a_{kj} < -1\} \\ \min\left\{\frac{a_{kj}-\lfloor a_{kj} \rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj} \rceil}{1-a_{k0}}\right\} & j \in J_1^< := \{j \in J_1 : -1 \leq a_{kj} \leq a_{k0}\} \\ \max\left\{\frac{a_{kj}}{a_{k0}}, \frac{-a_{kj}}{1-a_{k0}}\right\} & j \in (J \setminus J_1) \cup \{j \in J_1 : a_{kj} > a_{k0}\} \end{cases} \tag{11}$$

Proof. We give a detailed proof of the validity of the cut $\alpha^+ x \geq 1$. The proof of validity of $\alpha^- x \geq 1$ is analogous. First we notice that if $J^+ \cup \{j \in J_1 : a_{kj} < a_{k0} - 1\} = \emptyset$, then $\alpha^+ x \geq 1$ is the GMI cut, hence valid. If $J_1^+ = \emptyset$ but $\{j \in J_1 : a_k j < a_{k0} - 1\} \neq \emptyset$, then $\alpha^+ x \geq 1$ is still valid, but has weaker (larger) coefficients for $j \in J_1$ such that $a_{kj} < a_{k0} - 1$. On the other hand, if $J_1^+ \neq 0$, then $\alpha^+ x \geq 1$ has negative coefficients, i.e. smaller coefficients than the GMI cut, for all $j \in J_1^+$.
Now, suppose instead of (6) we consider the q-term disjunction

$$\left(\sum_{j \in J} a_{kj} x_j \geq a_{k0}\right) \vee \left(\sum_{j \in J} (-a_{kj}) x_j \geq 1 - a_{k0}\right) \vee \cdots \vee \left(\sum_{j \in J} (-a_{kj}) x_j \geq 1 - a_{k0}\right), \tag{12}$$

where the second term of (6) is repeated $q-1$ times. Adding new terms to a given disjunction in general weakens the latter, hence is a legitimate operation. If the new terms are just replicas of an existing term, then the operation leaves the solution set of the disjunction unchanged. The number q of terms does not affect this reasoning, and will be specified later. Since (12) is a special case of (1), a specialized version of the strengthened inequality $\bar{\beta} x \geq 1$ with coefficients $\bar{\beta}_j$ defined by (3) is valid. In our case, $a_{i0} - b_i = 1$ for all $i \in Q$ in (3), and thus (3) becomes

$$\bar{\beta}_j := \begin{cases} \min_{m_j \in M} \max\left\{\frac{a_{kj}+m_j^1}{a_{k0}}, \frac{-a_{kj}+m_j^2}{1-a_{k0}}, \ldots, \frac{-a_{kj}+m_j^q}{1-a_{k0}}\right\} & j \in J_1 \\ \max\left\{\frac{a_{kj}}{a_{k0}}, \frac{-a_{kj}}{1-a_{k0}}\right\} & j \in J \setminus J_1. \end{cases} \tag{13}$$

Now for $j \in J \setminus J_1$, the coefficient $\bar{\beta}_j$ is clearly the same as in (9), hence the same as α_j^+ of (10) in the Theorem. Furthermore, the coefficient $\bar{\beta}_j$ for $j \in J_1$ such that $a_{k0} - 1 \leq a_{kj} \leq 1$ is easily seen to be identical to the same coefficient in (9), since the extra terms in the expression for $\bar{\beta}_j$ in (13) do not offer any improvement over the choice between the two terms of (9). Hence, setting $m_j^i = 0$ for $i = 3, \ldots, q$, we find that the minimum value for $\bar{\beta}_j, j \in J_1^>$, is attained for the smaller of $\frac{a_{kj} - \lfloor a_{kj} \rfloor}{a_{k0}}$ and $-\frac{a_{kj} + \lceil a_{kj} \rceil}{1 - a_{k0}}$; i.e., in this case also $\bar{\beta}_j$ is equal to α_j^+ of (10).

Consider now the case $j \in J^+$. We want to choose $m_j^i, i = 1, \ldots, q$ so as to minimize the maximum of the expression in the brackets of (13) subject to the sole constraint that the m_j^i are integers whose sum is nonnegative. Assigning a negative value to m_j^1 we can reduce the value of the largest (and only positive) term, while assigning a positive value of 1 to each of $m_j^i, i = 2, \ldots, q$, we can enforce the condition $\sum_{i=1}^q m_j^i \geq 0$ at the cost of a small increase in the value of the other terms. Thus, if we choose $\bar{m}_j^1 = -\min \left\{ p \in \mathbb{Z} : \frac{a_{kj} - p}{a_{k0}} \leq \frac{-a_{kj} + 1}{1 - a_{k0}} \right\}$ we find that $\bar{m}_j^1 = -\left\lceil \frac{a_{kj} - a_{k0}}{1 - a_{k0}} \right\rceil$. Therefore, in order to offset this value by the sum of $m_j^i = 1, i = 2, \ldots, q$, we need the number of such terms to be $-\bar{m}_j^1$, in other words, we need a total of $q = -\bar{m}_j^1 + 1$ terms. But the size of $\bar{m}_j^1$ depends on j, whereas the number q of terms in the expression in the brackets of (13) is independent of j. Therefore q has to be chosen as

$$q = \max_{j \in J_1^+} \left\{ -\bar{m}_j^1 + 1 \right\} = \max_{j \in J_1^+} \left\{ \left\lceil \frac{a_{kj} - a_{k0}}{1 - a_{k0}} \right\rceil + 1 \right\}.$$

With this choice, the number of terms will be redundant for all those $j \in J_1^+$ for which $-\bar{m}_j^1$ is less than the maximum, but this simply means that the redundant m_j^i will have to be set to 0 instead of 1. Choosing for $\bar{m}_j^1$ the value specified above guarantees that $\frac{a_{kj} - \bar{m}_j^1}{a_{k0}} \leq \frac{-a_{kj} + 1}{1 - a_{k0}}$, and therefore the value of the j-th coefficient for $j \in J_1^+$ is $\alpha_j^+ = \frac{-a_{kj} + 1}{1 - a_{k0}}$.

We have proven correct all the coefficients α_j^+ except for those with $j \in J_1$ such that $a_{kj} < a_{k0} - 1$ which are treated like those of the continuous variables, i.e. are not strengthened at all. Therefore they are clearly valid.

This completes the proof of the validity of $\alpha^+ x \geq 1$. The proof of the validity of $\alpha^- x \geq 1$ is analogous, with the disjunction (12) replaced by

$$\left(\sum_{j \in J} (-a_{kj}) x_j \geq 1 - a_{k0} \right) \vee \left(\sum_{j \in J} a_{kj} x_j \geq a_{k0} \right) \vee \cdots \vee \left(\sum_{j \in J} a_{kj} x_j \geq a_{k0} \right) \tag{14}$$

where the second term is replicated $q - 1$ times. In this case $\bar{m}_j^1 = -\min \left\{ p \in \mathbb{Z} : \frac{-a_{kj} - p}{1 - a_{k0}} \leq \frac{a_{kj} + 1}{a_{k0}} \right\}$, which yields $\bar{m}_j^1 = -\left\lceil \frac{a_{k0} - a_{kj} - 1}{a_{k0}} \right\rceil = \left\lfloor \frac{a_{kj} + 1}{a_{k0}} \right\rfloor - 1$, and hence

$$q = \max_{j \in J_1^-} \left\{ -\bar{m}_j^1 + 1 \right\} = \max_{j \in J_1^-} \left\{ - \left\lfloor \frac{a_{kj} + 1}{a_{k0}} \right\rfloor \right\}.$$

□

As stated in the above proof, the coefficient a_j^+ for the case $j \in J_1$ such that $a_{kj} < a_{k0} - 1$ is clearly valid. However, it is also as small as possible. To see this, note that if $a_{kj} < a_{k0} - 1$, then the maximum in (13) is minimized for some $m_j^i \leq 0, i \in \{2, \ldots, q\}$. But any $m_j^i < 0$ would have to be offset by $m_j^1 > 0$. Thus if $m_j^i < 0, i \in \{2, \ldots, q\}$ then m_j^1 has to be greater than $q - 1$; and if at least one $m_j^i, i \in \{2,\dots,q\}$ is set to 0, then $\frac{-a_{kj}}{1-a_{k0}}$ is a lower bound on the value of $\bar{\beta}_j$. Thus coefficients $\bar{\beta}_j$ with $a_{kj} < a_{k0} - 1$ cannot be strengthened in this framework.

We call $\alpha_j^+ x \geq 1$ the *Right lopsided cut* and $\alpha_j^- x \geq 1$ the *Left lopsided cut*.
Our Theorem introduces two classes of cuts that are guaranteed to have negative coefficients for certain indices. However, in order to obtain these stronger coefficients, the standard strengthening of some other coefficients must be given up. The next Corollary identifies the situations in which our new cuts strictly dominate the corresponding GMI cut.

Corollary 2.2. *If $a_{kj} \geq a_{k0} - 1, j \in J_1$ and $J_1^+ \neq \emptyset$, the cut $\alpha^+ x \geq 1$ strictly dominates the corresponding GMI cut. If $a_{kj} \leq a_{k0}, j \in J_1$ and $J_1^- \neq \emptyset$, the cut $\alpha^- x \geq 1$ strictly dominates the GMI cut.*

Example To illustrate the derivation of a Right lopsided cut, consider the following row of a simplex tableau

$$x_k = 0.2 - 1.5x_1 + 0.3x_2 + 0.4x_3 + 0.6x_4 - 4.3x_5 - 0.1x_6. \tag{15}$$

subject to the additional constraints $x_k \in \{0,1\}$ and $x_j \in \mathbb{Z}, j \in J_1 = \{1,\dots,6\}$. The point $\bar{x}_k = 0; \bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \bar{x}_4 = 1; \bar{x}_5 = \bar{x}_6 = 0$ is a feasible integer solution. The GMI cut obtained from this row is

$$0.625x_1 + 0.375x_2 + 0.5x_3 + 0.75x_4 + 0.875x_5 + 0.5x_6 \geq 1. \tag{16}$$

Now we show that applying Theorem 2.1 we obtain a Right lopsided cut that strictly dominates (16). For (15) the index sets $J_1^+, J_1^>$ are respectively $J_1^+ = \{1,5\}$ and $J_1^> = \{2,3,4,6\}$ and the Right lopsided cut coefficients are

$$\begin{array}{lllr} \alpha_1^+ = & \frac{-a_{kj}+1}{1-a_{k0}} = \frac{-0.5}{0.8} & & = -0.625 \\ \alpha_2^+ = & \min\left\{\frac{a_{kj}-\lfloor a_{kj}\rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj}\rceil}{1-a_{k0}}\right\} & = \min\left\{\frac{0.7}{0.2}, \frac{0.3}{0.8}\right\} & = 0.375 \\ \alpha_3^+ = & \min\left\{\frac{a_{kj}-\lfloor a_{kj}\rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj}\rceil}{1-a_{k0}}\right\} & = \min\left\{\frac{0.6}{0.2}, \frac{0.4}{0.8}\right\} & = 0.5 \\ \alpha_4^+ = & \min\left\{\frac{a_{kj}-\lfloor a_{kj}\rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj}\rceil}{1-a_{k0}}\right\} & = \min\left\{\frac{0.4}{0.2}, \frac{0.6}{0.8}\right\} & = 0.75 \\ \alpha_5^+ = & \frac{-a_{kj}+1}{1-a_{k0}} = \frac{-3.3}{0.8} & & = -4.125 \\ \alpha_6^+ = & \min\left\{\frac{a_{kj}-\lfloor a_{kj}\rfloor}{a_{k0}}, \frac{-a_{kj}+\lceil a_{kj}\rceil}{1-a_{k0}}\right\} & = \min\left\{\frac{0.1}{0.2}, \frac{0.9}{0.8}\right\} & = 0.5. \end{array} \tag{17}$$

Therefore we obtain the Right lopsided cut

$$-0.625x_1 + 0.375x_2 + 0.5x_3 + 0.75x_4 - 4.125x_5 + 0.5x_6 \geq 1. \tag{18}$$

Note that (18) is tight for the solution $(\bar{x}_k, \bar{x}_1, \dots, \bar{x}_6)$ while the GMI cut (16) has a slack of 1.25.

In Figure 1 we illustrate graphically for an arbitrary tableau row the value of the cut coefficients for the GMI cut and the two lopsided cuts given by the closed form expressions (8), (10) and (11). The cut coefficients are shown on the vertical axis as a function of the tableau row coefficient values $(-a_{kj})$ shown on the horizontal axis. The graph illustrates the case for $a_{k0} = 0.3$.

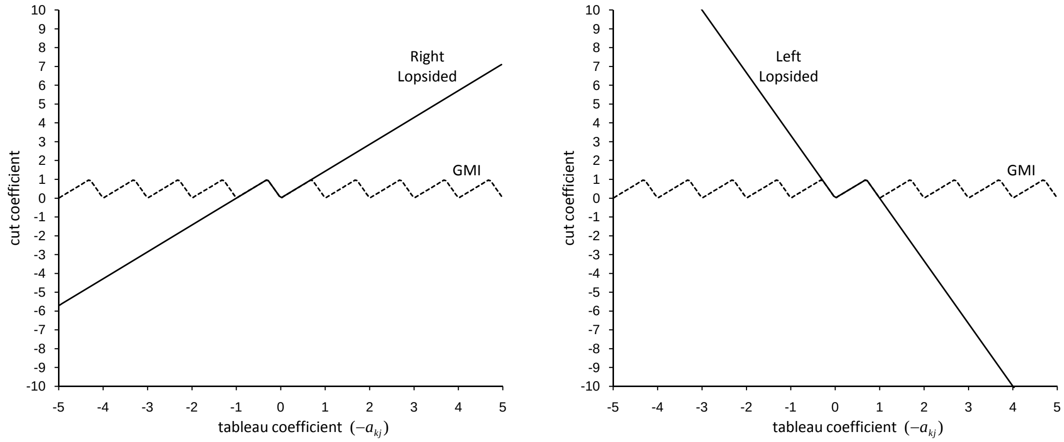

Figure 1: Coefficient values for the GMI and the lopsided cuts $x_k \in \{0;1\}, a_{k0} = 0.3$.

3 Practical considerations

Examining the cut coefficients defined by Theorem 2.1 we see that, if α_j denotes the j-th coefficient of the GMI cut, then

$$\alpha_j^+ \begin{cases} < \alpha_j & \text{for } j \in \{J_1 : a_{kj} > 1\} \\ = \alpha_j & \text{for } j \in \{J_1 : a_{k0} - 1 \leq a_{kj} \leq 1\} \cup \{J \setminus J_1\} \\ > \alpha_j & \text{for } j \in \{J_1 : a_{kj} < a_{k0} - 1\} \end{cases}$$

and

$$\alpha_j^- \begin{cases} < \alpha_j & \text{for } j \in \{J_1 : a_{kj} < -1\} \\ = \alpha_j & \text{for } j \in \{J_1^< := j \in J_1 : -1 \leq a_{kj} \leq a_{k0}\} \cup \{J \setminus J_1\} \\ > \alpha_j & \text{for } j \in \{J_1 : a_{kj} > a_{k0}\}. \end{cases}$$

The following practical questions arise in this context:

1. How often are coefficients with $\alpha_j^+ < \alpha_j$ (or $\alpha_j^- < \alpha_j$) present ?
2. How often does a lopsided cut strictly dominate the corresponding GMI cut ?
3. On average, how does the number of coefficients with $\alpha_j^+ < \alpha_j$ (or $\alpha_j^- < \alpha_j$) compare with the number of those such that $\alpha_j^+ > \alpha_j$ (or $\alpha_j^- > \alpha_j$) ?

To answer these questions, we collected data from the instances of MIPLIB3_C_V2 [6] and MIPLIB 2003 [7]. From the total of 96 instances that had 0-1 variables we removed those that (a) had their LP value equal to their IP value (7), or (b) had no known optimal IP value (4), or (c) had their LP optimum unchanged after adding a full round of GMI cuts (10). For the remaining 75 instances, the above three questions are answered by the following statistics. The averages below are taken over these 75 instances.

1. On average, 42.65% of the potential source rows contain at least one coefficient with $\alpha_j^+ < \alpha_j$, 40.64% contain at least one coefficient with $\alpha_j^- < \alpha_j$, and 48.49% contain at least one coefficient with $\min\{\alpha_j^+, \alpha_j^-\} < \alpha_j$.

2. On average, 3.24% of the potential source rows yield a lopsided cut that strictly dominates the corresponding GMI cut.

3. On average, the number of lopsided cut coefficients with $\alpha_j^+ < \alpha_j$ (or $\alpha_j^- < \alpha_j$) is 14.55% of the total, whereas the number of coefficients with $\alpha_j^+ > \alpha_j$ (or $\alpha_j^- > \alpha_j$) is 19.21% of the total. The number of weakened coefficients on average exceeds the number of improved ones, since improvement requires a coefficient with $a_{kj} > 1$ (or $a_{kj} < -1$), whereas weakening only requires $a_{kj} < a_{k0} - 1$ (or $a_{kj} > a_{k0}$).

These statistics suggest that while the lopsided cuts can be useful on particular problem instances, on average they are not likely to add much to the gap-closing power of the GMI cuts.

We implemented a lopsided cut generator [8] by modifying the GMI cut generator *CglGomoryTwo* code developed by Margot [4, 5]. The GMI cut generator includes a cut validator that discards a GMI cut if some conditions on the cut are not met. In our lopsided cut generator we use the same cut validator as that of *CglGomoryTwo* and it is configured with the same parameters. The code generates a lopsided cut only if $J^+ \cup J^- \neq \emptyset$, namely a Right one if $J^+ \neq \emptyset$ and a Left one if $J^- \neq \emptyset$.

We have used Clp 1.11.0 as Linear Programming Solver from COIN-OR [3]. The machine used is a 64 bit 2.66 GHz Intel Core 2 Duo E7300 processor, 4 GB of RAM, and Linux kernel 2.6.32.

We now present some numerical results meant to show the impact in practice of the new family of cuts. For 57 out of 75 instances our cut generator produced lopsided cuts that were added on top of the GMI cuts. For the remaining 18 instances there were no 0-1 basic variables present in the optimal solution to the LP relaxation or there were no candidate source rows with some entries $a_{kj} > 1 \vee a_{kj} < -1, j \in J_1$.

In Table 1 we compare the GMI relaxation (denoted by G) and the relaxation where both GMI cuts and lopsided cuts are generated (denoted by G+L). In both cases the cuts were applied for only 1 round. To measure the strength of the relaxations we consider the duality gap closed which is computed as

$$\text{Gap} = 100 \frac{C_{opt} - LP_{opt}}{IP_{opt} - LP_{opt}} \tag{19}$$

where IP_{opt}, LP_{opt} and C_{opt} are respectively the value of the optimal integer solution, the value of the linear relaxation and the value of the relaxation currently analyzed. The columns $G_{\#}$ and $G{+}L_{\#}$ indicate the number of cuts generated, $G_{\%}$ and $G{+}L_{\%}$ indicate the gap computed according to formula (19). The column imp shows the difference between $G_{\%}$ and $G + L_{\%}$, and the column imp$_{\%}$ indicates the percentage improvement produced by adding the lopsided cuts on top of the GMI relaxation. Table 1 shows only those instances for which the percentage improvement given by the lopsided cuts exceeds 1%.

On the set of 57 instances the lopsided cuts produce an average percentage improvement of 3.12%. The average number of GMI cuts generated is 99.60 and the average number of GMI+lopsided cuts generated is 239.32. According to Table 1 for the instance l152lav only 9 GMI cuts are used in contrast to 119 GMI+lopsided cuts. For that instance there are 55 candidate source rows of the simplex tableau, each one containing at least one entry $a_{kj} > 1, j \in J_1$ and at least one entry $a_{kj} < -1, j \in J_1$. These rows produce 55 GMI cuts and 110 lopsided cuts but the cut validator discarded 49 of the GMI cuts while it kept all the lopsided cuts.

The computational results show that the lopsided cuts produce some improvement in practice on 1 round of cuts when added on top of GMI cuts. However the benefit is marginal for the majority of instances.

Table 1: Computational results with lopsided cuts

Instance	$G_{\#}$	$G_{\%}$	$G+L_{\#}$	$G+L_{\%}$	imp	$\text{imp}_{\%}$
aflow40b	27	10.60	79	10.76	0.16	1.51
air04	156	8.44	582	8.53	0.09	1.07
blend2	5	15.98	11	16.17	0.19	1.19
dcmulti	45	47.69	53	48.36	0.67	1.40
gesa2	52	25.10	66	26.25	1.15	4.58
harp2	27	22.05	72	22.56	0.51	2.31
l152lav	9	12.80	119	15.18	2.38	18.59
mas76	9	6.36	25	6.53	0.17	2.67
mkc	126	1.83	330	4.25	2.42	132.24
modglob	16	13.32	29	14.05	0.73	5.48
vpm2	21	10.79	36	11.25	0.46	4.26

4 Conclusions

In conclusion, we have shown how sometimes weakening a disjunction can enhance the monoidal strengthening procedure applied to the resulting cut to the extent of producing a stronger cut. For the case of simple disjunctive cuts or GMI cuts, we have given a specific rule which takes maximum advantage of this possibility, and we have specified necessary and sufficient conditions for the resulting cuts to dominate the GMI cuts. We have conducted some computational experiments which show that while on most instances the improvement obtained – if any – is only marginal, on some instances the improvement is significant. Finally, we note that our procedure applies to other, more general types of disjunctive cuts.

References

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